The Mathematics of Voting:

Paradox, deception, and chaos

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http://xaravve.trentu.ca/voting.pdf

May 8, 2008

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Elections

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- ...Pick one dessert for the dinner party: Apple cobbler, Banana cream pie, Chocolate cake, or *Dulce de leche*.

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- ...An federal election with four candidates: LiberAl, Bloc Quebecois, Conservative, or New Democratic.

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We can describe people's preferences with a table:

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	23
$D \succ B \succ C \succ A$	25
Total	100

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Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	23
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We can describe people's preferences with a table: e.g. this means that 10% of the people prefer A to B, prefer B to C, and prefer C to D.
There is no unanimous favourite.
We must have a vote...

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$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
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$C \succ D \succ B \succ A$	23
$D \succ B \succ C \succ A$	25
Total	100

An election gone wrong....

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$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	23
$D \succ B \succ C \succ A$	25
Total	100

Consider an election with four candidates A, B, C, and D.
(e.g. 10% of the voters prefer A to B, prefer B to C, and prefer C to D.)

An election gone wrong....

Plurality Vote					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	10			
$A \succ C \succ D \succ B$	9	9			
$A \succ D \succ B \succ C$	11	11			
$B \succ C \succ D \succ A$	22		22		
$C \succ D \succ B \succ A$	23			23	
$D \succ B \succ C \succ A$	25				25
Total	100	30	22	23	25
Verdict:		A wins.			

Consider an election with four candidates A, B, C, and D.
(e.g. 10% of the voters prefer A to B, prefer B to C, and prefer C to D.)
Clearly A wins the election, with 30% of the vote.

An election gone wrong....

A versus B, C, and D							
Preferences	#	$A \succ B$	$B \succ A$	$A \succ C$	$C \succ A$	$A \succ D$	$D \succ A$
$A \succ B \succ C \succ D$	10	10		10		10	
$A \succ C \succ D \succ B$	9	9		9		9	
$A \succ D \succ B \succ C$	11	11		11		11	
$B \succ C \succ D \succ A$	22		22		22		22
$C \succ D \succ B \succ A$	23		23		23		23
$D \succ B \succ C \succ A$	25		25		25		25
Total	100	30	70	30	70	30	70
Verdict:		$B \succ A$		$C \succ A$		$D \succ A$	

Consider an election with four candidates A, B, C, and D.
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Clearly A wins the election, with 30% of the vote.
But most voters prefer *any other candidate* over A:

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$B \succ C \succ D \succ A$	22		22		22		22
$C \succ D \succ B \succ A$	23		23		23		23
$D \succ B \succ C \succ A$	25		25		25		25
Total	100	30	70	30	70	30	70
Verdict:		$B \succ A$		$C \succ A$		$D \succ A$	

Consider an election with four candidates A, B, C, and D.
(e.g. 10% of the voters prefer A to B, prefer B to C, and prefer C to D.)
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But most voters prefer *any other candidate* over A:

70% prefer B to A. . . . . .

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$C \succ D \succ B \succ A$	23		23		23		23
$D \succ B \succ C \succ A$	25		25		25		25
Total	100	30	70	30	70	30	70
Verdict:		$B \succ A$		$C \succ A$		$D \succ A$	

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(e.g. 10% of the voters prefer A to B, prefer B to C, and prefer C to D.)
Clearly A wins the election, with 30% of the vote.
But most voters prefer *any other candidate* over A:

70% prefer B to A. . . . . . 70% prefer C to A. . . . . .

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$A \succ C \succ D \succ B$	9	9		9		9	
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$B \succ C \succ D \succ A$	22		22		22		22
$C \succ D \succ B \succ A$	23		23		23		23
$D \succ B \succ C \succ A$	25		25		25		25
Total	100	30	70	30	70	30	70
Verdict:		$B \succ A$		$C \succ A$		$D \succ A$	

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Verdict:		$B \succ A$		$C \succ A$		$D \succ A$	

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70% prefer B to A. . . . . . 70% prefer C to A. . . . . . 70% prefer D to A.

How did A win?

Splitting the opposition

A versus B, C, and D							
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Total	100	30	70	30	70	30	70
Verdict:		$B \succ A$		$C \succ A$		$D \succ A$	

Idea: More than 50% of voters despise A, but the 'anti-A' vote is 'split' between candidates B, C, and D, so A still wins.

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Verdict:		$B \succ A$		$C \succ A$		$D \succ A$	

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Problem: With four candidates, no single candidate gets a clear majority.

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Solution? Have a 'run-off election' between A and the second-place candidate

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Verdict:		$B \succ A$		$C \succ A$		$D \succ A$	

Idea: More than 50% of voters despise A, but the 'anti-A' vote is 'split' between candidates B, C, and D, so A still wins.

Problem: With four candidates, no single candidate gets a clear majority. (We say A wins with a **plurality**, meaning she gets the biggest fraction of votes, but still a minority).

Solution? Have a 'run-off election' between A and the second-place candidate (In this case, this is D, who got 25% of the vote).

A run-off election

A versus D			
Preferences	#	$A \succ D$	$D \succ A$
$A \succ B \succ C \succ D$	10	10	
$A \succ C \succ D \succ B$	9	9	
$A \succ D \succ B \succ C$	11	11	
$B \succ C \succ D \succ A$	22		22
$C \succ D \succ B \succ A$	23		23
$D \succ B \succ C \succ A$	25		25
Total	100	30	70
Verdict:		$D \succ A$	

In the run-off election, D crushes A, winning with 70% of the vote.

A run-off election

A versus D versus C					
Preferences	#	$A \succ D$	$D \succ A$	$C \succ D$	$D \succ C$
$A \succ B \succ C \succ D$	10	10		10	
$A \succ C \succ D \succ B$	9	9		9	
$A \succ D \succ B \succ C$	11	11			11
$B \succ C \succ D \succ A$	22		22	22	
$C \succ D \succ B \succ A$	23		23	23	
$D \succ B \succ C \succ A$	25		25		25
Total	100	30	70	64	36
Verdict:		$D \succ A$		$C \succ D$	

In the run-off election, D crushes A, winning with 70% of the vote.

Problem: 64% of the voters prefer C to D!

A run-off election

A versus D versus C					
Preferences	#	$A \succ D$	$D \succ A$	$C \succ D$	$D \succ C$
$A \succ B \succ C \succ D$	10	10		10	
$A \succ C \succ D \succ B$	9	9		9	
$A \succ D \succ B \succ C$	11	11			11
$B \succ C \succ D \succ A$	22		22	22	
$C \succ D \succ B \succ A$	23		23	23	
$D \succ B \succ C \succ A$	25		25		25
Total	100	30	70	64	36
Verdict:		$D \succ A$		$C \succ D$	

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The 'wrong' candidate won again! How?

A run-off election

A versus D versus C					
Preferences	#	$A \succ D$	$D \succ A$	$C \succ D$	$D \succ C$
$A \succ B \succ C \succ D$	10	10		10	
$A \succ C \succ D \succ B$	9	9		9	
$A \succ D \succ B \succ C$	11	11			11
$B \succ C \succ D \succ A$	22		22	22	
$C \succ D \succ B \succ A$	23		23	23	
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Total	100	30	70	64	36
Verdict:		$D \succ A$		$C \succ D$	

In the run-off election, D crushes A, winning with 70% of the vote.

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Idea: The 'anti-D vote' was split between B and C.

A run-off election

A versus D versus C					
Preferences	#	$A \succ D$	$D \succ A$	$C \succ D$	$D \succ C$
$A \succ B \succ C \succ D$	10	10		10	
$A \succ C \succ D \succ B$	9	9		9	
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Verdict:		$D \succ A$		$C \succ D$	

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Problem: 64% of the voters prefer C to D!

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Idea: The 'anti-D vote' was split between B and C.

Thus, D obtained second place, even though most voters prefer C.

A run-off election

A versus D versus C					
Preferences	#	$A \succ D$	$D \succ A$	$C \succ D$	$D \succ C$
$A \succ B \succ C \succ D$	10	10		10	
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$B \succ C \succ D \succ A$	22		22	22	
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Verdict:		$D \succ A$		$C \succ D$	

In the run-off election, D crushes A, winning with 70% of the vote.

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Solution? Have a *sequence* of *two*-candidate elections.

A run-off election

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Verdict:		$D \succ A$		$C \succ D$	

In the run-off election, D crushes A, winning with 70% of the vote.

Problem: 64% of the voters prefer C to D!

The 'wrong' candidate won again! How?

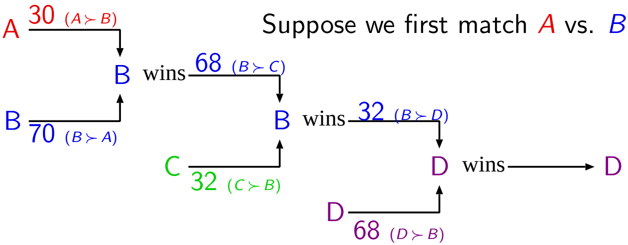
Idea: The 'anti-D vote' was split between B and C.

Thus, D obtained second place, even though most voters prefer C.

Solution? Have a *sequence* of *two*-candidate elections.

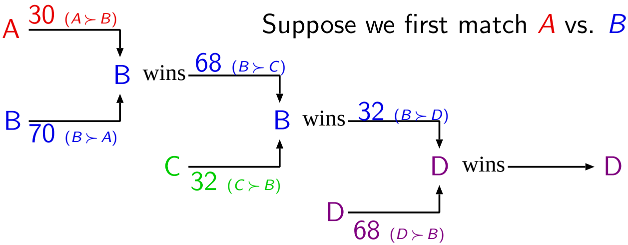
In each of these, the winner must have a clear majority.

An agenda of pairwise votes



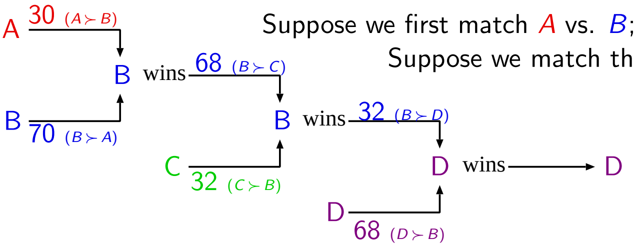
Suppose we first match A vs. B;

An agenda of pairwise votes



Suppose we first match A vs. B; then B wins with 70%.

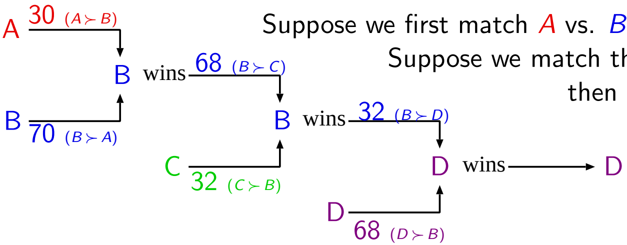
An agenda of pairwise votes



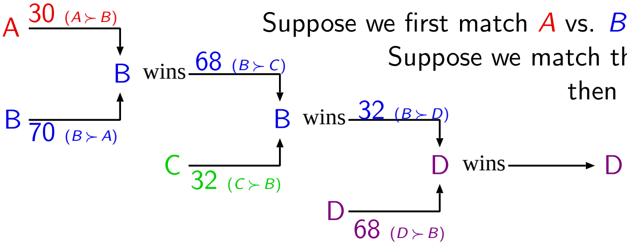
Suppose we first match A vs. B; then B wins with 70%.
Suppose we match the winner (B) against C,

An agenda of pairwise votes

Suppose we first match A vs. B; then B wins with 70%.
Suppose we match the winner (B) against C,
then B wins again, with 68%.



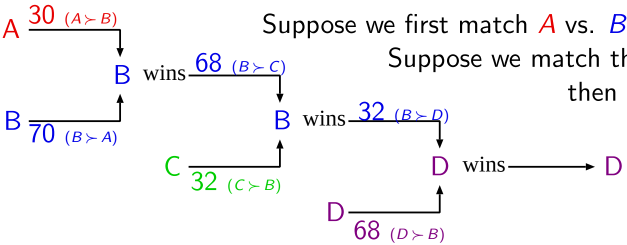
An agenda of pairwise votes



Suppose we first match A vs. B; then B wins with 70%.
Suppose we match the winner (B) against C,
then B wins again, with 68%.

Finally, if we match *this* winner (B) against D; then D wins, with 68%.

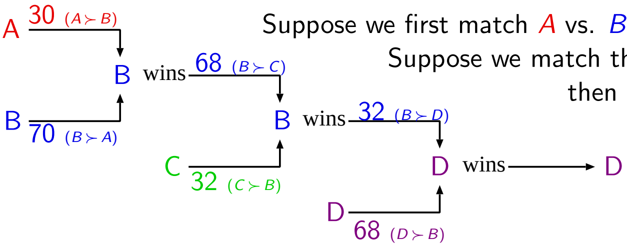
An agenda of pairwise votes



Suppose we first match A vs. B; then B wins with 70%.
Suppose we match the winner (B) against C, then B wins again, with 68%.

Finally, if we match *this* winner (B) against D; then D wins, with 68%.
Thus, D wins the election.

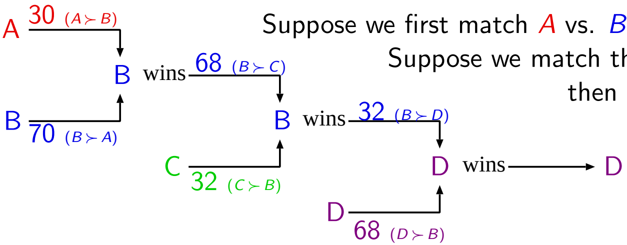
An agenda of pairwise votes



Suppose we first match A vs. B; then B wins with 70%.
Suppose we match the winner (B) against C,
then B wins again, with 68%.

Finally, if we match *this* winner (B) against D; then D wins, with 68%.
Thus, D wins the election. Such a sequence of pairwise votes is often used by committees to approve motions and amendments.

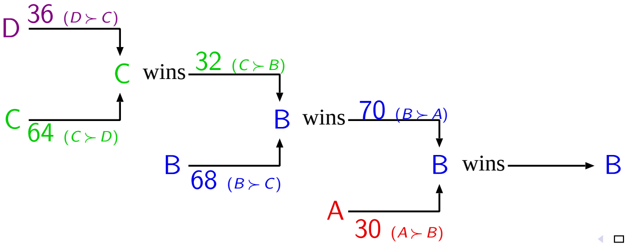
An agenda of pairwise votes



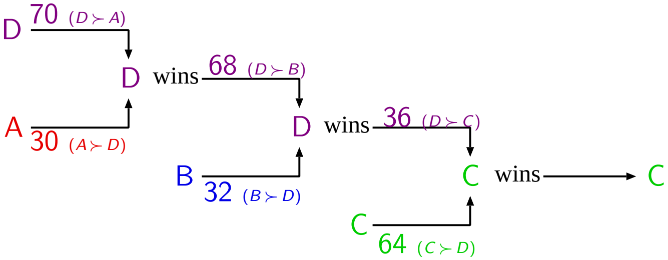
Suppose we first match A vs. B; then B wins with 70%. Suppose we match the winner (B) against C, then B wins again, with 68%.

Finally, if we match *this* winner (B) against D; then D wins, with 68%. Thus, D wins the election. Such a sequence of pairwise votes is often used by committees to approve motions and amendments.

Problem: With a different 'agenda' of matches, we get a different winner:

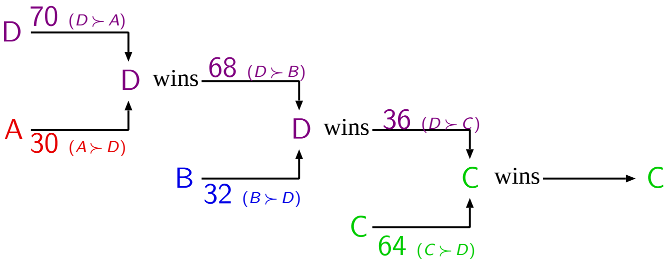


With yet another ‘agenda’, we can get a yet another winner.



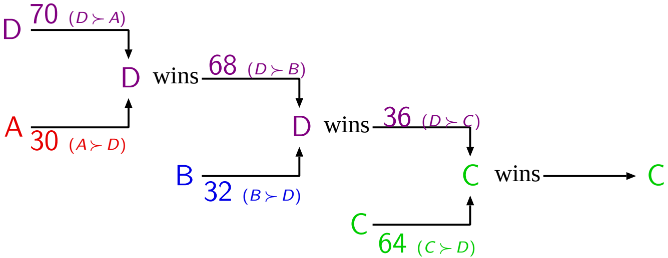
Another agenda of pairwise votes

With yet another 'agenda', we can get a yet another winner.



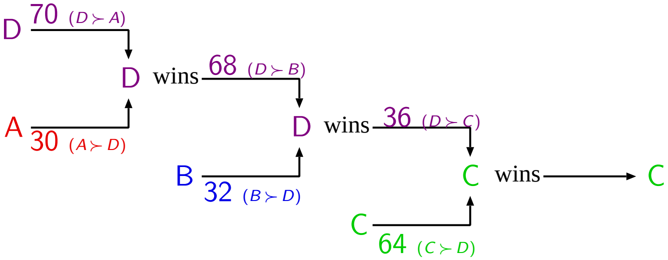
Problem: The winner depends upon the *order* in which we match the candidates against each other.

