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Each ‘slide’ consists of five to ten separate ‘overlays’, which appear as distinct ‘pages’ in the file. Thus, the document has over 330 pages in total, so printing it would be an enormous waste of paper
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Walrasian Equilibrium Theory

The foundation of modern mathematical microeconomics

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Peterborough, Ontario, Canada
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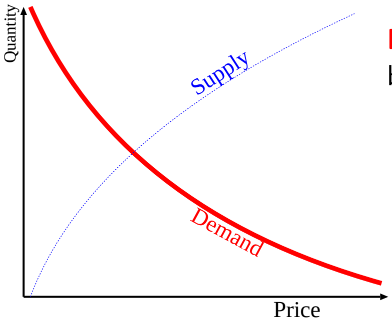
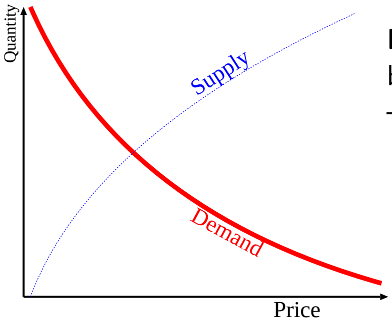
November 8, 2006

Supply and Demand



Demand: total quantity bought by all consumers.

Supply and Demand



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—*Decreasing* function of price.

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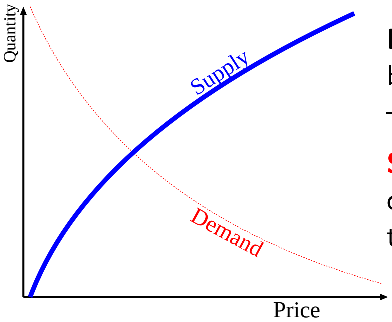

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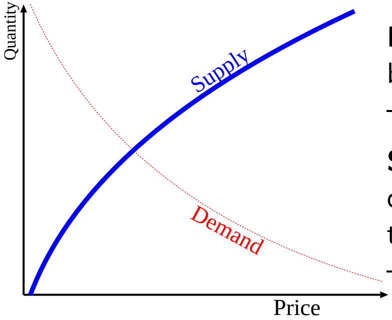

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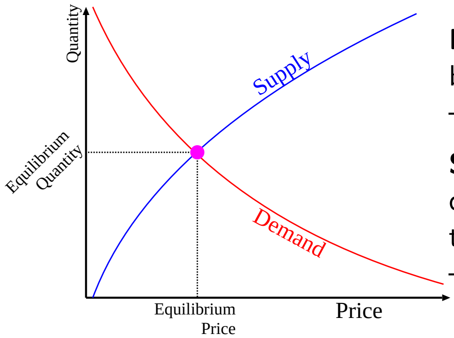

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Thus, there is neither *excess* nor *shortage*. The market 'clears'.

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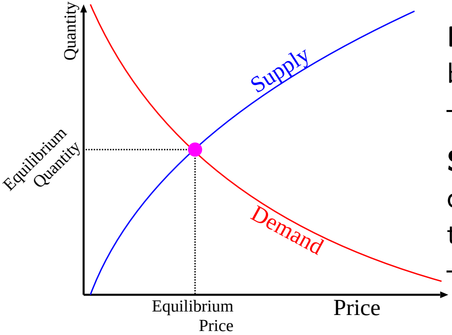

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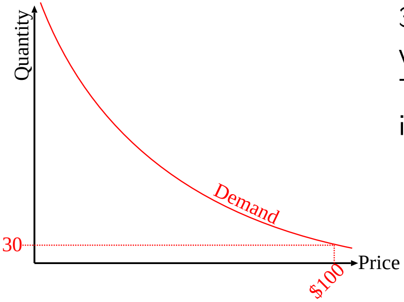
Fundamental assumption of microeconomics:

Markets rapidly converge to equilibrium, and stay there.

Demand curve

Why is demand a *decreasing* function of price?



30 consumers want the product very much.
They are willing to pay $100 for it.

Demand curve

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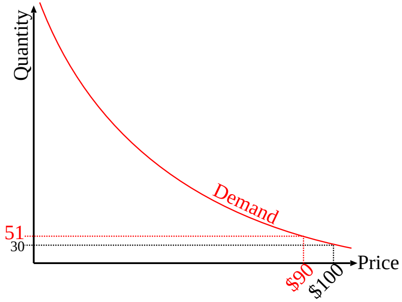
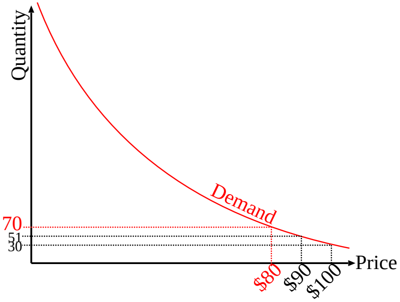


21 more consumers want the product slightly less.
They would buy it for $90

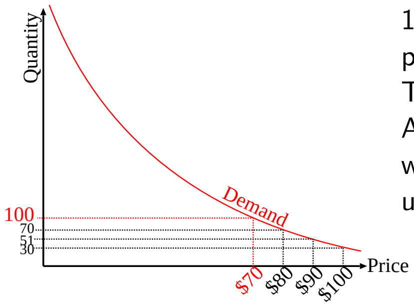
Demand curve

Why is demand a *decreasing* function of price?



19 more consumers want the product slightly less

They would buy it for $80



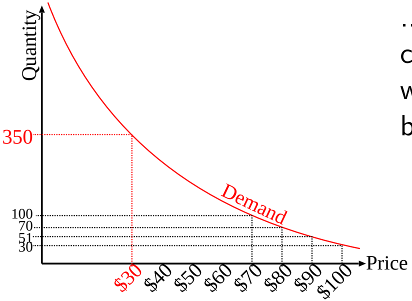
18 more consumers want the product slightly less

They would buy it for \$70.

Also, 12 out of the first 70 would be willing to buy *two* units at this price...

Demand curve

Why is demand a *decreasing* function of price?



....If the price was \$30, then 200 consumers would buy one, 60 would buy *two*, and 10 would buy *three*.

Demand curve

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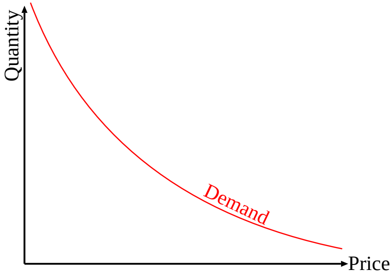
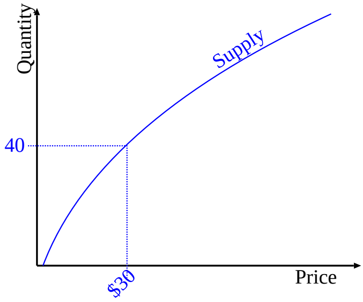


In this way, the ‘demand curve’ emerges from the aggregate purchasing behaviour of millions of consumers.

Supply curve

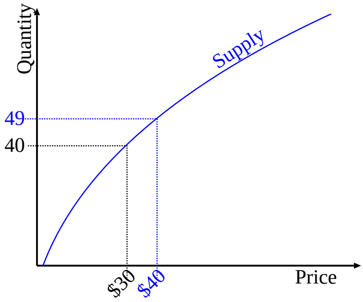
Why is supply an *increasing* function of price?



40 producers are very efficient. They can make the commodity for \$30/unit.

Supply curve

Why is supply an *increasing* function of price?

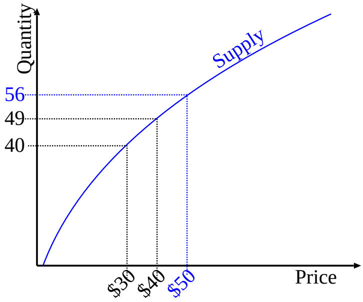


9 more producers are slightly less efficient.
They can make it for \$40/unit.

(They produce nothing if price is lower, because they would lose money)

Supply curve

Why is supply an *increasing* function of price?



7 more producers are even less efficient.

They will make the commodity if the price is at least \$50/unit.

