Kernels and Injectivity

Definition 1: *Kernel*

Let $f : \mathbb{V} \longrightarrow \mathbb{W}$ be a linear transformation. The **kernel** of f is the set of all vectors *mapped to zero* by f:

$$\ker[f] \;=\; \{\vec{v} \in \mathbb{V}\;;\; f(\vec{v}) = 0\}$$

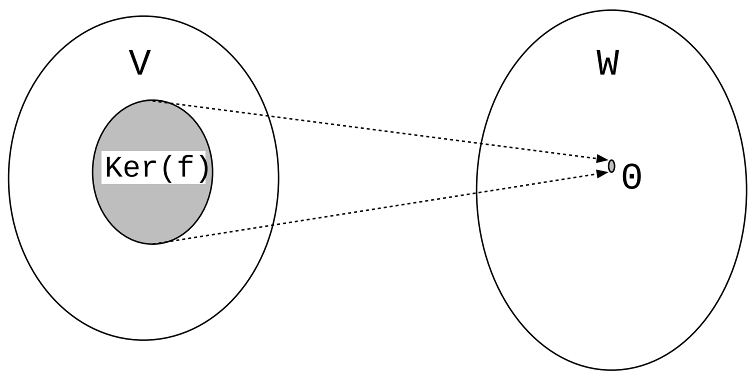

Figure 1: The **kernel** of f is the set of all vectors *mapped to zero* by f

Example 2: *Matrix Multiplication*

If $\boxed{\mathrm{A}} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1M} \\ a_{21} & a_{22} & \dots & a_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NM} \end{bmatrix}$, and $f : \mathbb{R}^M \longrightarrow \mathbb{R}^N$ is the map

$$f(\vec{v}) = \boxed{\mathrm{A}} \cdot \vec{v},$$

then

$$\ker[f] \\ = \left\{\vec{v} \in \mathbb{R}^M\;;\; \boxed{\mathrm{A}} \cdot \vec{v} = 0\right\}$$

$$= \text{null}\left[\boxed{A}\right]$$

$$= \left\{ \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_M \end{bmatrix} \; ; \; \text{solutions of} \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \ldots + a_{1M}v_M \\ a_{21}v_1 + a_{22}v_2 + \ldots + a_{2M}v_M \\ \vdots \\ a_{N1}v_1 + a_{N,2}v_2 + \ldots + a_{N,M}v_M \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

Example 3: *Orthogonal Projection* Suppose $\mathbb{V} \subset \mathbb{R}^N$ is a subspace. Then

$$\ker[\mathbf{pr}_{\mathbb{V}}] = \left\{ \vec{v} \in \mathbb{R}^M \; ; \; \mathbf{pr}_{\mathbb{V}}(\vec{v}) = 0 \right\} = \mathbb{V}^{\perp}$$

Proposition 4: The **kernel** of a linear transformation is always a **linear subspace**.

Proof: If $f : \mathbb{V} \longrightarrow \mathbb{W}$ is a linear transformation, and $\vec{x}, \vec{y} \in \ker[f]$, then

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y}) = 0 + 0 = 0,$$

thus, $\vec{x} + \vec{y} \in \ker[f]$. Also, for any $r \in \mathbb{R}$,

$$f(r.\vec{x}) = r.f(\vec{x}) = r.0 = 0,$$

thus $r.\vec{x} \in \ker[f]$.

□ [Proposition 4]

Proposition 5: *Kernel of Matrix Multiplication*
Suppose $\boxed{A}$ is a matrix, and $f : \mathbb{R}^M \longrightarrow \mathbb{R}^N$ is the map $f(\vec{v}) = \boxed{A} \cdot \vec{v}$.
The **kernel** of f is the **orthogonal complement** of the **row space** of $\boxed{A}$.

□

Definition 6: *Nullity*

Let $f : \mathbb{V} \longrightarrow \mathbb{W}$ be a linear transformation. The **nullity** of f is the **dimension** of $\ker[f]$.

Example 7: If $\boxed{\mathrm{A}}$ is a matrix, and $f : \mathbb{R}^M \longrightarrow \mathbb{R}^N$ is the map $f(\vec{v}) = \boxed{\mathrm{A}} \cdot \vec{v}$, then the **nullity** of f is the **nullity** of $\boxed{\mathrm{A}}$.