With yet another 'agenda', we can get a yet another winner.



Problem: The winner depends upon the *order* in which we match the candidates against each other.
With a suitable **agenda** of pairwise votes, we can make *any one* of B, C, or D the 'winner' of the election!

With yet another ‘agenda’, we can get a yet another winner.



Problem: The winner depends upon the *order* in which we match the candidates against each other.
With a suitable **agenda** of pairwise votes, we can make *any one* of B, C, or D the ‘winner’ of the election!
Solution? Have a *sequence* of run-off elections. Start with all candidates, and after each election, drop the lowest-ranked candidate.

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	23
$D \succ B \succ C \succ A$	25
Total	100

The instant runoff system (also called Hare's method) works as follows:

Thomas Hare's 'Instant Runoff' system.

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	23
$D \succ B \succ C \succ A$	25
Total	100

The instant runoff system (also called Hare's method) works as follows:
1. Each voter writes her *complete* preference ordering on her ballot.

Thus, we have all the information in the above table.

Thomas Hare's 'Instant Runoff' system.

Majority Vote					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	10			
$A \succ C \succ D \succ B$	9	9			
$A \succ D \succ B \succ C$	11	11			
$B \succ C \succ D \succ A$	22		22		
$C \succ D \succ B \succ A$	23			23	
$D \succ B \succ C \succ A$	25				25
Total	100	30	22	23	25
	Verdict:	No majority winner			

The instant runoff system (also called Hare's method) works as follows:

1. Each voter writes her *complete* preference ordering on her ballot.

Thus, we have all the information in the above table.

2. We count the number of voters who favour each candidate.

Thomas Hare's 'Instant Runoff' system.

Majority Vote					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	10			
$A \succ C \succ D \succ B$	9	9			
$A \succ D \succ B \succ C$	11	11			
$B \succ C \succ D \succ A$	22		22		
$C \succ D \succ B \succ A$	23			23	
$D \succ B \succ C \succ A$	25				25
Total	100	30	22	23	25
	Verdict:	No majority winner			

The instant runoff system (also called Hare's method) works as follows:

1. Each voter writes her *complete* preference ordering on her ballot.

Thus, we have all the information in the above table.

2. We count the number of voters who favour each candidate.

3(a). If some candidate has a strict majority of votes, she wins.

Majority Vote					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	10			
$A \succ C \succ D \succ B$	9	9			
$A \succ D \succ B \succ C$	11	11			
$B \succ C \succ D \succ A$	22		22		
$C \succ D \succ B \succ A$	23			23	
$D \succ B \succ C \succ A$	25				25
Total	100	30	22	23	25
	Verdict:	No majority winner			

The instant runoff system (also called Hare's method) works as follows:

1. Each voter writes her *complete* preference ordering on her ballot. Thus, we have all the information in the above table.

2. We count the number of voters who favour each candidate.

3(a). If some candidate has a strict majority of votes, she wins.

3(b). Otherwise, we remove the candidate who is favoured by the fewest voters —in this case, B.

Thomas Hare's 'Instant Runoff' system.

Removal of B:							
Preferences	#	$A \succ D \succ C$	$D \succ C \succ A$	$C \succ A \succ D$	$D \succ A \succ C$	$A \succ C \succ D$	$C \succ D \succ A$
$A \succ \cancel{B} \succ C \succ D$	10					10	
$A \succ C \succ D \succ \cancel{B}$	9					9	
$A \succ D \succ \cancel{B} \succ C$	11	11					
$\cancel{B} \succ C \succ D \succ A$	22						22
$C \succ D \succ \cancel{B} \succ A$	23						23
$D \succ \cancel{B} \succ C \succ A$	25		25				
Total	100	11	25	0	0	19	45

The instant runoff system (also called Hare's method) works as follows:

1. Each voter writes her *complete* preference ordering on her ballot. Thus, we have all the information in the above table.

2. We count the number of voters who favour each candidate.

3(a). If some candidate has a strict majority of votes, she wins.

3(b). Otherwise, we remove the candidate who is favoured by the fewest voters —in this case, B.

4. We reconstruct the voter's preference orders, with B removed.

Majority Vote				
Preferences	#	A	C	D
$A \succ C \succ D$	19	19		
$A \succ D \succ C$	11	11		
$C \succ D \succ A$	45		45	
$D \succ C \succ A$	25			25
Total	100	30	45	25
Verdict:		No winner.		

1. Each voter writes her *complete* preference ordering on her ballot. Thus, we have all the information in the above table.

2. We count the number of voters who favour each candidate.

3(a). If some candidate has a strict majority of votes, she wins.

3(b). Otherwise, we remove the candidate who is favoured by the fewest voters —in this case, B.

4. We reconstruct the voter's preference orders, with B removed.

5. Again, we count the number of voters who favour each candidate.

Majority Vote				
Preferences	#	A	C	D
$A \succ C \succ D$	19	19		
$A \succ D \succ C$	11	11		
$C \succ D \succ A$	45		45	
$D \succ C \succ A$	25			25
Total	100	30	45	25
Verdict:		No winner.		

Thus, we have all the information in the above table.

2. We count the number of voters who favour each candidate.

3(a). If some candidate has a strict majority of votes, she wins.

3(b). Otherwise, we remove the candidate who is favoured by the fewest voters —in this case, B.

4. We reconstruct the voter's preference orders, with B removed.

5. Again, we count the number of voters who favour each candidate.

6(a) Again, if some candidate has a strict majority of votes, she wins.

Majority Vote				
Preferences	#	A	C	D
$A \succ C \succ D$	19	19		
$A \succ D \succ C$	11	11		
$C \succ D \succ A$	45		45	
$D \succ C \succ A$	25			25
Total	100	30	45	25
Verdict:		No winner.		

2. We count the number of voters who favour each candidate.
3(a). If some candidate has a strict majority of votes, she wins.
3(b). Otherwise, we remove the candidate who is favoured by the fewest voters —in this case, B.
4. We reconstruct the voter's preference orders, with B removed.
5. Again, we count the number of voters who favour each candidate.
6(a) Again, if some candidate has a strict majority of votes, she wins.
6(b) Otherwise, we again remove the candidate who is favoured by the fewest voters —in this case, D.

Thomas Hare's 'Instant Runoff' system.

Removal of D			
Preferences	#	$A \succ C$	$C \succ A$
$A \succ \cancel{D} \succ C$	11	11	
$\cancel{D} \succ C \succ A$	25		25
$A \succ C \succ \cancel{D}$	19	19	
$C \succ \cancel{D} \succ A$	45		45
Total	100	30	70

3(a). If some candidate has a strict majority of votes, she wins.

3(b). Otherwise, we remove the candidate who is favoured by the fewest voters —in this case, B.

4. We reconstruct the voter's preference orders, with B removed.

5. Again, we count the number of voters who favour each candidate.

6(a) Again, if some candidate has a strict majority of votes, she wins.

6(b) Otherwise, we again remove the candidate who is favoured by the fewest voters —in this case, D.

7. We continue this process until some candidate wins a strict majority.

Electorate	
Preferences	#
$A \succ C$	30
$C \succ A$	70
Verdict:	C wins.

3(b). Otherwise, we remove the candidate who is favoured by the fewest voters —in this case, B.
4. We reconstruct the voter's preference orders, with B removed.
5. Again, we count the number of voters who favour each candidate.
6(a) Again, if some candidate has a strict majority of votes, she wins.
6(b) Otherwise, we again remove the candidate who is favoured by the fewest voters —in this case, D.
7. We continue this process until some candidate wins a strict majority.
....In this case, it is C.

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	23
$D \succ B \succ C \succ A$	25
Total	100

Hare's 'Instant Runoff' is used to elect the President of Ireland, the mayors of London and San Francisco, and the host city for the Olympic Games.

"The greatest improvement in government"?

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	23
$D \succ B \succ C \succ A$	25
Total	100

Hare's 'Instant Runoff' is used to elect the President of Ireland, the mayors of London and San Francisco, and the host city for the Olympic Games.

In 1860, John Stuart Mill called it, "among the very greatest improvements yet made in the theory and practice of government."

“The greatest improvement in government”?

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	23
$D \succ B \succ C \succ A$	25
Total	100

Hare’s ‘Instant Runoff’ is used to elect the President of Ireland, the mayors of London and San Francisco, and the host city for the Olympic Games.
In 1860, John Stuart Mill called it, “among the very greatest improvements yet made in the theory and practice of government.”
However, Instant Runoff has a little problem.

Problem: Monotonicity Failure

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	23
$D \succ B \succ C \succ A$	25
Total	100

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Suppose 4% of the "$D \succ B \succ C \succ A$" voters change to "$C \succ D \succ B \succ A$".

Problem: Monotonicity Failure

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	23
$D \succ B \succ C \succ A$	25
Total	100

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However, Instant Runoff has a little problem.
Suppose 4% of the "$D \succ B \succ C \succ A$" voters change to "$C \succ D \succ B \succ A$".
Thus, the (bottom) "$D \succ B \succ C \succ A$" tally decreases from 25% to 21%, while the (second last) "$C \succ D \succ B \succ A$" tally increases from 23% to 27%.

Problem: Monotonicity Failure

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	27
$D \succ B \succ C \succ A$	21
Total	100

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Suppose 4% of the "$D \succ B \succ C \succ A$" voters change to "$C \succ D \succ B \succ A$".
Thus, the (bottom) "$D \succ B \succ C \succ A$" tally decreases from 25% to 21%, while the (second last) "$C \succ D \succ B \succ A$" tally increases from 23% to 27%.
Note that this change in public opinion is strictly *favourable* towards C.

Problem: Monotonicity Failure

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	27
$D \succ B \succ C \succ A$	21
Total	100

In 1860, John Stuart Mill called it, "among the very greatest improvements yet made in the theory and practice of government."
However, Instant Runoff has a little problem.
Suppose 4% of the "$D \succ B \succ C \succ A$" voters change to "$C \succ D \succ B \succ A$".
Thus, the (bottom) "$D \succ B \succ C \succ A$" tally decreases from 25% to 21%, while the (second last) "$C \succ D \succ B \succ A$" tally increases from 23% to 27%.
Note that this change in public opinion is strictly *favourable* towards C.
C won the election before, so she should win again. But let's watch....

Majority Vote					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	10			
$A \succ C \succ D \succ B$	9	9			
$A \succ D \succ B \succ C$	11	11			
$B \succ C \succ D \succ A$	22		22		
$C \succ D \succ B \succ A$	27			27	
$D \succ B \succ C \succ A$	21				21
Total	100	30	22	27	21
Verdict:		No majority winner			

In 1860, John Stuart Mill called it, "among the very greatest improvements yet made in the theory and practice of government."

However, Instant Runoff has a little problem.

Suppose 4% of the "$D \succ B \succ C \succ A$" voters change to "$C \succ D \succ B \succ A$".

Thus, the (bottom) "$D \succ B \succ C \succ A$" tally decreases from 25% to 21%, while the (second last) "$C \succ D \succ B \succ A$" tally increases from 23% to 27%.

Note that this change in public opinion is strictly *favourable* towards C.

C won the election before, so she should win again. But let's watch....

During the first round, D has the lowest support, so she is eliminated.

Problem: Monotonicity Failure

Removal of D:							
Preferences	#	$A \succ B \succ C$	$B \succ C \succ A$	$C \succ A \succ B$	$B \succ A \succ C$	$A \succ C \succ B$	$C \succ B \succ A$
$A \succ B \succ C \succ \cancel{D}$	10	10					
$A \succ C \succ \cancel{D} \succ B$	9					9	
$A \succ \cancel{D} \succ B \succ C$	11	11					
$B \succ C \succ \cancel{D} \succ A$	22		22				
$C \succ \cancel{D} \succ B \succ A$	27						27
$\cancel{D} \succ B \succ C \succ A$	21		21				
Total	100	21	43	0	0	9	27

In 1860, John Stuart Mill called it, “among the very greatest improvements yet made in the theory and practice of government.”
However, Instant Runoff has a little problem.
Suppose 4% of the “$D \succ B \succ C \succ A$” voters change to “$C \succ D \succ B \succ A$”.
Thus, the (bottom) “$D \succ B \succ C \succ A$” tally decreases from 25% to 21%, while the (second last) “$C \succ D \succ B \succ A$” tally increases from 23% to 27%.
Note that this change in public opinion is strictly *favourable* towards C.
C won the election before, so she should win again. But let's watch....
During the first round, D has the lowest support, so she is eliminated.

Problem: Monotonicity Failure

Majority Vote				
Preferences	#	A	B	C
$A \succ B \succ C$	21	21		
$A \succ C \succ B$	9	9		
$B \succ C \succ A$	43		43	
$C \succ B \succ A$	27			27
Total	100	30	43	27
Verdict:		No winner.		

In 1860, John Stuart Mill called it, "among the very greatest improvements yet made in the theory and practice of government."

However, Instant Runoff has a little problem.

Suppose 4% of the "$D \succ B \succ C \succ A$" voters change to "$C \succ D \succ B \succ A$".

Thus, the (bottom) "$D \succ B \succ C \succ A$" tally decreases from 25% to 21%, while the (second last) "$C \succ D \succ B \succ A$" tally increases from 23% to 27%.

Note that this change in public opinion is strictly *favourable* towards C.

C won the election before, so she should win again. But let's watch....

During the first round, D has the lowest support, so she is eliminated.

During the next round, C is eliminated...

Problem: Monotonicity Failure

Removal of C			
Preferences	#	$A \succ B$	$B \succ A$
$A \succ \cancel{C} \succ B$	9	9	
$\cancel{C} \succ B \succ A$	27		27
$A \succ B \succ \cancel{C}$	21	21	
$B \succ \cancel{C} \succ A$	43		43
Total	100	30	70

In 1860, John Stuart Mill called it, "among the very greatest improvements yet made in the theory and practice of government."

However, Instant Runoff has a little problem.

Suppose 4% of the "$D \succ B \succ C \succ A$" voters change to "$C \succ D \succ B \succ A$".

Thus, the (bottom) "$D \succ B \succ C \succ A$" tally decreases from 25% to 21%, while the (second last) "$C \succ D \succ B \succ A$" tally increases from 23% to 27%.

Note that this change in public opinion is strictly *favourable* towards C.

C won the election before, so she should win again. But let's watch....

During the first round, D has the lowest support, so she is eliminated.

During the next round, C is eliminated...

Electorate	
Preferences	#
$A \succ B$	30
$B \succ A$	70
Verdict:	B wins.

In 1860, John Stuart Mill called it, "among the very greatest improvements yet made in the theory and practice of government."
However, Instant Runoff has a little problem.
Suppose 4% of the "$D \succ B \succ C \succ A$" voters change to "$C \succ D \succ B \succ A$".
Thus, the (bottom) "$D \succ B \succ C \succ A$" tally decreases from 25% to 21%, while the (second last) "$C \succ D \succ B \succ A$" tally increases from 23% to 27%.
Note that this change in public opinion is strictly *favourable* towards C.
C won the election before, so she should win again. But let's watch....
During the first round, D has the lowest support, so she is eliminated.
During the next round, C is eliminated...
In the final round, B (*not C*) is the winner.

Electorate	
Preferences	#
$A \succ B$	30
$B \succ A$	70
Verdict:	B wins.

However, Instant Runoff has a little problem.
Suppose 4% of the "$D \succ B \succ C \succ A$" voters change to "$C \succ D \succ B \succ A$".
Thus, the (bottom) "$D \succ B \succ C \succ A$" tally decreases from 25% to 21%, while the (second last) "$C \succ D \succ B \succ A$" tally increases from 23% to 27%.
Note that this change in public opinion is strictly *favourable* towards C.
C won the election before, so she should win again. But let's watch....
During the first round, D has the lowest support, so she is eliminated.
During the next round, C is eliminated...
In the final round, B (*not* C) is the winner.
Thus, a shift in public opinion that *favoured* C actually *destroyed* C's victory!

Electorate	
Preferences	#
$A \succ B$	30
$B \succ A$	70
Verdict:	B wins.

Suppose 4% of the "$D \succ B \succ C \succ A$" voters change to "$C \succ D \succ B \succ A$".
Thus, the (bottom) "$D \succ B \succ C \succ A$" tally decreases from 25% to 21%, while the (second last) "$C \succ D \succ B \succ A$" tally increases from 23% to 27%.
Note that this change in public opinion is strictly *favourable* towards C.
C won the election before, so she should win again. But let's watch....
During the first round, D has the lowest support, so she is eliminated.
During the next round, C is eliminated...
In the final round, B (*not* C) is the winner.
Thus, a shift in public opinion that *favoured* C actually *destroyed* C's victory!
Thus, "Instant Runoff" lacks a critical property: *monotonicity*.

Problem: Monotonicity Failure

Electorate	
Preferences	#
$A \succ B$	30
$B \succ A$	70
Verdict:	B wins.

Note that this change in public opinion is strictly *favourable* towards C.
C won the election before, so she should win again. But let's watch....
During the first round, D has the lowest support, so she is eliminated.
During the next round, C is eliminated...
In the final round, B (*not* C) is the winner.
Thus, a shift in public opinion that *favoured* C actually *destroyed* C's victory!
Thus, "Instant Runoff" lacks a critical property: *monotonicity*.

Solution? Have a *single* election involving *all four* candidates. But let each voter more clearly and completely express her preferences.

