Kůrka's Classifications of Cellular Automata

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1 Background in Symbolic Dynamics

Let $\mathcal{A}$ be a finite **alphabet** of symbols. Usually, $\mathcal{A} = [0..A) = \{0, 1, \ldots, A-1\}$, which we will sometimes identify with the group $\mathbb{Z}_{/A}$.

Let $\mathbb{L}$ be a lattice, such as $\mathbb{Z}$, $\mathbb{N}$, or $\mathbb{Z}^D$. Then $\mathcal{A}^{\mathbb{L}}$ is the set of all functions $\mathbb{L} \mapsto \mathcal{A}$; these are called **configurations**. For example, if $\mathbb{L} = \mathbb{N}$, then a configuration is just an infinite sequence $\mathbf{a} = [a_\ell|_{\ell=0}^{\infty}] = [a_0, a_1, a_2, a_3, \ldots]$.

1.1 Topology and Metric Structure

If $\mathcal{A}$ has the discrete topology, we can endow $\mathcal{A}^{\mathbb{L}} \cong \prod_{\ell \in \mathbb{L}} \mathcal{A}$ with the corresponding Tychonoff product topology. $\mathcal{A}^{\mathbb{L}}$ is then a compact, totally disconnected space.

Let $\mathbb{U} \subset \mathbb{L}$ be some finite subset, and let $\mathcal{A}^{\mathbb{U}}$ denote the set of all **fragments** of the form $\mathbf{w} = [w_u|_{u \in \mathbb{U}}]$. For example, if $\mathbb{L} = \mathbb{Z}$ or $\mathbb{N}$, and $\mathbb{U} = [n..m]$, then an element of $\mathcal{A}^{\mathbb{U}}$ is a **word** of the form $\mathbf{w} = [w_n, w_{n+1}, \ldots, w_m]$.

If $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$, then $\mathbf{a}_{|_{\mathbb{U}}} = [a_u|_{u \in \mathbb{U}}]$ denotes the **restriction** of $\mathbf{a}$ to an element of $\mathcal{A}^{\mathbb{W}}$. In particular, for any $R > 0$, let $\boxed{R} = [-R..R]^D$ if $\mathbb{L} = \mathbb{Z}^D$, and let $\boxed{R} = [0..R]^D$ if $\mathbb{L} = \mathbb{N}^D$. Then $\mathbf{a}_{|_{\boxed{R}}}$ is the restriction of $\mathbf{a}$ to a radius-R 'neighbourhood' of the origin. For example, if $\mathbb{L} = \mathbb{Z}$, then $\mathbf{a}_{|_{\boxed{R}}} = [a_{-R}, a_{1-R}, \ldots, a_R]$.

Suppose $\mathbb{L} = \mathbb{Z}$, and let $\mathbb{U} = [0..U]$. Let $\mathbf{w} = [w_0, w_1, \ldots, w_U]$ be an element of $\mathcal{A}^{\mathbb{U}}$; then $\mathbf{w}$ determines a **cylinder set**:

$$\triangleleft \mathbf{w} \triangleright \quad = \quad \left\{ \mathbf{a} \in \mathcal{A}^{\mathbb{Z}} \; ; \; a_0 = w_0, \quad a_1 = w_1, \; \ldots, \; a_U = w_U \right\}$$

(see Figure 1). More generally, if $\mathbb{U} \subset \mathbb{L}$ is any finite subset, and $\mathbf{w} \in \mathcal{A}^{\mathbb{U}}$, the corresponding **cylinder set** is defined:

$$\triangleleft \mathbf{w} \triangleright \quad = \quad \left\{ \mathbf{a} \in \mathcal{A}^{\mathbb{L}} \; ; \; \mathbf{a}_{|_{\mathbb{U}}} = \mathbf{w} \right\}$$

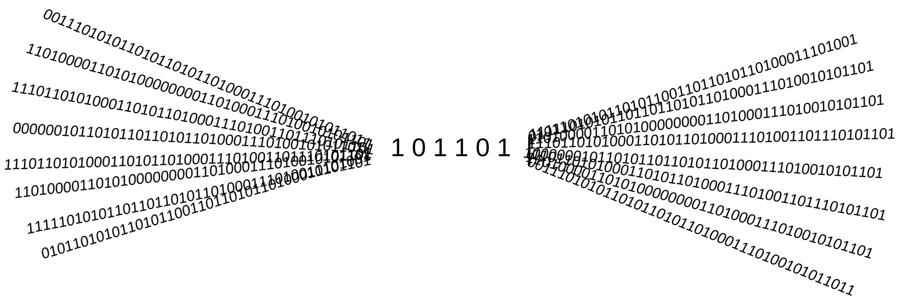

Figure 1: The cylinder set $\triangleleft 101101 \triangleright$; here, $\mathbb{U} = [0..5]$ and $\mathbf{w} = [101101]$.

The cylinder sets are **clopen** (both closed and open) in the Tychonoff topology, and form a basis for it. Thus, to establish results about this topology, it is usually sufficient to prove them for cylinder sets. For example, it is not hard to prove the following:

Lemma 1:

1. *Let $\phi : \mathcal{A}^{\mathbb{L}} \longrightarrow \mathcal{A}$ be continuous (w.r.t. the discrete topology on $\mathcal{A}$); then for any $a \in \mathcal{A}$, $\phi^{-1}\{a\}$ is a finite disjoint union of cylinder sets.*

2. *Let $\Phi : \mathcal{A}^{\mathbb{L}} \circlearrowleft$ be continuous. Then the preimage of any cylinder set is a finite disjoint union of cylinder sets.*

Proof: **Exercise 1** ______ □

The Tychonoff topology is metrizable. If $\mathbf{a}, \mathbf{b} \in \mathcal{A}^{\mathbb{L}}$, then the usual metric is defined[1]:

$$d(\mathbf{a}, \mathbf{b}) \;=\; 2^{-R} \qquad \text{where} \quad R = \min\{|\ell| \;;\; a_\ell \neq b_\ell\}$$

Thus, $\mathbf{a}$ and $\mathbf{b}$ are 'close' if they agree in a large region around the origin of $\mathbb{L}$.

As a consequence, a sequence $\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots$ of configurations **converges** to $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$ if elements of the sequence agree with $\mathbf{a}$ in larger and larger regions around the origin. For example, if $\mathbb{L} = \mathbb{Z}$, then

$$\left(\mathbf{a}^{(n)} \xrightarrow[n\to\infty]{} \mathbf{a} \right) \iff \left(\mathbf{a}^{(n)}\big|_{\boxed{R_n}} = \mathbf{a}\big|_{\boxed{R_n}}, \text{ where } R_n \xrightarrow[n\to\infty]{} \infty \right)$$

(see Figure 2).

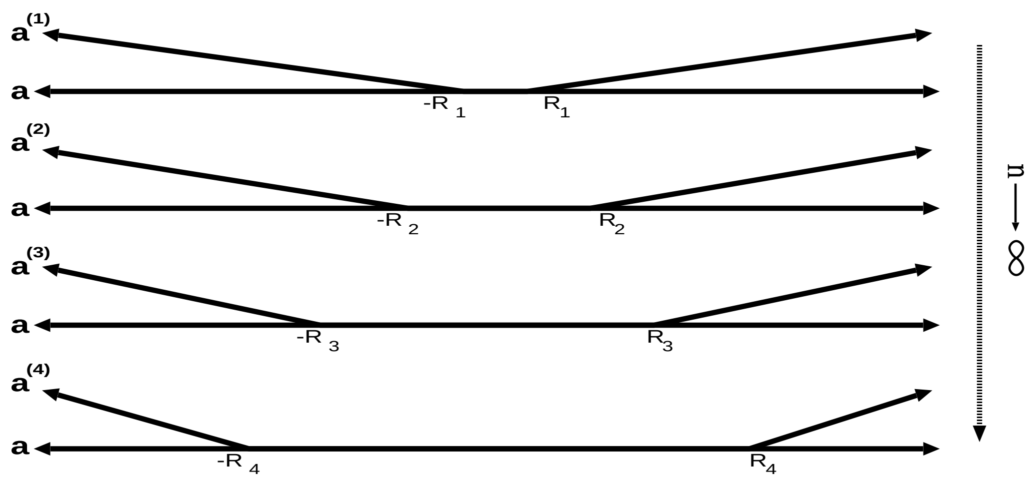

Figure 2: Convergence of configurations.

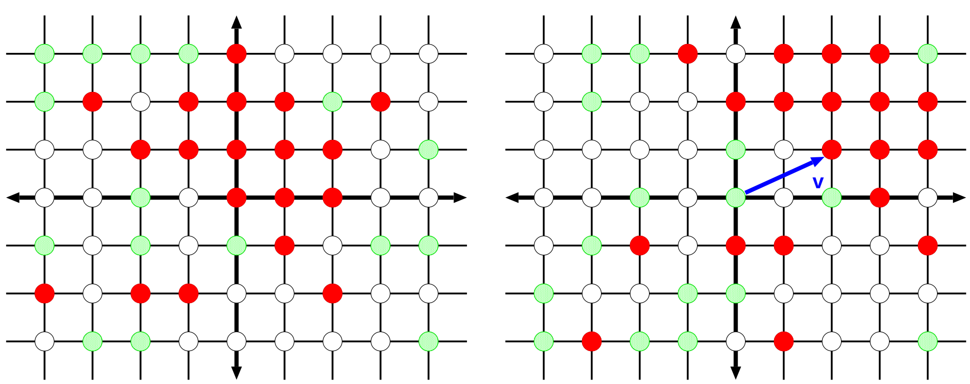

Figure 3: The shift map. Here, $\mathbb{L} = \mathbb{Z}^2$ and $v = (2, 1)$.

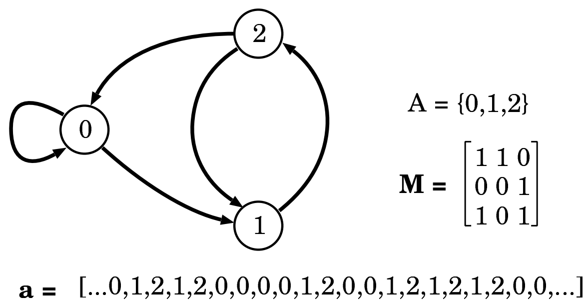

Figure 4: A topological Markov chain.

1.2 Shifts and Subshifts

If $v \in \mathbb{L}$, then the **shift by** v is the map $\boldsymbol{\sigma}^v : \mathcal{A}^{\mathbb{L}} \circlearrowleft$ defined:

$$\boldsymbol{\sigma}^v(\mathbf{a}) \ = \ \mathbf{b} \qquad \text{where } \ b_\ell = a_{\ell+v}, \ \ \forall \ell \in \mathbb{L}$$

These maps are continuous with respect to the topology of $\mathcal{A}^{\mathbb{L}}$, and determines an *action* of the monoid $\mathbb{L}$ on $\mathcal{A}^{\mathbb{L}}$.

If $\mathbb{L} = \mathbb{Z}$ or $\mathbb{N}$, then a configuration $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$ can be thought of as a bi-infinite 'ticker tape', and the map $\boldsymbol{\sigma} = \boldsymbol{\sigma}^1$ just moves the ticker tape forward by one symbol. Thus, $\boldsymbol{\sigma}$ can be thought of as a sort of 'time evolution' operator, and $(\mathcal{A}^{\mathbb{L}}, \boldsymbol{\sigma})$ is a compact topological dynamical system; it is called the **full shift**, and is an example of a **shift dynamical system**.

If $\mathbb{L} = \mathbb{Z}$, then $\boldsymbol{\sigma}$ is invertible, meaning that information about the past is preserved; $(\mathcal{A}^{\mathbb{Z}}, \boldsymbol{\sigma})$ is called the **two-sided** (or **bilateral**) **shift**. If $\mathbb{L} = \mathbb{N}$, then $\boldsymbol{\sigma}$ is many-to-one, meaning that information about the past is forgotten; $(\mathcal{A}^{\mathbb{N}}, \boldsymbol{\sigma})$ is called the **one-sided** (or **unilateral**) **shift**.

A **subshift** is any closed subset $\mathfrak{X} \subset \mathcal{A}^{\mathbb{L}}$ which is closed under all shifts —that is, $\boldsymbol{\sigma}^\ell(\mathfrak{X}) \subset \mathfrak{X}$ for any $\ell \in \mathbb{L}$. Since $\mathfrak{X}$ is compact, $(\mathfrak{X}, \boldsymbol{\sigma})$ is then a compact topological dynamical system; the study of such dynamical systems is called **symbolic dynamics**. Below are some important examples of subshifts. In what follows, "$\mathbb{L}$" refers to either $\mathbb{N}$ or $\mathbb{Z}$. We will usually specify the case for $\mathbb{L} = \mathbb{Z}$, and leave the case $\mathbb{L} = \mathbb{N}$ for the reader.

Topological Markov Chains: Consider a directed graph whose vertices are the elements of $\mathcal{A}$ (Figure 4). Any infinite, directed path through this graph determines a sequence $\mathbf{a} = [a_0, a_1, a_2, \ldots]$ in $\mathcal{A}^{\mathbb{N}}$; likewise, a bi-infinite directed path defines a sequence

[1]Here, if $\ell = (\ell_1, \ldots, \ell_D)$, then $|\ell| = \sup_{d \in [1..D]} |\ell_d|$.

A topologically equivalent metric can be defined using any decreasing function of R, and using any other 'reasonable' norm on $\mathbb{L}$.

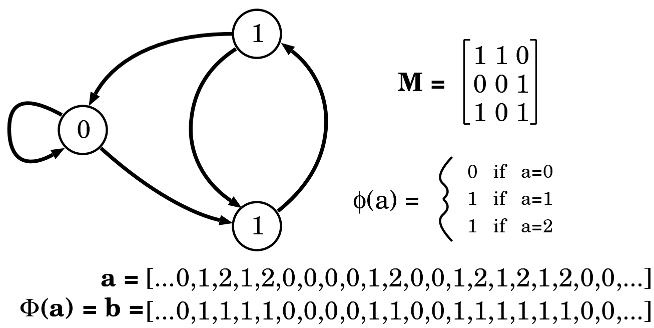

Figure 5: The **even shift** is an example of a sofic shift.

$\mathbf{a} = [\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots]$ in $\mathcal{A}^{\mathbb{Z}}$. For $\mathbb{L} = \mathbb{Z}$ or $\mathbb{N}$, let $\mathfrak{X} \subset \mathcal{A}^{\mathbb{L}}$ be the set of all sequences which arise in this fashion; then $\mathfrak{X}$ is a subshift of $\mathcal{A}^{\mathbb{L}}$.

Notice that the symbol a_n can only follow the symbol a_{n-1} if there is an edge $a_{n-1} \longrightarrow a_n$ in the graph. Let $\mathbf{M} = \left[m_{ab}|_{a,b\in\mathcal{A}}\right]$ be a matrix of 0's and 1's such that $m_{ab} = 1$ if and only if there is an edge $a \longrightarrow b$. Thus, symbol b can only follow the symbol a if $m_{ab} = 1$. If we imagine $\mathbf{M}$ as a matrix of Boolean 'transition probabilities', then sequences in $\mathfrak{X}$ can be thought of as arising from the corresponding Markov chain; because of this, $\mathfrak{X}$ is sometimes called a **topological Markov chain**.

Subshifts of finite Type: Let $\mathbb{U} = \{0, 1\}$, and let $\mathfrak{M} \subset \mathcal{A}^{\mathbb{U}}$ be the set of all pairs (a, b) so that there is an edge $a \longrightarrow b$ in the digraph. Then clearly, $\mathbf{a} \in \mathfrak{X}$ if and only if, for any $\ell \in \mathbb{L}$, the pair $[a_\ell, a_{\ell+1}]$ is an element of $\mathfrak{M}$.

This is an example of a more general construction. Let $\mathbb{U} = [0..U]$, and let $\mathfrak{M} \subset \mathcal{A}^{\mathbb{U}}$ be a set of **admissable words** on $\mathbb{U}$. Imagine a sequence $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$ so that, for any $\ell \in \mathbb{L}$, the word $[a_\ell, a_{\ell+1}, \ldots, a_{\ell+U}]$ is an element of $\mathfrak{M}$; such a sequence is called **$\mathfrak{M}$-admissable**. The set of all $\mathfrak{M}$-admissable sequences is denoted $\langle\mathfrak{M}\rangle \subset \mathcal{A}^{\mathbb{L}}$, and forms a **subshift of finite type** (SFT).

The previous example shows that any topological Markov chain is a subshift of finite type. Conversely, by 'recoding' each element of $\mathcal{A}^{\mathbb{L}}$ as an element of $\mathcal{B}^{\mathbb{L}}$ (where $\mathcal{B} = \mathcal{A}^{\mathbb{U}}$), we can represent any subshift of finite type on $\mathcal{A}$ as a topological Markov chain over $\mathcal{B}$. The details are an exercise.

Sofic Shifts: Consider again a labeled digraph, but now suppose that we assign the *same* label to more than one vertex, as in Figure 5. Again, any (bi)-infinite directed path through the graph yields a sequence of symbols in $\mathcal{A}^{\mathbb{N}}$ (or $\mathcal{A}^{\mathbb{Z}}$); this is an example of a sofic shift.