Definition 8: *One-to-one, injective*

$f : \mathbb{V} \longrightarrow \mathbb{W}$ is called **one-to-one** (or **injective**) if different elements of $\mathbb{V}$ always map to different elements of $\mathbb{W}$. Formally: for any $\vec{x}, \vec{y} \in \mathbb{V}$

$$\left(\vec{x} \neq \vec{y} \right) \Longrightarrow \left(f(\vec{x}) \neq f(\vec{y}) \right)$$

Example 9:

- The **identity map** $\mathbf{Id}_{\mathbb{V}} : \mathbb{V} \longrightarrow \mathbb{V}$ is injective.
- **Rotations and Reflections** in $\mathbb{R}^2$ are injective.
- If $\mathbb{V} \subset \mathbb{R}^D$, then **orthogonal projection** onto $\mathbb{V}$ *not* injective.

Proposition 10: Let $f : \mathbb{V} \longrightarrow \mathbb{W}$ be a linear transformation. Then

$$\left(f \text{ is } \textbf{one-to-one} \right) \iff \left(\text{nullity}[f] = 0 \right).$$

Proof:

Proof of "$\Longrightarrow$": Suppose $\vec{x} \in \ker[f]$. Then $f(\vec{x}) = 0 = f(0)$. But f is one-to-one, therefor $\vec{x} = 0$. Conclusion: $\ker[f] = \{0\}$.

Proof of "$\Longleftarrow$": Suppose $f(\vec{x}) = f(\vec{y})$. Then

$$f(\vec{x} - \vec{y}) \; = \; f(\vec{x}) - f(\vec{y}) \; = \; 0$$

Thus, $(\vec{x} - \vec{y}) \in \ker[f]$.

But $\mathsf{nullity}[f] = 0$, therefor $(\vec{x} - \vec{y})$ must be 0, so $\vec{x} = \vec{y}$. Conclusion: f is one-to-one.

□ [Proposition 10]

Images and Surjectivity

Definition 11: *Range*

Let $f : \mathbb{V} \longrightarrow \mathbb{W}$ be a linear transformation. The **range** of f is the set $\mathbb{W}$.

Remark 12: Notice that the "range" of a function is largely an artifact of how we define the function; in other words, it is a matter of perspective.

For example, suppose $\mathbb{P} \subset \mathbb{R}^3$ is a plane, and $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the **orthogonal projection** onto $\mathbb{P}$. Thus, the **range** of f is all of $\mathbb{R}^3$. However, we could just as easily have **defined** f with the expression "$f : \mathbb{R}^3 \longrightarrow \mathbb{P}$". In this case, the range of f would be $\mathbb{P}$, since "$\mathbb{P}$" is what appears on the right hand side of the arrow. We have changed our "perspective" on f, and the meaning of the word "range" must change in a corresponding fashion.

Definition 13: *Image*

Let $f : \mathbb{V} \longrightarrow \mathbb{W}$ be a linear transformation. The **image** of f is the set

$$\mathsf{image}[f] \; = \; \{\vec{w} \in \mathbb{W} \; ; \; \vec{w} = f(\vec{v}) \text{ for some } \vec{v} \in \mathbb{V}\}$$

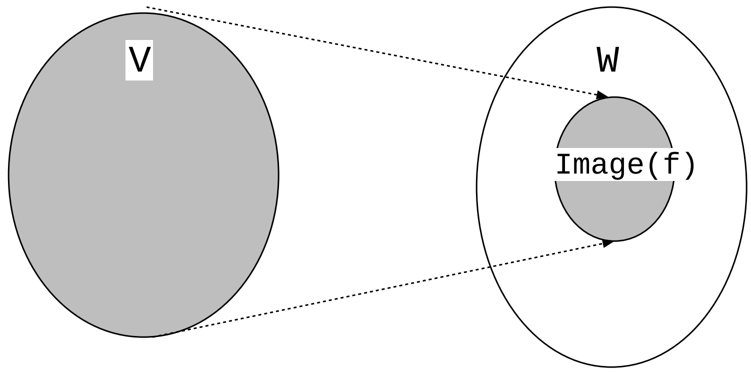

Figure 2: The **image** of f

Example 14: *Matrix Multiplication*

If $\boxed{\mathrm{A}} = \begin{bmatrix} \uparrow & \uparrow & \dots & \uparrow \\ a_1 & a_2 & \dots & a_M \\ \downarrow & \downarrow & \dots & \downarrow \end{bmatrix}$, and $f : \mathbb{R}^M \longrightarrow \mathbb{R}^N$ is the map

$$f(\vec{v}) = \boxed{\mathrm{A}} \cdot \vec{v},$$

then

$$\begin{aligned} \mathsf{image}\,[f] &= \left\{ \boxed{\mathrm{A}} \cdot \vec{v}\ ;\ \vec{v} \in \mathbb{R}^M \right\} \\ &= \left\{ v_1 \begin{bmatrix} \uparrow \\ \mathbf{a}_1 \\ \downarrow \end{bmatrix} + v_2 \begin{bmatrix} \uparrow \\ \mathbf{a}_2 \\ \downarrow \end{bmatrix} + \ldots + v_M \begin{bmatrix} \uparrow \\ \mathbf{a}_M \\ \downarrow \end{bmatrix}\ ;\ v_1, \ldots, v_M \in \mathbb{R} \right\} \\ &= \mathsf{colspace}\left[\boxed{\mathrm{A}}\right] \end{aligned}$$