Electorate	
Preferences	#
$A \succ B$	30
$B \succ A$	70
Verdict:	B wins.

Note that this change in public opinion is strictly *favourable* towards C.
C won the election before, so she should win again. But let's watch....
During the first round, D has the lowest support, so she is eliminated.
During the next round, C is eliminated...
In the final round, B (*not C*) is the winner.
Thus, a shift in public opinion that *favoured* C actually *destroyed* C's victory!
Thus, "Instant Runoff" lacks a critical property: *monotonicity*.

Solution? Have a *single* election involving *all four* candidates. But let each voter more clearly and completely express her preferences.
Example: Let each voter vote for her 'top two' candidates, or even her 'top three' candidates. Or let her 'rank' all four candidates.

Top two vs. top three candidates

Vote for top two					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	10	10		
$A \succ C \succ D \succ B$	9	9		9	
$A \succ D \succ B \succ C$	11	11			11
$B \succ C \succ D \succ A$	22		22	22	
$C \succ D \succ B \succ A$	23			23	23
$D \succ B \succ C \succ A$	25		25		25
Total	100	30	57	54	59
Verdict:		D wins.			

Suppose each voter votes for her 'top two' candidates.

Top two vs. top three candidates

Vote for top two					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	10	10		
$A \succ C \succ D \succ B$	9	9		9	
$A \succ D \succ B \succ C$	11	11			11
$B \succ C \succ D \succ A$	22		22	22	
$C \succ D \succ B \succ A$	23			23	23
$D \succ B \succ C \succ A$	25		25		25
Total	100	30	57	54	59
Verdict:		D wins.			

Suppose each voter votes for her 'top two' candidates.
Then D wins the election, with 59 points.

Top two vs. top three candidates

Vote for top two					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	10	10		
$A \succ C \succ D \succ B$	9	9		9	
$A \succ D \succ B \succ C$	11	11			11
$B \succ C \succ D \succ A$	22		22	22	
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But suppose instead we let each voter vote for her 'top three' candidates.
(Effectively, she 'votes against' her worst candidate; thus this is called *antiplurality* vote).

Antiplurality (vote for top three)					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	10	10	10	
$A \succ C \succ D \succ B$	9	9		9	9
$A \succ D \succ B \succ C$	11	11	11		11
$B \succ C \succ D \succ A$	22		22	22	22
$C \succ D \succ B \succ A$	23		23	23	23
$D \succ B \succ C \succ A$	25		25	25	25
Total	100	30	91	89	90
Verdict:		B wins.			

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$C \succ D \succ B \succ A$	23		23	23	23
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Who is the 'real' winner?

Top two vs. top three candidates

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$A \succ B \succ C \succ D$	10	10	10	10	
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Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	10	10	10	
$A \succ C \succ D \succ B$	9	9		9	9
$A \succ D \succ B \succ C$	11	11	11		11
$B \succ C \succ D \succ A$	22		22	22	22
$C \succ D \succ B \succ A$	23		23	23	23
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Total	100	30	91	89	90
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Suppose each voter votes for her 'top two' candidates.
Then D wins the election, with 59 points.
But suppose instead we let each voter vote for her 'top three' candidates.
(Effectively, she 'votes against' her worst candidate; thus this is called *antiplurality* vote). Then B wins the election, with 91 points.
Who is the 'real' winner? B(antiplurality)? D(vote-for-2)? or A(plurality)?

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Then we sum up the points, and the candidate with the highest sum wins.

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- 0 points to her least-favourite candidate.

Then we sum up the points, and the candidate with the highest sum wins.

Borda Count					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	3×10	2×10	10	
$A \succ C \succ D \succ B$	9	3×9		2×9	9
$A \succ D \succ B \succ C$	11	3×11	11		2×11
$B \succ C \succ D \succ A$	22		3×22	2×22	22
$C \succ D \succ B \succ A$	23		23	3×23	2×23
$D \succ B \succ C \succ A$	25		2×25	25	3×25
Total	100	90	170	166	174
Verdict:		D wins.			

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Borda Count					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	10	3×10	2×10	10	
$A \succ C \succ D \succ B$	9	3×9		2×9	9
$A \succ D \succ B \succ C$	11	3×11	11		2×11
$B \succ C \succ D \succ A$	22		3×22	2×22	22
$C \succ D \succ B \succ A$	23		23	3×23	2×23
$D \succ B \succ C \succ A$	25		2×25	25	3×25
Total	100	90	170	166	174
Verdict:	D wins.				

In this case, the winner is D, with 174 points.

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Now consider the following profile

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Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	22
$A \succ D \succ C \succ B$	22
$C \succ B \succ D \succ A$	23
$D \succ B \succ C \succ A$	33
Total	100

In this case, all four methods give different answers....

Now consider the following profile

Plurality Vote					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	22	22			
$A \succ D \succ C \succ B$	22	22			
$C \succ B \succ D \succ A$	23			23	
$D \succ B \succ C \succ A$	33				33
Total	100	44	0	23	33
Verdict:		A wins.			

In this case, all four methods give different answers....

- A wins the plurality election, with 44% of the vote.

All four methods can disagree

Now consider the following profile

Vote for top two					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	22	22	22		
$A \succ D \succ C \succ B$	22	22			22
$C \succ B \succ D \succ A$	23		23	23	
$D \succ B \succ C \succ A$	33		33		33
Total	100	44	78	23	55
Verdict:		B wins.			

In this case, all four methods give different answers....

- A wins the plurality election, with 44% of the vote.
- B wins the 'vote-for-two' election, with 78 points.

All four methods can disagree

Now consider the following profile

Antiplurality (vote for top three)						
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	22	22	22	22	
$A \succ D \succ C \succ B$	22	22		22	22
$C \succ B \succ D \succ A$	23		23	23	23
$D \succ B \succ C \succ A$	33		33	33	33
Total	100	44	78	100	78
Verdict:		C wins.			

In this case, all four methods give different answers....

- A wins the plurality election, with 44% of the vote.
- B wins the 'vote-for-two' election, with 78 points.
- C wins the antiplurality election, with 100 points.

Now consider the following profile

Borda Count					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	22	3×22	2×22	22	
$A \succ D \succ C \succ B$	22	3×22		22	2×22
$C \succ B \succ D \succ A$	23		2×23	3×23	23
$D \succ B \succ C \succ A$	33		2×33	33	3×33
Total	100	132	156	146	166
Verdict:		D wins.			

In this case, all four methods give different answers....

- A wins the plurality election, with 44% of the vote.
- B wins the 'vote-for-two' election, with 78 points.
- C wins the antiplurality election, with 100 points.
- D wins the Borda Count election, with 166 points.

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Now consider the following profile

Borda Count					
Preferences	#	A	B	C	D
$A \succ B \succ C \succ D$	22	3×22	2×22	22	
$A \succ D \succ C \succ B$	22	3×22		22	2×22
$C \succ B \succ D \succ A$	23		2×23	3×23	23
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Who is the real winner?

(Anti)plurality, vote-for-two, and Borda count are positional voting systems.

Positional Voting Systems

(Anti)plurality, vote-for-two, and Borda count are positional voting systems.

In a **positional voting system**, there is some sequence of ‘scores’

$$s_1 \geq s_2 \geq s_3 \geq s_4 \geq \cdots$$

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Unlike 'Instant runoff', positional systems are always **monotone**: a change in public opinion which favours candidate **X** will *always* benefit **X**.

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Unlike ‘Instant runoff’, positional systems are always **monotone**: a change in public opinion which favours candidate **X** will *always* benefit **X**.
Also, unlike agendas of pairwise votes, positional systems are **neutral**: they don’t systematically favour one candidate over others.

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The Borda count seems to be a good compromise between 'plurality', 'antiplurality', and 'vote-for-two' procedures.

It allows each voter to 'vote against' her worst candidate, but also assigns more 'weight' to her favourite than to her 2nd best, and more 'weight' to her 2nd best than her 3rd best, etc.

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But Borda has drawbacks. For example, consider the following profile:

Electorate	
Preferences	#
$A \succ B \succ C$	60
$B \succ C \succ A$	40
Total	100

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Borda Count				
Preferences	#	A	B	C
$A \succ B \succ C$	60	2×60	60	
$B \succ C \succ A$	40		2×40	40
Total	100	120	140	40
Verdict:		B wins.		

Clearly B wins the Borda Count election, with 140 points.

Borda Count was invented by Jean Charles de Borda (1733-1799), a French mathematician, military engineer, naval commander, and political theorist.

It has many advantages, and is still widely used.

But Borda has drawbacks. For example, consider the following profile:

Plurality Vote				
Preferences	#	A	B	C
$A \succ B \succ C$	60	60		
$B \succ C \succ A$	40		40	
Total	100	60	40	0
Verdict:		A wins.		

Clearly B wins the Borda Count election, with 140 points.

However, A is prefered by a *strict majority* (60%) of the voters.

The Borda Count

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Total	100	60	40	0
Verdict:		A wins.		

Clearly B wins the Borda Count election, with 140 points.
However, A is prefered by a *strict majority* (60%) of the voters.
Rationale: A is 'polarizing' candidate, loved by 60%, but *hated* by 40%. B is loved by only 40%, but hated by no one; she is a good 'compromise' candidate, and Borda Count detects this.

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Clearly B wins the Borda Count election, with 140 points.

However, A is prefered by a *strict majority* (60%) of the voters.

Rationale: A is ‘polarizing’ candidate, loved by 60%, but *hated* by 40%. B is loved by only 40%, but hated by no one; she is a good ‘compromise’ candidate, and Borda Count detects this.

But many ‘positional’ systems have this property; why use Borda’s?

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Condorcet's Criterion

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Thus, **Condorcet's Criterion** says:

A voting system should always choose the Condorcet winner, if one exists.

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(The last example shows that Borda count violates the Condorcet criteria.)

Condorcet's Paradox

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Condorcet Pairwise Votes							
Preferences	#	$A \succ B$	$B \succ A$	$B \succ C$	$C \succ B$	$A \succ C$	$C \succ A$
$A \succ B \succ C$	33	33		33		33	
$B \succ C \succ A$	33		33	33			33
$C \succ A \succ B$	34	34			34		34
Total	100	67	33	66	34	33	67
Verdict:		$A \succ B$		$B \succ C$		$C \succ A$	

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Thus, *there is no Condorcet winner*.

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Verdict:		$A \succ B$		$B \succ C$		$C \succ A$	

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- 67% of the voters prefer A to B.
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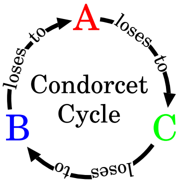
Thus, *there is no Condorcet winner*. This is called **Condorcet's Paradox**. The majority's apparently 'cyclical' preference ordering

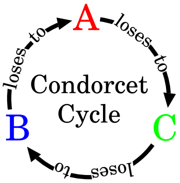
$$\cdots \succ A \succ B \succ C \succ A \succ B \succ C \succ A \succ B \succ C \succ A \succ B \succ C \succ \cdots$$

is called a **Condorcet cycle**.

Condorcet cycles cause lots of problems.



Condorcet cycles cause lots of problems. For example, they are the reason why different 'agendas' of pairwise votes can produce different winners:



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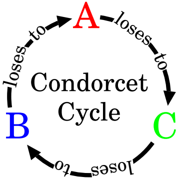

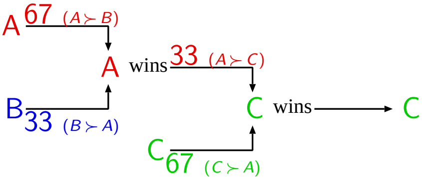

Condorcet cycles & Agenda Manipulation

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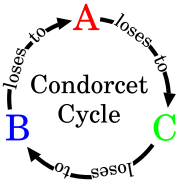

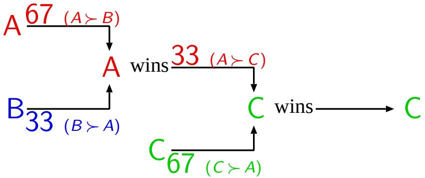

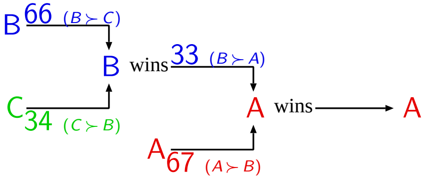

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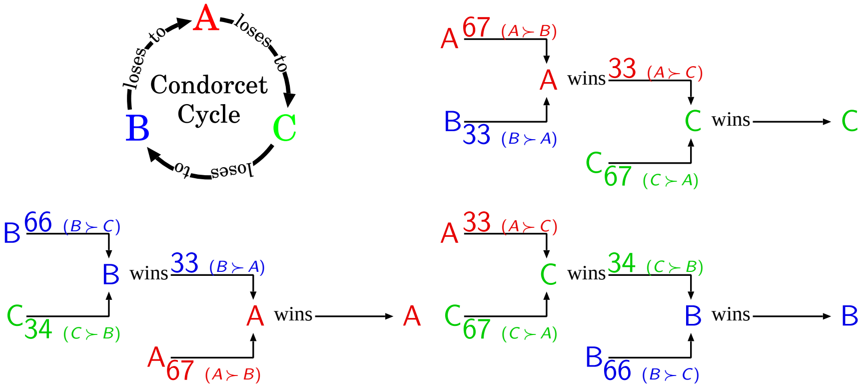

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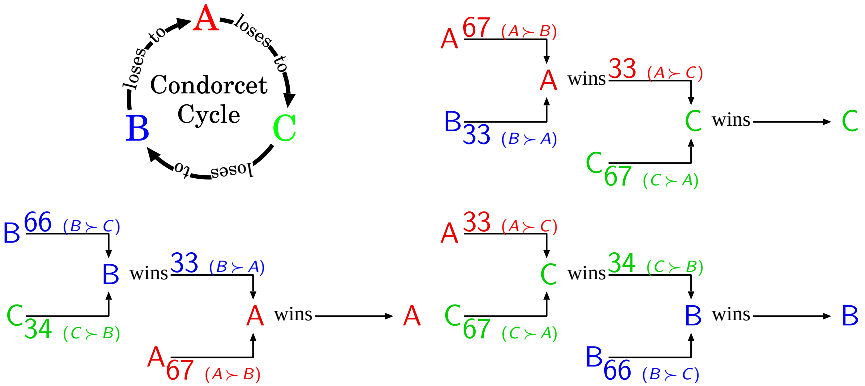

Problem: Whoever controls the agenda (e.g. the Chair of a committee, the head of the Election Commission) can control the outcome.

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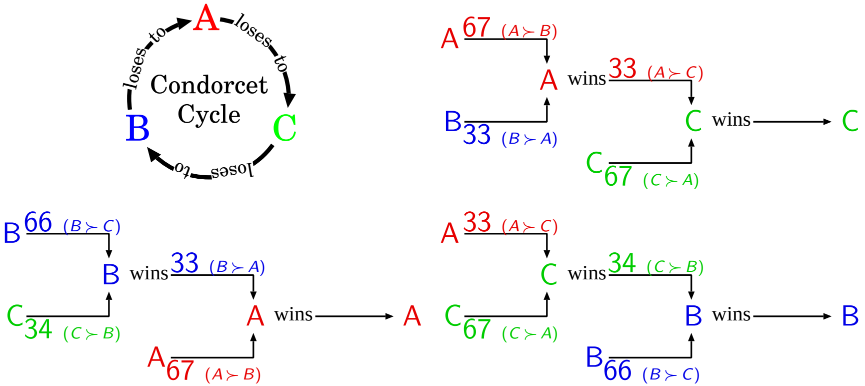
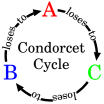
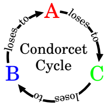


Problem: Whoever controls the agenda (e.g. the Chair of a committee, the head of the Election Commission) can control the outcome.
This is called **agenda manipulation**.

Political Instability & Condorcet Spirals



Condorcet cycles can also cause political instability.



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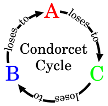
Suppose A wins an election against B.



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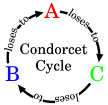
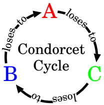
Suppose A wins an election against B.

In the *next* election, opponents of A can introduce C, who will beat A.

Political Instability & Condorcet Spirals



Condorcet cycles can also cause political instability.
Suppose A wins an election against B.
In the *next* election, opponents of A can introduce C, who will beat A.
In the *third* election, opponents of C can reintroduce B, who will beat C.



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Suppose A wins an election against B.
In the *next* election, opponents of A can introduce C, who will beat A.
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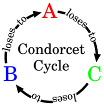
But now, in the *fourth* election, opponents of B bring back A, who beats B.
Then the cycle starts over.



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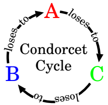
According to certain mathematical models of electoral politics developed by McKelvey (1976,1979) and Schofield (1978,1983), such 'voting chaos' is quite common (perhaps ubiquitous) in real democracies.



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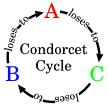
Worse yet: a sly 'electioneer' can construct a Condorcet spiral

$$A_1 \succ B_1 \succ C_1 \succ A_2 \succ B_2 \succ C_2 \succ A_3 \succ B_3 \succ C_3 \succ \cdots$$

which will converge towards any desired target in the 'political spectrum'.

Political Instability & Condorcet Spirals



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which will converge towards any desired target in the 'political spectrum'.