Supply curve

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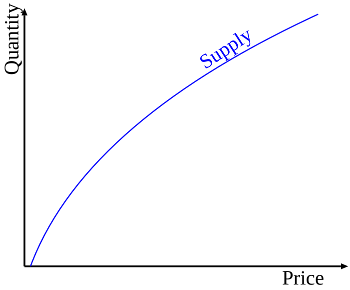
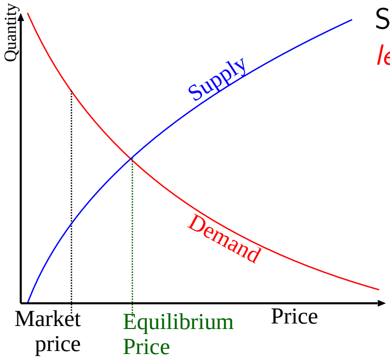


In this way, the ‘supply curve’ emerges from the aggregate production behaviour of hundreds or thousands of producers.

Why does market converge to equilibrium?



Suppose the market price was *less* than the equilibrium price...

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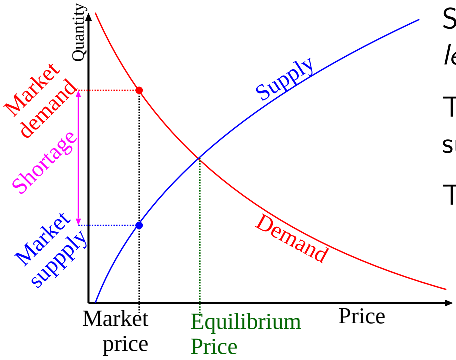

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Then demand would exceed supply...

There would be a *shortage*.

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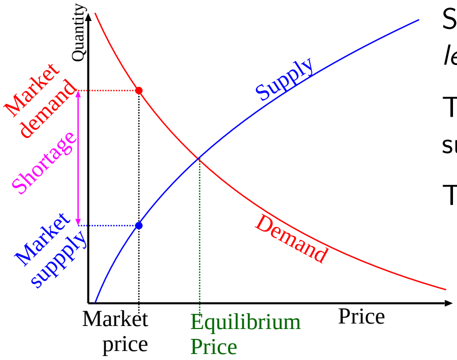

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Some buyers would be willing to pay more than market price for the scarce commodity.

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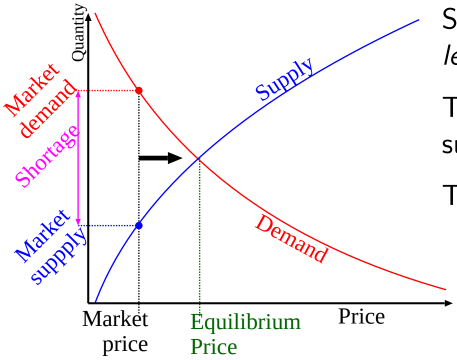

Suppose the market price was *less* than the equilibrium price...

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Some buyers would be willing to pay more than market price for the scarce commodity.

They would 'bid up' the price until it was at equilibrium.

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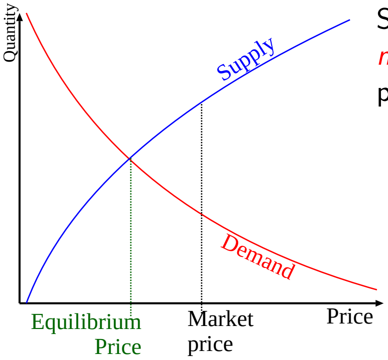

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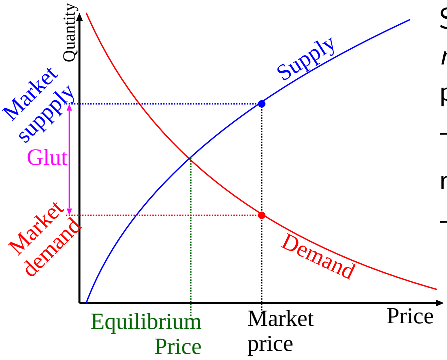
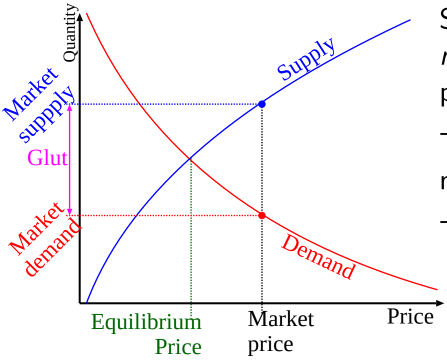


Suppose the market price was *more* than the equilibrium price...

Then supply would exceed demand....

There would be a *glut*.

Why does market converge to equilibrium?



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Some retailers will sell at lower price to clear excess stock.

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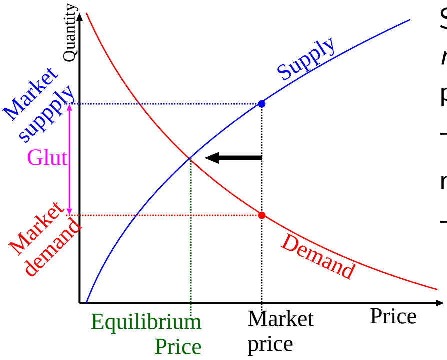

Suppose the market price was *more* than the equilibrium price...

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Less efficient producers will cut back production (or go out of business).

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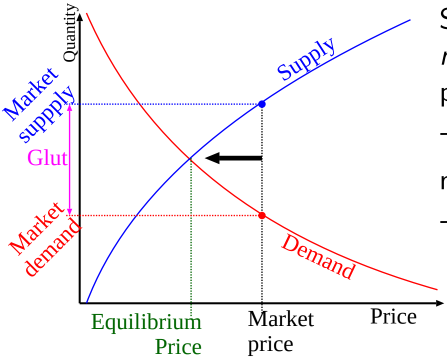

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This would 'drive down' the price until it was at equilibrium.

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  If people consume more margarine, they will consume less butter.

Problems and complications

Complementarities and substitution in commodity prices

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Thus, the price (and hence, consumption) of bread and margarine can affect the demand (and hence, the price) of butter.

A full economic model must account for this.

Consumption of commodity X can affect the supply for commodity Y, if X and Y require common *factors of production*.

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Factor price interactions in commodity prices

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Example: Wheat is a production factor for both bread and pasta.

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  This *decreases* supply (equivalently, raises price) of bread.

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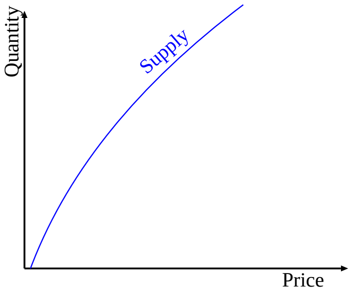
Problems and complications

Wealth effects: Perverse supply curves



Supply should be increasing function of price.

Problems and complications

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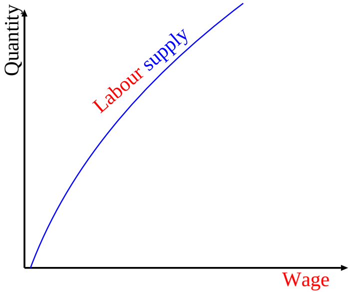
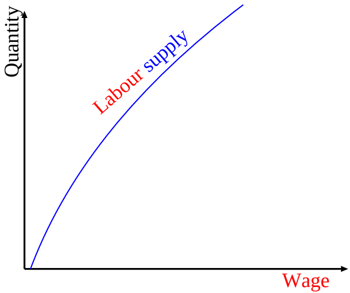


Labour Supply should be increasing function of wage.

