Eigenvalues and Eigenvectors

Prerequisites:

- Linear Functions
- Determinants

Definition 1: *Eigenvalue, Eigenvector, Spectrum*

Let $\mathbb{V}$ be a vector space, and $f : \mathbb{V} \longrightarrow \mathbb{V}$ a linear transformation. Let $\vec{v} \in \mathbb{V}$. $\vec{v}$ is an **eigenvector** of f if there is some $\lambda \in \mathbb{R}$ so that

$$f(\vec{v}) \quad = \quad \lambda.\vec{v}$$

The number λ is called the **eigenvalue** associated with $\vec{v}$.

(The eigenvectors are sometimes called the **characteristic** vectors of f; the eigenvalues are sometimes called the **characteristic** values of f.)

The set of *all* eigenvalues of f is called the **spectrum** of f.

If $\boxed{\mathsf{A}}$ is a $N \times N$ matrix, then the **eigenvalues**, **eigenvectors** and **spectrum** of $\boxed{\mathsf{A}}$ are simply those of linear transformation $f(\vec{v}) = \boxed{\mathsf{A}} \cdot \vec{v}$.

Example 2: *Identity map*
If $\mathbf{Id}_{\mathbb{V}} : \mathbb{V} \longrightarrow \mathbb{V}$ is the identity map, then for any $\vec{v} \in \mathbb{V}$, $\mathbf{Id}_{\mathbb{V}}(\vec{v}) = \vec{v}$. Thus, *every* $\vec{v} \in \mathbb{V}$ is an eigenvector, all with eigenvalue 1.

Example 3: *Diagonal Matrices*

Suppose $\boxed{\mathrm{A}} = \begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_N \end{bmatrix}$, and let $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be multiplication-by-$\boxed{\mathrm{A}}$. Thus, if

$$\begin{aligned} \text{if } \mathbf{e}_n &= \left(\overbrace{\underbrace{0, \dots, 0}_{(n-1)}, 1, 0, \dots, 0}^{N} \right), \\ \text{then } f(\mathbf{e}_n) &= \boxed{\mathrm{A}} \cdot \mathbf{e}_n \;=\; a_n \mathbf{e}_n \end{aligned}$$

Thus, $\mathbf{e}_n$ is an eigenvector of f, with eigenvalue a_n.

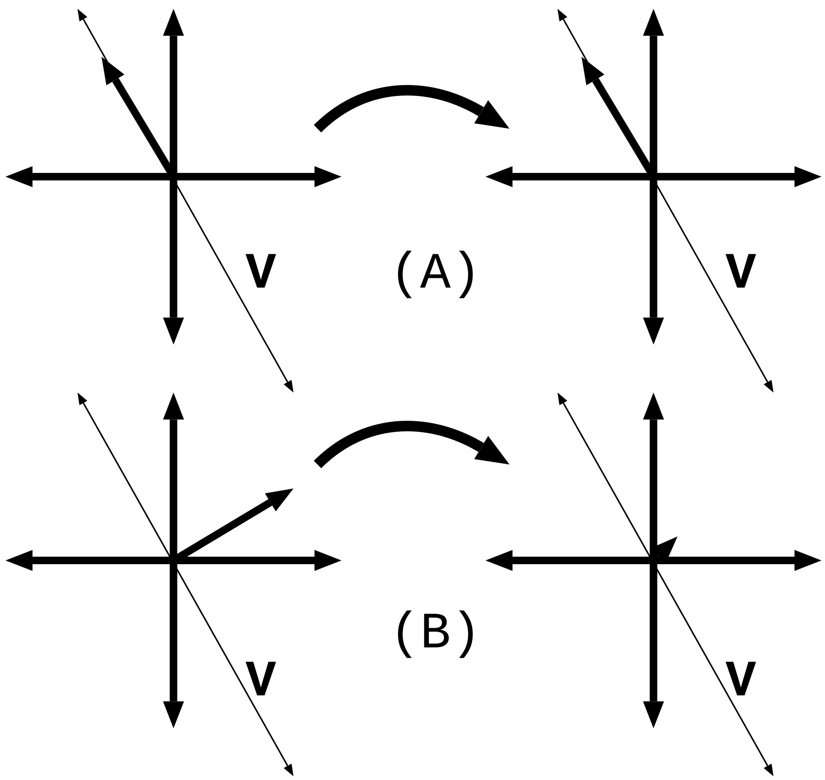

Figure 1: The Eigenvectors of projection

Example 4: *Orthogonal Projection*
Let $\mathbb{V} \subset \mathbb{R}^N$ be some subspace, and let $\mathbf{pr}_{\mathbb{V}} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be the **orthogonal projection** onto $\mathbb{V}$.