Thus, by deploying a suitable sequence of candidates, the electioneer can 'steer' the democracy wherever she wants.

The Borda Count fails the Condorcet Criterion.

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Borda Count				
Preferences	#	A	B	C
$A \succ B \succ C$	60	2×60	60	
$B \succ C \succ A$	40		2×40	40
Total	100	120	140	40
Verdict:		B wins.		

Recall: B wins the Borda Count election, with 140 points.

The Borda Count fails the Condorcet Criterion. Recall our earlier example:

Condorcet Pairwise Votes							
Preferences	#	$A \succ B$	$B \succ A$	$B \succ C$	$C \succ B$	$A \succ C$	$C \succ A$
$A \succ B \succ C$	60	60		60		60	
$B \succ C \succ A$	40		40	40			40
Total	100	60	40	100	0	60	40
Verdict:		$A \succ B$		$B \succ C$		$A \succ C$	

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$B \succ C \succ A$	40		40	40			40
Total	100	60	40	100	0	60	40
Verdict:		$A \succ B$		$B \succ C$		$A \succ C$	

Recall: B wins the Borda Count election, with 140 points.

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Problem: B wins the Borda only because the presence of the 3rd-place candidate C 'boosts' B's score (because B picks up '2nd place' points).

The Borda Count fails the Condorcet Criterion. Recall our earlier example:

2-way Borda count			
Preferences	#	$A \succ B$	$B \succ A$
$A \succ B$	60	$1\times$ 60	
$B \succ A$	40		$1\times$ 40
Total	100	60	40

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If C *withdrew*, then A would win the Borda Count instead of B!

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If C *withdrew*, then A would win the Borda Count instead of B!
Thus, the presence of an **irrelevant alternative** —a third-place candidate like C —can change the outcome of the contest between A and B.

This sensitivity to irrelevant alternatives plagues not only Borda Count, but many voting systems, including the conventional plurality vote.

Sensitivity to Irrelevant Alternatives: Plurality Vote

This sensitivity to irrelevant alternatives plagues not only Borda Count, but many voting systems, including the conventional plurality vote.
For example, consider the following election:

Electorate	
Preferences	#
$A \succ B \succ C$	40
$B \succ C \succ A$	35
$C \succ B \succ A$	25
Total	100

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For example, consider the following election:

Plurality Vote				
Preferences	#	A	B	C
$A \succ B \succ C$	40	40		
$B \succ C \succ A$	35		35	
$C \succ B \succ A$	25			25
Total	100	40	35	25
Verdict:		A wins.		

A wins , but only because the anti-A vote is 'split' between B and C.

This sensitivity to irrelevant alternatives plagues not only Borda Count, but many voting systems, including the conventional plurality vote.
For example, consider the following election:

A versus B (C excluded)			
Preferences	#	$A \succ B$	$B \succ A$
$A \succ B \succ C$	40	40	
$B \succ C \succ A$	35		35
$C \succ B \succ A$	25		25
Total	100	40	60
Verdict:		$B \succ A$	

A wins , but only because the anti-A vote is 'split' between B and C.
If the third-place C withdraws, then B wins with a majority of 60%.

This sensitivity to irrelevant alternatives plagues not only Borda Count, but many voting systems, including the conventional plurality vote.
For example, consider the following election:

A versus B (C excluded)			
Preferences	#	$A \succ B$	$B \succ A$
$A \succ B \succ C$	40	40	
$B \succ C \succ A$	35		35
$C \succ B \succ A$	25		25
Total	100	40	60
	Verdict:	$B \succ A$	

A wins , but only because the anti-A vote is 'split' between B and C.
If the third-place C withdraws, then B wins with a majority of 60%.
Thus, plurality vote is sensitive to the 'irrelevant alternative' C.

This sensitivity to irrelevant alternatives plagues not only Borda Count, but many voting systems, including the conventional plurality vote.
For example, consider the following election:

A versus B (C excluded)			
Preferences	#	$A \succ B$	$B \succ A$
$A \succ B \succ C$	40	40	
$B \succ C \succ A$	35		35
$C \succ B \succ A$	25		25
Total	100	40	60
Verdict:		$B \succ A$	

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Thus, plurality vote is sensitive to the 'irrelevant alternative' C.
In fact, Donald Saari (1989) has shown that almost *any* positional voting system (e.g. (anti)plurality, vote-for-two, etc.) is highly sensitive to irrelevant alternatives....

Sensitivity to irrelevant alternatives is a form of 'collective irrationality'.

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- Apple cobbler
- Banana cream pie
- Chocolate cake.

You think, "I prefer Apple cobbler to Banana pie, and I prefer Banana pie to Chocolate cake (i.e. $A \succ B \succ C$). So I will order the Apple cobbler."

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But then the waiter comes and says, "I'm sorry; the kitchen says there is no more Chocolate cake."

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But then the waiter comes and says, "I'm sorry; the kitchen says there is no more Chocolate cake."
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But then the waiter comes and says, "I'm sorry; the kitchen says there is no more Chocolate cake."
So you say: "Then I will order the Banana cream pie."
Does this make sense? No. But that is exactly what a voting procedure does if it is sensitive to irrelevant alternatives (in this case, Chocolate cake).

Irrelevant alternatives $\Longrightarrow$ electioneering

If a voting procedure is sensitive to 'irrelevant alternatives', then a sly 'electioneer' can manipulate the outcome by introducing 'fringe' candidates.

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A versus B			
Preferences	#	A	B
$A \succ B$	40	40	
$B \succ A$	60		60
Total	100	40	60
Verdict:		B wins	

If a voting procedure is sensitive to 'irrelevant alternatives', then a sly 'electioneer' can manipulate the outcome by introducing 'fringe' candidates. For example, in the following plurality election, the supporters of A might introduce an 'irrelevant' third candidate, C....

This splits the opposition, so A wins instead of B.

Plurality Vote				
Preferences	#	A	B	C
$A \succ B \succ C$	40	40		
$B \succ C \succ A$	35		35	
$C \succ B \succ A$	25			25
Total	100	40	35	25
Verdict:		A wins.		

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Plurality Vote				
Preferences	#	A	B	C
$A \succ B \succ C$	40	40		
$B \succ C \succ A$	35		35	
$C \succ B \succ A$	25			25
Total	100	40	35	25
Verdict:		A wins.		

2-way Borda count			
Preferences	#	$A \succ B$	$B \succ A$
$A \succ B$	60	$1\times$ 60	
$B \succ A$	40		$1\times$ 40
Total	100	60	40
Verdict:		A wins	

On the other hand, suppose the right-hand election was a Borda Count. The supporters of B might introduce an 'irrelevant' third candidate, C....

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Plurality Vote				
Preferences	#	A	B	C
$A \succ B \succ C$	40	40		
$B \succ C \succ A$	35		35	
$C \succ B \succ A$	25			25
Total	100	40	35	25
Verdict:		A wins.		

Borda Count				
Preferences	#	A	B	C
$A \succ B \succ C$	60	2×60	60	
$B \succ C \succ A$	40		2×40	40
Total	100	120	140	40
Verdict:		B wins.		

On the other hand, suppose the right-hand election was a Borda Count. The supporters of B might introduce an 'irrelevant' third candidate, C....
This inflates B's Borda score, so that B wins instead of A.

The inescapable influence of irrelevant alternatives

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Consider an election with three candidates A, B, and C.
There are *six* possible preference orderings a voter could have over these three candidates, namely:

$$A \succ B \succ C, \quad B \succ C \succ A, \quad C \succ A \succ B, \quad B \succ A \succ C, \quad A \succ C \succ B, \quad C \succ B \succ A.$$

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For example, here is one possible profile.

Electorate	
Preferences	#
$A \succ B \succ C$	10
$A \succ C \succ B$	15
$B \succ A \succ C$	12
$B \succ C \succ A$	18
$C \succ A \succ B$	20
$C \succ B \succ A$	25
Total	100

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A 3-candidate profile is a list of how many voters espouse each of these six preference orderings.

For example, here is one possible profile.

Likewise, there are 24 possible preference orderings over four candidates A, B, C, D.

Electorate	
Preferences	#
$A \succ B \succ C$	10
$A \succ C \succ B$	15
$B \succ A \succ C$	12
$B \succ C \succ A$	18
$C \succ A \succ B$	20
$C \succ B \succ A$	25
Total	100

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A 3-candidate profile is a list of how many voters espouse each of these six preference orderings.

For example, here is one possible profile.

Likewise, there are 24 possible preference orderings over four candidates A, B, C, D.

A 4-candidate profile lists how many voters espouse each of these 24 orderings.

Electorate	
Preferences	#
$A \succ B \succ C$	10
$A \succ C \succ B$	15
$B \succ A \succ C$	12
$B \succ C \succ A$	18
$C \succ A \succ B$	20
$C \succ B \succ A$	25
Total	100

For example, here is one possible 4-candidate profile.

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	1
$A \succ B \succ D \succ C$	2
$A \succ C \succ B \succ D$	1
$A \succ C \succ D \succ B$	1
$A \succ D \succ B \succ C$	3
$A \succ D \succ C \succ B$	17
$B \succ A \succ C \succ D$	2
$B \succ A \succ D \succ C$	6
$B \succ C \succ A \succ D$	1
$B \succ C \succ D \succ A$	5
$B \succ D \succ A \succ C$	1
$B \succ D \succ C \succ A$	15
$C \succ A \succ B \succ D$	3
$C \succ A \succ D \succ B$	2
$C \succ B \succ A \succ D$	6
$C \succ B \succ D \succ A$	1
$C \succ D \succ A \succ B$	1
$C \succ D \succ B \succ A$	1
$D \succ A \succ B \succ C$	2
$D \succ A \succ C \succ B$	1
$D \succ B \succ A \succ C$	1
$D \succ B \succ C \succ A$	24
$D \succ C \succ A \succ B$	2
$D \succ C \succ B \succ A$	1
Total	100

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	1
$A \succ B \succ D \succ C$	2
$A \succ C \succ B \succ D$	1
$A \succ C \succ D \succ B$	1
$A \succ D \succ B \succ C$	3
$A \succ D \succ C \succ B$	17
$B \succ A \succ C \succ D$	2
$B \succ A \succ D \succ C$	6
$B \succ C \succ A \succ D$	1
$B \succ C \succ D \succ A$	5
$B \succ D \succ A \succ C$	1
$B \succ D \succ C \succ A$	15
$C \succ A \succ B \succ D$	3
$C \succ A \succ D \succ B$	2
$C \succ B \succ A \succ D$	6
$C \succ B \succ D \succ A$	1
$C \succ D \succ A \succ B$	1
$C \succ D \succ B \succ A$	1
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$D \succ A \succ C \succ B$	1
$D \succ B \succ A \succ C$	1
$D \succ B \succ C \succ A$	24
$D \succ C \succ A \succ B$	2
$D \succ C \succ B \succ A$	1
Total	100

For example, here is one possible 4-candidate profile.

An ordinal voting procedure is a rule which, for any *profile* selects some candidate as the 'winner'.*

(*) This definition requires the procedure to be **anonymous**: all voters are treated exactly the same. This excludes 'weighted voting', or giving some voters 'tie-breaker' or 'veto' powers. It also excludes a **dictatorship**, where one 'voter' makes all the decisions. Anonymity is actually not necessary for what follows, but it makes it simpler.

Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	1
$A \succ B \succ D \succ C$	2
$A \succ C \succ B \succ D$	1
$A \succ C \succ D \succ B$	1
$A \succ D \succ B \succ C$	3
$A \succ D \succ C \succ B$	17
$B \succ A \succ C \succ D$	2
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$B \succ C \succ A \succ D$	1
$B \succ C \succ D \succ A$	5
$B \succ D \succ A \succ C$	1
$B \succ D \succ C \succ A$	15
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$C \succ B \succ D \succ A$	1
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$A \succ B \succ D \succ C$	2
$A \succ C \succ B \succ D$	1
$A \succ C \succ D \succ B$	1
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So is every possible 'agenda' of pairwise votes.

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So is 'plurality vote with a runoff election'.
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There are also many other, more exotic procedures.
Which one is right?

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Desideratum: Unanimity

A profile unanimously prefers candidate **X** if **X** is ranked *first* by 100% of the voters.

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$B \succ A \succ C \succ D$	10
$B \succ A \succ D \succ C$	18
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Total	100

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This rules out stupid procedures like "Always pick A", or "Always pick the candidate who has the *lowest* Borda count", or "Always pick whichever candidate gets the most votes, except for B".

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Example: Borda count, plurality vote, antiplurality vote, etc. all respect unanimity.

Desideratum: Independence of Irrelevant Alternatives

We say that two profiles agree about candidates **X** and **Y** if, in both profiles, exactly the same number of voters feel that $\mathbf{X} \succ \mathbf{Y}$, and exactly the same number feel that $\mathbf{Y} \succ \mathbf{X}$.
(However, voters might differ in how they rank **X** and/or **Y** relative to other candidates, or how they rank other candidates relative to each other.)

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For example, the following two profiles agree about A and B:

A versus B (C excluded)			
Preferences	#	$A \succ B$	$B \succ A$
$A \succ B \succ C$	20	20	
$A \succ C \succ B$	20	20	
$B \succ A \succ C$	15		15
$B \succ C \succ A$	15		15
$C \succ A \succ B$	5	5	
$C \succ B \succ A$	25		25
Total	100	45	55
Verdict:		$B \succ A$	

A versus B (C excluded)			
Preferences	#	$A \succ B$	$B \succ A$
$A \succ B \succ C$	15	15	
$A \succ C \succ B$	15	15	
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$B \succ C \succ A$	20		20
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$C \succ B \succ A$	25		25
Total	100	45	55
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Total	100	45	55
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A voting procedure V satisfies Independence of Irrelevant Alternatives if, whenever two profiles P_1 and P_2 agree about **X** and **Y**, and V makes **X** the winner in P_1, then V can't make **Y** the winner in P_2.

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Plurality vote does *not* satisfy IIA, as the following tables show:

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Preferences	#	$A \succ B$	$B \succ A$
$A \succ B \succ C$	20	20	
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Total	100	45	55
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$C \succ B \succ A$	15		15
Total	100	45	55
	Verdict:	$B \succ A$	

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The two profiles agree about A and B.

But on the left, A wins the plurality vote, whereas on the right, B does.

Plurality Vote				
Preferences	#	A	B	C
$A \succ B \succ C$	20	20		
$A \succ C \succ B$	20	20		
$B \succ A \succ C$	15		15	
$B \succ C \succ A$	15		15	
$C \succ A \succ B$	5			5
$C \succ B \succ A$	25			25
Total	100	40	30	30
Verdict:		A wins.		

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$C \succ A \succ B$	15			15
$C \succ B \succ A$	15			15
Total	100	30	40	30
Verdict:		B wins.		

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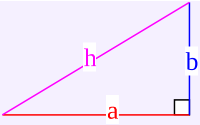

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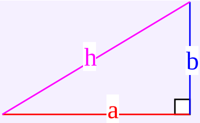

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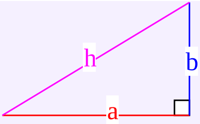

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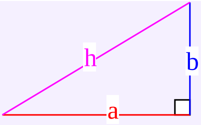

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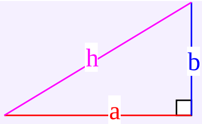

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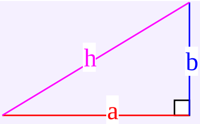

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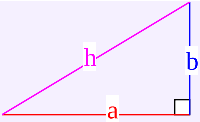

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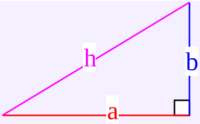

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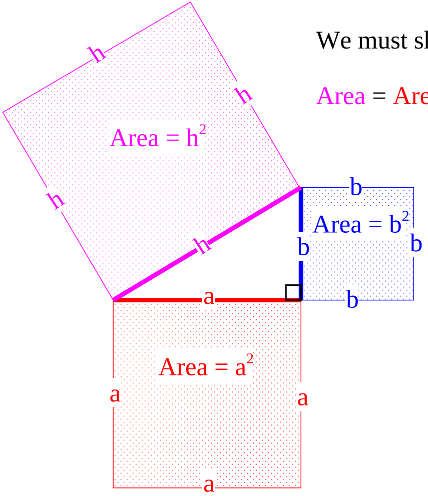

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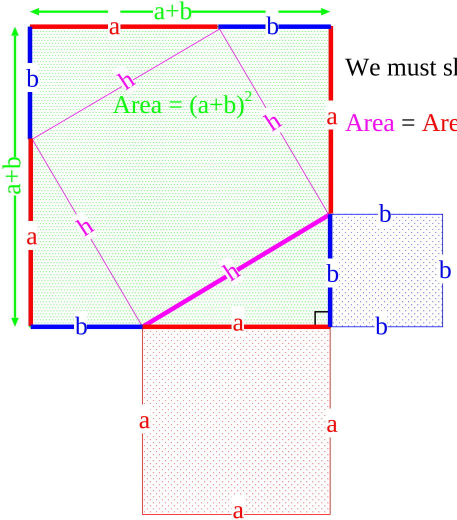