Problems and complications

Wealth effects: Perverse supply curves



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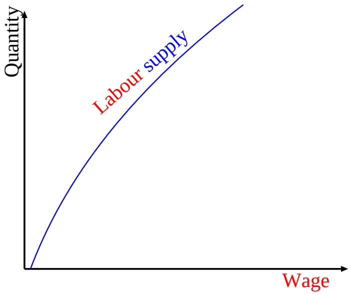

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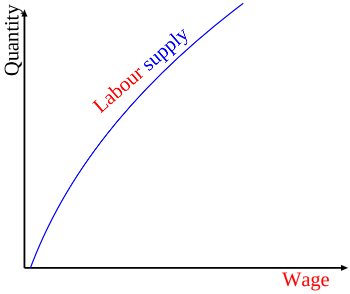

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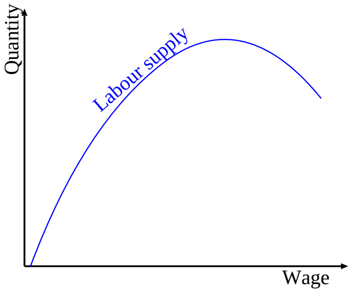

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Thus, labour supply curve turns *down*, due to a wealth effect.

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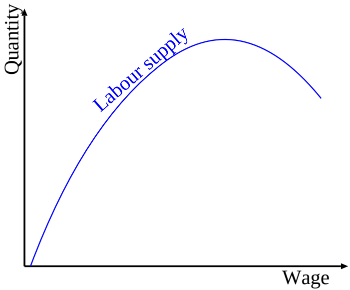

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The Law of One Price

Why should there be a single ‘global’ price for commodity X?

This model assumes the Law of One Price (LOP):

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...until $p_A = p_B$, and the 'arbitrage opportunity' disappears.

Thus, (Law of One Price) $\iff$ (Assumption of Zero Arbitrage).

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'Zero Arbitrage' holds in markets with *perfect information*, *zero transportation costs*, and *zero transaction costs*. (Not the real world).

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Partial solution: the definition of 'commodity *c*' must include *location*, *convenience*, *information vs. uncertainty level*, and *experience qualities* (e.g. 'ambience', 'marketing', 'branding'). For example:

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A proper economic model must represent such product differentiation.

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A production economy has commodities and consumers, and also has producers, called firms (e.g. megacorporations, small businesses, farmers, artisans, freelance consultants, contractors...).

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Commodities: endowments, prices, and trade

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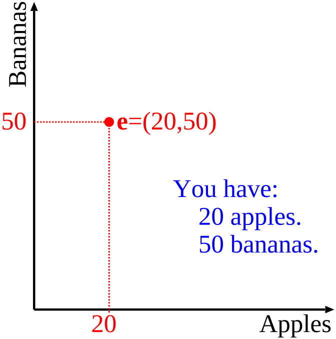

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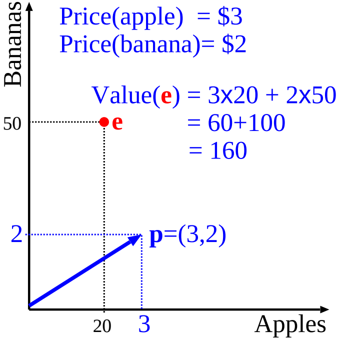

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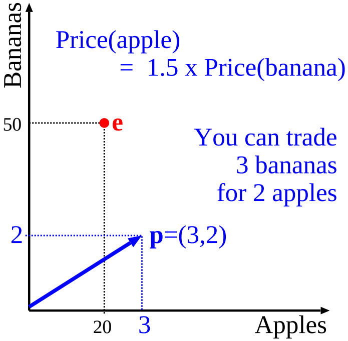

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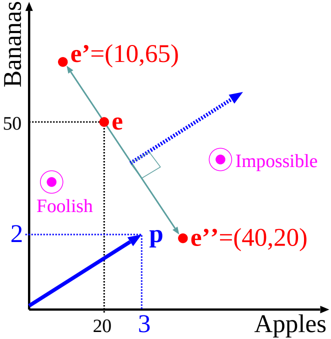

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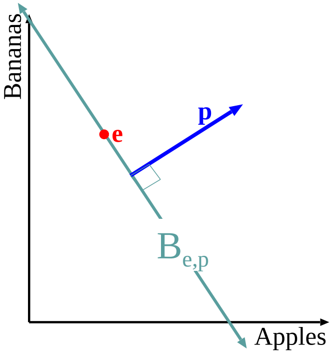

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The set $\mathcal{B}_{\mathbf{e},\mathbf{p}} := \{\mathbf{e}' \in \mathbb{R}^{\mathcal{C}} \; ; \; \mathbf{p} \bullet \mathbf{e}' = \mathbf{p} \bullet \mathbf{e}\}$ is budget hyperplane through $\mathbf{e}$ (orthogonal to $\mathbf{p}$). All trades must occur along $\mathcal{B}_{\mathbf{e},\mathbf{p}}$.

The Consumer: Utility functions and indifference curves

Each consumer i has a (continuous, concave) utility function $u_i : \mathbb{R}^{\mathcal{C}} \longrightarrow \mathbb{R}$, such that

$$\Big(u_i(\mathbf{e}') > u_i(\mathbf{e})\Big) \iff \Big(i \text{ prefers bundle } \mathbf{e}' \text{ to bundle } \mathbf{e}\Big).$$

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For all $r \in \mathbb{R}$, we define the indifference set:

$$\mathcal{S}_r \quad := \quad \left\{\mathbf{e} \in \mathbb{R}^{\mathcal{C}} \; ; \; u_i(\mathbf{e}) = r\right\}.$$

(usually a smooth hypersurface).

The *geometry* of the indifference sets encodes substitute/complement relationships between commodities.

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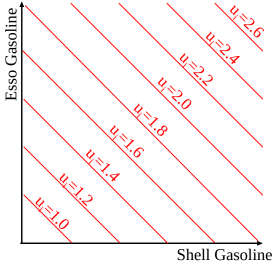

Commodities c and d are perfect substitutes

$\iff$

(c, d) indifference curves are straight lines.

Example: Esso gasoline vs. Shell gasoline.

Formal model: u is *linear function*

$$u(e_c, e_d) \quad = \quad k_c e_c + k_d e_d.$$

for some constants $k_c, k_d > 0$.

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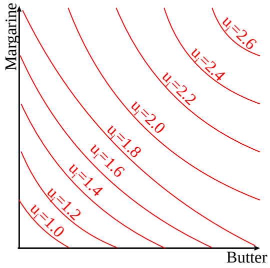

Commodities c and d are good substitutes $\iff$ (c, d) indifference curves are almost linear.

Example: Butter vs. margarine.

Formal model: u is *CES function:*

$$u(e_c, e_d) \quad = \quad (e_c^{\alpha} + e_d^{\alpha})^{1/\alpha}.$$

for some constant $0 < \alpha \leq 1$.

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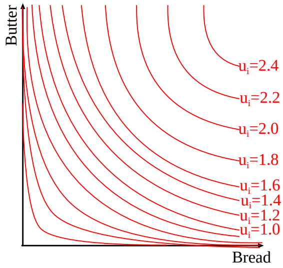

Commodities c and d are complements

$\iff$

(c, d) indifference curves are almost square.

Example: Bread vs. butter.

Formal model: *Cobb-Douglas function:*

$$u(e_c, e_d) \quad = \quad e_c^{\alpha_c} \cdot e_d^{\alpha_d}.$$

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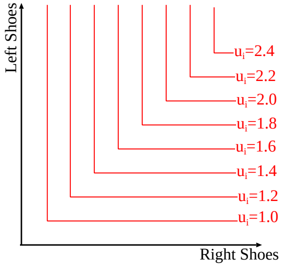

c and d are perfect complements

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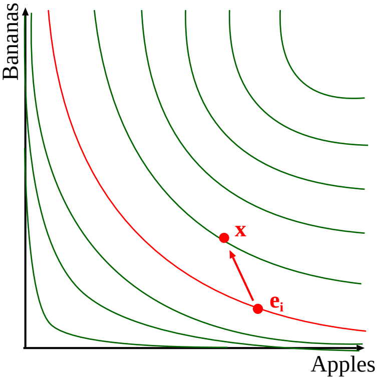
(c, d) indifference curves are square.

Example: Left shoes vs. right shoes.

Formal model: $u(e_c, e_d) = \min\{e_c, e_d\}$.