Example 15: *Orthogonal Projection*

Suppose $\mathbb{V} \subset \mathbb{R}^N$ is a subspace. Then

$$\mathsf{image}\,[\mathbf{pr}_{\mathbb{V}}] = \left\{ \mathbf{pr}_{\mathbb{V}}(\vec{v})\ ;\ \vec{v} \in \mathbb{R}^M \right\} = \mathbb{V}$$

Proposition 16: The **image** of a linear transformation is always a linear subspace.

Proof: If $f : \mathbb{V} \longrightarrow \mathbb{W}$ is a linear transformation, and $\vec{x}, \vec{y} \in \mathsf{image}[f]$, then

$$f(\vec{x}) + f(\vec{y}) \;=\; f(\vec{x}+\vec{y}) \in \mathsf{image}[f]\,.$$

and, for any $r \in \mathbb{R}$,

$$r.f(\vec{x}) \;=\; f(r.\vec{x}) \in \mathsf{image}[f]\,.$$

$\Box$ [Proposition 16]

Definition 17: *Rank*

Let $f : \mathbb{V} \longrightarrow \mathbb{W}$ be a linear transformation. The **rank** of f is the **dimension** of $\mathsf{image}[f]$.

Remark 18: If $\{\mathbf{b}_1, \ldots, \mathbf{b}_N\}$ is a **basis** for $\mathbb{V}$, then $\{f(\mathbf{b}_1), \ldots, f(\mathbf{b}_N)\}$ is a **spanning set** for $\mathsf{image}[f]$ (check this). Hence

$$\mathsf{rank}[f] \leq \dim[\mathbb{V}].$$

Example 19: If $\boxed{\mathrm{A}}$ is a matrix, and $f : \mathbb{R}^M \longrightarrow \mathbb{R}^N$ is the map $f(\vec{v}) = \boxed{\mathrm{A}} \cdot \vec{v}$, then the **rank** of f is the **rank** of $\boxed{\mathrm{A}}$.

Definition 20: *Onto, surjective*

$f : \mathbb{V} \longrightarrow \mathbb{W}$ is called **onto** (or **surjective**) if *every* element of $\mathbb{W}$ is in the image of f. Formally: for any $\vec{w} \in \mathbb{W}$ there is some $\vec{v} \in \mathbb{V}$ so that $f(\vec{v}) = \vec{w}$.

In other words, a function is **onto** if its *image* is equal to its *range*. Hence, whether f is onto depends on what we have defined its range to be. To avoid ambiguity, we sometimes say, "f is **onto** $\mathbb{W}$", to make it explicit that we regard $\mathbb{W}$ as the range, rather than some superset of $\mathbb{W}$.

Example 21:

- The **identity map** $\mathbf{Id}_{\mathbb{V}} : \mathbb{V} \longrightarrow \mathbb{V}$ is surjective.
- **Rotations and Reflections** in $\mathbb{R}^2$ are surjective.

Remark 22: Suppose $f : \mathbb{V} \longrightarrow \mathbb{W}$ is linear.

1. f is **onto** if, and only if $\mathsf{rank}\,[f] = \dim[\mathbb{W}]$.
2. If f is equivalent to multiplication by the matrix $\boxed{\mathrm{A}}$, then f is **onto** if, and only if $\mathsf{rank}\left[\boxed{\mathrm{A}}\right] = \dim[\mathbb{W}]$.
3. But $\mathsf{rank}\left[\boxed{\mathrm{A}}\right] \leq \dim[\mathbb{V}]$ so if $\dim[\mathbb{V}] < \dim[\mathbb{W}]$, then f can never be onto.