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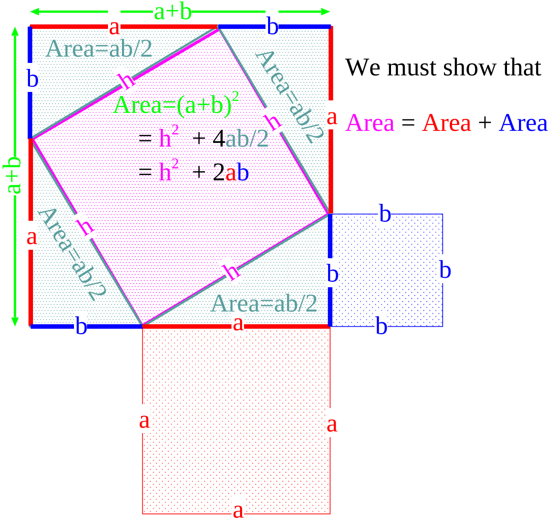

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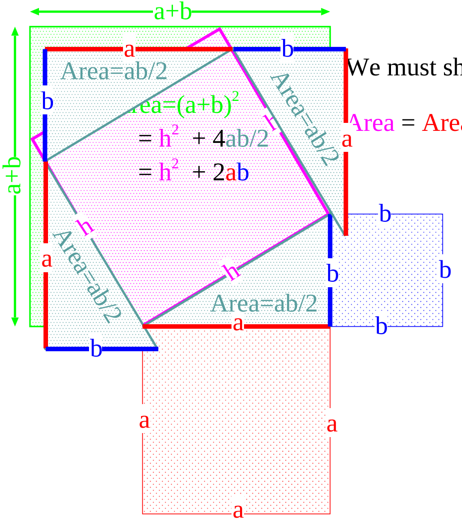

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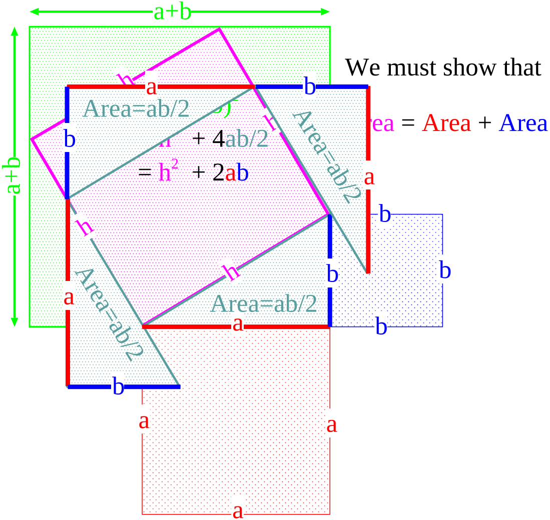

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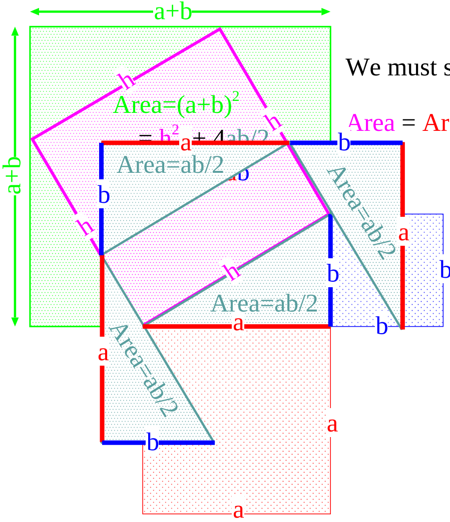

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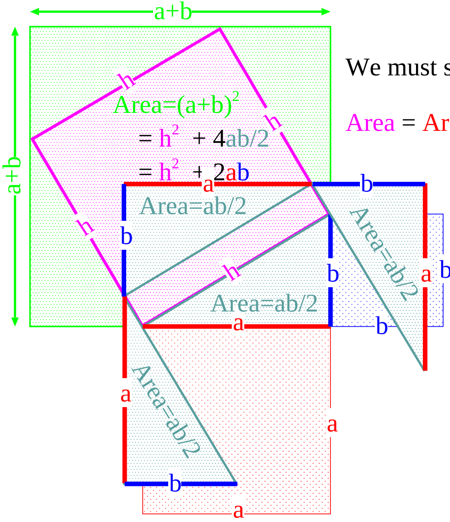

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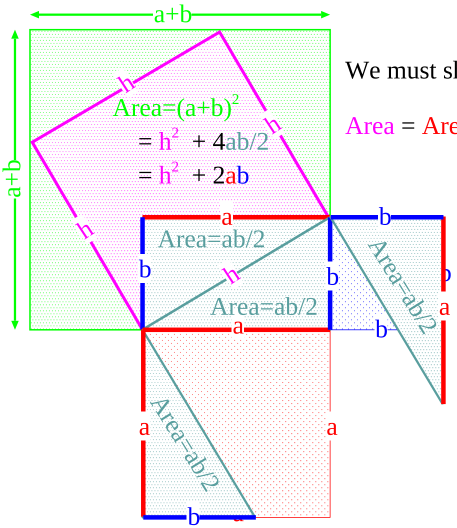

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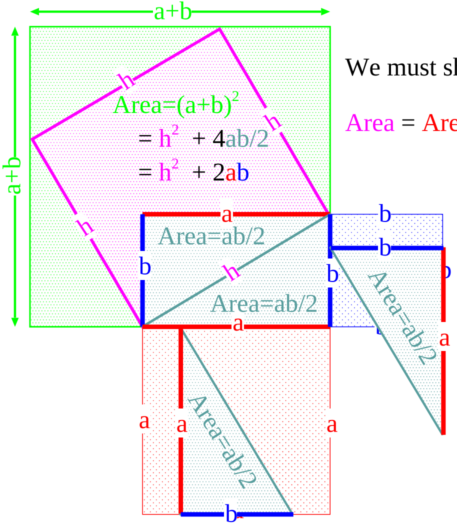

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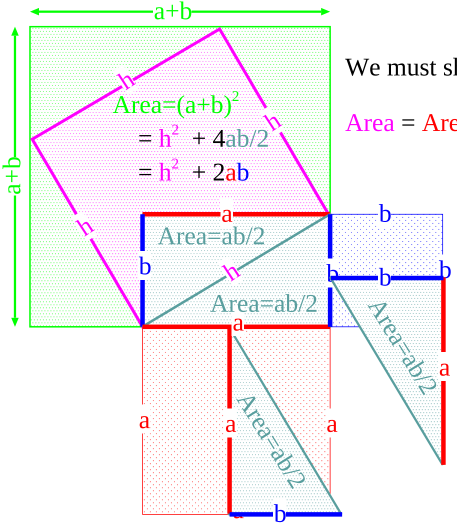

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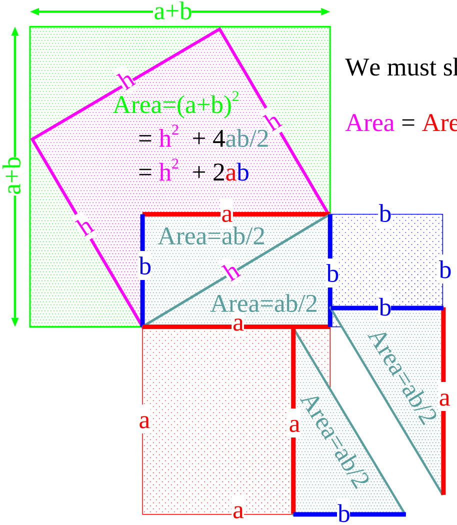

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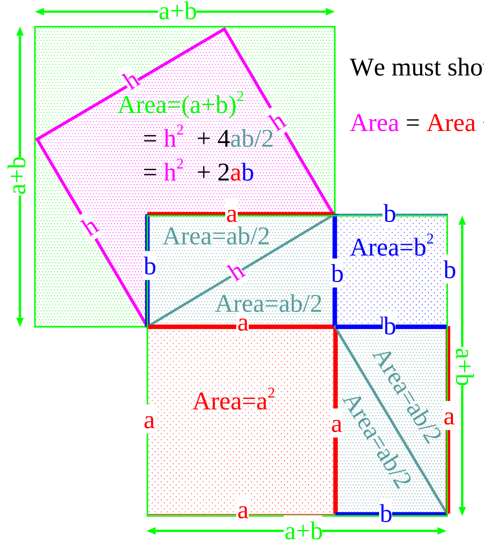

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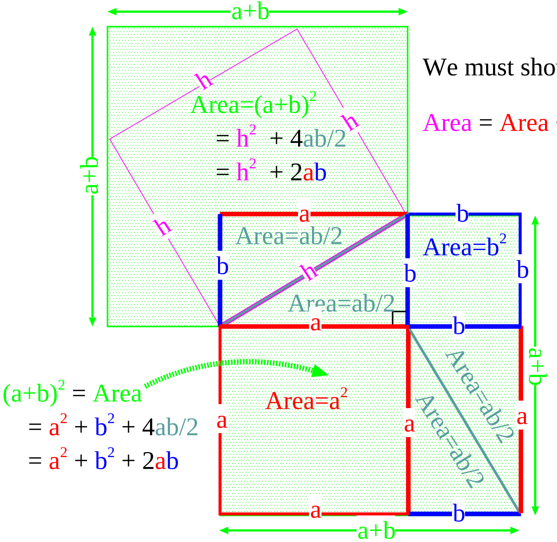

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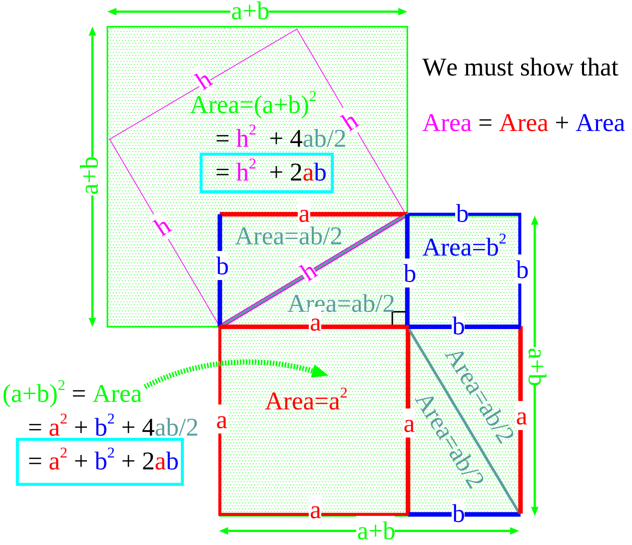

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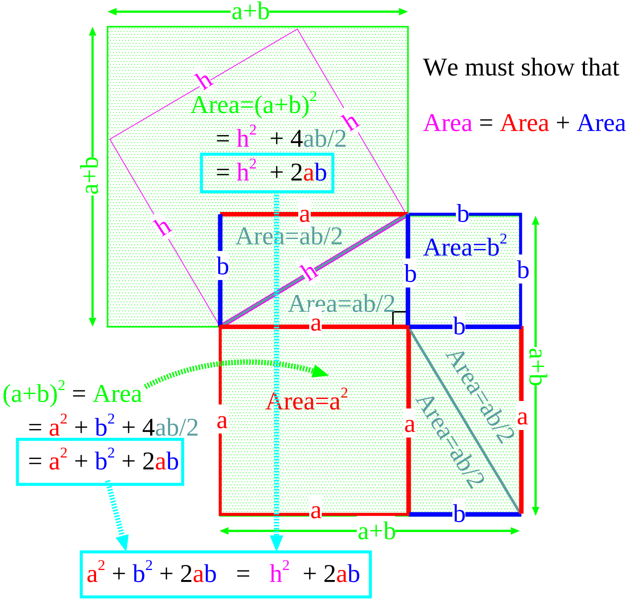

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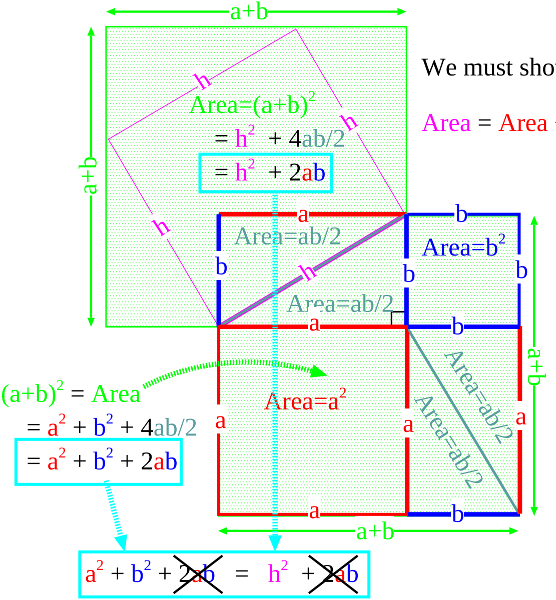

We must show that

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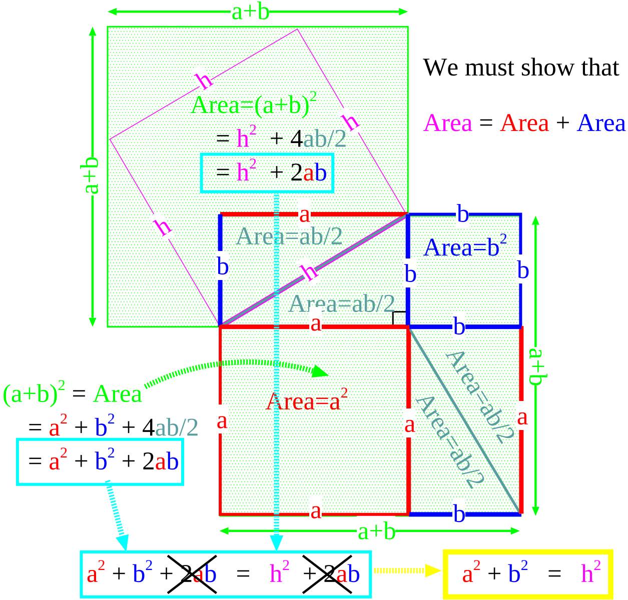

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Proof of Arrow's Theorem: Defeating thresholds

Given a voting rule V, we say that A *can defeat* B under V *with p% support* if there exists a profile of voter preferences where:

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- and where V chooses A as the winner.

For example, if V is plurality vote, then A can defeat B with 55% support, because A is the winner of the following profile:

Plurality Vote				
Preferences	#	A	B	C
$A \succ B \succ C$	55	55		
$B \succ A \succ C$	45		45	
Total	100	55	45	0
Verdict:	A wins.			

(Here, "C" represents all the other candidates besides A and B.)

Given a voting rule V, we say that A *can defeat* B under V *with* $p\%$ *support* if there exists a profile of voter preferences where:

- $p\%$ of the voters believe $A \succ B$;
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- and where V chooses A as the winner.

If V is *any* voting rule respecting unanimity, then A can defeat B with 100% support, because A is the winner of the following profile:

Preferences	%
$A \succ B \succ C$	100
Unanimous verdict:	A

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If A can defeat B with p% support, and V satisfies IIA, then in *any* profile of voter preferences where p% of the voters believe that $A \succ B$, the procedure V *cannot* choose B as the winner.

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In this case, we say that *A always defeats B with $p\%$ support*.

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(That is: if *some* candidate *can* defeat *some* other candidate with p% support, then *any* candidate *always* defeats *any* other candidate with p% support.)

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Preferences	%				
$A \succ B \succ D \succ \mathbf{X}$	60%				
$B \succ D \succ A \succ \mathbf{X}$	40%				
Total:	100%				

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But V must pick *someone* as the winner.

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But V must pick *someone* as the winner. A is the only choice left, so V picks A. Hence A wins. □(Claim 1).

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Preferences	%				$A \succ D$
$A \succ B \succ D \succ \mathbf{X}$	60%				60%
$B \succ D \succ A \succ \mathbf{X}$	40%				0
Total:	100%				60%

(Here, **X**= all other candidates.)

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: 60% of voters think that $A \succ B$. But A *can* defeat B with 60% (by hypothesis); hence A *always* defeats B with 60% (by IIA).
D can't win: 100% of voters think $B \succ D$. But B *can* defeat D with 100% (V respects unanimity); hence B *always* defeats D with 100% (by IIA).
Finally, any other candidate **X** can't win: 100% of voters think that $B \succ \mathbf{X}$. But B *can* defeat **X** with 100% (because V respects unanimity); hence B *always* defeats **X** with 100% (by IIA).
But V must pick *someone* as the winner. A is the only choice left, so V picks A. Hence A wins. $\Box$(Claim 1).

Note: Claim 1 means that A can defeat D with 60% support.

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything.

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$.

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%				
$A \succ B \succ D \succ \mathbf{X}$	$p\%$				
$B \succ D \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%				

(Here, **X**= all other candidates.)

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%				
$A \succ B \succ D \succ \mathbf{X}$	$p\%$				
$B \succ D \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%				

(Here, **X**= all other candidates.)

Claim 1: *V makes A the winner for this profile.*

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%				
$A \succ B \succ D \succ \mathbf{X}$	$p\%$				
$B \succ D \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%				

$\left(\begin{array}{l} \text{Here, } \mathbf{X} = \text{all} \\ \text{other candidates.} \end{array} \right)$

Claim 1: *V makes A the winner for this profile.*

Proof:

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%				
$A \succ B \succ D \succ \mathbf{X}$	$p\%$				
$B \succ D \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%				

(Here, **X**= all other candidates.)