The Consumer: Utility maximization through trade



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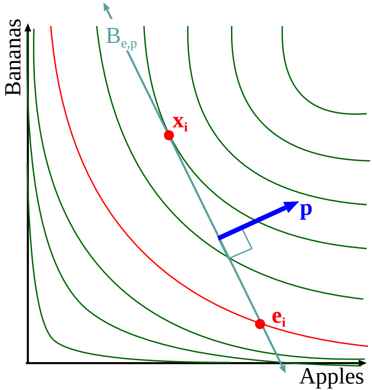

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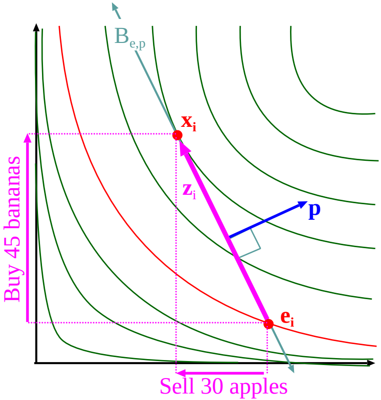

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$\mathbf{x}_i = \mathbf{x}_i(\mathbf{p})$ is i's consumption plan.

$\mathbf{z}_i(\mathbf{p}) := \mathbf{x}_i - \mathbf{e}_i$ is i's excess demand. It encodes how much of each commodity i must buy or sell to maximize her utility at the market price $\mathbf{p}$.

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- Given $\mathbf{p}$, each $i \in \mathcal{I}$ chooses utility-maximizing consumption plan $\mathbf{x}_i(\mathbf{p})$ in her budget set $\mathcal{B}_i(\mathbf{p})$.

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- A set of *initial endowments* $\{\mathbf{e}_i\}_{i\in\mathcal{I}}$ (where $\mathbf{e}_i \in \mathbb{R}^{\mathcal{C}}$ for all $i \in \mathcal{I}$).
- A set of *utility functions* $\{u_i\}_{i\in\mathcal{I}}$ (where $u_i : \mathbb{R}^{\mathcal{C}} \longrightarrow \mathbb{R}$ for all $i \in \mathcal{I}$).

Let $\mathbf{p} \in \mathbb{R}^{\mathcal{C}}$ be a price vector.

- Given $\mathbf{p}$, each $i \in \mathcal{I}$ chooses utility-maximizing consumption plan $\mathbf{x}_i(\mathbf{p})$ in her budget set $\mathcal{B}_i(\mathbf{p})$.
- The aggregate excess demand of the market (at price $\mathbf{p}$) is defined:

$$\mathbf{z}(\mathbf{p}) \quad := \quad \sum_{i\in\mathcal{I}} \Big(\mathbf{x}_i(\mathbf{p}) - \mathbf{e}_i \Big) \,.$$

$\mathbf{z}(\mathbf{p})$ encodes the *net quantity* of every commodity which the entire market demands (or supplies) at price $\mathbf{p}$.

Aggregate excess demand & Walrasian Equilibrium

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Definition: The price vector $\mathbf{p}$ is a Walrasian equilibrium if $\mathbf{z}(\mathbf{p}) = \mathbf{0}$.

If $\mathbf{p}$ is a Walrasian equilibrium, then supply exactly matches demand (i.e. 'the market clears') for every commodity, simultaneously.

Existence of Walrasian Equilibria

Theorem: (Arrow and Debreu, 1954)

- *For all* $i \in \mathcal{I}$*, suppose that the utility function* $u_i : \mathbb{R}^{\mathcal{C}} \longrightarrow \mathbb{R}$ *is:*

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    (i.e. every consumer always prefers more of every commodity —i.e. $\partial_c u_i > 0$ for every $i \in \mathcal{I}$ and $c \in \mathcal{C}$).

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Then a Walrasian equilibrium exists in this economy.

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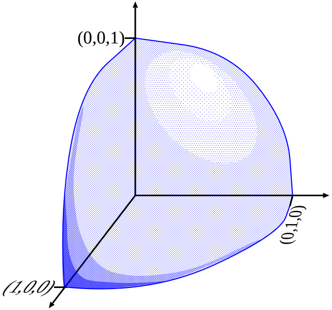

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Consequence: We can always normalize $\mathbf{p}$ to lie on the price semisphere

$$\mathbb{S}_+ \; := \; \left\{ \mathbf{p} \in \mathbb{R}_+^{\mathcal{C}} \; ; \; \sum_{c \in \mathcal{C}} p_c^2 = 1 \right\}.$$

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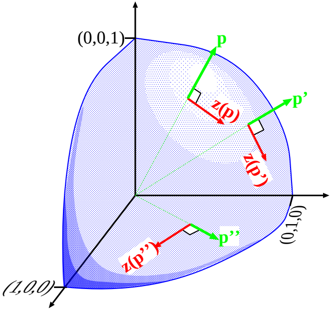

- *0-Homogeneous:* Can assume $\mathbf{p} \in \mathbb{S}_+$ (price semisphere).
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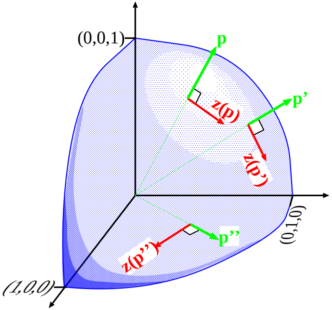

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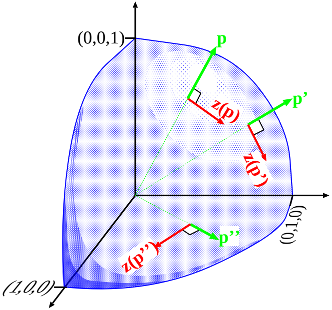

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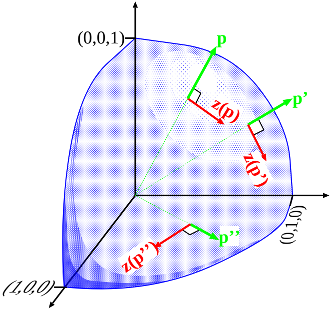

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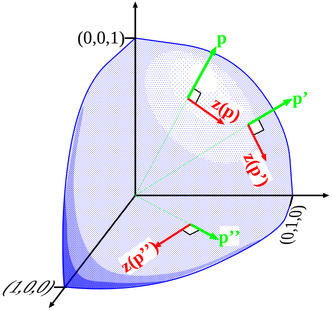

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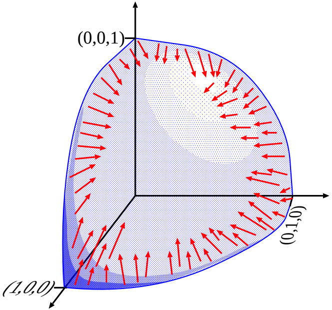

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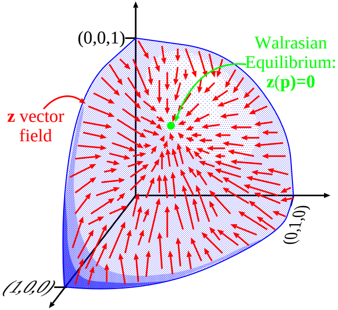

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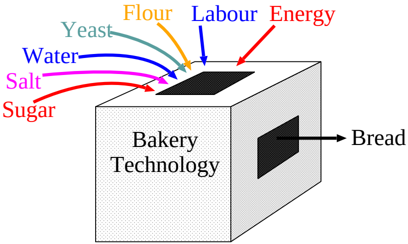
These properties, combined with *Brouwer's Fixed Point Theorem*, imply that $\mathbf{z}(\mathbf{p}^*) = 0$ for some $\mathbf{p}^* \in \mathbb{S}_+$.

This $\mathbf{p}^*$ is a Walrasian equilibrium. $\square$.

The Firm: Technology



- Each firm has a technology, which eats a bundle of commodities ('factors') and yields bundle of some other commodity ('merchandise').