Formally, a **sofic shift** is the image $\mathfrak{Y} = \Phi(\mathfrak{X})$ of a subshift of finite type $\mathfrak{X} \subset \mathcal{A}^{\mathbb{L}}$ under a cellular automaton Φ.

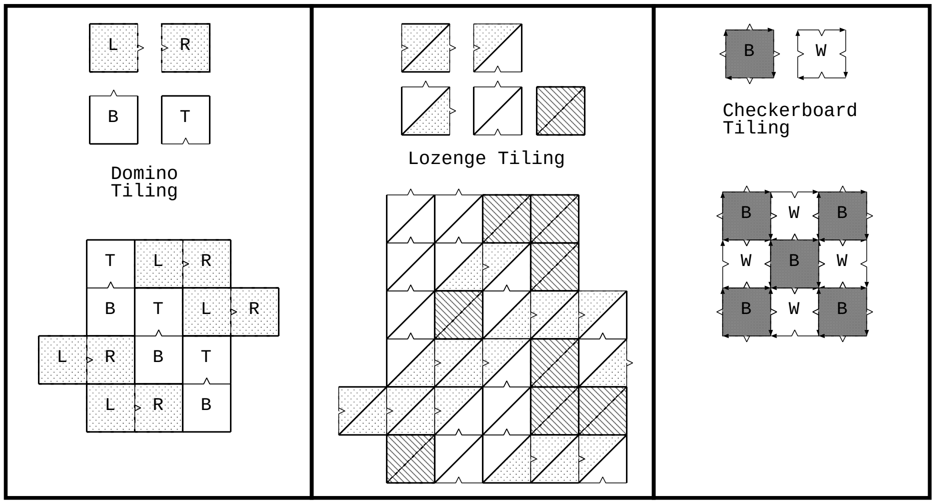

Figure 6: Some 2-dimensional subshifts of finite type, visualized as tilings.

For example, given any topological Markov chain $\mathfrak{X}$, and an endomorphism $\phi : \mathcal{A}\circlearrowleft$, we can apply ϕ componentwise to an element $\mathbf{x} \in \mathfrak{X}$ to get an element $\mathbf{y} = \phi(\mathbf{x}) \in \mathfrak{Y}$. For example, the sofic shift in Figure 5 is obtained from the SFT in Figure 4 by applying the map ϕ defined: $\phi(0) = 0$ and $\phi(1) = \phi(2) = 1$. This is called the **Even Shift**, because any contiguous block of 1's must be of even length.

Multidimensional Subshifts of Finite Type: The definition of SFTs generalizes to higher dimensional lattices. Let $\mathbb{L} = \mathbb{Z}^D$ or $\mathbb{N}^D$, and let $\mathbb{U} \subset \mathbb{L}$ be some finite subset. Let $\mathfrak{M} \subset \mathcal{A}^{\mathbb{U}}$ be a set of **admissable fragments** on **U**; a configuration $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$ is called **$\mathfrak{M}$-admissable** if for any $\ell \in \mathbb{L}$, the fragment $[a_{\ell+u}|_{u\in\mathbb{U}}]$ is an element of $\mathfrak{M}$. The set of all

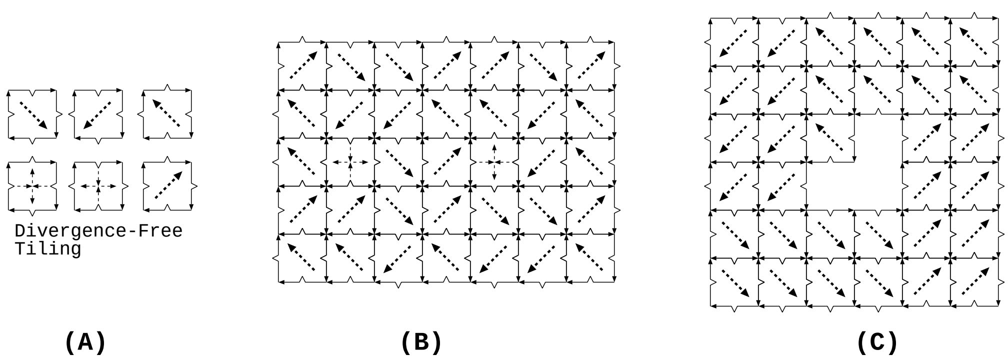

Figure 7: Another 2-dimensional subshift of finite type. **(c)** shows a defect.

$\mathfrak{M}$-admissable sequences is again denoted $\langle \mathfrak{M} \rangle \subset \mathcal{A}^{\mathbb{L}}$, and is (multidimensional) **subshift of finite type**.

For example, the set of all possible spacetime diagrams for a D-dimensional CA defines a $(D+1)$-dimensional subshift of finite type.

A one-dimensional SFT can be recoded as a topological Markov chain, determined by a **transition matrix** of 0's and 1's; in the same way, a D dimensional SFT can be recoded as a 'topological Markov mesh', determined by D distinct transition matrices of 0's and 1's, which encode the allowed transitions in each of the D directions of the lattice.

Another way to interpret this is in terms of *tiling* (Figures 6 and 7). If $D = 2$, we visualize the symbols of the alphabet as square tiles; each tile's edges are toothed or notched so as to impose *edge-matching constraints* on tiles which can appear above, below, right, or left of that tile. The transition matrices then encode these constraints. (If $D \geq 3$, visualize the symbols as (hyper)cubical building blocks, whose (hyper)faces are grooved to introduce 'face-matching' constraints.)

The theory of one-dimensional SFTs is pretty complete, but higher dimensional SFTs are very poorly understood. Indeed, most interesting questions can be shown to be *formally undecidable*, meaning that there cannot exist any single, general-purpose theorem or algorithm for answering them (although in specific cases, the answer may be available through *ad hoc* methods). For example, consider the **Nontriviality problem:**

Given a domain $\mathbb{U} \subset \mathbb{L}$ and a set $\mathfrak{M} \subset \mathcal{A}^{\mathbb{U}}$ of allowed fragments, is the resulting SFT $\langle \mathfrak{M} \rangle$ nonempty?

In one dimension, this problem is easy: to build a sequence inside $\mathfrak{X}$, we begin with a single-symbol word, and extend this word by adding one symbol at a time, so that each symbol is an admissable successor of the previous symbol. In higher dimensions, however, building an $\mathfrak{X}$-admissable configuration is highly nontrivial. Figure 7 illustrates the problem: it is possible to accidentally construct 'defects' which cannot be filled in any admissable fashion. The Nontriviality Problem basically asks: is it possible to avoid such defects, or are they inevitable?

Theorem 2: *(Robinson)* [26]
If $D \geq 2$, then the nontriviality problem is formally undecidable. □

Further Reading

An excellent introduction to symbolic dynamics is [17]; another reference is [14]. The connections between symbolic dynamics and formal languages are explored in [1]. The extensive and beautiful theory of symbolic systems with additional algebraic structure is exposited in [28]. Symbolic systems can be used to represent other topological dynamical

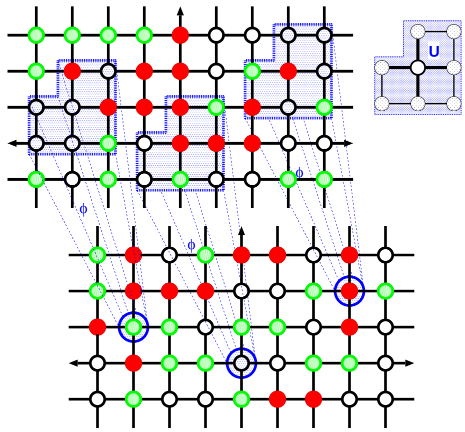

Figure 8: The Curtis-Hedlund-Lyndon theorem. In this case, $\mathbb{L} = \mathbb{Z}^2$, and $\mathbb{U} = \{(-1,0),\ (-1,-1);\ (0,1),\ (0,0),\ (0,-1);\ (1,1),\ (1,0),\ (1,-1)\}$.

systems via the *thermodynamic formalism*; approachable expositions can be found in [30, 19].

2 Cellular Automata

A **cellular automaton** (CA) is a continuous transformation $\Phi : \mathcal{A}^{\mathbb{L}} \circlearrowleft$ that commutes with all shift maps —that is,

$$\Phi \circ \boldsymbol{\sigma}^{\ell}(\mathbf{a}) \ = \ \boldsymbol{\sigma}^{\ell} \circ \Phi(\mathbf{a})$$

for any $\ell \in \mathbb{L}$ and $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$. For example, clearly, all shift maps are cellular automata.

Proposition 3: *(Curtis, Hedlund, Lyndon)* [9]
$\Phi : \mathcal{A}^{\mathbb{L}} \circlearrowleft$ is a cellular automaton if and only if there is some finite subset $\mathbb{U} \subset \mathbb{L}$ and a map $\phi : \mathcal{A}^{\mathbb{U}} \longrightarrow \mathcal{A}$ so that, for any $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$, and any $\ell \in \mathbb{L}$,

$$\Phi(\mathbf{a})_{\ell} \ = \ \phi\left(\mathbf{a}_{|_{\mathbb{U}+\ell}}\right)$$

Proof: Combine Lemma 1 with the requirement that Φ commutes with shifts. ____□

$\mathbb{U}$ can be thought of as a 'neighbourhood of the origin' in $\mathbb{L}$, and ϕ is called the **local map** associated with the cellular automaton. Because of this, cellular automata are thought of as *spatially extended dynamical systems governed by local interactions*. Here, the 'space' is $\mathbb{L}$, and the 'local interactions' are defined by the map ϕ.

Example 4:

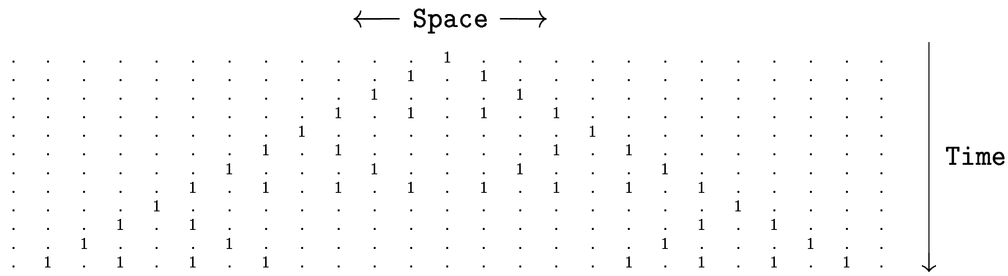

Figure 9: A spacetime diagram for the nearest neighbour XOR map. Here, '.' represents a '0'.

(a) **Shift map:** Suppose $\mathbb{L} = \mathbb{Z}$, and let $\Phi = \boldsymbol{\sigma}$ be the shift map. Then $\mathbb{U} = \{0, 1\}$, and $\phi : \mathcal{A}^{\{0,1\}} \longrightarrow \mathcal{A}$ is just defined: $\phi(a_0, a_1) = a_1$.

(b) **Nearest neighbour XOR:** Let $\mathbb{L} = \mathbb{Z}$ and $\mathbb{U} = \{-1, 0, 1\}$. Suppose $\mathcal{A} = \mathbb{Z}_{/2}$, regarded as a cyclic group, and define $\phi(a_{-1}, a_0, a_1) = a_{-1} + a_1 \pmod 2$. Figure 9 shows a spacetime diagram for this automaton.

(c) **Linear Cellular Automata:** If $\mathcal{A} = \mathbb{Z}_{/A}$ is regarded as a ring under multiplication and addition, mod A, then a **linear** cellular automaton is one with a local rule of the form:

$$\phi(\mathbf{a}) = \sum_{u \in \mathbb{U}} \varphi_u a_u \pmod A$$

where $\{\varphi_u\}_{u \in \mathbb{U}}$ are constants in $\mathbb{Z}_{/A}$. Thus, for example, the Nearest Neighbour XOR is a linear CA, with $\varphi_{-1} = \varphi_1 = 1$ and $\varphi_0 = 0$.

(d) **John H. Conway's Game of Life:** Now $\mathbb{L} = \mathbb{Z}^2$; $\mathbb{U} = [-1..1] \times [-1..1]$, and $\mathcal{A} = \{0, 1\}$; The local map is defined:

$$\phi(\mathbf{a}) = \begin{cases} 1 & \text{if } a_0 = 1 \text{ and } \sum_{0 \neq u \in \mathbb{U}} a_u = 3, 4 \\ 1 & \text{if } a_0 = 0 \text{ and } \sum_{0 \neq u \in \mathbb{U}} a_u = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 10 shows some examples of 'organisms' from Life.

(e) **Majority Voter Automata:** Let $\mathcal{A} = \{0, 1\}$ and $\mathbb{U} = [-L..R]$; let $U = \mathsf{Card}\,[\mathbb{U}] = R + L + 1$. For any $\mathbf{a} = (a_{-L}, a_{1-L}, \ldots, a_R) \in \mathcal{A}^{\mathbb{U}}$, let $\#(\mathbf{a}) = \sum_{\ell=-L}^{R} a_\ell$ be the number of

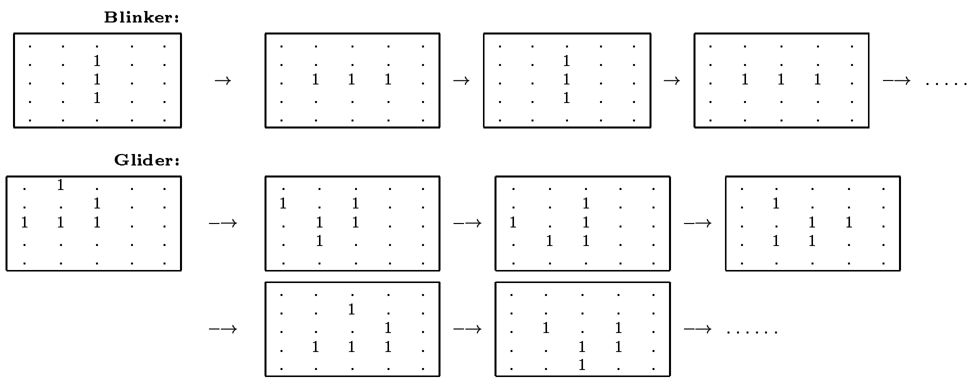

Figure 10: Some ‘organisms’ from the Game of Life

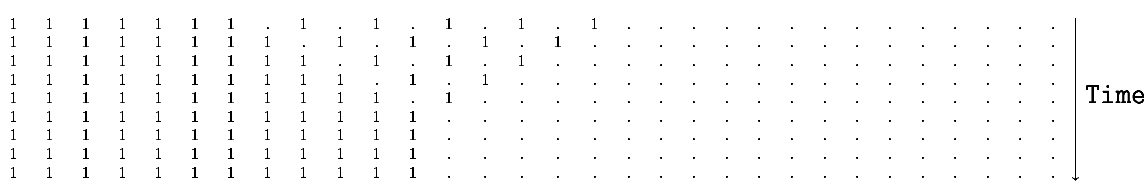

Figure 11: The nearest neighbour voter rule.

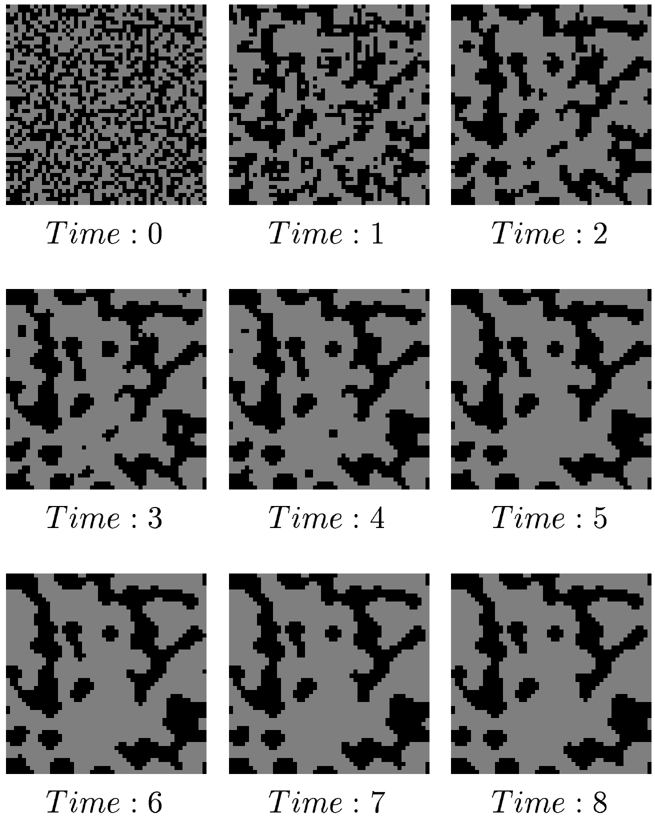

Figure 12: The two-dimensional nearest neighbour voter converges to a 'cow patch' configuration.