- For every $\vec{v} \in \mathbb{V}$, $\mathbf{pr}_{\mathbb{V}}(\vec{v}) = \vec{v}$, so $\vec{v}$ is an eigenvector of $\mathbf{pr}_{\mathbb{V}}$, with eigenvalue 1. (see **Part (A)** of Figure 1 on the facing page)
- For every $\vec{w} \in \mathbb{V}^{\perp}$, $\mathbf{pr}_{\mathbb{V}}(\vec{w}) = 0$, so $\vec{w}$ is an eigenvector of $\mathbf{pr}_{\mathbb{V}}$, with eigenvalue 0. (see **Part (B)** of Figure 1 on the preceding page)

Example 5: *Reflection*
Let $\mathbb{V} \subset \mathbb{R}^N$ be some subspace, and let $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be the **reflection** across $\mathbb{V}$.

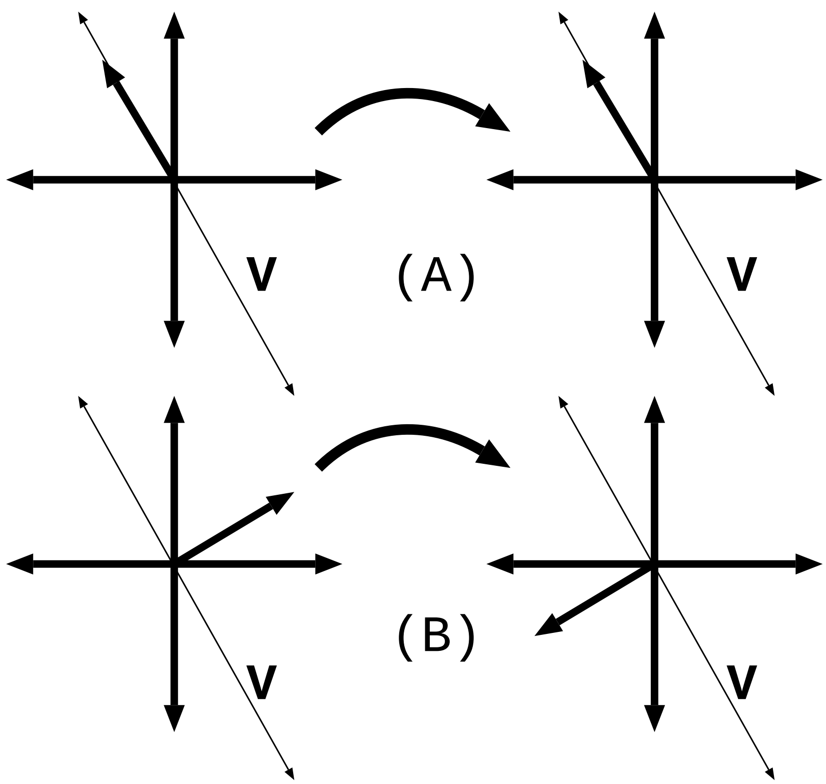

Figure 2: The Eigenvectors of reflection

- For every $\vec{v} \in \mathbb{V}$, $f(\vec{v}) = \vec{v}$, so $\vec{v}$ is an eigenvector of f, with eigenvalue 1. (see **Part (A)** of Figure 2)
- For every $\vec{w} \in \mathbb{V}^{\perp}$, $f(\vec{w}) = -\vec{w}$, so $\vec{w}$ is an eigenvector of f, with eigenvalue -1. (see **Part (B)** of Figure 2)

Example 6: *Permutation*

Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be the function corresponding to multiplication by the matrix

$$\boxed{\mathrm{A}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and let $\vec{v} = (1,\ -1,\ 1,\ -1\ )$. Then $\boxed{\mathrm{A}} \cdot \vec{v} = -\vec{v}$, so $\vec{v}$ is an eigenvector of f with eigenvalue -1.