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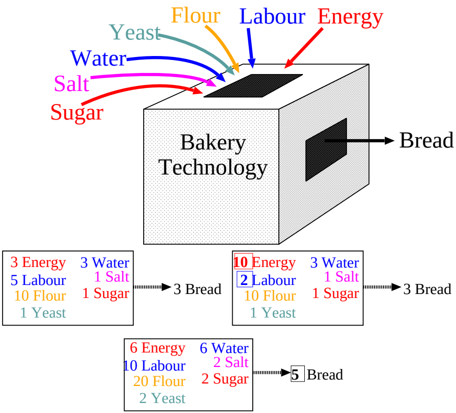

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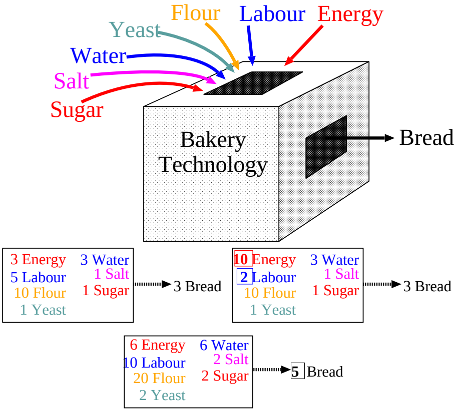

- Each firm has a technology, which eats a bundle of commodities ('factors') and yields bundle of some other commodity ('merchandise').
- Different combinations of factors yield different output quantities of merchandise.
- We describe this with (nonlinear) function $f : \mathbb{R}^{\mathcal{C}}_{-} \longrightarrow \mathbb{R}^{\mathcal{C}}_{+}$.

Suppose $\mathbf{i} = (i_1, \ldots, i_C) \in \mathbb{R}^{\mathcal{C}}_{+}$, and $f(-\mathbf{i}) = \mathbf{o} = (o_1, \ldots, o_C) \in \mathbb{R}^{\mathcal{C}}_{+}$.
This means: if the *input* is i_1 units of factor 1, i_2 units of factor 2, etc....
then *output* is o_1 units of merchandise 1, o_2 units of merchandise 2, etc.
(Note: Factor inputs are measured in *negative* units.)

The Firm: Single-factor, single-output technology

$f: \mathbb{R}_- \longrightarrow \mathbb{R}$. If $f(-i) = o$, then i units of input yield o units of output. Assume $f(0) \leq 0$ ('No Free Lunch'). Define production possibility set:

$$\mathcal{Y} \quad := \quad \{(-i, o) \in \mathbb{R}^2 \; ; \; o \leq f(-i)\}$$

The geometry of $\mathcal{Y}$ describes the 'scale economies' of the technology:

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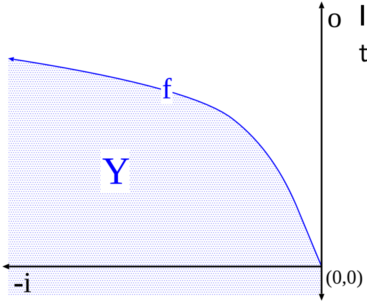

If $\mathcal{Y}$ is convex (e.g. f' is *decreasing*), then get diminishing returns to scale.

- Ubiquitous assumption in most economic models.
- Reasonable model of a single factory.
- Perhaps applies to long-term global economy? ('Fundamental resource constraints').

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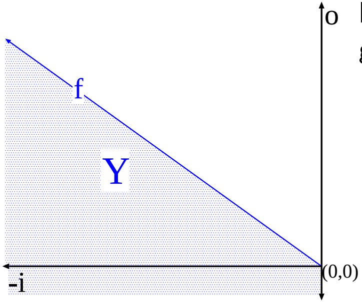

If $\mathcal{Y}$ is a cone (e.g. f is *linear*), then get constant returns to scale.

- Reasonable assumption for model of an entire industry, due to 'replication' argument.

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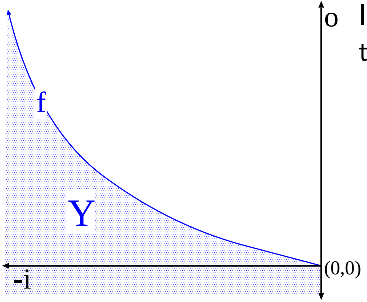

If $\mathcal{Y}$ is concave (e.g. f' is *increasing*) then get increasing returns to scale.

- 'Intellectual property' industries (zero marginal cost of production).
- 'Network effects' (e.g. transport/communication networks, software standards).
- Leads to 'natural monopolies' (e.g. telephone, electricity).

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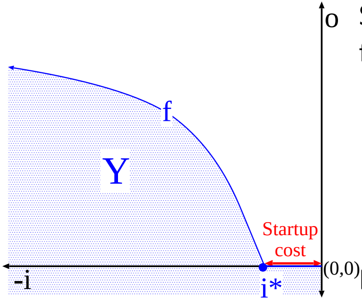

Suppose $\exists\, i^* < 0$ such that $f(i) = 0$ for all $i > i_*$. Then get startup costs.

- Firm must invest minimal amount to produce anything.
- Increasing returns on small scale.
- Complete shutdown is possible.

Example: Electricity for factory lighting.

The Firm: Single-factor, single-output technology

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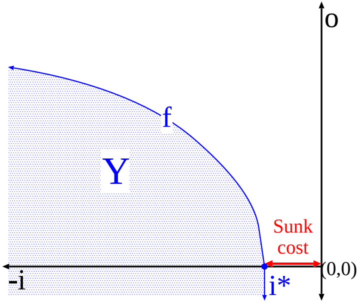


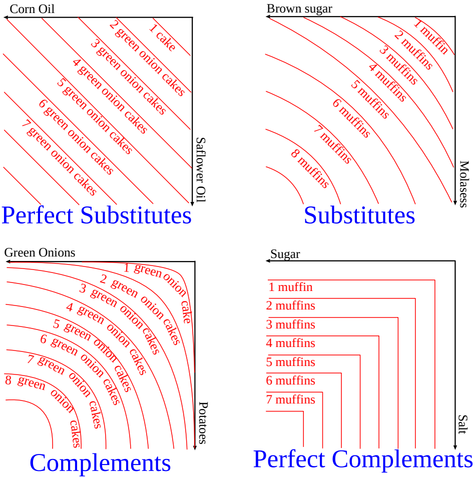
Suppose exists $i^* < 0$ such that $f(i) = -\infty$ for all $i > i_*$. Then get sunk costs.

- Firm must invest minimal amount just to exist.
- Increasing returns on small scale.
- Complete shutdown is impossible.

Example: Construction of factory. Rental of land and equipment.

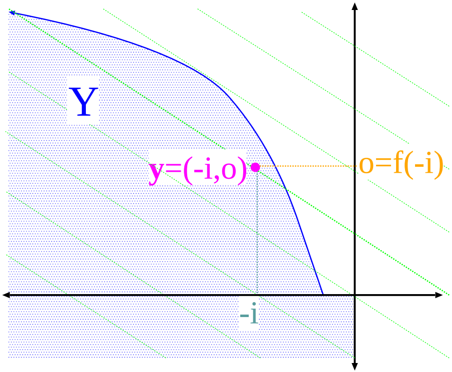
The Firm: Multi-factor, single-output technology

$f : \mathbb{R}^{\mathcal{C}}_{-} \longrightarrow \mathbb{R}$. If $f(-\mathbf{i}) = o$, then input vector $\mathbf{i}$ yields o units of output.
Geometry of f describes *substitution* or *complementarity* between factors:



The Firm: Profit maximization

A production plan is vector $\mathbf{y} := f(-\mathbf{i}) - \mathbf{i} \in \mathbb{R}^{\mathcal{C}}$ (for some $-\mathbf{i} \in \mathbb{R}^{\mathcal{C}}_{-}$).