1's in **a**, and define the **majority voter rule:**

$$\phi(\mathbf{a}) \;=\; \begin{cases} 1 & \text{if } \#(\mathbf{a}) > \lfloor U/2 \rfloor \\ 0 & \text{if } \#(\mathbf{a}) \leq \lfloor U/2 \rfloor \end{cases}$$

For example, if $\mathbb{L} = \mathbb{Z}$ and $\mathbb{U} = \{-1, 0, 1\}$, then we get the **one-dimensional nearest neighbour voter rule** (see Figure 11)

$$\phi(\mathbf{a}) \;=\; \begin{cases} 1 & \text{if } \#(\mathbf{a}) > 1 \\ 0 & \text{if } \#(\mathbf{a}) \leq 1 \end{cases}$$

If $\mathbb{L} = \mathbb{Z}^2$ and $\mathbb{U} = [-1..1]^2 = \{(-1,-1);\ (-1,0);\ (-1,1);\ \ (0,-1);\ (0,0);\ (0,1);\ (1,-1);\ (1,0);\ (1,1)\}$, then we get the **two-dimensional nearest neighbour voter rule**

$$\phi(\mathbf{a}) \;=\; \begin{cases} 1 & \text{if } \#(\mathbf{a}) > 4 \\ 0 & \text{if } \#(\mathbf{a}) \leq 4 \end{cases}$$

As Figure 12 illustrates, the 2D voter rule evolves initial configurations into 'cow patch' patterns, which is are stable fixed point for the dynamics.

(f) **Totalistic automata** If $\mathcal{A} = \{0, 1, \ldots, N\}$, then Φ is **totalistic** if with the local map ϕ has the form:

$$\phi(\mathbf{a}) \;=\; \varphi(\#(\mathbf{a}))$$

where $\#(\mathbf{a}) = \sum_{u \in \mathbb{U}} a_u$, and $\varphi : [0...NU] \longrightarrow \mathcal{A}$ is some function.

Thus, for example, voter models are totalistic.

(g) **Digital Multiplication:** Let $\mathcal{A} = [0..K)$, where $K = k_1 \cdot k_2$. For every $a \in \mathcal{A}$, there are thus unique $q_a \in [0..k_2)$ and $r_a \in [0..k_1)$ so that $a = q_a \cdot k_1 + r_a$.

If $\mathbb{L} = \mathbb{N}$, then an element $\mathbf{a} = [a_0 a_1 a_2 \ldots]$ can be thought of as the K-ary expansion of the number $\alpha \in [0, 1]$ defined: $\alpha = \sum_{n=0}^{\infty} a_n K^{-n}$.

Let $\mathbb{U} = \{0, 1\}$, and consider the local map:

$$\phi(a_0, a_1) \;=\; k_1 \cdot r_{(a_0)} \;+\; q_{(a_1)}$$

It is not hard to verify (**Exercise 2**) that, if $\mathbf{b} = \Phi(\mathbf{a})$, then $\mathbf{b}$ is the K-ary expansion of the number $\beta = k_1 \cdot \alpha \pmod 1$. In other words, the automaton Φ 'simulates' the action of multiplication-by-k_1 on the unit interval.

More generally, for *any* integer J such that J divides some power of K, it is possible to construct a CA on $\mathcal{A}^{\mathbb{N}}$ which simulates multiplication-by-J; see [2].

(h) **Particle-preserving CA:** In particle-preserving cellular automata, the elements of $\mathcal{A}$ play the roles of 'particles', which propagate through space and interact with one another. We will give a specific example of such an automaton, due to Choffrut and Čulick [3, 5].

Here, $\mathcal{A} = \{0, 1, 2a, 2b, 3a, 3b, 3c\}$ and $\mathbb{L} = \mathbb{Z}$. The rules are as follows:

(0) ...represents the 'vacuum' or background state.

(1) ...represents a 'barrier' particle with velocity 0.

(2) ...represents a 'bouncing' particle with velocity 1. The **(2a)** particle moves to the *right*, until it encounters a **(2b)** or a **(3)** particle, at which point it 'reflects' and becomes a **(2b)** particle, which propagates to the *left*, until it bounces off another particle and becomes a **(2a)** again.

If a **(2a)** particle collides with a **(1)** particle, it is annihilated.

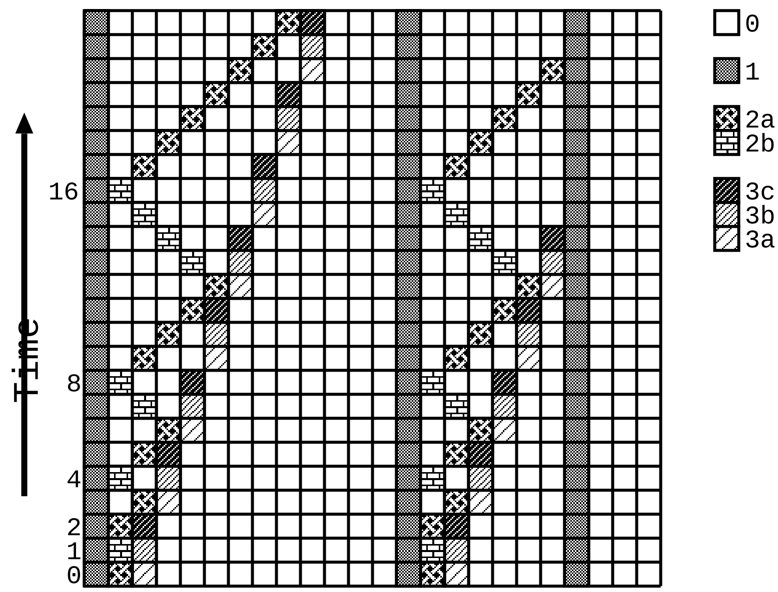

Figure 13: A particle-preserving cellular automaton. Notice that the times at which the **(2)** particle strikes the **(1)** form powers of two.

(3) ...represents a particle with velocity 1/3. The **(3)** particle cycles through three states, **(3a)**, **(3b)** and **(3c)**. A **(3a)** particle moves one unit to the *right*, and becomes a **(3b)**. A **(3b)** particle remains where it is, and becomes **(3c)**, and a **(3c)** particle remains where it is, but becomes a **(3a)**. Thus, the **(3)** particle effectively moves one unit every three time intervals.

If a **(3)** particle collides with a **(1)** particle, it is annihilated.

Notice that, starting from the initial configuration in Figure 13, the times at which the **(2)** particle strikes the **(1)** form powers of two.

One convenient way of visualizing cellular automata is through a spacetime diagrams. If $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$, then for all $\ell \in \mathbb{L}$ and $n \in \mathbb{N}$, define $a_\ell^n = \Phi^n(\mathbf{a})_\ell$. Then the $D+1$-dimensional matrix $\mathbf{A} = \left[a_\ell^n|_{\ell\in\mathbb{L}}^{n\in\mathbb{N}}\right]$ is called the **spacetime diagram** with initial conditions **a**. If $\mathbb{L} = \mathbb{Z}$ or $\mathbb{N}$, then this is a 2-dimensional array such as in Figure 14. Sometimes the arrow of time points upwards, and sometimes downwards. For example, Figure 9 is a spacetime diagram for the Nearest Neighbour XOR automaton.

Cellular automata are approached from several directions:

1. **Physics Perspective:** Cellular automata are used to model spatially extended physical systems, in a manner analogous to partial differential equations. In a sense, CA

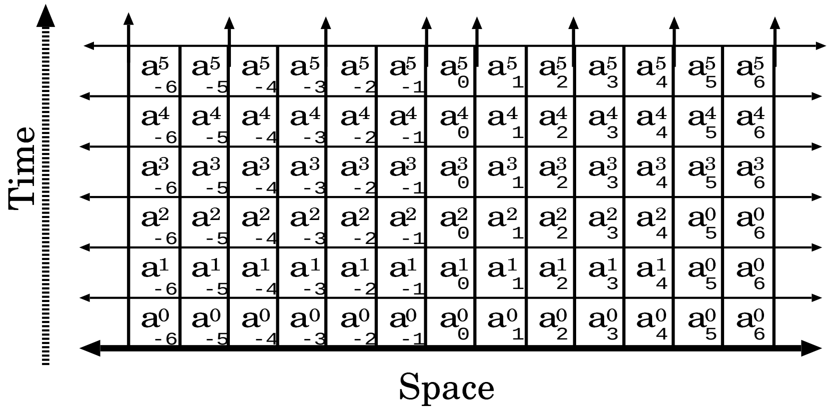

Figure 14: A schematic spacetime diagram.

are 'totally discretized' PDEs. In a PDE, time, space, and the local state information are all continuously parameterized, while in a CA, all these quantities are discrete.

For example, in *particle-preserving* CA such as Example 4h, the cellular automaton simulates the interaction of particles. Alternately, the lattice elements ('cells') in a CA can be thought of as immobile atoms of a crystal lattice; the local map ϕ then describes the 'physics' by which these atoms interact.

2. **Computation Perspective:** Each of 'cells' in a CA can be thought of as a miniature computer, whose memory is so small that it is only capable of storing a single symbol from the alphabet $\mathcal{A}$. These computers interact with one another, and each executes an identical program, changing its memory state in response to the states of its immediate neighbours. The local map ϕ then describes the program executed by each cell. Thus, a CA can be seen as a model of massively parallel computation. For example, the particle-preserving CA of Example 4h can be seen as a machine for enumerating the powers of two. Meanwhile, the CA in Example 4g performs an infinite-precision arithmetic operation (multiplication by k_1).

3. **Symbolic Dynamics & Coding Perspective:** If $(\mathcal{A}^{\mathbb{L}}, \boldsymbol{\sigma})$ is a dynamical system, then $\Phi : \mathcal{A}^{\mathbb{L}} \circlearrowleft$ is an endomorphism of this dynamical system. If a sequence $\mathbf{a} = [\ldots, a_{-1}, a_0, a_1, \ldots]$ is thought of as a 'ticker tape' containing some message, then $\Phi(\mathbf{a})$ is a way of 'encoding' this message, for the data compression, error detection, error correction, or encryption. In this context, the linear cellular automata of Example 4c are sometimes called *convolutional encoders*.

4. **Dynamical Systems Perspective:** $(\mathcal{A}^{\mathbb{L}}, \Phi)$ is a discrete-time topological dynamical system, so it makes sense to try to characterize its fixed points, periodic points,

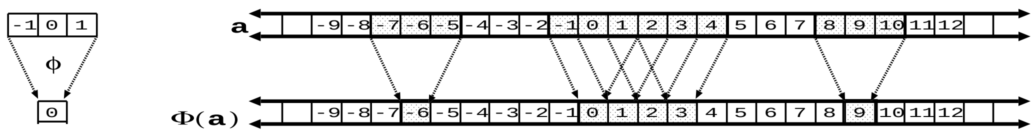

Figure 15: A nearest-neighbour, one-dimensional CA.

attractors, invariant measures, etc.

If $\mathbb{L} = \mathbb{Z}$ or $\mathbb{N}$, then Φ is called a **one-dimensional cellular automaton**. If $\mathbb{L} = \mathbb{Z}$, then we can assume $\mathbb{U} = [-L..R]$, and the local map takes the form $\phi : \mathcal{A}^{[-L..R]} \longrightarrow \mathcal{A}$. Then Φ is called a **two-sided** (or **bilateral**) CA, and L and R are its **left radius** and **right radius**.

If $\mathbb{L} = \mathbb{N}$, then we can assume $\mathbb{U} = [0..R]$, and the local map takes the form $\phi : \mathcal{A}^{[0..R]} \longrightarrow \mathcal{A}$. Then Φ is called a **one-sided** (or **unilateral**) CA, and R is its **radius**. To simplify the discussion, we will sometimes treat one-sided CA as two-sided CA, with $L = 0$.

If $R = L = 1$, then Φ is said to be a **nearest neighbour** CA (see Figure 15). Through suitable recoding, *any* CA can be represented by a nearest neighbour CA. To be precise, suppose $\mathbb{L} = \mathbb{Z}$. Let $U = \mathsf{Card}[\mathbb{U}] = R + L + 1$, and $\mathcal{B} = \mathcal{A}^U$, and define the map $\Psi : \mathcal{A}^{\mathbb{L}} \longrightarrow \mathcal{B}^{\mathbb{L}}$ by

$$\Psi(\mathbf{a}) \;=\; \left[\ldots, \begin{bmatrix} a_{-U-L} \\ a_{1-U-L} \\ \vdots \\ a_{-1-L} \end{bmatrix}, \begin{bmatrix} a_{-L} \\ a_{1-L} \\ \vdots \\ a_R \end{bmatrix}, \begin{bmatrix} a_{R+1} \\ a_{R+2} \\ \vdots \\ a_{R+U} \end{bmatrix}, \ldots \right]$$

(This is called a **higher power presentation**. If $\mathbb{L} = \mathbb{N}$, set $L = 0$ and perform a similar construction). Then it is not hard to verify:

Proposition 5:

1. *$\Psi : \mathcal{A}^{\mathbb{L}} \longrightarrow \mathcal{B}^{\mathbb{L}}$ is a bijection, and* $\boldsymbol{\sigma} \circ \Psi \;=\; \Psi \circ \boldsymbol{\sigma}^U$.

2. *If $\Phi : \mathcal{A}^{\mathbb{M}} \circlearrowleft$ is any cellular automaton, then there exists a unique cellular automaton $\widetilde{\Phi} : \mathcal{B}^{\mathbb{M}} \circlearrowleft$ so that* $\widetilde{\Phi} \circ \Psi = \Psi \circ \Phi$.

   Thus, the dynamical system $(\mathcal{A}^{\mathbb{L}}, \Phi)$ is isomorphic to the dynamical system $(\mathcal{B}^{\mathbb{L}}, \widetilde{\Phi})$.

Proof: **Exercise 3** ______ □

Because of this, we will assume from now on that all one-dimensional CA are of nearest-neighbour type, unless the contrary is explicitly specified.

3 Surjectivity and Injectivity

We say Φ is **surjective** if $\Phi(\mathcal{A}^{\mathbb{L}}) = \mathcal{A}^{\mathbb{L}}$. We say Φ is **finite-to-one** if $\Phi^{-1}\{\mathbf{a}\}$ is a finite set for any $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$.

The **uniform measure** is the probability measure η on $\mathcal{A}^{\mathbb{L}}$ so that, for any subset $\mathbb{U} \subset \mathbb{L}$ with $\mathsf{Card}\,[\mathbb{U}]$, and any $\mathbf{w} \in \mathcal{A}^{\mathbb{U}}$, the cylinder set $\lhd\mathbf{w}\rhd$ has probability A^{-U}.

Proposition 6:

1. *If Φ is finite-to-one, then it is surjective.*

2. *If Φ is surjective, then the uniform measure is Φ-invariant.*

Proof: (1) See [17]. (2) See [9]. ____ □

If Φ is not surjective, then there are some configurations $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$ such that $\Phi^{-1}\{\mathbf{a}\} = \emptyset$. These are sometimes (poetically) called **garden-of-Eden** configurations (since they can only occur at the "beginning of time").

Proposition 7:

1. *The only injective cellular automata on $\mathcal{A}^{\mathbb{N}}$ are the trivial ones —ie. those of radius* 0.

2. *If $\Phi : \mathcal{A}^{\mathbb{L}} \circlearrowleft$ is injective, then it is bijective, and Φ^{-1} is also a cellular automaton.*

Proof: **Exercise 4** ____ □

Injectivity and surjectivity for one-dimensional CA are well-understood, and have been completely characterized [27]. For higher dimensional CA, however, the problem is unsolvable...

Proposition 8: *(J. Kari)* [13] *Let $\mathbb{L}$ be a lattice of dimension $D \geq 2$. There exist cellular automata on $\mathcal{A}^{\mathbb{L}}$ whose injectivity and/or surjectivity is formally undecidable.* □

4 Equicontinuity

Let $(\mathbf{X}, d)$ be a metric space. If $x, y \in \mathbf{X}$, write $x \underset{\epsilon}{\sim} y$ if $d(x, y) < \epsilon$.

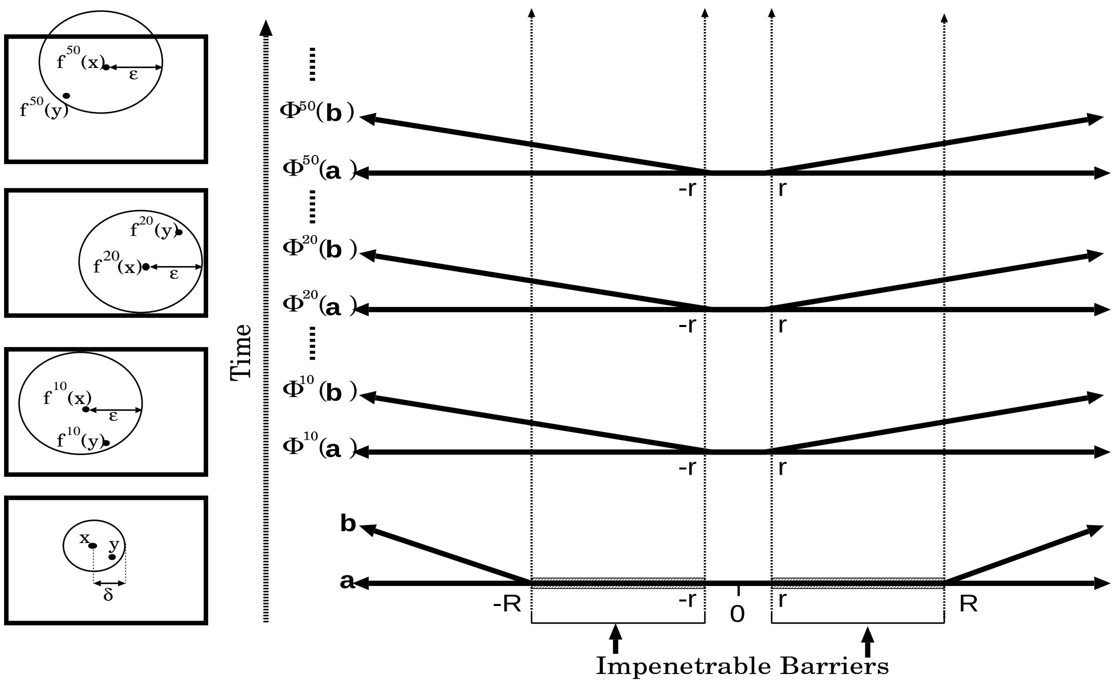

Figure 16: Equicontinuity.