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win:

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$			
$A \succ B \succ D \succ \mathbf{X}$	$p\%$	$p\%$			
$B \succ D \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%	$p\%$			

$\left(\begin{array}{l} \text{Here, } \mathbf{X} = \text{all} \\ \text{other candidates.} \end{array} \right)$

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$.

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$			
$A \succ B \succ D \succ \mathbf{X}$	$p\%$	$p\%$			
$B \succ D \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%	$p\%$			

(Here, **X**= all other candidates.)

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis);

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$			
$A \succ B \succ D \succ \mathbf{X}$	$p\%$	$p\%$			
$B \succ D \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%	$p\%$			

(Here, **X**= all other candidates.)

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis); hence A *always* defeats B with p% (by IIA).

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$			
$A \succ B \succ D \succ \mathbf{X}$	$p\%$	$p\%$			
$B \succ D \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%	$p\%$			

(Here, $\mathbf{X}$= all other candidates.)

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis); hence A *always* defeats B with p% (by IIA).

D can't win:

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$	$B \succ D$		
$A \succ B \succ D \succ \mathbf{X}$	$p\%$	$p\%$	$p\%$		
$B \succ D \succ A \succ \mathbf{X}$	$q\%$		$q\%$		
Total:	100%	$p\%$	100%		

(Here, **X**= all other candidates.)

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis); hence A *always* defeats B with p% (by IIA).
D can't win: 100% of voters think $B \succ D$.

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$	$B \succ D$		
$A \succ B \succ D \succ \mathbf{X}$	$p\%$	$p\%$	$p\%$		
$B \succ D \succ A \succ \mathbf{X}$	$q\%$		$q\%$		
Total:	100%	$p\%$	100%		

(Here, $\mathbf{X}$= all other candidates.)

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis); hence A *always* defeats B with p% (by IIA).
D can't win: 100% of voters think $B \succ D$. But B *can* defeat D with 100% (V respects unanimity);

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$	$B \succ D$		
$A \succ B \succ D \succ \mathbf{X}$	$p\%$	$p\%$	$p\%$		
$B \succ D \succ A \succ \mathbf{X}$	$q\%$		$q\%$		
Total:	100%	$p\%$	100%		

(Here, **X**= all other candidates.)

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis); hence A *always* defeats B with p% (by IIA).
D can't win: 100% of voters think $B \succ D$. But B *can* defeat D with 100% (V respects unanimity); hence B *always* defeats D with 100% (by IIA).

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$	$B \succ D$		
$A \succ B \succ D \succ \mathbf{X}$	$p\%$	$p\%$	$p\%$		
$B \succ D \succ A \succ \mathbf{X}$	$q\%$		$q\%$		
Total:	100%	$p\%$	100%		

(Here, $\mathbf{X}$= all other candidates.)

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis); hence A *always* defeats B with p% (by IIA).
D can't win: 100% of voters think $B \succ D$. But B *can* defeat D with 100% (V respects unanimity); hence B *always* defeats D with 100% (by IIA).
Finally, any other candidate $\mathbf{X}$ can't win:

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$	$B \succ D$	$B \succ \mathbf{X}$	
$A \succ B \succ D \succ \mathbf{X}$ $B \succ D \succ A \succ \mathbf{X}$	$p\%$ $q\%$	$p\%$	$p\%$ $q\%$	$p\%$ $q\%$	
Total:	100%	$p\%$	100%	100%	

(Here, $\mathbf{X}$= all other candidates.

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis); hence A *always* defeats B with p% (by IIA).
D can't win: 100% of voters think $B \succ D$. But B *can* defeat D with 100% (V respects unanimity); hence B *always* defeats D with 100% (by IIA).
Finally, any other candidate $\mathbf{X}$ can't win: 100% of voters think that $B \succ \mathbf{X}$.

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$	$B \succ D$	$B \succ \mathbf{X}$	
$A \succ B \succ D \succ \mathbf{X}$	$p\%$	$p\%$	$p\%$	$p\%$	
$B \succ D \succ A \succ \mathbf{X}$	$q\%$		$q\%$	$q\%$	
Total:	100%	$p\%$	100%	100%	

(Here, $\mathbf{X}$= all other candidates.

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis); hence A *always* defeats B with p% (by IIA).

D can't win: 100% of voters think $B \succ D$. But B *can* defeat D with 100% (V respects unanimity); hence B *always* defeats D with 100% (by IIA).

Finally, any other candidate $\mathbf{X}$ can't win: 100% of voters think that $B \succ \mathbf{X}$. But B *can* defeat $\mathbf{X}$ with 100% (because V respects unanimity);

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$	$B \succ D$	$B \succ \mathbf{X}$	
$A \succ B \succ D \succ \mathbf{X}$	$p\%$	$p\%$	$p\%$	$p\%$	
$B \succ D \succ A \succ \mathbf{X}$	$q\%$		$q\%$	$q\%$	
Total:	100%	$p\%$	100%	100%	

(Here, $\mathbf{X}$= all other candidates.

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis); hence A *always* defeats B with p% (by IIA).
D can't win: 100% of voters think $B \succ D$. But B *can* defeat D with 100% (V respects unanimity); hence B *always* defeats D with 100% (by IIA).
Finally, any other candidate $\mathbf{X}$ can't win: 100% of voters think that $B \succ \mathbf{X}$. But B *can* defeat $\mathbf{X}$ with 100% (because V respects unanimity); hence B *always* defeats $\mathbf{X}$ with 100% (by IIA).

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$	$B \succ D$	$B \succ \mathbf{X}$	
$A \succ B \succ D \succ \mathbf{X}$	$p\%$	$p\%$	$p\%$	$p\%$	
$B \succ D \succ A \succ \mathbf{X}$	$q\%$		$q\%$	$q\%$	
Total:	100%	$p\%$	100%	100%	

(Here, $\mathbf{X}$= all other candidates.

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis); hence A *always* defeats B with p% (by IIA).

D can't win: 100% of voters think $B \succ D$. But B *can* defeat D with 100% (V respects unanimity); hence B *always* defeats D with 100% (by IIA).

Finally, any other candidate $\mathbf{X}$ can't win: 100% of voters think that $B \succ \mathbf{X}$. But B *can* defeat $\mathbf{X}$ with 100% (because V respects unanimity); hence B *always* defeats $\mathbf{X}$ with 100% (by IIA).

V must pick *someone*.

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% \; := \; 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%	$A \succ B$	$B \succ D$	$B \succ \mathbf{X}$	
$A \succ B \succ D \succ \mathbf{X}$ $B \succ D \succ A \succ \mathbf{X}$	$p\%$ $q\%$	$p\%$	$p\%$ $q\%$	$p\%$ $q\%$	
Total:	100%	$p\%$	100%	100%	

(Here, $\mathbf{X}$= all other candidates.

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: $p\%$ of voters think that $A \succ B$. But A *can* defeat B with $p\%$ (by hypothesis); hence A *always* defeats B with p% (by IIA).
D can't win: 100% of voters think $B \succ D$. But B *can* defeat D with 100% (V respects unanimity); hence B *always* defeats D with 100% (by IIA).
Finally, any other candidate $\mathbf{X}$ can't win: 100% of voters think that $B \succ \mathbf{X}$. But B *can* defeat $\mathbf{X}$ with 100% (because V respects unanimity); hence B *always* defeats $\mathbf{X}$ with 100% (by IIA).
V must pick *someone*. A is the only choice left, so V picks A. $\square$(Claim 1).

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Now let p be anything. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$). Consider the following profile:

Preferences	%				$A \succ D$
$A \succ B \succ D \succ \mathbf{X}$	p%				p%
$B \succ D \succ A \succ \mathbf{X}$	q%				
Total:	100%				p%

(Here, $\mathbf{X}$= all other candidates.)

Claim 1: *V makes A the winner for this profile.*

Proof: B can't win: p% of voters think that $A \succ B$. But A *can* defeat B with p% (by hypothesis); hence A *always* defeats B with p% (by IIA).
D can't win: 100% of voters think $B \succ D$. But B *can* defeat D with 100% (V respects unanimity); hence B *always* defeats D with 100% (by IIA).
Finally, any other candidate $\mathbf{X}$ can't win: 100% of voters think that $B \succ \mathbf{X}$. But B *can* defeat $\mathbf{X}$ with 100% (because V respects unanimity); hence B *always* defeats $\mathbf{X}$ with 100% (by IIA).
V must pick *someone*. A is the only choice left, so V picks A. $\square_{\text{(Claim 1)}}$.

Note: Claim 1 means that A can defeat D with p% support.

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Now consider the following profile

Preferences	%				
$C \succ A \succ D \succ \mathbf{X}$	$p\%$				
$D \succ C \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%				

(Here, **X**= all other candidates.)

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Now consider the following profile

Preferences	%				
$C \succ A \succ D \succ \mathbf{X}$	$p\%$				
$D \succ C \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%				

(Here, **X**= all other candidates.)

Claim 2: *V makes C the winner for this profile.*

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Now consider the following profile

Preferences	%				
$C \succ A \succ D \succ \mathbf{X}$	$p\%$				
$D \succ C \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%				

(Here, $\mathbf{X}$= all other candidates.)

Claim 2: *V makes C the winner for this profile.*

Proof:

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Now consider the following profile

Preferences	%				
$C \succ A \succ D \succ \mathbf{X}$	$p\%$				
$D \succ C \succ A \succ \mathbf{X}$	$q\%$				
Total:	100%				

(Here, **X**= all other candidates.)

Claim 2: *V makes C the winner for this profile.*

Proof: D can't win:

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Now consider the following profile

Preferences	%	$A \succ D$			
$C \succ A \succ D \succ \mathbf{X}$	$p\%$	$p\%$			
$D \succ C \succ A \succ \mathbf{X}$	$q\%$	0			
Total:	100%	$p\%$			

(Here, $\mathbf{X}$= all other candidates.)

Claim 2: *V makes C the winner for this profile.*

Proof: D can't win: $p\%$ of voters think $A \succ D$.

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Now consider the following profile

Preferences	%	$A \succ D$			
$C \succ A \succ D \succ \mathbf{X}$	$p\%$	$p\%$			
$D \succ C \succ A \succ \mathbf{X}$	$q\%$	0			
Total:	100%	$p\%$			

(Here, $\mathbf{X}$= all other candidates.)

Claim 2: *V makes C the winner for this profile.*

Proof: D can't win: $p\%$ of voters think $A \succ D$. But A *can* defeat D with $p\%$ support (by Claim 1).

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Now consider the following profile

Preferences	%	$A \succ D$			
$C \succ A \succ D \succ \mathbf{X}$	$p\%$	$p\%$			
$D \succ C \succ A \succ \mathbf{X}$	$q\%$	0			
Total:	100%	$p\%$			

(Here, $\mathbf{X}$= all other candidates.)

Claim 2: *V makes C the winner for this profile.*

Proof: D can't win: $p\%$ of voters think $A \succ D$. But A *can* defeat D with $p\%$ support (by Claim 1). Hence A always defeats D with $p\%$ (by IIA).

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Now consider the following profile

Preferences	%	$A \succ D$			
$C \succ A \succ D \succ \mathbf{X}$	$p\%$	$p\%$			
$D \succ C \succ A \succ \mathbf{X}$	$q\%$	0			
Total:	100%	$p\%$			

(Here, **X**= all other candidates.)

Claim 2: *V makes C the winner for this profile.*

Proof: D can't win: $p\%$ of voters think $A \succ D$. But A *can* defeat D with $p\%$ support (by Claim 1). Hence A always defeats D with $p\%$ (by IIA).

A can't win:

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Now consider the following profile

Preferences	%	$A \succ D$	$C \succ A$		
$C \succ A \succ D \succ \mathbf{X}$	$p\%$	$p\%$	$p\%$		
$D \succ C \succ A \succ \mathbf{X}$	$q\%$	0	$q\%$		
Total:	100%	$p\%$	100%		

(Here, $\mathbf{X}$= all other candidates.)

Claim 2: *V makes C the winner for this profile.*

Proof: D can't win: $p\%$ of voters think $A \succ D$. But A *can* defeat D with $p\%$ support (by Claim 1). Hence A always defeats D with $p\%$ (by IIA).

A can't win: 100% of voters think $C \succ A$.

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

Now consider the following profile

Preferences	%	$A \succ D$	$C \succ A$		
$C \succ A \succ D \succ \mathbf{X}$	$p\%$	$p\%$	$p\%$		
$D \succ C \succ A \succ \mathbf{X}$	$q\%$	0	$q\%$		
Total:	100%	$p\%$	100%		

(Here, **X**= all other candidates.)

Claim 2: *V makes C the winner for this profile.*

Proof: D can't win: $p\%$ of voters think $A \succ D$. But A *can* defeat D with $p\%$ support (by Claim 1). Hence A always defeats D with $p\%$ (by IIA).
A can't win: 100% of voters think $C \succ A$. But C *can* defeat A with 100% (V respects unanimity);

Lemma 1. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates. Suppose A can defeat B with p% support. Then for any candidates C and D, C always defeats D with p% support.*

Proof. Let $q\% := 100 - p\%$. (e.g. if $p = 60\%$ then $q = 40\%$).

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V must pick *someone*.

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V must pick *someone*. C is the only choice left, so V picks C. $\square_{\text{(Claim 2)}}$.

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But this is what we wanted to prove. □

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In other words: *increasing* the number of voters who prefer A over B can never cause A to *lose* to B.

Proof of Arrow's Theorem: Monotonicity

Recall that *Hare's method* suffered from nonmonotonicity: Candidate A can go from being a winner to a loser when we *increase* A's support.
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In other words: *increasing* the number of voters who prefer A over B can never cause A to *lose* to B.

Example: In plurality vote, if A defeats B with 55% support, then A also defeats B with 60% support.

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Lemma 2. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates, and suppose A can defeat B with 60% support. Then A always defeats B with 70% support.*

Proof. For simplicity, first suppose $p := 60$ and $P := 70$.

Lemma 2. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates, and suppose A can defeat B with 60% support. Then A always defeats B with 70% support.*

Proof. For simplicity, first suppose $p := 60$ and $P := 70$. Let C be some third candidate, and consider the following profile:

Preferences	%				
$A \succ C \succ B \succ \mathbf{X}$	60%				
$A \succ B \succ C \succ \mathbf{X}$	10%				
$B \succ A \succ C \succ \mathbf{X}$	30%				
Total:	100%				

$\left(\begin{array}{l}\text{Here, } \mathbf{X} = \text{all} \\ \text{other candidates.}\end{array}\right)$

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Claim: *V makes A the winner for this profile.*

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$A \succ B \succ C \succ \mathbf{X}$	10%		10%	10%	
$B \succ A \succ C \succ \mathbf{X}$	30%		30%	30%	
Total:	100%	60%	100%	100%	

$\left(\text{Here, } \mathbf{X} = \text{all other candidates.} \right)$

Claim: *V makes A the winner for this profile.*

Proof: B can't win: 60% of voters think $C \succ B$. But C always defeats B with 60% by Lemma 1, because A *can* defeat B with 60% (by hypothesis).
C can't win: 100% of voters think $A \succ C$. But A always defeats C with 100% support (because V respects unanimity and IIA).
Finally, any other candidate $\mathbf{X}$ can't win: 100% of voters think $A \succ \mathbf{X}$. But A always defeat $\mathbf{X}$ with 100% (because V respects unanimity and IIA).

Lemma 2. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates, and suppose A can defeat B with 60% support. Then A always defeats B with 70% support.*

Proof. For simplicity, first suppose $p := 60$ and $P := 70$. Let C be some third candidate, and consider the following profile:

Preferences	%	$C \succ B$	$A \succ C$	$A \succ \mathbf{X}$	
$A \succ C \succ B \succ \mathbf{X}$	60%	60%	60%	60%	
$A \succ B \succ C \succ \mathbf{X}$	10%		10%	10%	
$B \succ A \succ C \succ \mathbf{X}$	30%		30%	30%	
Total:	100%	60%	100%	100%	

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$A \succ B \succ C \succ \mathbf{X}$	10%		10%	10%	
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V must pick *someone* as winner. But A is the only choice left.
Thus, V picks A. $\square$(Claim).

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$A \succ B \succ C \succ \mathbf{X}$	10%				10%
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Lemma 2. *Let V be a voting rule which respects unanimity and IIA. Let A and B be two candidates, and suppose A can defeat B with $p\%$ support. Then for* <u>*any*</u> *$P > p$, A always defeats B with $P\%$ support.*

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Proof. Now let p and $P > p$ be arbitrary. Let C be some third candidate, and consider the following profile:

Preferences	%				
$A \succ C \succ B \succ \mathbf{X}$	$p\%$				
$A \succ B \succ C \succ \mathbf{X}$	$P - p\%$				
$B \succ A \succ C \succ \mathbf{X}$	$100 - P\%$				
Total:	100%				

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Total:	100%				

(**X**= all other candidates.)

Claim: *V makes A the winner for this profile.*

Proof:

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Preferences	%	$C \succ B$			
$A \succ C \succ B \succ \mathbf{X}$	$p\%$	$p\%$			
$A \succ B \succ C \succ \mathbf{X}$	$P-p\%$				
$B \succ A \succ C \succ \mathbf{X}$	$100-P\%$				
Total:	100%	$p\%$			

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$B \succ A \succ C \succ \mathbf{X}$	$100 - P\%$		$100 - P\%$	$100 - P\%$	
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$A \succ B \succ C \succ \mathbf{X}$	$P - p\%$		$P - p\%$	$P - p\%$	
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$A \succ B \succ C \succ \mathbf{X}$	$P - p\%$				$P - p\%$
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But this is what we wanted to prove. □

Recall that Condorcet said the winner of an election should be able to defeat any other candidate in a two-way race.