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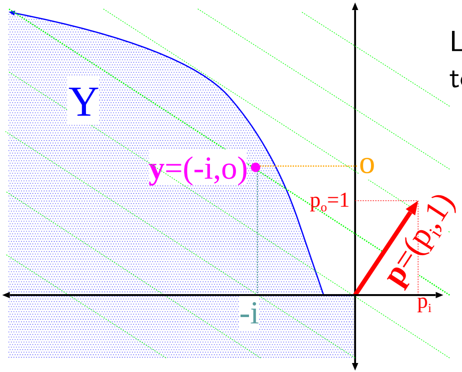

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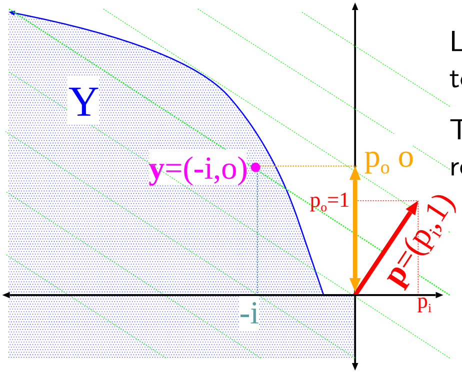

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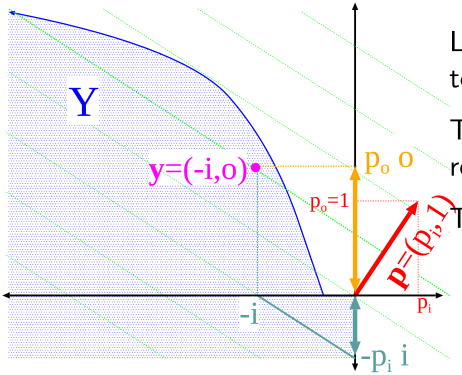

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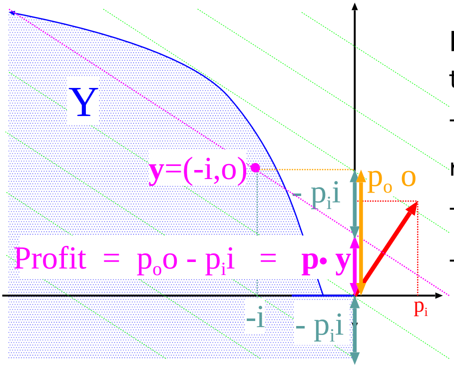

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The Firm: Profit maximization

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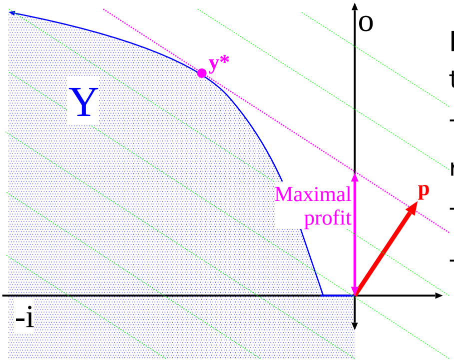

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- Assume firm chooses production plan $\mathbf{y}^*$ which *maximizes profit*.
- At $\mathbf{y}^*$, boundary of production possibility set $\mathcal{Y}$ is *orthogonal** to $\mathbf{p}$.

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  Thus, i's total dividend income is $\sum_{j \in \mathcal{J}} \theta_{ij}\, \mathbf{p} \bullet \mathbf{y}_j^*$.
- If i had initial endowment $\mathbf{e}_i \in \mathbb{R}^{\mathcal{C}}$, then i's effective budget set is now

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Definition: The price vector $\mathbf{p}$ is a Walrasian equilibrium if $\mathbf{z}(\mathbf{p}) = \mathbf{0}$.

If $\mathbf{p}$ is a Walrasian equilibrium, then supply exactly matches demand (i.e. 'the market clears') for every commodity.

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Fact: This game has a *Nash equilibrium*, $\mathbf{p}'$, where each player's action is her best response to the actions of all other players. If $\mathbf{z}' = \mathbf{z}(\mathbf{p})$, then $\mathbf{z}' \bullet \mathbf{p}' = 0$. Thus, $\forall\, c \in \mathcal{C}$, either $z'_c = 0$ or $p'_c = 0$ (excess goods are 'free'). From $\mathbf{p}'$, derive $\mathbf{p}$ with $\mathbf{z}(\mathbf{p}) = 0$ —i.e. $\mathbf{p}$ is a Walrasian equilibrium. $\square$

Remark: The 'tangent field' strategy from 'pure exchange economy' is inapplicable, because *Walras' Law* ($\mathbf{p} \perp \mathbf{z}(\mathbf{p})$) is false.
In general $\mathbf{z}(\mathbf{p}) \bullet \mathbf{p} < 0$ because $\mathbf{y}_j(\mathbf{p}) \bullet \mathbf{p} \geq 0$ for each $j \in \mathcal{J}$
(i.e. the profit-maximizing production of firms 'creates value' in economy.)

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This is called comparative statics.

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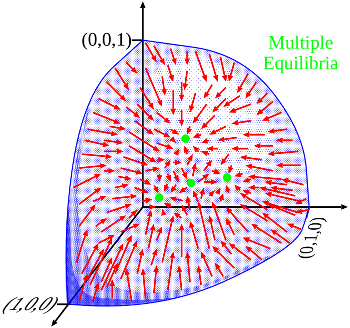

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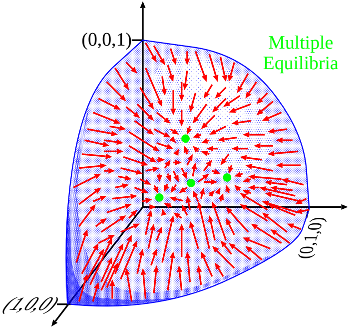

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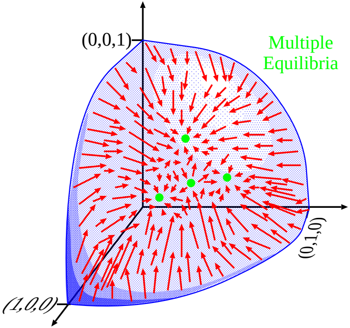

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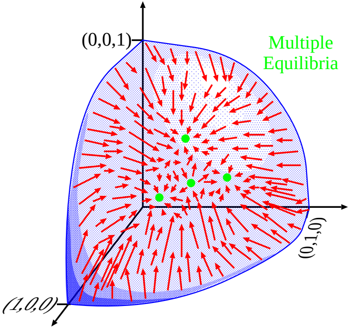

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Depending on initial conditions, market might choose different equilibria.

This is called path-dependence or lock-in.

Walras Problem #2: LOP & Monopolistic Competition

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In such a monopolistic competition, each firm is a *price setter* for its product, not a *price taker*. This contradicts the 'price-taker' assumption of Walrasian model.

Problems with Walrasian equilibria #3: Price Setting?

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Price setting problem: Walras model assumes each consumer and firm is a price taker, who treats market prices as exogenous and 'fixed', and simply chooses her own individual 'best response' to these prices. That is:

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"In the long run, we are all dead." —J.M. Keynes

Extreme 'long run' predictions are irrelevant for crafting economic policy.

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Simplest choice: $\boldsymbol{\alpha}(\mathbf{z}) = \mathbf{z}$.

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Suppose $|\mathcal{I}| \geq |\mathcal{C}|$. *Let* $\epsilon > 0$. *Let* $\mathbb{S}_\epsilon := \{\mathbf{p} \in \mathbb{S}_+ \; ; \; p_c \geq \epsilon, \ \forall c \in \mathcal{C}\}$.
Let $\mathbf{V} : \mathbb{S}_\epsilon \longrightarrow \mathbb{R}^{\mathcal{C}}$ *be any crazy tangent vector field you want.*
There exist 'reasonable' utility functions $u_i : \mathbb{R}^{\mathcal{C}} \longrightarrow \mathbb{R}$ *and initial endowments* $\mathbf{e}_i \in \mathbb{R}^{\mathcal{C}}$ (for $i \in \mathcal{I}$) *such that, if* $\mathbf{z} : \mathbb{S}_+ \longrightarrow \mathbb{R}^{\mathcal{C}}$ *is the resulting excess demand vector field, then* $\mathbf{z}_{|\mathbb{S}_\epsilon} \equiv \mathbf{V}$.

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- Generalizing "$\dot{\mathbf{p}} = \mathbf{z}(\mathbf{p})$" to "$\dot{\mathbf{p}} = \alpha[\mathbf{z}(\mathbf{p})]$" also doesn't work, for 'reasonable' choices of α.

Tâtonnement Problems #2: Nonfeasibility & Rationing

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In short, the whole Walrasian model falls apart outside of equilibrium.