Let $f : \mathbf{X} \circlearrowleft$ be a continuous self-map. A point $x \in \mathbf{X}$ is called a **point of equicontinuity** of f if, given any y sufficiently close to x, the f-orbits of x and y will remain close together for all time. Formally: for every $\epsilon > 0$, there is some $\delta > 0$ so that, for any $y \in \mathbf{X}$,

$$\left(y \underset{\delta}{\sim} x \right) \Longrightarrow \left(f^n(y) \underset{\epsilon}{\sim} f^n(x) \text{ for all } n \in \mathbb{N} \right)$$

If $\mathbf{X} = \mathcal{A}^{\mathbb{L}}$ and $f = \Phi$ is a CA, then a configuration $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$ is a point of equicontinuity if, for every $r > 0$, there is some $R > 0$ so that, for any $\mathbf{b} \in \mathcal{A}^{\mathbb{L}}$,

$$\left(\mathbf{b}\big|_{\boxed{R}} = \mathbf{a}\big|_{\boxed{R}} \right) \Longrightarrow \left(\Phi^n(\mathbf{b})\big|_{\boxed{r}} = \Phi^N(\mathbf{a})\big|_{\boxed{r}} \text{ for all } n \in \mathbb{N} \right)$$

For example, for a two-sided, one-dimensional CA, we have:

$$\left([b_{-R}, b_{1-R}, \ldots, b_R] \;=\; [a_{-R}, a_{1-R}, \ldots, a_R] \right) \Longrightarrow \left(\Phi^n(\mathbf{b})\big|_{[-r..r]} \;=\; \Phi^N(\mathbf{a})\big|_{[-r..r]} \text{ for all } n \in \mathbb{N} \right) \quad (1)$$

Heuristically speaking, (1) means that the word $[a_{-R}, a_{1-R}, \ldots, a_{-r}]$ constitutes an 'impenetrable barrier'; no alteration of $\mathbf{a}$ to the left of this barrier can ever affect what happens to the right of it. Likewise, $[a_r, a_{r+1} \ldots, a_R]$ is an impenetrable barrier, which blocks all perturbations from the right. Hence, equicontinuity in a one-dimensional CA manifests as the *blocking of information-transmission* through the lattice.

Let $\mathcal{E}q[\Phi] \subset \mathcal{A}^{\mathbb{L}}$ be the set of all points of equicontinuity for $\mathcal{A}^{\mathbb{L}}$. We say that Φ is **equicontinuous** if $\mathcal{E}q[\Phi] = \mathcal{A}^{\mathbb{L}}$.

Example 9:

(a) The identity map is equicontinuous, as is any constant map.

(b) If $R = L = 0$, then Φ is a function of the form $\Phi(\mathbf{a})_\ell = \phi(a_\ell)$, where $\phi : \mathcal{A} \longrightarrow \mathcal{A}$ is some self-map. Such cellular automata are clearly equicontinuous.

CA which halt in finite time

A configuration $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$ is a **fixed point** of Φ if $\Phi(\mathbf{a}) = \mathbf{a}$; the set of all fixed points is denoted $\mathsf{Fix}[\Phi]$. We say Φ **halts in finite time** if $\Phi^M(\mathcal{A}^{\mathbb{L}}) = \mathsf{Fix}[\Phi]$ for some M. It follows easily:

Lemma 10: *If Φ halts in finite time, then Φ is equicontinuous.*

Proof: **Exercise 5** □

Eventual Periodicity

If $P \in \mathbb{N}$, then $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$ is a P**-periodic** point of Φ if $\Phi^P(\mathbf{a}) = \mathbf{a}$; the set of all such points is denoted $\mathsf{Per}^P[\Phi]$. Thus $\mathsf{Per}^P[\Phi] = \mathsf{Fix}\left[\Phi^P\right]$.

Lemma 11: $\mathsf{Fix}[\Phi]$ *is a subshift of finite type.*
For any P, $\mathsf{Per}^P[\Phi]$ is a subshift of finite type.

Proof: **Exercise 6** □

Φ is called **eventually periodic** if there are some $M, P \in \mathbb{N}$ so that, for any $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$, $\Phi^M(\mathbf{a})$ is P-periodic; thus $\Phi^{m+P}(\mathbf{a}) = \Phi^m(\mathbf{a})$, for any $m > M$. Equivalently, $\Phi^M(\mathcal{A}^{\mathbb{L}}) \subset \mathsf{Per}^P[\Phi]$.

Proposition 12: *Let Φ be a one-dimensional CA. Then:*

$$\Big(\Phi \text{ is eventually periodic} \Big) \iff \Big(\Phi \text{ is equicontinuous} \Big)$$

Proof: **'$\Longrightarrow$': (first proof)** Clearly, Φ is eventually P-periodic iff Φ^P halts in finite time. Thus, Φ^P is equicontinuous; it follows from this that Φ is equicontinous.

(second proof) Let $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$, and let $a_0^n = \Phi^n(\mathbf{a})_0$, for $n \in \mathbb{N}$. Since Φ is eventually periodic, the sequence $[a_0^0, a_0^1, a_0^2 \ldots]$ is wholly determined by its first $N = M+P$ elements. But the elements $[a_0^1, \ldots, a_0^N]$, in turn, are determined by $\mathbf{a}_{|\mathbb{W}}$, where $\mathbb{W} = \underbrace{\mathbb{U} + \ldots + \mathbb{U}}_{N}$.

For example, if $\mathbb{U} = \boxed{1} = \{-1, 0, 1\}$, then $\mathbb{W} = [-N...N]$; thus, $[a_0^0, a_0^1, a_0^2 \ldots]$ is determined by $[a_{-N}, a_{1-N}, \ldots, a_N]$.

By shift-invariance, this means that, for any $\ell \in \mathbb{L}$, the sequence $a_\ell^n = \Phi^n(\mathbf{a})_\ell$, for $n \in \mathbb{Z}$, is entirely determined by $[a_{\ell-N}, a_{1+\ell-N}, \ldots, a_{\ell+N}]$. Thus, for any r, the sequence $\Phi(\mathbf{a})_{|\boxed{r}}$, $\Phi^2(\mathbf{a})_{|\boxed{r}}$, $\Phi^3(\mathbf{a})_{|\boxed{r}}, \ldots$ is determined by $[a_{-R}, a_{1-R}, \ldots, a_R]$, where $R = r + N$. In other words, if $\mathbf{b}_{|\boxed{R}} = \mathbf{a}_{|\boxed{R}}$, then $\Phi^n(\mathbf{a})_{|\boxed{r}} = \Phi^n(\mathbf{a})_{|\boxed{r}}$ for all $n \in \mathbb{N}$.

'$\Longleftarrow$': Find $R > 0$ so that, for any $\mathbf{a}, \mathbf{b} \in \mathcal{A}^{\mathbb{L}}$, if $\mathbf{a}_{|\boxed{R}} = \mathbf{b}_{|\boxed{R}}$, then $a_0^n = b_0^n$ for all $n \in \mathbb{N}$.

Let $\mathbf{w} = \mathbf{a}_{|\boxed{R}}$, and let $\mathbf{W} = {}^{\infty}\mathbf{w}^{\infty} = [\ldots \mathbf{www} \ldots]$. It follows that, for any $\mathbf{b} \in \lhd \mathbf{w} \rhd$, $b_0^n = w_0^n$ for all $n \in \mathbb{N}$.

Claim 1: *The sequence $\{w_0^n\}_{n=0}^{\infty}$ is eventually periodic.*

Thus, there is some $M_\mathbf{w}$ and $P_\mathbf{w}$ so that $w_0^{m+P_\mathbf{w}} = w_0^m$ for any $m > M_\mathbf{w}$. Hence $b_0^{m+P_\mathbf{w}} = b_0^m$ for any $\mathbf{b} \in \lhd \mathbf{w} \rhd$ and $m > M_\mathbf{w}$.

Now define

$$M = \max_{\mathbf{w} \in \mathcal{A}^{\boxed{R}}} M_\mathbf{w} \qquad \text{and} \quad P = \operatorname{lcm}_{\mathbf{w} \in \mathcal{A}^{\boxed{R}}} P_\mathbf{w}$$

Then for any $\mathbf{b} \in \mathcal{A}^{\mathbb{L}}$, and $m > M$, we have $b_0^{m+P} = b_0^m$. For any $\ell \in \mathbb{L}$, we can replace $\mathbf{b}$ by $\mathbf{c} = \boldsymbol{\sigma}^\ell(\mathbf{b})$, to alsoget $b_\ell^{m+P} = c_0^{m+P} = c_0^m = b_\ell^m$. Hence, $\Phi^{m+P}(\mathbf{b}) = \Phi^m(\mathbf{b})$ for any $\mathbf{b} \in \mathcal{A}^{\mathbb{L}}$ and any $m > M$. ☐

CA that eventually halt

Note that $\mathcal{A} \supset \Phi(\mathcal{A}^{\mathbb{L}}) \supset \Phi^2(\mathcal{A}^{\mathbb{L}}) \supset \ldots$. The **eventual image** of Φ is defined:

$$\Phi^\infty(\mathcal{A}^{\mathbb{L}}) \;=\; \bigcap_{n=1}^{\infty} \Phi^n(\mathcal{A}^{\mathbb{L}})$$

In other words, $\Phi^\infty(\mathcal{A}^{\mathbb{L}})$ is the global attracting set for Φ.

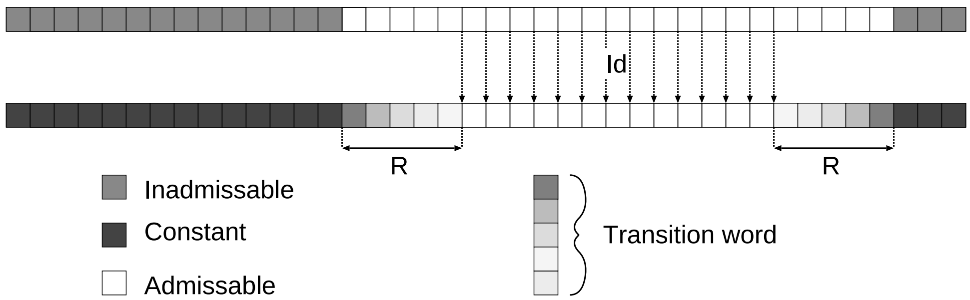

Figure 17: The Maass-Boyle automaton

Lemma 13: *(Lyman Hurd)* [11, 12]
Let $\mathfrak{X} \subset \mathcal{A}^{\mathbb{L}}$ be an SFT, and Φ a CA.
If $\Phi^{\infty}(\mathcal{A}^{\mathbb{L}}) = \mathfrak{X}$, then there is some N so that $\Phi^{N}(\mathcal{A}^{\mathbb{L}}) = \mathfrak{X}$.

Proof: **Exercise 7** ________ □

Note that $\mathsf{Fix}[\Phi] \subset \Phi^{\infty}(\mathcal{A}^{\mathbb{L}})$; say that Φ **eventually halts** if $\Phi^{\infty}(\mathcal{A}^{\mathbb{L}}) = \mathsf{Fix}[\Phi]$. The set $\mathfrak{X} = \mathsf{Fix}[\Phi]$ is a subshift of finite type, called the **halting set** for Φ.

Proposition 14: *If Φ eventually halts, then it halts in finite time, and Φ is equicontinuous.*

Proof: Combine Lemma 13 with Lemma 11 on page 18 and Lemma 10 on page 18. _□

When is an SFT $\mathfrak{X}$ be the halting sets of some CA? In one dimension, the answer is known.

A configuration $\mathbf{c} \in \mathcal{A}^{\mathbb{L}}$ is **spatially constant** if $c_\ell = a$ for all $\ell \in \mathbb{L}$, where $a \in \mathcal{A}$ is some fixed element. Notice that, if Φ is a CA, and $\mathbf{c}$ is a constant configuration, then $\Phi(\mathbf{c})$ is also constant.

Theorem 15: *(Mike Boyle; Alejandro Maass)* [18]
If $\mathfrak{X} \subset \mathcal{A}^{\mathbb{Z}}$ is an SFT, then $\mathfrak{X}$ is the halting set of a cellular automaton if and only if:

- *$\mathfrak{X}$ has a spatially constant configuration.*
- *$\mathfrak{X}$ is topologically mixing.*

Proof: By Hurd's lemma, $\mathfrak{X}$ is the eventual image of Φ iff $\Phi^{N}(\mathcal{A}^{\mathbb{L}}) = \mathfrak{X}$ for some finite N. By replacing Φ with Φ^{N}, we can assume WOLOG that $\Phi(\mathcal{A}^{\mathbb{L}}) = \mathfrak{X}$.

'$\Longrightarrow$': Since $\mathcal{A}^{\mathbb{L}}$ contains constant configurations, $\mathfrak{X}$ must, also. Likewise, since $\mathcal{A}^{\mathbb{L}}$ is topologically mixing, $\mathfrak{X}$ must also be mixing.

'$\Longleftarrow$': Suppose that $\mathfrak{X}$ is a topological Markov chain. Let $\mathfrak{X}$ have mixing radius R and constant configuration $\mathbf{c} = [\ldots cccc \ldots]$.

For any any $y, z \in \mathfrak{X}$, topological mixing implies that there is a *transition word* $\mathbf{w}(y, z) = [w_1 w_2 \ldots w_R]$ so that the fragment $[y w_1 w_2 \ldots w_R z]$ is $\mathfrak{X}$-admissable.

We define the local map $\phi : \mathcal{A}^{[-R-1 \ldots R+1]} \longrightarrow \mathcal{A}$ so that, for any $\mathbf{a}$ in $\mathcal{A}^{[-R-1..R+1]}$, $\phi(\mathbf{a})$ is given by the following rules (see Figure 17):

1. If $\mathbf{a}$ is $\mathfrak{X}$-admissable at r for every $r \in \boxed{R}$, then $\phi(\mathbf{a}) = a_0$.
2. If $\mathbf{a}$ is $\mathfrak{X}$-inadmissable at 0, then $\phi(\mathbf{a}) = c$.
3. If $\mathbf{a}$ is $\mathfrak{X}$-admissable at 0, but inadmissable at some $n_0 \in [-R..0)$ and some $n_1 \in (0..R]$, then $\phi(\mathbf{a}) = c$.
4. If $\mathbf{a}$ is $\mathfrak{X}$-admissable for every $r \in [-n..R]$, but inadmissable at $-n$, where $n > 0$, then $\phi(\mathbf{a}) = w_m$, where $m = R - n$ and $\mathbf{w} = [w_1 w_2 \ldots w_R] = \mathbf{w}(c, a_m)$.
5. If $\mathbf{a}$ is $\mathfrak{X}$-admissable for every $r \in [-R..n]$, but inadmissable at n, where $n > 0$, then $\phi(\mathbf{a}) = w_m$, where $m = R - n$ and $\mathbf{w} = [w_1 w_2 \ldots w_R] = \mathbf{w}(a_{-m}, c)$.

□

Remark: An extension of this result to higher dimensions has recently been announced by Sam Lightwood [16].

5 Partial Equicontinuity

Φ is **partially equicontinuous** if $\emptyset \neq \mathcal{E}q[\Phi] \neq \mathcal{A}^{\mathbb{L}}$. For example, if Φ contains at a stable fixed (or periodic) point $\mathbf{a}$ and an unstable fixed (or periodic) point $\mathbf{b}$, then it is partially equicontinuous, because $\mathbf{a} \in \mathcal{E}q[\Phi]$, but $\mathbf{b} \notin \mathcal{E}q[\Phi]$.

Example 16:

(a) **Contagion:** Let $\mathcal{A} = \{0, 1\}$and $\mathbb{U} = [-1..1] \subset \mathbb{L} = \mathbb{Z}$, and define

$$\phi(a_{-1}, a_0, a_1) = a_{-1} \cdot a_0 \cdot a_1$$

Then ${}^{\infty}0^{\infty} = (\ldots 00000 \ldots)$ is a stable fixed point, but ${}^{\infty}1^{\infty} = (\ldots 11111 \ldots)$ is unstable, because it can be 'invaded' by ${}^{\infty}0^{\infty}$ (see Figure 18).

```
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 . . . . . 1 1 1 1 1 1 1 1 |
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 . . . . . . . 1 1 1 1 1 1 1 |
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 . . . . . . . . . 1 1 1 1 1 1 |
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1 1 1 1 1 1 1 1 1 1 1 1 . . . . . . . . . . . . . . . 1 1 1 |
1 1 1 1 1 1 1 1 1 1 1 . . . . . . . . . . . . . . . . . 1 1 |
1 1 1 1 1 1 1 1 1 1 . . . . . . . . . . . . . . . . . . . 1 |
1 1 1 1 1 1 1 1 1 . . . . . . . . . . . . . . . . . . . . . ↓
```

Figure 18: The contagion rule.