Proof of Arrow's Theorem: Condorcet property

Recall that Condorcet said the winner of an election should be able to defeat any other candidate in a two-way race.

Lemma 3: *Let V be a rule which respects unanimity and IIA. Let $p > 50\%$.*

Recall that Condorcet said the winner of an election should be able to defeat any other candidate in a two-way race.

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Proof.

Lemma 3: *Let V be a rule which respects unanimity and IIA.*
If A are B are any candidates, then A always defeats B with 51% support.

Proof. First suppose $p := 51\%$

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Proof. First suppose $p := 51\%$ Consider the following profile:

Preferences	%			
$A \succ B \succ \mathbf{X}$	51%			
$B \succ A \succ \mathbf{X}$	49%			
Total:	100%			

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Lemma 3: *Let V be a rule which respects unanimity and IIA.*
If A are B are any candidates, then A always defeats B with 51% support.

Proof. First suppose $p := 51\%$ Consider the following profile:

Preferences	%			
$A \succ B \succ \mathbf{X}$	51%			
$B \succ A \succ \mathbf{X}$	49%			
Total:	100%			

(Here, **X**= all other candidates.)

Note that some other candidate **X** can't win:

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Thus, for any $Q > 49$, Lemma 2 says that B defeats A with $Q\%$ support.

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Proof of Arrow's Theorem: Condorcet property

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Proof of Arrow's Theorem: Condorcet property

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Lemma 3: *Let V be a rule which respects unanimity and IIA. Let $p > 50\%$. If A are B are any candidates, then A always defeats B with p% support.*

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Arrow's Theorem: *If an election has three or more candidates, then there is* <u>*no*</u> *ordinal voting procedure which respects both Unanimity and IIA.*

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Preferences	%				
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$				
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$				
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$				
Total:	100%				

(Here, $\mathbf{X}$= all other candidates.)

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Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%				
$A \succ B \succ C \succ \mathbf{X}$ $B \succ C \succ A \succ \mathbf{X}$ $C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$ $33\frac{1}{3}\%$ $33\frac{1}{3}\%$				
Total:	100%				

$\left(\begin{array}{l}\text{Here, } \mathbf{X} = \text{all} \\ \text{other candidates.}\end{array}\right)$

B can't win:

Arrow's Theorem: *If an election has three or more candidates, then there is no ordinal voting procedure which respects both Unanimity and IIA.*

Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$			
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$			
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%			
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$			
Total:	100%	$66\frac{2}{3}\%$			

(Here, $\mathbf{X}$= all other candidates.)

B can't win: $66\frac{2}{3}\%$ of voters think $A \succ B$.

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Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$			
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$			
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%			
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$			
Total:	100%	$66\frac{2}{3}\%$			

(Here, $\mathbf{X}$= all other candidates.)

B can't win: $66\frac{2}{3}\%$ of voters think $A \succ B$. But A always defeats B with $66\frac{2}{3}\%$ support, by Lemma 3.

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$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$			
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%			
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$			
Total:	100%	$66\frac{2}{3}\%$			

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C can't win:

Arrow's Theorem: *If an election has three or more candidates, then there is no ordinal voting procedure which respects both Unanimity and IIA.*

Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$	$B \succ C$		
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$		
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$		
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%		
Total:	100%	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$		

(Here, $\mathbf{X}$= all other candidates.)

B can't win: $66\frac{2}{3}\%$ of voters think $A \succ B$. But A always defeats B with $66\frac{2}{3}\%$ support, by Lemma 3.

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Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$	$B \succ C$		
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$		
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$		
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%		
Total:	100%	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$		

(Here, **X**= all other candidates.)

B can't win: $66\frac{2}{3}\%$ of voters think $A \succ B$. But A always defeats B with $66\frac{2}{3}\%$ support, by Lemma 3.

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Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$	$B \succ C$		
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$		
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$		
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%		
Total:	100%	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$		

(Here, $\mathbf{X}$= all other candidates.)

B can't win: $66\frac{2}{3}\%$ of voters think $A \succ B$. But A always defeats B with $66\frac{2}{3}\%$ support, by Lemma 3.

C can't win: $66\frac{2}{3}\%$ of voters think $B \succ C$. But B always defeats C with $66\frac{2}{3}\%$ support, by Lemma 3.

A can't win:

Arrow's Theorem: *If an election has three or more candidates, then there is no ordinal voting procedure which respects both Unanimity and IIA.*

Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$	$B \succ C$	$C \succ A$	
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	
Total:	100%	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	

(Here, **X**= all other candidates.)

B can't win: $66\frac{2}{3}\%$ of voters think $A \succ B$. But A always defeats B with $66\frac{2}{3}\%$ support, by Lemma 3.

C can't win: $66\frac{2}{3}\%$ of voters think $B \succ C$. But B always defeats C with $66\frac{2}{3}\%$ support, by Lemma 3.

A can't win: $66\frac{2}{3}\%$ of voters think $C \succ A$.

Arrow's Theorem: *If an election has three or more candidates, then there is no ordinal voting procedure which respects both Unanimity and IIA.*

Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$	$B \succ C$	$C \succ A$	
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	
Total:	100%	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	

(Here, **X**= all other candidates.)

B can't win: $66\frac{2}{3}\%$ of voters think $A \succ B$. But A always defeats B with $66\frac{2}{3}\%$ support, by Lemma 3.

C can't win: $66\frac{2}{3}\%$ of voters think $B \succ C$. But B always defeats C with $66\frac{2}{3}\%$ support, by Lemma 3.

A can't win: $66\frac{2}{3}\%$ of voters think $C \succ A$. But C always defeats A with $66\frac{2}{3}\%$ support, by Lemma 3.

Arrow's Theorem: *If an election has three or more candidates, then there is no ordinal voting procedure which respects both Unanimity and IIA.*

Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$	$B \succ C$	$C \succ A$	
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	
Total:	100%	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	

(Here, **X**= all other candidates.)

B can't win: $66\frac{2}{3}\%$ of voters think $A \succ B$. But A always defeats B with $66\frac{2}{3}\%$ support, by Lemma 3.

C can't win: $66\frac{2}{3}\%$ of voters think $B \succ C$. But B always defeats C with $66\frac{2}{3}\%$ support, by Lemma 3.

A can't win: $66\frac{2}{3}\%$ of voters think $C \succ A$. But C always defeats A with $66\frac{2}{3}\%$ support, by Lemma 3.

Finally, no other candidate **X** can win:

Arrow's Theorem: *If an election has three or more candidates, then there is* <u>*no*</u> *ordinal voting procedure which respects both Unanimity and IIA.*

Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$	$B \succ C$	$C \succ A$	$A \succ \mathbf{X}$
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Total:	100%	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	100%

(Here, $\mathbf{X}$= all other candidates.)

B can't win: $66\frac{2}{3}\%$ of voters think $A \succ B$. But A always defeats B with $66\frac{2}{3}\%$ support, by Lemma 3.

C can't win: $66\frac{2}{3}\%$ of voters think $B \succ C$. But B always defeats C with $66\frac{2}{3}\%$ support, by Lemma 3.

A can't win: $66\frac{2}{3}\%$ of voters think $C \succ A$. But C always defeats A with $66\frac{2}{3}\%$ support, by Lemma 3.

Finally, no other candidate $\mathbf{X}$ can win: 100% of voters think $A \succ \mathbf{X}$.

Arrow's Theorem: *If an election has three or more candidates, then there is no ordinal voting procedure which respects both Unanimity and IIA.*

Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$	$B \succ C$	$C \succ A$	$A \succ \mathbf{X}$
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Total:	100%	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	100%

(Here, $\mathbf{X}$= all other candidates.)

B can't win: $66\frac{2}{3}\%$ of voters think $A \succ B$. But A always defeats B with $66\frac{2}{3}\%$ support, by Lemma 3.

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A can't win: $66\frac{2}{3}\%$ of voters think $C \succ A$. But C always defeats A with $66\frac{2}{3}\%$ support, by Lemma 3.

Finally, no other candidate $\mathbf{X}$ can win: 100% of voters think $A \succ \mathbf{X}$. But A always defeats $\mathbf{X}$ with 100% support, by unanimity and IIA.

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Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$	$B \succ C$	$C \succ A$	$A \succ \mathbf{X}$
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Total:	100%	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	100%

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Finally, no other candidate **X** can win: 100% of voters think $A \succ \mathbf{X}$. But A always defeats **X** with 100% support, by unanimity and IIA.

Thus, *no candidate can win* in this profile.

Arrow's Theorem: *If an election has three or more candidates, then there is* <u>*no*</u> *ordinal voting procedure which respects both Unanimity and IIA.*

Proof. (by contradiction) Suppose V was such a procedure. Consider profile:

Preferences	%	$A \succ B$	$B \succ C$	$C \succ A$	$A \succ \mathbf{X}$
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Total:	100%	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	100%

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Thus, *no candidate can win* in this profile. But V is supposed to *always* pick a winner.

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Preferences	%	$A \succ B$	$B \succ C$	$C \succ A$	$A \succ \mathbf{X}$
$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
Total:	100%	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	$66\frac{2}{3}\%$	100%

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$A \succ B \succ C \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$
$B \succ C \succ A \succ \mathbf{X}$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
$C \succ A \succ B \succ \mathbf{X}$	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$	0%	$33\frac{1}{3}\%$	$33\frac{1}{3}\%$
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Thus, *no candidate can win* in this profile. But V is supposed to *always* pick a winner. Thus, we have a contradiction. Thus, no such voting procedure V can exist. $\square$

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The general version of Arrow's Theorem states that the only 'voting procedure' which respects unanimity and IIA is a *dictatorship*, where *one* voter has *all* the power.

Arrow's Theorem: Is the problem 'egalitarianism'?

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The general version of Arrow's Theorem states that the only 'voting procedure' which respects unanimity and IIA is a *dictatorship*, where *one* voter has *all* the power. This is hardly a desirable form of 'democracy.'

Arrow's Theorem: Indecisive and non-ordinal procedures

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Answer: Yes. We will next consider several 'non-ordinal' voting procedures.

Next:

- Escape from Arrow? Nonordinal voting systems.
- Strategic Voting: The Gibbard-Satterthwaite Theorem
- Representative democracy: Paradoxes.
- Voting power indices.
- What is democracy? ‘Liberalism vs. Populism’.
- Social choice and social welfare functions.

Non-ordinal Voting Systems

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AV is *not* an 'ordinal voting system', so Arrow's Theorem doesn't apply.

Approval Voting

For example, suppose 5 voters rate 4 candidates on a scale from 0 to 1:

Candidate ⇒		A		B		C		D	
Voter ⇓	Threshold	Rating	Vote	Rating	Vote	Rating	Vote	Rating	Vote
#1		0.95		0.75		0.31		0.04	
#2		0.94		0.05		0.33		0.73	
#3		0.06		0.73		0.98		0.32	
#4		0.04		0.65		0.31		0.91	
#5		0.01		0.92		0.25		0.75	
Total: ⇒									
Outcome ⇒									

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#5	0.5	0.01		0.92	*	0.25		0.75	*
Total: ⇒			2		4		2		2
Outcome ⇒		B wins, when each person approves above some threshold.							

Suppose each voter approves all candidates whom she rates at or above some personal threshold.... In this case, B wins the election.

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Approval Voting

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Candidate ⇒		A		B		C		D	
Voter ⇓	Threshold	Rating	Vote	Rating	Vote	Rating	Vote	Rating	Vote
#1	0.9	0.95	*	0.75		0.31		0.04	
#2	0.9	0.94	*	0.05		0.33		0.73	
#3	0.9	0.06		0.73		0.98	*	0.32	
#4	0.9	0.04		0.65		0.31		0.91	*
#5	0.9	0.01		0.92	*	0.25		0.75	
Total: ⇒			2		1		1		1
Outcome ⇒		A wins the *de facto* 'plurality vote'							

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For example, suppose 5 voters rate 4 candidates on a scale from 0 to 1:

Candidate ⇒		A		B		C		D	
Voter ⇓	Threshold	Rating	Vote	Rating	Vote	Rating	Vote	Rating	Vote
#1	0.1	0.95	*	0.75	*	0.31	*	0.04	
#2	0.1	0.94	*	0.05		0.33	*	0.73	*
#3	0.1	0.06		0.73	*	0.98	*	0.32	*
#4	0.1	0.04		0.65	*	0.31	*	0.91	*
#5	0.1	0.01		0.92	*	0.25	*	0.75	*
Total: ⇒			2		4		5		4
Outcome ⇒		C wins the *de facto* 'antiplurality vote'							

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Total: ⇒			2		4		5		4
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Thus, in reality, Approval Voting will devolve into either a *de facto* plurality vote or antiplurality vote, with all the weaknesses of these methods.

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Again, CV is not an 'ordinal' system, so Arrow's Theorem doesn't apply.

Cumulative Voting

For example, again suppose that 5 voters each rate 4 candidates on a scale from 0 to 1:

Candidate ⇒		A		B		C		D	
Voter ⇓	Strategy	Rating	Vote	Rating	Vote	Rating	Vote	Rating	Vote
#1		0.95		0.75		0.31		0.04	
#2		0.94		0.05		0.33		0.73	
#3		0.06		0.73		0.98		0.32	
#4		0.04		0.65		0.31		0.91	
#5		0.01		0.92		0.25		0.75	
Total: ⇒									
Outcome ⇒									

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Candidate ⇒		A		B		C		D	
Voter ⇓	Strategy	Rating	Vote	Rating	Vote	Rating	Vote	Rating	Vote
#1	Borda	0.95	4	0.75	3	0.31	2	0.04	1
#2	Vote for 2	0.94	5	0.05	0	0.33	0	0.73	5
#3	Vote for 1	0.06	0	0.73	0	0.98	10	0.32	0
#4	Vote for 3	0.04	0	0.65	3.33	0.31	3.33	0.91	3.33
#5	Vote for 3	0.01	0	0.92	3.33	0.25	3.33	0.75	3.33
Total: ⇒									
Outcome ⇒									

Suppose each voter has 10 ‘points’, and the voters adopt various point-allocation strategies, as shown....

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#5	Vote for 3	0.01	0	0.92	3.33	0.25	3.33	0.75	3.33
Total: ⇒			9		9.66		18.66		12.66
Outcome ⇒		C wins, when voters use various strategies.							

Suppose each voter has 10 'points', and the voters adopt various point-allocation strategies, as shown.... ...then C will win.

For example, again suppose that 5 voters each rate 4 candidates on a scale from 0 to 1:

Candidate ⇒		A		B		C		D	
Voter ⇓	Strategy	Rating	Vote	Rating	Vote	Rating	Vote	Rating	Vote
#1	Concentrate	0.95	10	0.75	0	0.31	0	0.04	0
#2	Concentrate	0.94	10	0.05	0	0.33	0	0.73	0
#3	Concentrate	0.06	0	0.73	0	0.98	10	0.32	0
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Problem: Each voters will maximize her impact by concentrating *all ten points* on her favourite (amongst those who has any chance of winning).

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#5	Concentrate	0.01	0	0.92	10	0.25	0	0.75	0
Total: ⇒			20		10		10		10
Outcome ⇒		A wins the *de facto* 'plurality vote'							

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Problem: Each voters will maximize her impact by concentrating *all ten points* on her favourite (amongst those who has any chance of winning).

Thus, in reality, CV will function just like plurality vote.

In Relative Utilitarianism (RU, also called range voting, ratings summation, or score system) each voter gives each candidate a fractional numerical 'score' on a scale of 0 to 1 (where 1=best and 0=worst).
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- Her most favourite candidate would get a score of 1.0.
- Her 2nd best candidate might get a score of 0.95, etc.

Non-ordinal voting systems: Relative Utilitarianism

In Relative Utilitarianism (RU, also called range voting, ratings summation, or score system) each voter gives each candidate a fractional numerical 'score' on a scale of 0 to 1 (where 1=best and 0=worst).
For example:

- Her most favourite candidate would get a score of 1.0.
- Her 2nd best candidate might get a score of 0.95, etc.
- If she is ambivalent about candidate **X**, she might give **X** a score of 0.5.

Non-ordinal voting systems: Relative Utilitarianism

In Relative Utilitarianism (RU, also called range voting, ratings summation, or score system) each voter gives each candidate a fractional numerical 'score' on a scale of 0 to 1 (where 1=best and 0=worst).
For example:

- Her most favourite candidate would get a score of 1.0.
- Her 2nd best candidate might get a score of 0.95, etc.
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Again, RU is not an 'ordinal' voting system (rather, it is a 'cardinal' voting system), so Arrow's Theorem doesn't apply.