A proper disequilibrium model must describes how scarce goods are 'rationed' in the market, and must describe how consumers and firms react when their consumption/production plans are not feasible.

Tâtonnement Problems #3: One price or many?

Like Walras, the tâtonnement model assumes the *Law of One Price* (LOP):

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A proper disequilibrium model must allow each market agent to assign her own 'price' to each good, and evolve these individual prices over time. (Presumably they eventually converge to a common value.)

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Hahn and Negishi's (1962) model somewhat obviates these problems.

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Explanation: "$c \underset{t}{\sim} d$" means that *some* consumer owns nonzero amounts both of c and d. Economy doesn't 'decompose' into subeconomies with disjoint commodities held by disjoint populations. Thus, 'price comparisons' between commodities are meaningful, because they all trade in same market.

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"Every buyer can always find a seller, if one exists."
(And vice versa).

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Let $\mathbf{Z}(t) := \sum_{i \in \mathcal{I}} \mathbf{z}^i(t)$ be aggregate excess demand of the market at time t.

Finally, the key assumption in Hahn-Negishi models is:

- *Orderly markets*: For all $i \in \mathcal{I}$ and $c \in \mathcal{C}$, if $z_c^i(t) \neq 0$, then $\mathrm{sign}\left[z_c^i(t)\right] = \mathrm{sign}\left[Z_c(t)\right]$.

Explanation: We are assuming it is *impossible* for $z_c^j(t) < 0 < z_c^j(t)$ for some $i \neq j$. Why? Because in an 'orderly market', i and j would meet and trade until at least one of them was satisfied. In other words,

"Every buyer can always find a seller, if one exists."
(And vice versa).

Consequence: If $\mathbf{p}$ is a Walrasian equilibrium, then not only is $\mathbf{Z}(\mathbf{p}) = \mathbf{0}$, but $\mathbf{z}^i(\mathbf{p}) = \mathbf{0}$ for all $i \in \mathcal{I}$.

The Hahn-Negishi Stability Theorem

Assumptions:

(a) *Law of One Price.*

(b) *Tâtonnement price adjustment*: $\dot{\mathbf{p}}(t) \;=\; \boldsymbol{\alpha}\,[\mathbf{p}(t)]$.

(c) *Disequilibrium trade*: $\dot{\mathbf{e}}^i(t) \;=\; \boldsymbol{\tau}^i\left[\mathbf{e}^1(t),\ldots,\mathbf{e}^{\mathrm{I}}(t);\; \mathbf{p}(t)\right]$.

(d) *Dawn of time:* No consumption before equilibrium.

(e) *Pure exchange:* For all $t \in \mathbb{R}_+$, $\sum_{i\in\mathcal{I}} \dot{\mathbf{e}}^i(t) \;=\; 0$.

(f) *No swindling:* For all $t \in \mathbb{R}_+$, $\mathbf{p}(t) \bullet \dot{\mathbf{e}}^i(t) \;=\; 0$, for all $i \in \mathcal{I}$.

(g) *Indecomposability:* $\forall\; c, d \in \mathcal{C}$, write $c \underset{t}{\sim} d$ if $\exists\; i \in \mathcal{I}$ such that $e^i_c(t) \neq 0 \neq e^i_d(t)$. Then $(\mathcal{C}, \underset{t}{\sim})$ is a connected graph for all $t \in \mathbb{R}_+$.

(h) *Orderly markets*: For all $i \in \mathcal{I}$ and $c \in \mathcal{C}$, if $z^i_c(t) \neq 0$, then $\operatorname{sign}\left[z^i_c(t)\right] \;=\; \operatorname{sign}\left[Z_c(t)\right]$.

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This guarantees convergence to a *local minimum* of $U(t)$. This will be an equilibrium. $\square$

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With additional technical assumptions, Fisher proved a Stability Theorem for this more general model.

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Other disequilibrium models include Uzawa's (1962) 'Edgeworth trading' model and Benassy's (1984, 2002) 'quantity-constrained rationing' model.

Sonnenschein-Mantel-Debreu Theorem

Recall: In a 'pure exchange economy', we can assume $\mathbf{p} \in \mathbb{S}_+$ (the 'price semisphere'), because $\mathbf{z}$ is homogeneous of degree zero.
Then $\mathbf{z}(\mathbf{p})$ is *tangent* to $\mathbb{S}_+$ (by Walras' law).
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There exist utility functions $u_i : \mathbb{R}^{\mathcal{C}} \longrightarrow \mathbb{R}$ *and initial endowments* $\mathbf{e}_i \in \mathbb{R}^{\mathcal{C}}$ (for $i \in \mathcal{I}$) *such that, if* $\mathbf{z} : \mathbb{S}_+ \longrightarrow \mathbb{R}^{\mathcal{C}}$ *is the resulting excess demand vector field, then* $\mathbf{z}_{|\mathbb{S}_\epsilon} \equiv \mathbf{V}$.

Proof background for SMD Theorem

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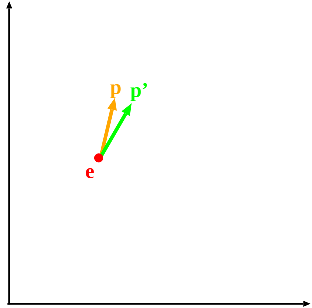

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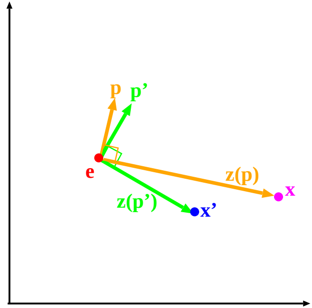

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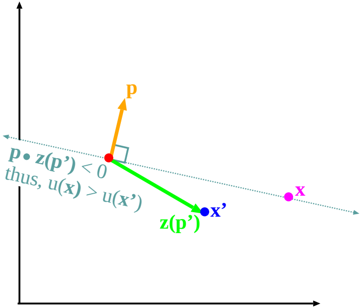

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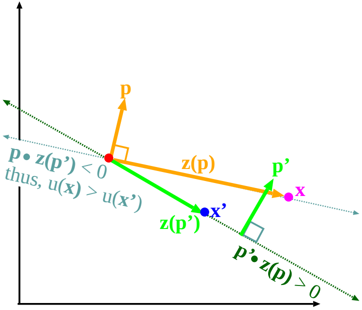

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Proof background for SMD Theorem

Let $\mathbf{z} : \mathbb{S}_+ \longrightarrow \mathbb{R}^{\mathcal{C}}$ be some tangent vector field on $\mathbb{S}_+$.

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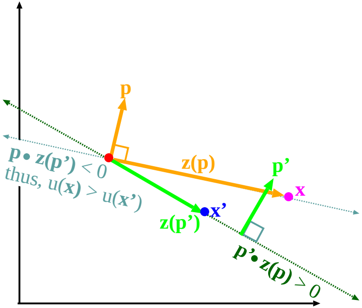

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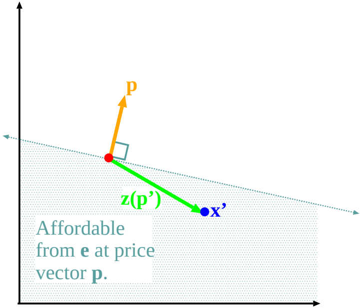

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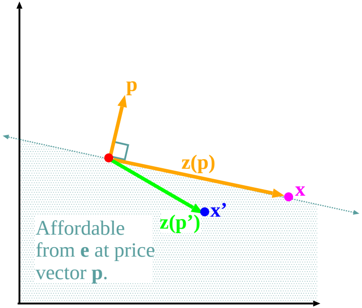

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Proof background for SMD Theorem

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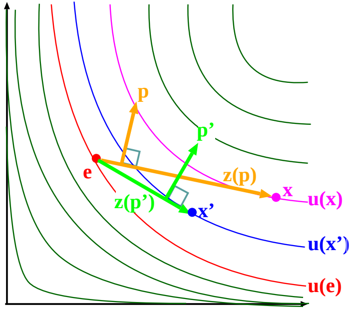