(b) **Voter rules:** The majority voter rules from Example 4e are partially equicontinuous. To see this, consider the one-dimensional nearest neighbour voter model. Notice that the sequence ${}^{\infty}1^{\infty} = (\ldots 11111 \ldots)$ is a stable fixed point, as is the point ${}^{\infty}0^{\infty} = (\ldots 00000 \ldots)$. Hence, both are points of equicontinuity for the system. For the nearest neighbour voter model, the point ${}^{\infty}(01)^{\infty} = (\ldots 0101010101 \ldots)$ is a 2-periodic point, but is unstable, because it can be 'invaded' from either side by either ${}^{\infty}1^{\infty}$ or ${}^{\infty}0^{\infty}$ (see Figure 11). Hence, it is not a point of equicontinuity. For other, longer range voter models, other unstable points can be found. ______________________

Recall that voter rules are totalistic.

Question: *What are the equicontinuity properties of other totalistic cellular automata?*

6 Positive Expansiveness

Let $(\mathbf{X}, d)$ be a metric space, and let $f : \mathbf{X} \circlearrowleft$ be a continuous self-map. The dynamical system $(\mathbf{X}, f)$ is **positively expansive** if any two points in $\mathbf{X}$, no matter how close, eventually diverge in their trajectories. Formally, there is some **expansion constant** $\epsilon > 0$ so that, for any two $x, y \in \mathbb{X}$, there is some $n \in \mathbb{N}$ such that $d\left(f^n(x),\ f^n(y)\right) > \epsilon$.

If you were trying to 'predict' the future trajectory of x, this means that any initial measurement error (mistaking y for x), no matter how small, would eventually blow up to produce an error of magnitude at least ϵ.

If $\mathbf{X} = \mathcal{A}^{\mathbb{L}}$ and $f = \Phi$ is a cellular automaton, then Φ is positively expansive if there is some **expansion radius** $r > 0$ so that, for any $\mathbf{a}, \mathbf{b} \in \mathcal{A}^{\mathbb{L}}$, there is some $n \in \mathbb{N}$ with

$$\Phi^n(\mathbf{a})_{|\mathbb{B}(r)} \neq \Phi^N(\mathbf{b})_{|\mathbb{B}(r)}$$

Heuristically speaking, this means that any difference between $\mathbf{a}$ and $\mathbf{b}$, no matter how far away from the origin of $\mathbb{L}$, will eventually 'propagate' through $\mathbb{L}$ until it creates a disturbance within distance r of the origin. Thus, positive expansiveness manifests in CA as an ability for 'information' to inexorably propagate through the space.

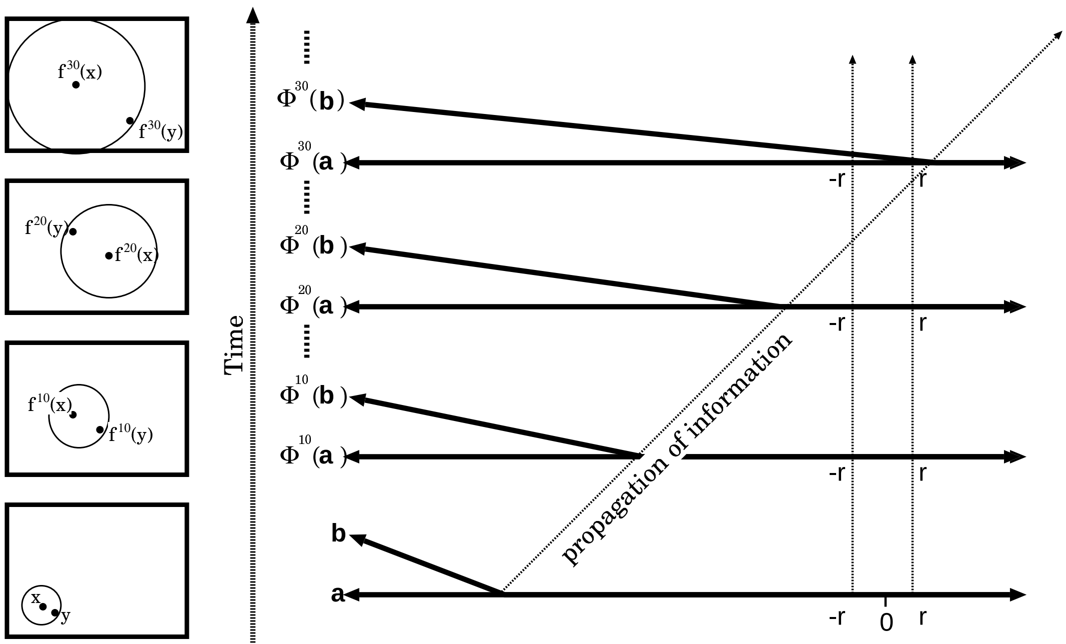

Figure 19: Positive expansiveness.

Lemma 17: *If Φ is a one-sided CA with radius R, then $r \leq R$.*
If Φ is a two-sided CA with left radius L and right radius R, then $r \leq \min\{L, R\}$.

Proof: **Exercise 8** □

Here are some examples:

6.1 Linear CA

Suppose $\mathcal{A} = \mathbb{Z}_{/2}$, regarded as a cyclic group, and recall the **nearest neighbour XOR** automaton, with local map $\phi(a_{-1}, a_0, a_1) = a_{-1} + a_1 \pmod 2$. Figure 9 on page 9 shows how information propagates through the lattice, and suggests that this map is expansive.

More generally, suppose $\mathcal{A} = \mathbb{Z}_{/N}$, as a cyclic group. A **linear cellular automaton** (LCA) is one with a local map of the form:

$$\phi(a_{-L}, a_{1-L}, \ldots, a_R) \;=\; \sum_{\ell=-L}^{R} c_\ell a_\ell \pmod N \tag{2}$$

where $c_{-L}, \ldots, c_R \in \mathbb{Z}_{/N}$ are constants. If c_{-L} and c_{+R} are relatively prime to N, then such a cellular automaton will be positively expansive (**Exercise 9**).

6.2 Multiplicative CA

Suppose $(\mathcal{A}, \cdot)$ is some (nonabelian) group, and consider a cellular automaton with local map

$$\phi(a_{-1}, a_0, a_1) \;=\; a_{-1}^2 \cdot (c^{-1} \cdot a_0 \cdot c) \cdot a_1^{-1} \tag{3}$$

Assuming $\mathcal{A}$ has no elements of order 2, this automaton is positively expansive.

More generally, let $u_0, u_1, \ldots, u_J \in \mathbb{U} = [-L..R]$, and let $\varphi_0, \varphi_1, \ldots, \varphi_J \in \mathbf{End}[\mathcal{A}]$ be group endomorphisms of $\mathcal{A}$. A **multiplicative cellular automaton** is one with a local map of the form

$$\phi(a_{-L}, a_{1-L}, \ldots, a_R) \;=\; \varphi_0(a_{u_0}) \cdot \varphi_1(a_{u_1}) \cdot \ldots \varphi_J(a_{u_J}) \cdot$$

For example, the MCA (3) is recovered by setting $u_0 = u_1 = -1$, $u_2 = 0$, and $u_3 = u_4 = \ldots = u_{N+2} = 1$, where $\mathcal{A}$ is a group of order N (so that $a^{N-1} = a^{-1}$ for any $a \in \mathcal{A}$). The endomorphisms φ_j are the identity maps for all $j \neq 2$, and $\varphi_2(a) = c^{-1} \cdot a \cdot c$.

If $(\mathcal{A}, +)$ was a cyclic group, then a the LCA (2) is obtained by defining $J = L + R + 1$, and defining $u_j = j - L$ and $\varphi_j(a) = c_{j-L} \cdot a$ for all $j \in [1...J]$.

Suppose that $u_j = -L$ and $u_i = R$, and no other u_k take on either value; if ϕ_j and ϕ_i are *automorphisms* of $\mathcal{A}$, then Φ is positively expansive (**Exercise 10**).

6.3 Permutative CA

If $\mathbf{b} \in \mathcal{A}^{(-L..R]}$, and $a \in \mathcal{A}$, then let $a.\mathbf{b}$ denote the element $[a, b_{1-L}, b_{2-L}, \ldots, b_R]$, and define $\phi_{\mathbf{b}} : \mathcal{A} \longrightarrow \mathcal{A}$ by:

$$\phi_{\mathbf{b}}(a) = \phi(a.\mathbf{b})$$

The cellular automaton Φ is called **left-permutative** if $\phi_{\mathbf{b}} : \mathcal{A} \circlearrowleft$ is a bijection for every $\mathbf{b} \in \mathcal{A}^{(-L..R]}$. For example, the LCA (2) is left-permutative iff c_{-L} is relatively prime to N.

Likewise, if $\mathbf{b} \in \mathcal{A}^{[-L..R)}$, and $c \in \mathcal{A}$, then let $\mathbf{b}.c$ denote the element $[b_{-L}, b_{1-L}, \ldots, b_{R-1}, c]$. If $\phi : \mathcal{A}^{[-L..R]} \longrightarrow \mathcal{A}$ is a local map define ${}_{\mathbf{b}}\phi : \mathcal{A} \longrightarrow \mathcal{A}$ by:

$${}_{\mathbf{b}}\phi(c) = \phi(\mathbf{b}.c)$$

Then Φ is **right-permutative** if ${}_{\mathbf{b}}\phi : \mathcal{A} \circlearrowleft$ is a bijection for every $\mathbf{b} \in \mathcal{A}^{[-L..R)}$. For example, the LCA (2) is right-permutative iff c_R is relatively prime to N.

If $\mathbb{L} = \mathbb{Z}$, then Φ is **bipermutative** if it is both left- and right-permutative. For example, the preceeding examples of positively expansive LCA and MCA are all bipermutative.

If $\mathbb{L} = \mathbb{N}$, then Φ is called **bipermutative** if it is *right*-permutative. We will develop the theory of bipermutative automata below for the case $\mathbb{L} = \mathbb{Z}$; the case $\mathbb{L} = \mathbb{N}$ is similar. Note that, when $\mathbb{L} = \mathbb{N}$, the correct analog of bipermutativity is *right*-permutativity; the consequences of *left*-permutativity for one-sided automata are much different.

Any bipermutative cellular automaton is positively expansive. Indeed, not only do disturbances propagate inexorably through space, but they do so with constant speed. To be precise, it is not hard to prove:

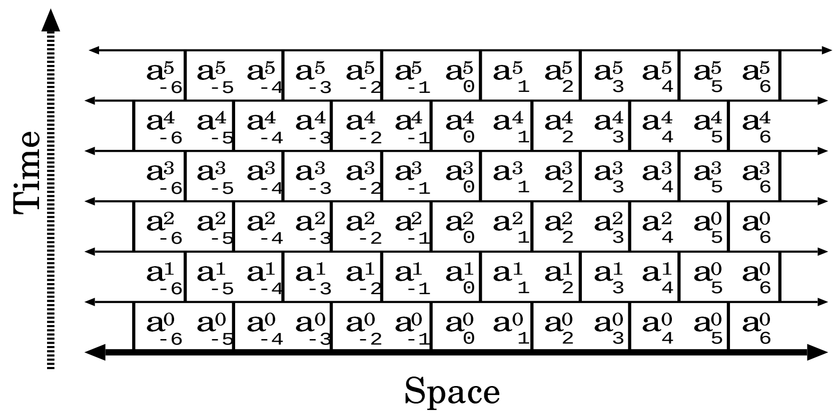

Figure 20: The brickwall decomposition.

Lemma 18: *Let* $\mathbb{L} = \mathbb{Z}$

- *Suppose* $\mathbf{a}\big|_{(-\infty..nR)} = \mathbf{b}\big|_{(-\infty..nR)}$, *but* $a_{nR} \neq b_{nR}$. *If* Φ *is right-permutative, then* $\Phi^n(\mathbf{a})_0 \neq \Phi^n(\mathbf{b})_0$.
- *Likewise, if* $\mathbf{a}\big|_{(-n\cdot L..\infty)} = \mathbf{b}\big|_{(-nL..\infty)}$, *but* $a_{-nL} \neq b_{-nL}$, *and* Φ *is left-permutative, then* $\Phi^n(\mathbf{a})_0 \neq \Phi^n(\mathbf{b})_0$.

Proof: **Exercise 11** ______ □

Multiplicative CA provide natural examples of bipermutative CA. Conversely, any permutative CA can be recoded to be 'quasimultiplicative'. For simplicity, suppose $L = R = 1$. Let $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$, and let $a_z^n = \Phi^n(\mathbf{a})_z$ for any $z \in \mathbb{Z}$ and $n \in \mathbb{N}$. Let $\mathbf{A} = \left[a_z^n\big|_{z\in\mathbb{Z}}^{n\in\mathbb{N}}\right]$ be the space-time diagram for $\mathbf{a}$. Suppose we decompose $\mathbf{A}$ into two-element blocks as in Figure 20. This **brickwall decomposition** represents a recoding of Φ as a cellular automaton over the alphabet $\mathcal{B} = \mathcal{A}^2$ as follows:

Define the maps $\Psi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{B}^{\mathbb{Z}}$ and $\Xi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{B}^{\mathbb{Z}}$ by

$$\begin{aligned} \Psi(\mathbf{a}) &= \left[\ldots, \begin{bmatrix} a_{-2} \\ a_{-1} \end{bmatrix}, \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ a_3 \end{bmatrix}, \ldots\right], \\ \text{and} \quad \Xi(\mathbf{a}) &= \left[\ldots, \begin{bmatrix} a_{-3} \\ a_{-2} \end{bmatrix}, \begin{bmatrix} a_{-1} \\ a_0 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \ldots\right]. \end{aligned}$$

Intuitively, the brickwall decomposition of $\mathbf{A}$ is obtained by applying Ψ and Ξ to alternate rows of $\mathbf{A}$.

Now, define $\widetilde{\phi} : \mathcal{B}^2 \longrightarrow \mathcal{B}$ by

$$\widetilde{\phi}\left(\left[\begin{array}{c} b_{-1} \\ b_0 \end{array}\right], \left[\begin{array}{c} b_1 \\ b_2 \end{array}\right]\right) \quad = \quad \left[\begin{array}{c} \phi(b_{-1}, b_0, b_1) \\ \phi(b_0, b_1, b_2) \end{array}\right]$$

and let $\widetilde{\Phi} : \mathcal{B}^{\mathbb{Z}} \circlearrowleft$ be the corresponding cellular automaton. Then for any $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$, we have:

$$\widetilde{\Phi} \circ \Psi(\mathbf{a}) \quad = \quad \Xi \circ \Phi(\mathbf{a})$$

Via the brickwall recoding, we can represent Φ by $\widetilde{\phi}$. We then interpret $\widetilde{\phi}$ as a binary operator $\star$ on $\mathcal{B}$, defined $a \star b = \widetilde{\phi}(a, b)$.

Let $\mathcal{B}$ be a set with binary operator $\star$. The pair $(\mathcal{B}, \star)$ is called a **loop** if $\star$-multiplication is **left-** and **right-invertible**; in other words, given any b and d in $\mathcal{B}$, there is a unique a so that $a \star b = d$, and a unique c so that $b \star c = d$.

The **multiplication table** for $(\mathcal{B}, \star)$ is just the $\mathcal{B} \times \mathcal{B}$-indexed matrix $\mathbf{M} = \left[m_{ab}|_{a,b \in \mathcal{B}}\right]$ where $m_{ab} = a \star b$. A $\mathcal{B} \times \mathcal{B}$-indexed matrix $\mathbf{M}$ of elements in $\mathcal{B}$ is called a **Latin square** if every element of $\mathcal{B}$ appears exactly once in each row of $\mathbf{M}$ and exactly once in each column of $\mathbf{M}$. For example, the following is a Latin Square on $\mathcal{B} = \{0, 1, 2, 3\}$.

$\star$	0	1	2	3
0	1	2	0	3
1	3	1	2	0
2	2	0	3	1
3	0	3	1	2

Proposition 19: *Let $\Phi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}$ be an arbitrary one-dimensional CA, and let $\widetilde{\Phi}$ be the brickwall recoding of Φ, with local map $\widetilde{\phi} : \mathcal{B}^2 \longrightarrow \mathcal{B}$, defining binary operator $\star$. Then the following are equivalent:*

1. *Φ is bipermutative.*
2. *$\widetilde{\Phi}$ is bipermutative.*
3. *$(\mathcal{B}, \star)$ is a loop.*
4. *The multiplication table of $(\mathcal{B}, \star)$ is a Latin square.*

Proof: Exercise 12 ______________________________ □

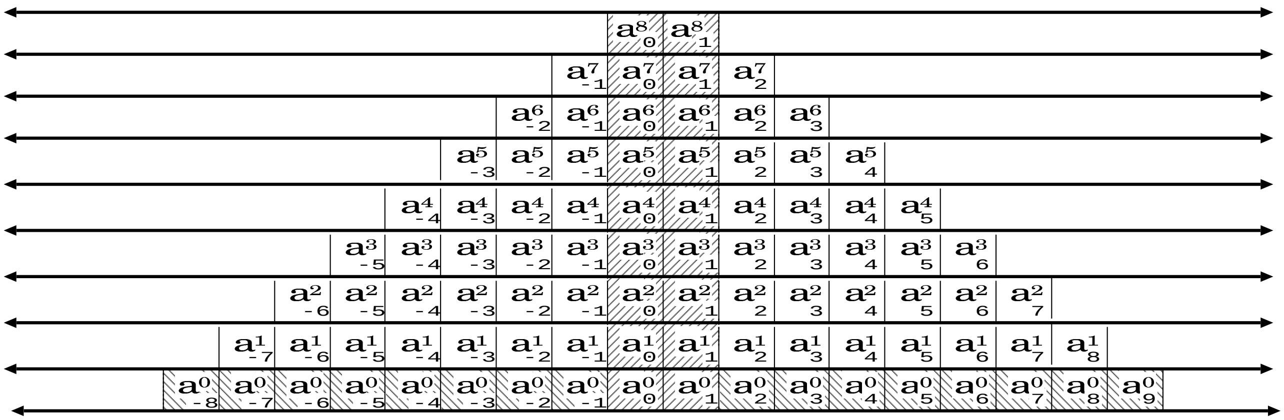

Figure 21: The conjugacy between a bipermutative CA and the full shift.