For example, suppose 5 voters each rate 4 candidates from 0 to 1:

Candidate ⇒		A		B		C		D	
Voter ⇓	Strategy	Rating	Vote	Rating	Vote	Rating	Vote	Rating	Vote
#1		0.95		0.75		0.31		0.04	
#2		0.94		0.05		0.33		0.73	
#3		0.06		0.73		0.98		0.32	
#4		0.04		0.65		0.31		0.91	
#5		0.01		0.92		0.25		0.75	
Total: ⇒									
Outcome ⇒									

Relative Utilitarianism

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Candidate ⇒		A		B		C		D	
Voter ⇓	Strategy	Rating	Vote	Rating	Vote	Rating	Vote	Rating	Vote
#1	Honest	0.95	0.95	0.75	0.75	0.31	0.31	0.04	0.04
#2	Honest	0.94	0.94	0.05	0.05	0.33	0.33	0.73	0.75
#3	Honest	0.06	0.06	0.73	0.73	0.98	0.98	0.32	0.32
#4	Honest	0.04	0.04	0.65	0.65	0.31	0.31	0.91	0.91
#5	Honest	0.01	0.01	0.92	0.92	0.25	0.25	0.75	0.75
Total: ⇒									
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#2	Honest	0.94	0.94	0.05	0.05	0.33	0.33	0.73	0.75
#3	Honest	0.06	0.06	0.73	0.73	0.98	0.98	0.32	0.32
#4	Honest	0.04	0.04	0.65	0.65	0.31	0.31	0.91	0.91
#5	Honest	0.01	0.01	0.92	0.92	0.25	0.25	0.75	0.75
Total: ⇒			2.00		3.10		2.18		2.77
Outcome ⇒		B wins, when each person votes honestly.							

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#3	Honest	0.06	0.06	0.73	0.73	0.98	0.98	0.32	0.32
#4	Honest	0.04	0.04	0.65	0.65	0.31	0.31	0.91	0.91
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Problem: Each voters will maximize her impact by giving a score of 1.0 to her favourite(s), and 0 to everyone else.

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For example, suppose 5 voters each rate 4 candidates from 0 to 1:

Candidate ⇒		A		B		C		D	
Voter ⇓	Strategy	Rating	Vote	Rating	Vote	Rating	Vote	Rating	Vote
#1	Plurality	0.95	1.00	0.75	0.00	0.31	0.00	0.04	0.00
#2	Plurality	0.94	1.00	0.05	0.00	0.33	0.00	0.73	0.00
#3	Plurality	0.06	0.00	0.73	0.00	0.98	1.00	0.32	0.00
#4	Plurality	0.04	0.00	0.65	0.00	0.31	0.00	0.91	1.00
#5	Plurality	0.01	0.00	0.92	1.00	0.25	0.00	0.75	0.00
Total: ⇒			2.00		1.00		1.00		1.00
Outcome ⇒		A wins the *de facto* 'plurality vote'							

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Thus, in reality, RU will function just like Approval Voting.

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Candidate ⇒		A		B		C		D	
Voter ⇓	Strategy	Rating	Vote	Rating	Vote	Rating	Vote	Rating	Vote
#1	Antiplurality	0.95	1.00	0.75	1.00	0.31	1.00	0.04	0.00
#2	Antiplurality	0.94	1.00	0.05	0.00	0.33	1.00	0.73	1.00
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#5	Antiplurality	0.01	0.00	0.92	1.00	0.25	1.00	0.75	1.00
Total: ⇒			2.00		4.00		5.00		4.00
Outcome ⇒		C wins the *de facto* 'antiplurality vote'							

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Strategic Voting

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For example, consider this election:

Plurality Vote				
Preferences	#	A	B	C
$A \succ B \succ C$	45	45		
$B \succ C \succ A$	40		40	
$C \succ B \succ A$	15			15
Total	100	45	40	15
Verdict:		A wins.		

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Total	100	45	50	5
Verdict:		B wins.		

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But if the outcome is the result of strategic voting, how can we say it really reflects the 'Will of the People'?

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Thus, public opinion polls actually *facilitate* strategic voting.

Susceptibility to Strategic Voting

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For example, suppose there are three candidates, A, B, and C, so voters are distributed over six possible preference orders:

$$A \succ B \succ C, \quad B \succ C \succ A, \quad C \succ A \succ B, \quad B \succ A \succ C, \quad A \succ C \succ B, \quad C \succ B \succ A.$$

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Theorem. (Donald Saari, 1990)
Suppose all possible distributions of voters over these six orders are equally likely. Then, amongst all positional voting systems, the Borda Count is the least susceptible to strategic voting.

Representative Democracy and Compound-Majority Paradoxes

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  In the open list version, people can vote for individual list members.

Hybrid Proportional Systems

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Regardless of how they are chosen, the use of delegates introduces additional paradoxes and pathologies into democracy.

Regional Representation: Single-Party Domination

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	District
Preference	X
$A \succ B \succ C \succ D$	26%
$B \succ C \succ D \succ A$	24%
$C \succ D \succ B \succ A$	25%
$D \succ B \succ C \succ A$	25%
Verdict:	A wins

As we've seen, if there are four parties (e.g. LiberAl, Bloc, Conservative and New Democratic), then it's possible for the A candidate in District X to get elected with only 26% of the votes —even if the A candidate is despised by the other 74% of the voters.

Regional Representation: Single-Party Domination

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	District					
Preference	1	2	3	4	5	Nationwide
$A \succ B \succ C \succ D$	26%	26%	26%	26%	26%	26%
$B \succ C \succ D \succ A$	24%	24%	24%	24%	24%	24%
$C \succ D \succ B \succ A$	25%	25%	25%	25%	25%	25%
$D \succ B \succ C \succ A$	25%	25%	25%	25%	25%	25%
Verdict:	A wins	A wins	A wins	A wins	A wins	A gets all seats

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However, stranger things can happen, even when there are only *two* candidates, and one has a strict majority....

Regional Representation: The Referendum Paradox

Consider a referendum on some proposal.

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Suppose there are 100 voters living in five districts with 20 voters each.
Suppose popular support for the proposal is distributed as follows:

	District					
Preference	1	2	3	4	5	Total
Yes	20	20	8	8	8	64
No	0	0	12	12	12	36
People's verdict:						

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64% of all voters say Yes, so the proposal is easily approved in a referendum.

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Parliamentary verdict:	No					

64% of all voters say Yes, so the proposal is easily approved in a referendum.
But instead, suppose the question is decided by a Parliament chosen through Regional Representation.
Then 3 out of 5 Parliamentarians can each honestly say that a majority of her constituents reject the proposal.
Thus, if each Parliamentarian obeys the 'wishes' of her constituents, the proposal would be rejected by a vote of 3 to 2.

Even with Proportional Representation, there can be problems.

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Suppose there are two parties, A and B, which propose different policies on three distinct 'issues'.

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Voters	Issue 1	Issue 2	Issue 3	Votes for
20%	A	B	B	
20%	B	A	B	
20%	B	B	A	
20%	A	A	A	
20%	A	A	A	

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20%	A	B	B	B
20%	B	A	B	B
20%	B	B	A	B
20%	A	A	A	A
20%	A	A	A	A

Suppose each person votes for a party if she agrees with it on most issues.

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20%	B	B	A	B
20%	A	A	A	A
20%	A	A	A	A
Parliamentary Majority:				B (60%)

Suppose each person votes for a party if she agrees with it on most issues.
Then 60% will vote for B, so the B party will control a majority in Parliament, and will implement B party policies.

Proportional Representation: Ostrogorski's Paradox

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20%	A	B	B	B
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20%	B	B	A	B
20%	A	A	A	A
20%	A	A	A	A
Majority Position:	A	A	A	
Parliamentary Majority:				B (60%)

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But on *every issue*, a majority of voters prefer A's policy to B's policy.

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20%	B	A	B	B
20%	B	B	A	B
20%	A	A	A	A
20%	A	A	A	A
Majority Position:	A	A	A	
Parliamentary Majority:				B (60%)

Suppose each person votes for a party if she agrees with it on most issues. Then 60% will vote for B, so the B party will control a majority in Parliament, and will implement B party policies.

But on *every issue*, a majority of voters prefer A's policy to B's policy.

This paradox was discovered by Moise Ostrogorski (1902), who was highly critical of the role of political parties in democratic politics.

Anscombe's Paradox

Ostrogorski's paradox happens because people can't vote for individual *policies*; instead, they must vote for a party's 'platform' of policies.

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Voters	Issue 1		Issue 2		Issue 3		
	Policy		Policy		Policy		
20%	A		C		F		
20%	B		D		F		
20%	B		C		E		
20%	A		D		E		
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Verdict:	A (60%)		D (60%)		E (60%)		

In referenda, A, D, and E will be chosen, each with 60% support.

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Voters	Issue 1		Issue 2		Issue 3		
	Policy	Happy?	Policy	Happy?	Policy	Happy?	
20%	A	☺	C	☹	F	☹	
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20%	B	☹	C	☹	E	☺	
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Voters	Issue 1		Issue 2		Issue 3		Overall
	Policy	Happy?	Policy	Happy?	Policy	Happy?	Satisfaction
20%	A	☺	C	☹	F	☹	☹
20%	B	☹	D	☺	F	☹	☹
20%	B	☹	C	☹	E	☺	☹
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Suppose a voter is 'satisfied overall' if she is happy with 2 out of 3 referenda.

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Each referendum outcome will make some voters happy and others unhappy.
Suppose a voter is 'satisfied overall' if she is happy with 2 out of 3 referenda. Then 60% of the voters are dissatisfied overall!

Possible solution: Consensus via creative compromise

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The only solution is to attempt some ‘creative compromise’, and replace A and/or B with new proposals (say, A_1 and B_1), one of which is likely to satisfy 3/4 of the voters. (The same goes for C vs. D, and E vs. F).

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How can we find this 'creative compromise'? Only through widespread dialogue and deliberation....

Voting Power

All animals are equal...

Suppose a 100-seat Parliament is spit between parties A, B, C and D:

	Parliament Breakdown			
Party	A	B	C	D
#Seats	28	26	26	20

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If 'power' $\approx$ ($\#$ of seats), then all four parties have roughly the same power.

Maybe A has a little more, and D has a little less.

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	Partisan Bloc Votes						
Party	A	B	C	D	Total Score		
#Seats	28	26	26	20	Yes	No	Outcome
	No	No	No	No	0	100	No
	No	No	No	Yes	20	80	No
	Yes	No	No	No	28	72	No
	Yes	No	No	Yes	48	52	No
	No	Yes	No	No	26	74	No
	No	Yes	No	Yes	46	54	No
	Yes	Yes	No	No	54	46	Yes
	Yes	Yes	No	Yes	74	26	Yes
	No	Yes	Yes	No	52	48	Yes
	No	Yes	Yes	Yes	72	28	Yes
	Yes	Yes	Yes	No	80	20	Yes
	Yes	Yes	Yes	Yes	100	0	Yes

Assume each party votes as a 'bloc' —either because all members have identical ideologies, or because of strong 'party discipline'.

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Answer: Never.

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Answer: Never.

Conclusion: Although party D has 20% of the seats, D has *zero* power.

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Suppose a 100-seat Parliament is spit between parties A, B, C and D:

If 'power' $\approx$ (# of seats), then all four parties have roughly the same power.

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	Yes	Yes	Yes	Yes	100	0	Yes

Assume each party votes as a 'bloc' —either because all members have identical ideologies, or because of strong 'party discipline'.

Question: How often does party D's vote actually change the outcome?

Answer: Never.

Conclusion: Although party D has 20% of the seats, D has *zero* power.

Also, parties A, B and C all have *exactly the same* power, even though A has slightly more seats.

Idea: The correct measure of your 'voting power' is not the percentage of seats you control.....

Voting Power Indices

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	Parliament Breakdown		
Party	A	B	C
#Seats	48	47	5

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Example: Suppose three parties split Parliament as shown above.

Voting Power Indices

	Partisan Bloc Votes			Total Score		
Party	A	B	C			
#Seats	48	47	5	Yes	No	Outcome
	No	No	No	0	100	No
	No	No	Yes	5	95	No
	No	Yes	No	47	53	No
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Example: Suppose three parties split Parliament as shown above.

No *one* party can dictate the outcome, but any team of *two* parties can.

Voting Power Indices

	Partisan Bloc Votes			Total Score		
Party	A	B	C			
#Seats	48	47	5	Yes	No	Outcome
	No	No	No	0	100	No
	No	No	Yes	5	95	No
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	No	Yes	Yes	52	48	Yes
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	Yes	Yes	No	95	5	Yes
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A **voting power index** (VPI) is an *estimate* of this probability, based on assumptions about how 'voting coalitions' can form.
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Voting Power Indices

	Partisan Bloc Votes					
Party	A	B	C	Total Score		
#Seats	48	47	5	Yes	No	Outcome
	No	No	No	0	100	No
	No	No	Yes	5	95	No
	No	Yes	No	47	53	No
	No	Yes	Yes	52	48	Yes
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In particular C has exactly *the same* power as A and B, even though C has only 5% of the votes (C holds "the balance of power").

The Shapley-Shubik Index

Suppose Parliament votes on a bill. We can linearly order the parties from those *most in favour* of the bill to those *most opposed* to it:

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$$SSI(p) \;:=\; \frac{\#(\text{orderings where } p \text{ is pivotal})}{\#(\text{orderings of } N \text{ parties})} \;=\; \frac{\#(\text{orderings where } p \text{ is pivotal})}{N!}$$

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(Here, $N! := N \cdot (N-1) \cdots 3 \cdot 2 \cdot 1$. Example: $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.)

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When combined with the previous 'voting paradoxes', the power of a party may be even more wildly disproportionate to its share of the popular vote.

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Liberalism, Populism, and Social Choice

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Thus, Riker advocates the more pessimistic 'Liberal' view of democracy.

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Each of these social choice functions satisfies different (and mutually exclusive) mathematical axioms (which encode philosophical ideals).

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Thus, it is *never* optimal to exaggerate her preferences in the CPM (you can mathematically prove this, if you assume each voter is 'risk neutral').

Even if we can identify the 'Will of the People' (e.g. via a social choice function), how can we measure it, without being misled by strategic voting?

For example: suppose we implement Bentham's Utilitarianism by asking each voter to assign a 'utility' to each candidate. Then each voter will exaggerate her preferences (e.g. *overstate* the utility of her favourite candidates, and *understate* the utility of her least prefered candidates).

Each voting system is like a 'game', and sometimes a voter's best 'strategy' is to be dishonest. We want to design a 'voting game' where each voter's best strategy is always to be honest. This is the subject of a branch of mathematical economics called mechanism design.

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Problem: People *aren't* 'risk neutral'. Also, CPM favours rich voters.

Some other topics we haven't even touched upon:

May's Theorem says that, if there are only two candidates, then simple Majority Vote really is the 'best' way to aggregate preferences.

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Public choice theory applies methods of economics to political science; e.g. the role of campaign finance in elections and policy formation; the corruption of legislators and bureaucrats by special interests, etc.

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There are no simple solutions to these problems.
But mathematical analysis can help us to identify and mitigate them.

Further Reading I

All slides for this lecture are available at
http://xaravve.trentu.ca/voting.pdf

Further Reading

Basic Voting Theory (general audience)

- William Poundstone *Gaming the vote: Why elections aren't fair (and what we can do about it)*. Hill & Wang, New York, 2008. 338 pages. ISBN:978-0-8090-4893-9
- Ya-Ping Yee's colour 'visualizations' of various voting methods: http://zesty.ca/voting/sim/
- *Electorama* website: http://wiki.electorama.com/wiki/Main_Page

Basic Voting Theory (middle school level)

- Saari, Donald G. *Chaotic elections! A mathematician looks at voting*. American Mathematical Society, Providence, RI, 2001. 159 pages. ISBN: 0-8218-2847-991-01
- Nurmi, Hannu. *Voting paradoxes and how to deal with them*. Springer-Verlag, Berlin, 1999. 153 pages. ISBN: 3-540-66236-791B12

Voting Theory (highschool or intro college level)

Further Reading II

- Taylor, Alan D. *Mathematics and politics. Strategy, voting, power and proof.* Springer-Verlag, New York, 1995. 284 pages. ISBN: 0-387-94500-8
- Hodge, Jonathan K. and Klima, Richard E. *The mathematics of voting and elections: a hands-on approach.* American Mathematical Society, Providence, RI, 2005. 226 pages. ISBN: 0-8218-3798-2
- Riker, William H. *Liberalism against Populism*, Waveland Press, Prospect Heights, IL, 1982. 311 pages. ISBN: 0-88133-367-0.

Advanced Voting Theory

- Saari, Donald G. *Geometry of voting*. Springer-Verlag, Berlin, 1994. 372 pages. ISBN: 3-540-57199-X

Social Choice Theory

- Moulin, Hervé. *Axioms of cooperative decision making*. Cambridge University Press, Cambridge, U.K. 1988. 332 pages. ISBN: 0-521-36055-2
- Roemer, John E. *Theories of Distributive Justice*. Harvard University Press, Cambridge, MA. 1996, 342 pages. ISBN: 0-674-87920-1

Further Reading III

Special Topics

Approval Voting.

- Steven J. Brams and Peter C. Fishburn *Approval voting* (2nd edition). Springer-Verlag, 2007. 198 pages. ISBN: 978-387-49895-9.
- *Citizens for Approval Voting:* `http://www.approvalvoting.org`
- *Americans for Approval Voting:* `http://www.approvalvoting.com`

Range Voting. (a.k.a. 'relative utilitarianism')

- *Centre for Range Voting* website: `http://rangevoting.org/`

Proportional Representation v.s. Single Transferable Vote.

- *FairVote:* `http://www.fairvote.org`
- *Electoral Reform Society* `http://www.electoral-reform.org.uk`

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