Theorem 23: *Dimension Theorem*
Suppose $f : \mathbb{V} \longrightarrow \mathbb{W}$ is linear, and that $\ker[f]$ and $\mathsf{image}\,[f]$ are both finite-dimensional. Then

1. $\mathbb{V}$ is finite-dimensional.
2. $\dim[\mathbb{V}] \;=\; \dim[\ker[f]] + \dim[\mathsf{image}\,[f]] \;=\; \mathsf{nullity}\,[f] + \mathsf{rank}\,[f]$

Proof: (When $\dim[\mathbb{V}]$ is finite)

Since $\mathbb{V}$ is finite-dimensional, every linear transformation corresponds to multiplication by some matrix. So, let $\boxed{\mathrm{A}}$ be the matrix corresponding to f. Then

$$\mathsf{rank}\,[f] \;=\; \mathsf{rank}\left[\boxed{\mathrm{A}}\right]$$

$$\mathsf{nullity}\,[f] \;=\; \dim\left[\mathsf{null}\left[\boxed{\mathrm{A}}\right]\right]$$

But we know that $M = \dim\left[\mathsf{null}\left[\boxed{\mathrm{A}}\right]\right] + \mathsf{rank}\left[\boxed{\mathrm{A}}\right]$. (Theorem 5, section 5.5, p.234) In other words, $\dim[\mathbb{R}^M] = \mathsf{nullity}\,[f] + \mathsf{rank}\,[f]$.

□ [Theorem 23]

Example 24: Define $f : \mathcal{M}_{N\times N} \longrightarrow \mathcal{M}_{N\times N}$ by

$$f\left(\boxed{\text{A}}\right) = \boxed{\text{A}} + \boxed{\text{A}}^t$$

$$\begin{aligned}\text{Note:}\quad \ker[f] &= \left\{\boxed{\text{A}}\,;\ \boxed{\text{A}}^t = -\boxed{\text{A}}\right\} \\ &= \{\textbf{antisymmetric}\text{ matrices}\}.\end{aligned}$$

I claim that $\mathsf{image}[f] = \{$**symmetric** matrices$\}$. To see this, note:

1. If $\boxed{\text{B}} \in \mathsf{image}[f]$, then $\boxed{\text{B}} = \boxed{\text{A}} + \boxed{\text{A}}^t$ for some $\boxed{\text{A}}$. But then $\boxed{\text{B}}^t = \boxed{\text{A}}^t + \boxed{\text{A}} = \boxed{\text{B}}$. Hence, $\boxed{\text{B}}$ is symmetric.
2. Suppose $\boxed{\text{B}}$ is a symmetric matrix. Then so is $\frac{1}{2}\boxed{\text{B}}$. But then

$$\begin{aligned}\boxed{\text{B}} &= \frac{1}{2}\boxed{\text{B}} + \frac{1}{2}\boxed{\text{B}} \\ &= \frac{1}{2}\boxed{\text{B}} + \frac{1}{2}\boxed{\text{B}}^t \\ &= f\left(\frac{1}{2}\boxed{\text{B}}\right)\end{aligned}$$

so $\boxed{\text{B}} \in \mathsf{image}[f]$.

Consequence: If $\mathbf{Symm}_N$ is the set of symmetric matrices, and $\mathbf{Antisymm}_N$ is the set of antisymmetric matrices, then both $\mathbf{Symm}_N$ and $\mathbf{Antisymm}_N$ are linear subspaces of $\mathcal{M}_{N\times N}$, and

$$\dim[\mathbf{Symm}_N] + \dim[\mathbf{Antisymm}_N] = \dim[\mathcal{M}_{N\times N}] = N^2$$

Example 25:

1. Define $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^5$ by $f(x_1, x_2, x_3) = (x_1, x_2, x_3, 0, 0)$ Then f is **one-to-one** but *not* **onto**.

$$3 = 0 + 3 = \dim[\ker[f]] + \dim[\mathsf{image}[f]].$$

2. Define $f : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ by $f(x_1, \ldots, x_5) = (x_1, x_2, x_3)$. Then f is **onto** but *not* **one-to-one**, and

$$5 \;=\; 2+3 \;=\; \dim[\ker[f]] + \dim[\mathsf{image}\,[f]].$$

Isomorphisms

Definition 26: *Isomorphism, bijective*

$f : \mathbb{V} \longrightarrow \mathbb{W}$ is called a **linear isomorphism** (or **bijective**) if f is *both* **one-to-one** *and* **onto**.

Examples 27:

1. If $\mathbf{Id}_{\mathbb{V}} : \mathbb{V} \longrightarrow \mathbb{V}$ is the **identity map**, then $\mathbf{Id}_{\mathbb{V}}$ is an **isomorphism**.
2. **Rotation about the origin** in $\mathbb{R}^2$ by any angle is an **isomorphism**.
3. **Compression/Reflection/Expansion** in some dimension of $\mathbb{R}^N$ is an isomorphism (unless you **annihilate** the dimension).
4. **Reflection** across a subspace of $\mathbb{R}^N$ is an isomorphism.
5. **The transposition map:** If $f : \mathcal{M}_{N\times M} \longrightarrow \mathcal{M}_{M\times N}$ is the map $f\left(\boxed{\mathrm{A}}\right) = \boxed{\mathrm{A}}^{\,t}$, then f is an isomorphism.