In this sense, every permutative CA is a sort of 'generalized multiplicative cellular automaton'. Note that $(\mathcal{B}, \star)$ is not necessarily associative. Indeed, it is not hard to show:

Any associative finite loop must actually be a group. (**Exercise 13**)

It is easy to find loops which are not groups; simply construct a Latin square. The number of such loops grows quickly with the size of $\mathcal{B}$:

If $\mathsf{Card}[\mathcal{B}] = B$*, then the number of* $\mathcal{B} \times \mathcal{B}$ *Latin squares is* $B!! = B! \cdot (B-1)! \cdot (B-2)! \cdot \ldots 2 \cdot 1.$ (**Exercise 14**)

Permutative cellular automata were first studied by Hedlund [9], who proved the following:

Proposition 20: [9] *Let* $\Phi : \mathcal{A}^{\mathbb{Z}} \circlearrowleft$ *be a one-dimensional, bipermutative cellular automaton with local map* $\phi : \mathcal{A}^{[-L..R]} \longrightarrow \mathcal{A}$*. Let* $\mathcal{B} = \mathcal{A}^{[-L..R)}$*. Then* $(\mathcal{A}^{\mathbb{Z}}, \Phi)$ *is topologically conjugate to the one-sided shift* $(\mathcal{B}^{\mathbb{N}}, \boldsymbol{\sigma})$*, via the map* $\Psi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{B}^{\mathbb{N}}$ *defined:*

$$\Psi(\mathbf{a}) \quad = \quad \left[\mathbf{a}\big|_{[-L..R)};\ \ \Phi(\mathbf{a})\big|_{[-L..R)};\ \ \Phi^2(\mathbf{a})\big|_{[-L..R)};\ \ \Phi^3(\mathbf{a})\big|_{[-L..R)};\ldots\right]$$

Proof: For simplicity, suppose that $L = R = 1$ so that $\mathcal{B} = \mathcal{A}^{\{-1,0\}} \cong \mathcal{A}^2$. Consider the fragmentary spacetime diagram shown in Figure 21. The horizontal shaded strip represents an element $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$, and the vertical shaded strip is the image $\Psi(\mathbf{a}) \in \mathcal{B}^{\mathbb{N}}$.

We claim that the map Ψ from $\mathcal{A}^{\mathbb{Z}}$ to $\mathcal{B}^{\mathbb{N}}$ is bijective. We will leave the details as an exercise, but sketch the general idea here.

First, note that, because of right-permutativity, the elements a_1^8 and a_1^7 uniquely determine a_2^7; similarly, by left-permutativity, a_0^8 and a_0^7 uniquely determine a_{-1}^7. Proceeding inductively in this fashion, it is possible to reconstruct the entire triangle in the picture, just from the data in the vertical shaded strip. In particular, it is possible to reconstruct the horizontal shaded strip from the vertical shaded strip in a unique fashion. This demonstrates two things:

1. The vertical shaded strip (that is, $\Psi(\mathbf{a})$) uniquely determines the horizontal strip (that is, $\mathbf{a}$); hence, Ψ is injective.
2. Any possible choice of vertical strip can be used to build a horizontal preimage in this manner. Hence, Ψ is surjective.

To see that Ψ is a conjugacy of Φ with $\boldsymbol{\sigma}$, notice that applying Ψ to $\mathbf{a}$ is equivalent to shifting the whole picture downwards by one row; this, in turn, is equivalent to applying the shift operator $\boldsymbol{\sigma}$ to the central shaded column. □

Corollary 21: *Let Φ be a bipermutative cellular automaton. Then the dynamical system $(\mathcal{A}^{\mathbb{Z}}, \Phi)$ is topologically transitive and mixing. The uniformly distributed measure on $\mathcal{A}^{\mathbb{Z}}$ is Φ-invariant, and the measurable system $(\mathcal{A}^{\mathbb{Z}}, \Phi, \mu)$ is measurable isomorphic to a Bernoulli shift.* □

Further reading:

If $\mathcal{A}$ is a group, then multiplicative CA over $\mathcal{A}$ are amenable to analysis via group-theoretic methods. For example, if $\mathcal{A}$ is abelian, we can apply harmonic analysis [20, 21] to study the evolution of measures on $\mathcal{A}^{\mathbb{L}}$. Even if $\mathcal{A}$ is nonabelian, we can still use the structure theory of $\mathcal{A}$ to yield a structural decomposition [25] of MCA on $\mathcal{A}^{\mathbb{L}}$, which can be applied to measurable asymptotics. Algebraic methods can also be used to gauge the computational complexity the 'prediction problem' for such automata [23]. However, when $\mathcal{A}$ is an arbitrary (nonassociative) loop, these methods break down very quickly, and little is known. The ergodic properties of general bipermutative CA were explored by Shereshevsky [29].

A theory very similar to the one above can be developed for *right*-permutative CA on $\mathcal{A}^{\mathbb{N}}$. However, *left*-permutative CA on $\mathcal{A}^{\mathbb{N}}$ are quite different; in general, they may not even be expansive. One family of such automata is studied in [4].

6.4 Quasipermutative CA

Let $\mathcal{B}$ be a set with binary operator $\star$. We call $(\mathcal{B}, \star)$ a **quasiloop** if $\mathcal{B} = \bigsqcup_{j=1}^{J} \mathcal{B}_j$, where, $\mathcal{B}_1, \ldots, \mathcal{B}_J$ are $\star$-closed subsets, and for each j, $(\mathcal{B}_j, \star)$ is a loop. For example, suppose $\mathcal{B} = \{0, 1, \ldots, 6\}$, and $\star$ has multiplication table:

$\star$	0	1	2	3	4	5	6
0	1	2	0	3	0	0	0
1	3	1	2	0	0	6	0
2	2	0	3	1	0	6	0
3	0	3	1	2	0	0	0
4	6	6	6	6	4	5	6
5	6	0	0	6	5	6	4
6	6	6	6	6	6	4	5

In this case, the subsets $\mathcal{B}_1 = \{0, 1, 2, 3\}$ and $\mathcal{B}_2 = \{4, 5, 6\}$ are loops, but $\mathcal{B}$ itself is not.

If Φ is a one-dimensional cellular automaton, we can perform a brickwall coding to represent Φ via a binary operator $(\mathcal{B}, \star)$. We say Φ is **quasipermutative** if $(\mathcal{B}, \star)$ is a quasiloop.

Given some $\mathbf{a} \in \mathcal{B}^{\mathbb{Z}}$, we divide $\mathbb{Z}$ up into domains $\mathbb{I}_k = [n_k ... n_{k+1})$, where $\ldots < n_{-1} < n_0 < n_1 < \ldots$, so that a_i belong to the same subalphabet $\mathcal{B}_{j_k}$ for all $i \in \mathbb{I}_k$. Within each domain, the cellular automaton looks like a permutative one; in particular, information propagates at a constant speed, until it reaches a domain boundary. Information may or may not be able to propagate across boundaries, depending upon the interaction between the left-hand subalphabet and the right-hand subalphabet. Also, the domains themselves may migrate over time, as one domain 'conquers territory' from another. Eloranta [7, 6] showed that, under some conditons the boundaries will move with uniform speed, while under others, they will perform random walks.

Question: *What are the expansiveness properties of quasipermutative CA?*

6.5 Nonpermutative Expansive Cellular Automata:

Recall the **Digital multiplier** automata of Example 4g. In that example, $\mathcal{A} = [0..K)$, where $K = k_1 \cdot k_2$, and $\Phi : \mathcal{A}^{\mathbb{N}} \circlearrowleft$ is the automaton implementing multiplication-by-k_1 on the K-ary representations of numbers in $[0, 1]$. Blanchard and Maass [8] have proved the following:

Proposition 22: $\Big($ *Φ is positively expansive* $\Big) \iff \Big($ *k_2 divides some integer power of k_1.* $\Big)$

6.6 Column Factors

Fix $W > 0$, and let $\mathcal{B} = \mathcal{A}^W$. The **column factor** of **width** W is the map $\mathcal{H}_W : \mathcal{A}^{\mathbb{L}} \longrightarrow \mathcal{B}^{\mathbb{N}}$ defined:

$$\mathcal{H}_W(\mathbf{a}) \;=\; \Big[\mathbf{a}_{|\boxed{W}};\;\; \Phi(\mathbf{a})_{|\boxed{W}};\;\; \Phi^2(\mathbf{a})_{|\boxed{W}};\;\; \Phi^3(\mathbf{a})_{|\boxed{W}}; \ldots\Big]$$

For example, if $\mathbb{L} = \mathbb{Z}$, then

$$\mathcal{H}_W(\mathbf{a}) \;=\; \Big[\mathbf{a}\big|_{[-W..W]};\;\; \Phi(\mathbf{a})\big|_{[-W..W]};\;\; \Phi^2(\mathbf{a})\big|_{[-W..W]};\;\; \Phi^3(\mathbf{a})\big|_{[-W..W]}; \ldots\Big]$$

The image $\mathbf{\Sigma}_W[\Phi] = \mathcal{H}_W(\mathcal{A}^{\mathbb{L}})$ is called the **column shift** of **width** W, and is a subshift of $\mathcal{B}^{\mathbb{N}}$. It is not hard to show:

$$\mathcal{H}_W \circ \Phi \;=\; \boldsymbol{\sigma} \circ \mathcal{H}_W$$

Thus, $(\mathbf{\Sigma}_W[\Phi], \boldsymbol{\sigma})$ is a factor of $(\mathcal{A}^{\mathbb{L}}, \Phi)$. It is not hard to show:

Proposition 23: *The following are equivalent:*

1. *Φ is positively expansive, with expansion factor r.*
2. *$\mathcal{H}_r$ is injective.*
3. *$(\mathbf{\Sigma}_r[\Phi], \boldsymbol{\sigma})$ and $(\mathcal{A}^{\mathbb{L}}, \Phi)$ are isomorphic, as topological dynamical systems, via the map $\mathcal{H}_r$.*

Proof: **Exercise 15**[2]. □

If Φ is a *bipermutative* one-dimensional CA, then this result just restates Proposition 20. However, Proposition 20 goes further, in stating:

If Φ is bipermutative, then $\mathbf{\Sigma}_R[\Phi] = \mathcal{B}^{\mathbb{N}}$ is the full shift.

If Φ is expansive, but not permutative, what sort of subshift is $\mathbf{\Sigma}_R[\Phi]$?

Theorem 24: *If Φ is a one-dimensional, positively expansive CA, with expansion radius r. Then $\mathbf{\Sigma}_r[\Phi]$ is a subshift of finite type. Furthermore*

1. *(Blanchard, Maass) If Φ is one-sided (ie. $\mathbb{L} = \mathbb{N}$), then $\mathbf{\Sigma}_r[\Phi]$ is mixing.* [8]
2. *(Nasu, Kůrka) if Φ is two-sided (ie. $\mathbb{L} = \mathbb{Z}$), then $\mathbf{\Sigma}_r[\Phi]$ is conjugate to a full shift.* [24, 15] □

[2] This is actually a special case of a result of Ruelle, true for any expansive topological dynamical system

Corollary 25: *If Φ is expansive, then it is surjective, and the uniform measure is Φ-invariant.*

Proof: Since it is conjugate to a subshift of finite-type, Φ is finite-to-one; this guarantees that it is surjective onto $\mathcal{A}^{\mathbb{L}}$ by Proposition 6 on page 16. ☐

Remark: The result of Nasu and Kůrka for two-sided automata is stronger than that of Blanchard and Maass for one-sided, as it asserts conjugacy with the full shift; one wonders whether this can be extended to the one-sided case. Boyle, Fiebig and Fiebig [22] have a counterexample: an expansive one-sided CA such that $\boldsymbol{\Sigma}_R[\Phi]$ is *not* conjugate to a full shift.

Question: *Can* any *subshift of finite type appear as a column shift of some expansive one-sided cellular automaton? If not, what are necessary conditions?*

7 Sensitivity

Let $(\mathbf{X}, d)$ be a metric space, and let $f : \mathbf{X}\circlearrowleft$ be a continuous self-map. The dynamical system $(\mathbf{X}, f)$ is **sensitive to initial conditions** (or **sensitive**, for short) if there is some $\epsilon > 0$ so that, for any $x \in \mathbf{X}$ and any δ, there is some $y \underset{\delta}{\sim} x$ such that $d\left(f^n(x),\ f^n(y)\right) > \epsilon$ for some $n \in \mathbb{N}$.

A cellular automaton Φ is sensitive if there is some $r > 0$ such that, for any $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$, and any $R > 0$, there is some $\mathbf{b} \in \mathcal{A}^{\mathbb{L}}$ such that $\mathbf{a}\big|_{[-R..R]} = \mathbf{b}\big|_{[-R..R]}$, but $\Phi^n(\mathbf{a})\big|_{\boxed{r}} \neq \Phi^n(\mathbf{b})\big|_{\boxed{r}}$ for some $n \in \mathbb{N}$.

Example 26:

(a) Any positively expansive CA is sensitive.

(b) The shift map $\Phi = \boldsymbol{\sigma}$ is sensitive, but not expansive. If $\mathbf{a}\big|_{[-N..\infty)} = \mathbf{b}\big|_{[-N..\infty)}$, then the $\boldsymbol{\sigma}$-orbits of $\mathbf{a}$ and $\mathbf{b}$ will converge; however, if instead $a_n \neq b_n$ for all $n > N$, then their $\boldsymbol{\sigma}$-orbits will diverge, even if $\mathbf{a}\big|_{[-N..N]} = \mathbf{b}\big|_{[-N..N]}$ for some large N.

(c) Suppose Φ is right-permutative, but not left-permutative. For example, suppose Φ is the linear CA with local map $\phi(a_0, a_1) = a_0 + a_1$. Then Φ is sensitive.

(d) Let $\Phi : \mathcal{A}^{\mathbb{L}}\circlearrowleft$ be an expansive CA, and let $\Psi : \mathcal{B}^{\mathbb{L}}\circlearrowleft$ be equicontinuous. Let $\mathcal{C} = \mathcal{A} \times \mathcal{B}$; then there is a canonical identification $\mathcal{C}^{\mathbb{L}} \cong \mathcal{A}^{\mathbb{L}} \times \mathcal{B}^{\mathbb{L}}$. Let $\Xi = \Phi \times \Psi : \circlearrowleft \mathcal{C}^{\mathbb{L}}$; then Ξ is a sensitive cellular automaton.

Clearly, if a topological dynamical system $(\mathbf{X}, f)$ is sensitive, then $\mathcal{E}q[\mathbf{X}, f]$ must be empty. The converse is not true, in general, but we will see that it does hold for one-dimensional cellular automata.

Recall that a subset of a topological space is **comeager** if it is a countable intersection of dense, open sets. A probability measure on $\mathcal{A}^{\mathbb{M}}$ has **full support** if it assigns nonzero probability to every cylinder set.

Proposition 27: *Let Φ be a one-dimensional cellular automaton. If Φ is not sensitive, then $\mathcal{E}q[\Phi]$ is a comeager set, and, if μ is any probability measure on $\mathcal{A}^{\mathbb{L}}$ with full support, then $\mu(\mathcal{E}q[\Phi]) = 1$.*

Proof: Since Φ is not sensitive, there is some $\mathbf{a} \in \mathcal{A}^{\mathbb{L}}$ and some $R > 0$ so that, if $\mathbf{w} = \mathbf{a}\big|_{[-R..R]}$, then for any $\mathbf{b} \in [\mathbf{w}]$, $\Phi^n(\mathbf{b})\big|_{[-1..1]} = \Phi^n(\mathbf{a})\big|_{[-1..1]}$ for all $n \in \mathbb{N}$.

Now, for any N, let $\mathfrak{E}_N$ be the set of all $\mathbf{b} \in \mathcal{A}^{\mathbb{L}}$ so that $\mathbf{b}\big|_{[n-R..n+R]} = \mathbf{w}$ and $\mathbf{b}\big|_{[m-R..m+R]} = \mathbf{w}$ for some $m < -N$ and some $m > N$. Let $\mathfrak{E}_\infty = \bigcap_{N=1}^{\infty} \mathfrak{E}_N$. The following facts are not hard to verify:

1. $\mathfrak{E}_N$ is open and dense.
2. Thus, $\mathfrak{E}_\infty$ is comeager.
3. If μ has full support, then $\mu(\mathfrak{E}_N) = 1$.
4. Hence, $\mu(\mathfrak{E}_\infty) = 1$.
5. If $\mathbf{b} \in \mathfrak{E}_N$, and $\mathbf{c}\big|_{[-N-R...N+R]} = \mathbf{b}\big|_{[-N-R...N+R]}$, then $\Phi^n(\mathbf{c})\big|_{[-N-1...N+1]} = \Phi^n(\mathbf{b})\big|_{[-N-1...N+1]}$ for all $n \in \mathbb{N}$.
6. Hence, any element of $\mathfrak{E}$ is a point of equicontinuity.