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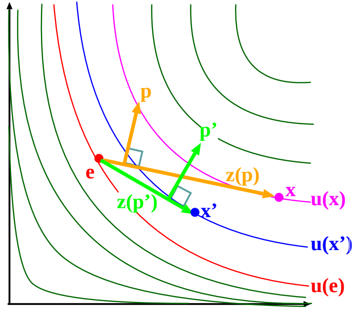

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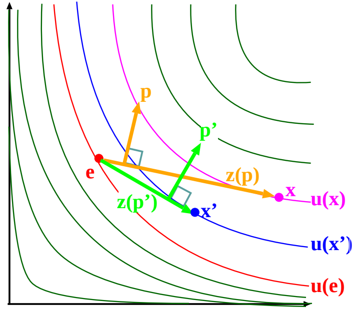

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Proof background for SMD Theorem

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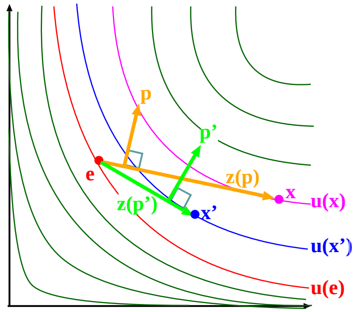

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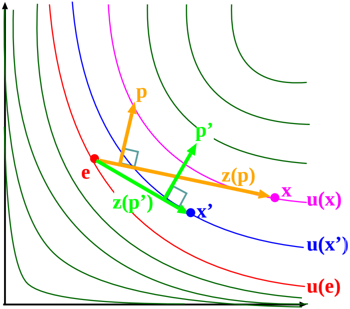

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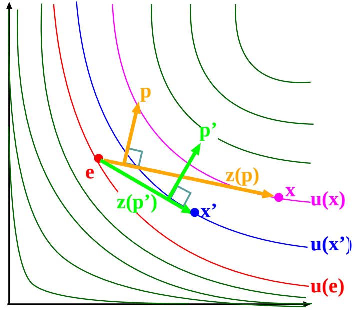

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Proof background for SMD Theorem

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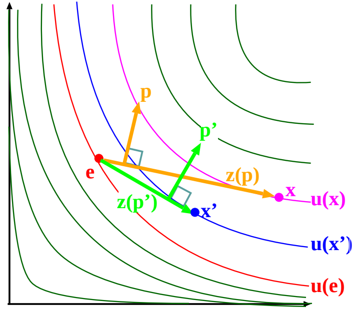

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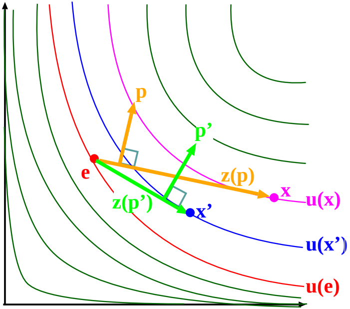

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Note: Part (c) says that the relation "$\succ$" is asymmetric.

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SARP Theorem: *$\mathbf{z}$ is the excess demand vector field for some utility function u and endowment $\mathbf{e}$ if and only if $\mathbf{z}$ satisfies SARP.*

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Say $\mathbf{z}$ is *proportionally injective* (PI) if, for all $\mathbf{p}, \mathbf{p}' \in \mathbb{S}_+$,

$$\Big(\mathbf{p} \neq \mathbf{p}'\Big) \implies \Big(\mathbf{z}(\mathbf{p}) \text{ and } \mathbf{z}(\mathbf{p}') \text{ are linearly independent.}\Big)$$

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Say $\mathbf{z}$ is *proportionally injective* (PI) if, for all $\mathbf{p}, \mathbf{p}' \in \mathbb{S}_+$,

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Let $\mathbf{z} : \mathbb{S}_+ \longrightarrow \mathbb{R}^{\mathcal{C}}$ be tangent vector field on $\mathbb{S}_+$. Recall:

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Corollary: *$\mathbf{z}'$ is also the excess demand field for some u' and $\mathbf{e}'$.*

(because of SARP Theorem.)

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- Thus, $\mathbf{z} = r_1\mathbf{z}_1 + \cdots + r_C\mathbf{z}_C$ is the aggregate excess demand field for this set of consumers, as desired.

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Lemma 1 says $\mathbf{z}_c$ satisfies SARP $\ (\mathbf{z}_c(\mathbf{p})$ is u_c-maximizing choice for all $\mathbf{p}$).

Also, $\mathbf{z}_c$ is proportionally injective (exercise). $\Diamond_{\text{Claim 1}}$.

Claim 2: *There exist continuous, scalar functions* $r_1, r_2, \ldots, r_C : \mathbb{S}_\epsilon \longrightarrow \mathbb{R}_+$ *such that, for all* $\mathbf{p} \in \mathbb{S}_\epsilon$, $\quad \mathbf{z}(\mathbf{p}) \; = \; r_1(\mathbf{p})\mathbf{z}_1(\mathbf{p}) + \cdots + r_C(\mathbf{p})\mathbf{z}_C(\mathbf{p})$.

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Thus, $\exists\ R > 0$ such that, for all $c \in \mathcal{C}$ and $\mathbf{p} \in \mathbb{S}_\epsilon$, $\ z_c(\mathbf{p}) + Rp_c > 0$.

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$$\begin{aligned}
\sum_{c\in\mathcal{C}} r_c \mathbf{z}_c(\mathbf{p}) &= \sum_{c\in\mathcal{C}} (z_c(\mathbf{p}) + Rp_c)(\mathbf{e}_c - p_c\mathbf{p}) \\
&= \sum_{c\in\mathcal{C}} z_c(\mathbf{p})\mathbf{e}_c + \sum_{c\in\mathcal{C}} Rp_c\mathbf{e}_c - \sum_{c\in\mathcal{C}} z_c(\mathbf{p})p_c\mathbf{p} - \sum_{c\in\mathcal{C}} Rp_c p_c \mathbf{p} \\
&= \mathbf{z}(\mathbf{p}) + R\mathbf{p} - \underbrace{(\mathbf{z}(\mathbf{p}) \bullet \mathbf{p})}_{=0 \text{ (Walras)}} \mathbf{p} - R \underbrace{\|\mathbf{p}\|_2^2}_{=1\ (\mathbf{p}\in\mathbb{S})} \mathbf{p} \\
&= \mathbf{z}(\mathbf{p}) + R\mathbf{p} - R\mathbf{p} = \mathbf{z}(\mathbf{p}).
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as desired. $\Diamond_{\text{Claim 2}}$.

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This completes the proof of the Sonnenschein-Mantel-Debreu Theorem. $\Box$.

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This is a major shortcoming.
'Real' economies are subjected to constant shocks.
Thus, 'real' economies are *often* (*usually*? *always*?) in disequilibrium.

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It cannot predict *which* equilibrium will be chosen, how the economy will *get to* this equilibrium, how long this process might take, and how the economy will function in the mean time.

This is a major shortcoming.
'Real' economies are subjected to constant shocks.
Thus, 'real' economies are *often* (*usually*? *always*?) in disequilibrium.

Needed: A mathematically rigorous, all-encompassing model of *disequilibrium dynamics* in microeconomics.

Further reading

These slides: `http://xaravve.trentu.ca/pivato/Teaching/walras.pdf`

Microeconomics (Walrasian Equilibria, etc.)

- Andreu Mas-Colell, Michaeal D. Whinston, and Jerry R. Green, *Microeconomic Theory*. (1995, Oxford UP).
- Hal Varian, *Microeconomic Analysis*, 3rd edition. (1992, W.H.Norton).

Tâtonnement and Sonnenschein-Mantel-Debreu Theorem

- §17.F of Mas-Colell, Whinston & Green.
- Donald Saari, "Mathematical complexity of simple economics", *Notices of the American Mathematical Society* **42**(#2) (February, 1995).

Disequilibrium dynamics (price-adjustment models, Hahn-Negishi, etc.)

- Jean-Pascal Benassy, *The economics of market disequilibrium*. (1982, Academic Press).
- Franklin M. Fisher, *Disequilibrium foundations of equilibrium economics*. (1983, Cambridge UP).
- Jean-Pascal Benassy, *The macroeconomics of imperfect competition and nonclearing markets*. (2002, MIT Press).