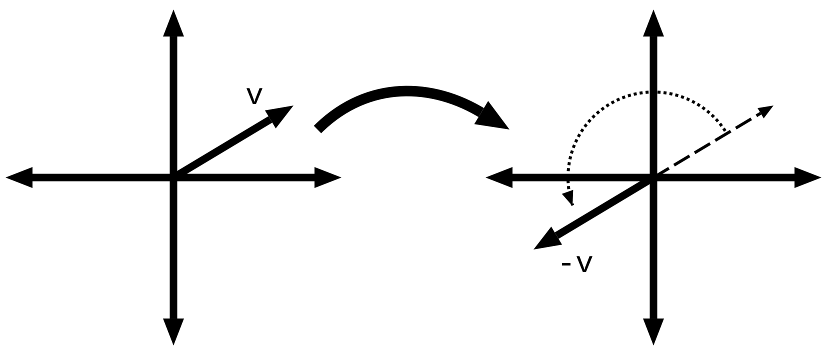

Figure 3: The Eigenvectors of rotation

Example 7: *Rotation*

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be **rotation by 180 degrees**. Then for every $\vec{v} \in \mathbb{R}^2$, $f(\vec{v}) = -\vec{v}$, so *every* element of $\mathbb{R}^2$ is an eigenvector of f, all with eigenvalue -1. (see Figure 3)

Definition 8: *Eigenspace*

Suppose that $f : \mathbb{V} \longrightarrow \mathbb{V}$ is a linear transformation, and $\lambda \in \mathbb{R}$ is an eigenvalue of f. The **eigenspace** of λ and f is the set of *all eigenvectors* of f which have λ as their eigenvalue:

$$\mathbb{E}_\lambda(f) = \{\vec{v} \in \mathbb{V}\ ;\ f(\vec{v}) = \lambda.\vec{v}\}$$

Proposition 9: If $f : \mathbb{V} \longrightarrow \mathbb{V}$ is a linear transformation, and $\lambda \in \mathbb{R}$ is an eigenvalue of f, then the eigenspace of f and λ is a **linear subspace**

Proof: Suppose $\vec{v}$ and $\vec{w}$ are λ-eigenvectors, and $r \in \mathbb{R}$. We want to show that $\vec{w} + \vec{w}$ and $r.\vec{v}$ are also λ-eigenvectors. But

$$\begin{array}{rcccccc} f(\vec{v}+\vec{w}) & = & f(\vec{v})+f(\vec{w}) & = & \lambda.\vec{v}+\lambda.\vec{w} & = & \lambda.(\vec{v}+\vec{w}), \\ \text{and } f(r.\vec{v}) & = & r.f(\vec{v}) & = & r.\lambda.\vec{v} & = & \lambda.(r.\vec{v}). \end{array}$$

□ [Proposition 9]

Example 10: *Orthogonal Projection*
Let $\mathbb{V} \subset \mathbb{R}^N$ be some subspace, and let $\mathbf{pr}_{\mathbb{V}} : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be the **orthogonal projection** onto $\mathbb{R}^N$.
The eigenvalues of $\mathbf{pr}_{\mathbb{V}}$ are 0 and 1.

- The **eigenspace** corresponding to 1 is $\mathbb{V}$.
- The **eigenspace** corresponding to 0 is $\mathbb{V}^{\perp}$.

There is a close relationship between eigenvalues and determinants.

Theorem 11: Let $f : \mathbb{V} \longrightarrow \mathbb{V}$ be a linear transformation, and $\lambda \in \mathbb{R}$. The following are equivalent:

1. λ is an *eigenvalue* of f.
2. $f(\vec{v}) - \lambda.\vec{v} = 0$ for some nonzero $\vec{v} \in \mathbb{V}$.
3. The linear transformation $(f - \lambda.\mathbf{Id}_{\mathbb{V}})$ has a **nontrivial kernel**, and is therefore **not invertible**.
4. The linear transformation $(f - \lambda.\mathbf{Id}_{\mathbb{V}})$ has **zero determinant**.

Thus, if f corresponds to multiplication by the matrix $\boxed{\mathsf{A}}$, then, for any $\lambda \in \mathbb{R}$,

$$\Big(\lambda \text{ is an eigenvalue of } f \Big) \iff \Big(\det\left(\boxed{\mathsf{A}} - \lambda.\boxed{\mathbf{Id}}\right) \;=\; 0. \Big)$$

Proof: Exercise

□ [Theorem 11]

Now, for any fixed real number λ, the determinant $\det\left(\lambda.\boxed{\mathbf{Id}} - \boxed{\mathsf{A}}\right)$ is another real number. This allows us to define a function from $\mathbb{R}$ to $\mathbb{R}$....