Thus $\mathcal{E}q[\Phi] \supset \mathfrak{E}_\infty$ is comeager and of measure 1. ______________________________ □

Remark: Note that the one-dimensional nature of the automaton is crucial to this proof.

Corollary 28: *Kůrka's Topological Classification*
Every one-dimensional cellular automata belongs to exactly one of the following four classes:

(E1) *Equicontinuous.*

(E2) *Partially equicontinuous.*

(E3) *Sensitive, but not positively expansive.*

(E4) *Positively expansive.* □

8 Formal Languages

If $\mathcal{A}$ is a finite alphabet, let $\mathcal{A}^n$ denote all **words** of length n, and let $\mathcal{A}^* = \bigcup_{n=1}^{\infty} \mathcal{A}^n$ be the set of all finite words. A **formal language** is some subset of $\mathcal{A}^*$.

Example 29:

(a) If $\mathbb{L} = \mathbb{Z}$ or $\mathbb{N}$, and $\mathfrak{X} \subset \mathcal{A}^{\mathbb{L}}$ is a subshift, then the **language** of $\mathfrak{X}$ is the set

$$\mathcal{L}(\mathfrak{X}) \;=\; \left\{ \mathbf{w} \in \mathcal{A}^* \;;\; \mathbf{w} = \mathbf{a}\big|_{[n..m]} \;\text{ for some }\; \mathbf{x} \in \mathfrak{X} \text{ and } \; n, m \in \mathbb{L} \right\}$$

(b) For example, if $\mathfrak{X}$ is a subshift of finite type or sofic shift, then $\mathcal{L}(\mathfrak{X})$ consists of all words $\mathbf{w} = [w_1 w_2 \ldots w_N]$ such that $w_1 \to w_2 \to \ldots \to w_N$ is the labeling of some finite directed path in the labeled digraph defining $\mathfrak{X}$.

(c) In particular, if Φ is a cellular automaton, and $\mathbf{\Sigma}_W[\Phi]$ is a *column shift* of Φ (see §6.6), then the corresponding **column language** of Φ is defined: $\mathcal{L}_W[\Phi] = \mathcal{L}(\mathbf{\Sigma}_W[\Phi]) \subset \mathcal{B}^*$, where $\mathcal{B} = \mathcal{A}^{\boxed{W}}$.

(d) **Powers of Two:** Let $\mathcal{A} = \{0, 1, 2\}$, and let $\mathcal{L} \subset \mathcal{A}^*$ be the set of all words such that any two consecutive instances of symbol '2' are separated by a distance that is a power of 2. Thus, '012 00 2 1010 2 11011011 2 10 2' is an element of $\mathcal{L}$, but '012 010 2 111010 2 110011 2 110 2' is not.

(e) Let $\mathcal{A} = \{0, 1\}$, and let $\mathcal{L} \subset \mathcal{A}^*$ be the set of all words $[w_0 w_1 w_2 \ldots w_L]$ so that, if $w_{N-1} = 1$ and $w_N = 0$, then $w_n = 0$ for all $n \in [N..M]$, where $M = \min\{2N, L\}$. Thus, for example, '111 000 11 00000000 1 0000000000000000000 1' is in $\mathcal{L}$, but '111 000 11 000 1' is not.

(f) **Arithmetic Expressions:** Let $\mathcal{A} = \{\langle, \rangle, +, \bullet, a, b, c, d\}$, and let $\mathcal{M}$ be the set of all valid 'arithmetic expressions' in the variables a, b, c, d, with the operators $+$ and $\bullet$, and with "$\langle$" and "$\rangle$" playing the role of brackets. Thus, "$\langle a + b\rangle \bullet \langle c + a \bullet d\rangle$" is an element of $\mathcal{M}$, but "$\langle\langle ab + \rangle c \bullet \rangle \langle$" is not. In particular, note that, in any element of $\mathcal{M}$, each "$\langle$" symbol must be matched by a "$\rangle$".

One way to generate valid elements of $\mathcal{M}$ is through a *substitution* procedure, as follows. Let x represent an indeterminate expression, and consider the following substitution rules:

$$\begin{array}{llll} (0) & x \Longrightarrow \langle x \rangle & (4) & x \Longrightarrow b \\ (1) & x \Longrightarrow x + x & (5) & x \Longrightarrow c \\ (2) & x \Longrightarrow x \bullet x & (6) & x \Longrightarrow d \\ (3) & x \Longrightarrow a & & \end{array}$$

For example, we obtain "$\langle a + b \rangle \bullet \langle c + a \bullet d \rangle$" through the following sequence of substitutions:

$$\begin{array}{lllll} x & \xrightarrow{(3)} x \bullet x & \xrightarrow{(0),(0)} \langle x \rangle \bullet \langle x \rangle & \xrightarrow{(1),(1)} \langle x + x \rangle \bullet \langle x + x \rangle \\ & \xrightarrow{(2)} \langle x + x \rangle \bullet \langle x + x \bullet x \rangle & \xrightarrow{(3)} \langle a + x \rangle \bullet \langle x + x \bullet x \rangle \\ & \xrightarrow{(4)} \langle a + b \rangle \bullet \langle x + x \bullet x \rangle & \xrightarrow{(5)} \langle a + b \rangle \bullet \langle c + x \bullet x \rangle \\ & \xrightarrow{(3)} \langle a + b \rangle \bullet \langle c + a \bullet x \rangle & \xrightarrow{(6)} \langle a + b \rangle \bullet \langle c + a \bullet d \rangle \end{array}$$

Beginning with the symbol x, any valid element of $\mathcal{M}$ can be constructed in this manner.

8.1 Phrase Structure Grammars

Formal languages are usually arise through some computational process. One way to specify a formal language is via a **phrase structure grammar** (PSG). A PSG is a 4-tuple $(\mathcal{V}, \mathcal{A}, \mathcal{S}, \mathcal{I})$. Here, $\mathcal{V}$ is a set of symbols called **variables**, $\mathcal{A}$ is a set of symbols called **terminals**, and $\mathcal{S}$ is a collection of **substitution rules** of the form

$$\mathbf{a} \Longrightarrow \mathbf{b}$$

where $\mathbf{a}, \mathbf{b} \in (\mathcal{V} \sqcup \mathcal{A})^*$. Finally $\mathcal{I}$ is some initial element of $\mathcal{V}$. The language $\mathcal{L}$ is then the set of all elements of $\mathcal{A}^*$ which can be obtained by any chain of substitution rules, beginning from $\mathcal{I}$.

For example, the previous example gave a phrase structure grammar for the language $\mathcal{M}$. In this case, $\mathcal{V} = \{x\}$, $\mathcal{A} = \{\langle, \rangle, +, \bullet, a, b, c, d\}$, $\mathcal{S}$ is the set of rules (0) to (6) indicated above, and $\mathcal{I} = x$ is the initial state.

We can restrict the complexity of $\mathcal{L}$ by placing restrictions on the sort of substitutions allowed in its PSG. This leads the **Chomsky Heirarchy** of linguistic complexity:

(Reg) Regular languages.

(CFL) Context-Free languages.

(CSL) Context-Sensitive languages.

(Phr) Phrase Structure languages.

We will also introduce the subclass (**BP**) of **bounded periodic** languages[3]. We then have: (**BP**) $\subset$ (**Reg**) $\subset$ (**CFL**) $\subset$ (**CSL**) $\subset$ (**Phr**) $\subset \mathcal{P}(\mathcal{A}^*)$, and each inclusion is proper.

Bounded Periodic languages: The language $\mathcal{L}$ is **bounded periodic** if there are $M, P \in \mathbb{N}$ so that, for any $\mathbf{w} = [w_1 w_2 \ldots w_N] \in \mathcal{L}$, if $N > P + M$, then $w_{m+P} = w_m$ for all $m > M$. In other words, every word in $\mathcal{L}$ longer than $M + P$ has the structure $\mathbf{w} = \mathbf{uvvv}\ldots\mathbf{v}$, where $\mathbf{u}$ is a 'prefix' of length at most M, and $\mathbf{v}$ is a word of length P.

Regular languages ...are those whose substitution rules must all take the form $v \Longrightarrow \mathbf{a}v'$ or $v \Longrightarrow \mathbf{a}$, where $v, v' \in \mathcal{V}$ and $\mathbf{a} \in \mathcal{A}^*$.

Example: Let $\mathfrak{X} \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift. Then it is not hard to show:

$$\Big(\mathcal{L}(\mathfrak{X}) \text{ is regular} \Big) \Longleftrightarrow \Big(\mathfrak{X} \text{ is sofic} \Big)$$

Thus, in a certain sense, the computational theory of regular languages is 'equivalent' to the symbolic dynamical theory of sofic shifts.

Context-Free languages ...are those whose substitution rules must all take the form $v \Longrightarrow \mathbf{x}$ where $v \in \mathcal{V}$ and $\mathbf{x}$ is some finite word in $\mathcal{A} \sqcup \mathcal{V}$.

For example, the 'arithmetic expression' language $\mathcal{M}$ of Example 29f is context-free, as can be seen from the substitution scheme described in that example.

Context-Sensitive languages ...are those whose substitution rules must all take the form $\mathbf{x}v\mathbf{x} \Longrightarrow \mathbf{xyx}$ where $v \in \mathcal{V}$ and $\mathbf{x}, \mathbf{y}$ are finite words in $\mathcal{A} \sqcup \mathcal{V}$.

For example, the 'powers-of-two' language from Example 29d, is context-sensitive (see [10], Example 9.5, p. 224), as is Example 29e.

Phrase Structure languages ...are languages arising from any phrase structure grammar.

[3]These are not officially part of the Chomsky heirarchy.

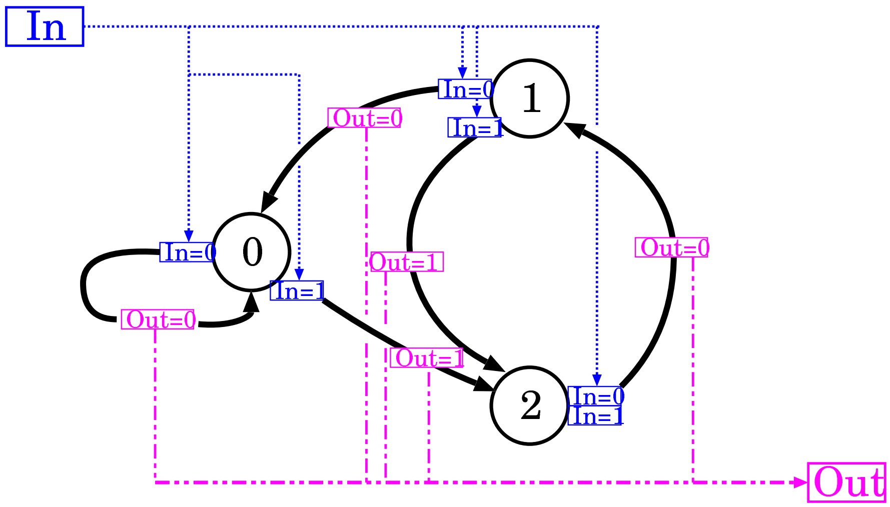

Figure 22: A finite state machine. Here, $\mathcal{S} = \{0, 1, 2\}$, while $\mathcal{I} = \mathcal{O} = \{0, 1\}$.

8.2 Machine Models

Various mathematical models of computation exist, with different levels of computational power. The four classes we will discuss, in increasing order of computational power, are:

(FSM) Finite state machines.

(Stack) Stack machines.

(LBM) Linear bounded machines.

(Turing) Turing machines.

Finite State Machines

A **finite state machine** (FSM) has a finite set of internal **states** $\mathcal{S}$, finite alphabets of **input** symbols $\mathcal{I}$ and **output** symbols $\mathcal{O}$, and a **transition rule** of the form

$$\Phi : \mathcal{I} \times \mathcal{S} \longrightarrow \mathcal{S} \times \mathcal{O}$$

If the machine begins in state s_0, and receives the input stream $i_0, i_1, i_2, \ldots, i_{N-1}$, then it proceeds progressively through states $s_1, s_2, \ldots, s_N$ and produces output $o_1, o_1, \ldots, o_N$, where, for every $n \in [0..N)$

$$(s_{n+1}, o_{n+1}) = \phi(s_n, i_n)$$

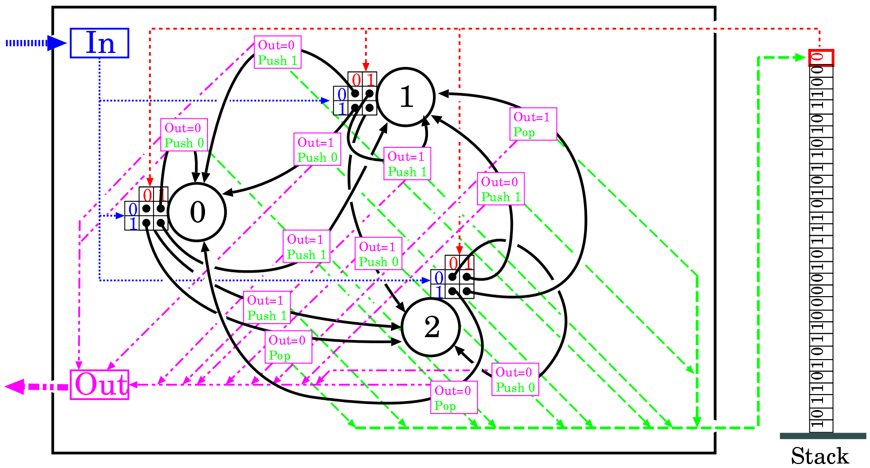

Figure 23: A stack machine.

Diagramatically:

$$
\begin{array}{ccccccccccccccc}
i_0 & & i_1 & & i_2 & & i_3 & \dots & \dots & \dots & i_{N-1} & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
s_0 & \Longrightarrow & s_1 & \Longrightarrow & s_2 & \Longrightarrow & s_3 & \Longrightarrow & \dots & \Longrightarrow & s_{N-1} & \Longrightarrow & s_N \\
 & \searrow & & \searrow & & \searrow & & \searrow & & & & \searrow & \\
 & o_1 & & o_2 & & o_3 & & o_4 & \dots & \dots & \dots & o_N &
\end{array}
$$

Finite state machines are the ‘weakest’ model of computation. The finite memory size of an FSM strictly limits the complexity of computations it can perform.

Stack Machines

A **stack machine** (or **pushdown automaton**) is a finite state machine augmented with a semi-infinite memory system called a **stack** (Figure 23). The stack is capable of holding an arbitrarily long finite string of symbols, which we imagine to be arranged vertically, with the first element at the ‘bottom’, and the last at the ‘top’. During a computation step, the machine can read the symbol at the *top* of the stack (but *not* those underneath), and use this symbol to decide what to do next. During any computation step, the machine can also decide to *add* (“push”) a new symbol onto the top of the stack (thereby obscuring the symbol below it), or to *remove* (“pop”) the symbol currently at the top (thereby revealing the symbol below).

Formally, a stack automaton is described by a function

$$\Phi : \mathcal{I} \times \mathcal{S} \times \mathcal{P} \longrightarrow \mathcal{O} \times \mathcal{S} \times \widetilde{\mathcal{P}}$$

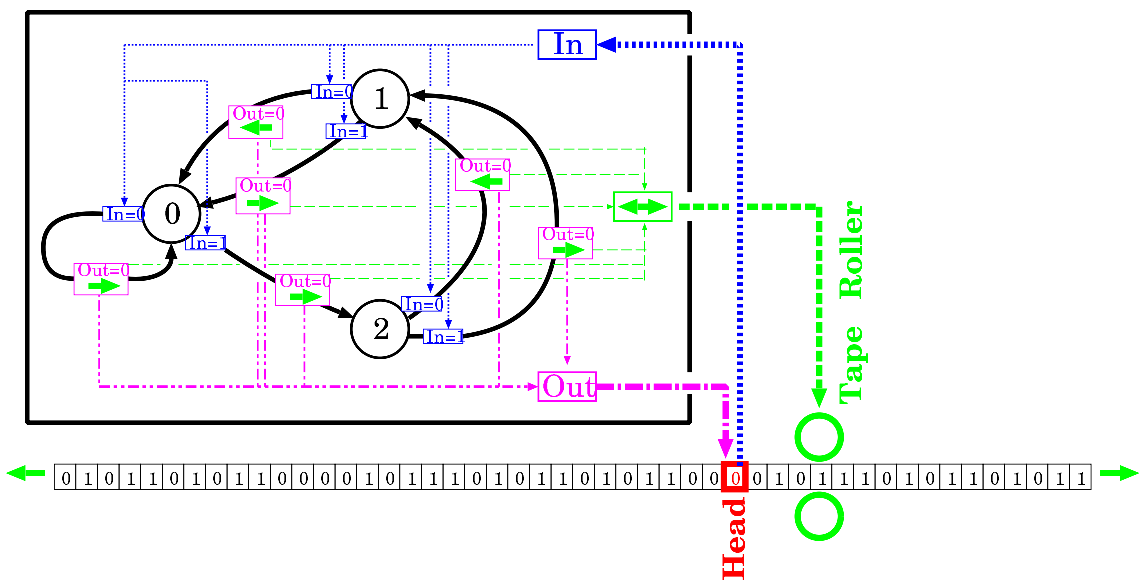

Figure 24: A Turing machine. Here, $\mathcal{S} = \{0, 1, 3\}$ and $\mathcal{A} = \{0, 1\}$.

Here, $\mathcal{I}$ and $\mathcal{O}$ are the input and output alphabets, and $\mathcal{S}$ is the internal statespace of the machine. $\mathcal{P}$ is the alphabet of stack symbols, and $\widetilde{\mathcal{P}}$ is $\mathcal{P}$ augmented with two special symbols '$\emptyset$' and '$\triangle$'. If the current machine state is $s_n \in \mathcal{S}$, the current input is $i \in \mathcal{I}$, and the top of the stack is $p \in \mathcal{P}$, and $\Phi(i, s_n, p) = (o, s', p')$, then the machine will output symbol o and change to internal state s'. If $p' = \triangle$, it will pop a symbol off the stack; if $p' \in \mathcal{P}$, it will push the symbol p' onto the stack, and if $p' = \emptyset$, it will do nothing to the stack.

Stack machines are more computationally powerful than finite state machines, because the stack allows computatations of arbitrary complexity, as long as these computations can be structured in a *recursive*[4] manner. For example, computations involving 'tree-like' data structures can usually be handled with stack automata.

Turing Machines

A **Turing machine** is a finite state machine equipped with an infinite memory storage system called a **tape**. The tape is a bi-infinite sequence of symbols, which the machine can read and write upon with a **head**, which, at any moment in time, is located at a specific position in the tape (see Figure 24).

Instead of having separate input/output channels, we normally assume the machine reads

[4]Recursion is the computational equivalent of mathematical induction: it is a technique whereby a complex problem is solved by systematically reducing it to simpler ones, through a process guaranteed to halt in a finite amount of time.

its input from the tape, and writes output to it. It can also use the tape as a memory store for computations. During any computation step, the Turing machine can change its internal state, overwrite the symbol under the head with a new symbol, and/or move the head one space forward or back along the tape.

Formally, let $\mathcal{A}$ be the tape alphabet, and $\mathcal{S}$ the internal statespace of the machine. Then the machine behaviour is described by a function $\Phi : \mathcal{S} \times \mathcal{A} \longrightarrow \mathcal{S} \times \widetilde{\mathcal{A}} \times \{\lhd, \Box, \rhd\}$, where $\widetilde{\mathcal{A}}$ is $\mathcal{A}$ augmented with the symbol '$\emptyset$'. If the machine is in state s and the current tape symbol is a, and $\Phi(s, a) = (s', a', m)$, then the machine switches to state s' and writes symbol a' on the tape, unless if $a' = \emptyset$, in which case it writes nothing. If $m = \lhd$ (resp. $\rhd$) the machine moves the head to the left (resp. right), while if $m = \Box$, it remains fixed.

A finite state machine or stack machine will only continue computing until it runs out of input; a Turing machine, on the other hand, can potentially keep going forever. Because of this, the statespace $\mathcal{S}$ must contain special **halt states**; if $\mathbf{M}$ enters a halt state, it stops moving, and the computation is finished. The 'output' of $\mathbf{M}$ then has two components: the final string written on the tape, and the particular state $\mathbf{M}$ halts in (for example, there might be two halt states, one meaning "yes", and the other one, "no".)

Of course, starting from a particular, there is no guarantee that $\mathbf{M}$ will *ever* halt. The question, "Will machine $\mathbf{M}$ ever halt if it starts with tape $\mathbf{a}$?" is called the *Halting Problem*, and Turing showed it to be formally undecidable.

Turing machines are the most powerful model of computation[5].

Linear Bounded Machines

A **linear bounded machine** (LBM) is like a Turing machine, but it has a tape of *finite* length. An LBM is more powerful than a finite state machine because the length of the tape is allowed to grow with the complexity of the input.

Formally, a LBM is a Turing machine where the tape alphabet $\mathcal{A}$ contains two special symbols, '$\langle$' and '$\rangle$'. The **input** of the machine is an arbitrarily long (but finite) stretch of tape, bounded by '$\langle$' on one end, and '$\rangle$' on the other. The machine cannot move its head past either one of these 'endmarker' symbols, but can range freely between them.

8.3 Languages and Machines

The complexity of a language is proportional to the computational power necessary to 'understand' that language. Thus, we can relate the heirarchy of linguistic complexity to the heirarchy of computational power. There are two ways in which machine $\mathbf{M}$ can be said to 'understand' language $\mathcal{L}$:

[5] That is, the most powerful model not involving 'supernatural' devices like oracles.

- **M recognizes** $\mathcal{L}$ if **M** has two special states, called `Accept` and `Reject`, and given any word $\mathbf{w} \in \mathcal{I}^*$ as input, **M** halts in state `Accept` if $\mathbf{w} \in \mathcal{L}$, and halts in state `Reject` otherwise.

- **M enumerates** $\mathcal{L}$ if, given the any word $\mathbf{w} \in \mathcal{I}^*$ as input, **M** outputs an element $\phi(\mathbf{w}) \in \mathcal{L}$ as output; furthermore, the function $\phi : \mathcal{I}^* \longrightarrow \mathcal{L}$ implemented by **M** is surjective.

  This is called *enumeration* because we imagine a correspondence between $\mathbb{N}$ and $\mathcal{I}^*$ (for example, if $\mathcal{I} = \{0, 1\}$, then we can represent each element of $\mathbb{N}$ by its binary expansion), so that the function ϕ implicitly implements a surjection $\varphi : \mathbb{N} \longrightarrow \mathcal{L}$.

If $\mathfrak{M}$ is a machine class (eg. one of **(FSM)**, **(Stack)**, **(LBM)**, or **(Turing)**), then we say language $\mathcal{L}$ is **$\mathfrak{M}$-recognizable** (resp. **$\mathfrak{M}$-enumerable**) if there is a machine $\mathbf{M} \in \mathfrak{M}$ that recognizes (resp. enumerates) $\mathcal{L}$.

If $\mathcal{L} \subset \mathcal{A}^*$ is $\mathfrak{M}$-recognizable, then it is $\mathfrak{M}$-enumerable. To see this, suppose $\mathbf{M} \in \mathfrak{M}$ recognizes $\mathcal{L}$; then to enumerate $\mathcal{L}$, we can build a machine $\mathbf{M}' \in \mathfrak{M}$ which searches through all elements of $\mathcal{A}^*$, in order, but only yields, as output, those elements which **M** recognizes as being in $\mathcal{L}$.

In the case of Turing machines, there is potentially another complication: **M** may 'recognize' words in $\mathcal{L}$, but fail to 'recognize' words *not* in $\mathcal{L}$, in the sense that, on some words not in $\mathcal{L}$, **M** *never halts*. Thus, we may end up waiting an infinite amount of time for **M** to give us a definitive answer. Clearly, this is not really a satisfactory form of 'recognition'. We say that **M accepts** language $\mathcal{L}$ if **M** halts in state `Accept` on any input in $\mathcal{L}$, and either never halts, or halts in state `Reject` on any input not in $\mathcal{L}$. Clearly, acceptance is weaker than recognition. However, it is equivalent to enumeration. To see this, observe that, if **M** is a machine that enumerates $\mathcal{L}$, then we can build a machine $\mathbf{M}'$ which accepts $\mathcal{L}$ very simply: given any word $\mathbf{w}$, the machine $\mathbf{M}'$ simply runs **M**, and waits to see if $\mathbf{w}$ eventually appears as an output of **M**. If it does, then $\mathbf{M}'$ accepts $\mathbf{w}$; if it doesn't, then $\mathbf{M}'$ just keeps waiting.

The machine classes are related to the Chomsky heirarchy as follows:

Theorem 30: *Let $\mathcal{L} \subset \mathcal{A}^*$ be a formal language.*

1. *$\mathcal{L}$ is* regular *iff $\mathcal{L}$ is* **(FSM)***-recognizable iff $\mathcal{L}$ is* **(FSM)***-enumerable.*

2. *$\mathcal{L}$ is* context-free *iff $\mathcal{L}$ is* **(Stack)***-recognizable.*

3. *$\mathcal{L}$ is* context-senstive *iff $\mathcal{L}$ is is* **(LBM)***-enumerable.*

4. *$\mathcal{L}$ is* a phrase structure language *iff $\mathcal{L}$ is* **(Turing)***-enumerable* □

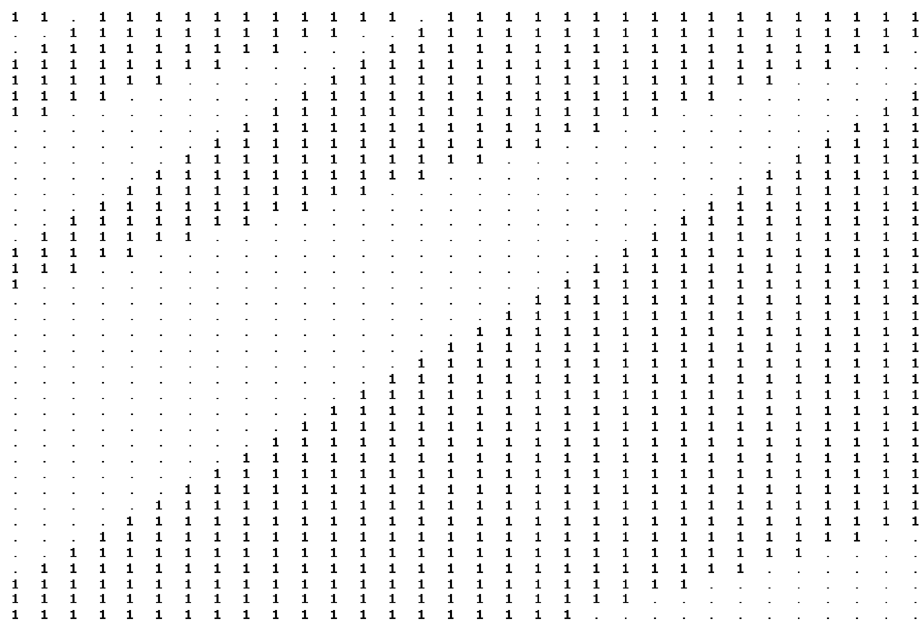

Figure 25: The Gilman automaton

Proof: (1) See [10], Theorems 2.3 and 2.4, p. 30.

(2) See [10], Theorems 5.3 and 5.4, p. 115.

(3) See [10], Theorems 9.5 and 9.6, p. 225.

(4) See [10], Theorems 9.3 and 9.4, p. 221. □

The term *recursively enumerable* is often used to mean **(Turing)**-enumerable; note that this has nothing to do with the kind of 'recursion' manifested by **(Stack)** machines.

8.4 The Column Languages of Cellular Automata

One way to characterize the 'complexity' of a cellular automaton is by the complexity of its column languages.

Theorem 31: *(Gilman)*
The column language of any cellular automaton is context-sensitive. □

There are CA whose languages are context-sensitive, but not context free.

Example 32: *Gilman's Automaton*

Let $\mathcal{A} = \{0, 1\}$, and let $\Phi : \mathcal{A}^{\mathbb{N}}\circlearrowleft$ have local map $\phi(a_0, a_1, a_2) = a_1 \cdot a_2$ (see Figure 25) . Then $\mathcal{L}_0[\Phi]$ is the context-sensitive language of Example 29e.

Kůrka's Linguistic Classification Let $\mathfrak{L}$ be one of the classes (**BP**), (**Reg**), (**CFL**), or (**CSL**) of formal languages. Let Φ be a cellular automaton. We say Φ is in **language class** $\mathfrak{L}$ if $\mathcal{L}_W[\Phi] \in \mathfrak{L}$ for all $W > 0$, and there is some W_0 so that $\mathcal{L}_{W_0}[\Phi] \notin \mathfrak{L}'$ for any language subclass $\mathfrak{L}' \subset \mathfrak{L}$.

By definition, each cellular automaton is in exactly one of the classes (**BP**), (**Reg**), (**CFL**), or (**CSL**). We have the following results:

Proposition 33: *Let Φ be a one-dimensional CA. Then*

1. *Φ is in class (**BP**) iff Φ is equicontinuous.*

2. *If Φ is positively expansive, then Φ is in class (**Reg**).*

Proof: (1) follows from Proposition 12 on page 19, while (2) follows from Theorem 24 on page 30. □

Gilman's automaton is in class **(CSL)**. It is unknown whether the class **(CFL)** is nonempty —that is, whether there is a CA whose languages are context-free but not regular.

Question: *Can* any *context-sensitive language appear as a column language of some cellular automaton? If not, what are necessary conditions?*

References

[1] François Blanchard Alejandro Maass A. Nogueira, editor. *Topics in Symbolic Dynamics and Applications*, volume 279 of *London Mathematical Society Lecture Note*. Cambridge University Press, Cambridge, MA, 2000.

[2] François Blanchard Bernard Host Alejandro Maass. Représentation par automate de fonctions continues du tore. *Journal de Théorie des Nombres de Bordeaux*, 8(1):205–214, 1996.

[3] C. Choffrut and Karl Čulick II. On real-time cellular automata and trellis automata. *Acta Informatica*, 21:393–407, 1984.

[4] Ethan Coven. Topological entropy of block maps. *Proceedings of the American Mathematical Society*, 78:590–594, 1980.

[5] M. Delorme. An introduction to cellular automata. In M. Delorme and J. Mazoyer, editors, *Cellular Automata: A Parallel Model*, volume 460 of *Mathematics and its Applications*. Kluwer, Dordrecht, 1999.

[6] Kari Eloranta. Partially permutive cellular automata. *Nonlinearity*, 6(6):1009–1023, 1993.

[7] Kari Eloranta. Random walks in cellular automata. *Nonlinearity*, 6(6):1025–1036, 1993.

[8] Alejandro Maass François Blanchard. Dynamical properties of expansive one-sided cellular automata. *Israel Journal of Mathematics*, 99:149–174, 1997.

[9] G. Hedlund. Endomorphisms and automorphisms of the shift dynamical systems. *Mathematical System Theory*, 3:320–375, 1969.

[10] John E. Hopcroft and Jeffrey D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley, New York, 1979.

[11] Lyman P. Hurd. Nonrecursive cellular automata invariant sets. *Complex Systems*, 4(2):131–138, 1990.

[12] Lyman P. Hurd. Recursive cellular automata invariant sets. *Complex Systems*, 4(2):119–129, 1990.

[13] Jaarko Kari. *Decision problems concerning cellular automata*. PhD thesis, University of Turku, 1990.

[14] Bruce P. Kitchens. *Symbolic dynamics : one-sided, two-sided, and countable state Markov shifts*. Springer-Verlag, New York, 1998.

[15] Petr Kůrka. Languages, equicontinuity and attractors in cellular automata. *Ergodic Theory and Dynamical Systems*, 17:417–433, 1997.

[16] Samuel J. Lightwood. in progress.

[17] Douglas Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, New York, 1995.

[18] Alejandro Maass. On the sofic limit sets of cellular automata. *Ergodic Theory and Dynamical Systems*, 15:663–684, 1995.

[19] Ricardo Mañe. *Ergodic Theory and Differentiable Dynamics.* Springer-Verlag, New York, 1987.

[20] Marcus Pivato and Reem Yassawi. Limit measures for affine cellular automata. To appear in *Ergodic Theory and Dynamical Systems;* preprint available at: `http://arXiv.org/abs/math.DS/0108082`, 2001.

[21] Marcus Pivato and Reem Yassawi. Limit measures for affine cellular automata II. Submitted to *Ergodic Theory and Dynamical Systems;* preprint available at: `http://arXiv.org/abs/math.DS/0108083`, April 2001.

[22] Doris Fiebig Mike Boyle and Ulf-Rainer Fiebig. A dimension group for local homeomorphisms and endomorphisms of onesided shifts of finite type. *J. Reine Angew. Math.*, 487:27–59, 1997.

[23] Cris Moore. Quasi-linear cellular automata. *Physica D*, 103:100–132, 1997.

[24] Masakazu Nasu. *Textile Systems for endomorphisms and automorphisms of the shift*, volume 546 of *Memoirs of the AMS.* American Mathematical Society, 1995.

[25] Marcus Pivato. Nonabelian multiplicative cellular automata. Submitted to *Journal of Statistical Physics;* preprint available at: `http://arXiv.org/abs/math.DS/0108084`, August 2001.

[26] Raphael M. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Invent. Math*, 12:177–209, 1971.

[27] S. Amoroso and Y. Patt. Decision procedure for surjectivity and injectivity of parallel maps for tesselation structures. *Journal of Computer System Science*, 6:448–464, 1975.

[28] Klaus Schmidt. *Dynamical Systems of Algebraic Origin.* Birkhäuser Verlag, Boston, Massachusetts, 1995.

[29] Mark A. Shereshevsky. Ergodic properties of certain surjective cellular automata. *Monatshefte für Mathematik*, 114:305–316, 1992.

[30] Peter Walters. *An Introduction to Ergodic Theory.* Springer-Verlag, New York, 1982.