Definition 12: *Characteristic Polynomial*

Suppose $\boxed{\mathsf{A}}$ is an $N \times N$ matrix. Define the function $c_{\boxed{A}} : \mathbb{R} \longrightarrow \mathbb{R}$ as follows: for any $x \in \mathbb{R}$,

$$c_{\boxed{A}}(x) = \det\left(x.\boxed{\mathbf{Id}} - \boxed{\mathsf{A}}\right)$$

It turns out that this function is a *polynomial* in the variable x (convince yourself of this with some examples). It is called the **characteristic polynomial** of the matrix $\boxed{\mathsf{A}}$.

Thus, the preceeding theorem can be rephrased:

Corollary 13: Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a linear transformation determined by multiplication by the matrix $\boxed{\mathsf{A}}$. Then for any $\lambda \in \mathbb{R}$,

$$\left(\lambda \text{ is an } \textbf{eigenvalue} \text{ of } f\right) \iff \left(\lambda \text{ is a } \textbf{root} \text{ of } c_{\boxed{A}}(x).\right)$$

The **multiplicity** of eigenvalue λ is its multiplicity as a root of the characteristic polynomial —ie. the number of times the factor $(x - \lambda)$ divides $c_{\boxed{A}}(x)$.

Example 14: *The Identity Matrix*
The characteristic polynomial of the $N \times N$ identity matrix is given:

$$\begin{aligned}
c_{\boxed{\mathbf{Id}_N}}(x) &= \det\left(x.\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} x & 0 & 0 & \dots & 0 \\ 0 & x & 0 & \dots & 0 \\ 0 & 0 & x & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}\right) \\
&= \det\begin{bmatrix} x-1 & 0 & 0 & \dots & 0 \\ 0 & x-1 & 0 & \dots & 0 \\ 0 & 0 & x-1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x-1 \end{bmatrix} \\
&= (x-1)^N
\end{aligned}$$

Thus, we can see that the only eigenvalue of $\boxed{\mathbf{Id}_N}$ is 1, and this eigenvalue has **multiplicity** N. Thus, the **spectrum** of $\boxed{\mathbf{Id}_N}$ is the singleton set $\{1\}$.

Example 15: *Reflection*
Suppose $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is **reflection across the horizontal axis**. Thus, f has matrix

$$\boxed{\mathrm{A}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The characteristic polynomial of $\boxed{\mathrm{A}}$ is thus:

$$\begin{aligned} c_{\boxed{A}}(x) &= \det\left(x.\boxed{\mathbf{Id}} - \boxed{\mathrm{A}}\right) \\ &= \det\begin{bmatrix} x-1 & 0 \\ 0 & x+1 \end{bmatrix} \\ &= (x-1)\cdot(x+1) \\ &= x^2 - 1. \end{aligned}$$

The only eigenvalues of $\boxed{\mathrm{A}}$ are therefor $+1$ and -1; each having **multiplicity** 1. The **spectrum** of f is the set $\{+1, -1\}$.

Example 16: *Upper - Triangular Matrix*
Suppose that $\boxed{\mathrm{A}}$ is a **diagonal matrix** of the form:

$$\boxed{\mathrm{A}} = \begin{bmatrix} a_1 & * & * & * & \ldots & * \\ 0 & a_2 & * & * & \ldots & * \\ 0 & 0 & a_3 & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_N \end{bmatrix}$$

Thus,

$$\begin{aligned} c_{\boxed{A}}(x) &= \det\left(x.\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} - \begin{bmatrix} a_1 & * & * & * & \ldots & * \\ 0 & a_2 & * & * & \ldots & * \\ 0 & 0 & a_3 & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_N \end{bmatrix} \right) \\ &= \det\begin{bmatrix} x-a_1 & * & * & * & \ldots & * \\ 0 & x-a_2 & * & * & \ldots & * \\ 0 & 0 & x-a_3 & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & x-a_N \end{bmatrix} \end{aligned}$$

$$= \quad (x - a_1) \cdot (x - a_2) \cdot \ldots \cdot (x - a_N)$$

Thus, the **eigenvalues** of $\boxed{\text{A}}$ are just $a_1,\ a_2, \ldots, a_N$. The **multiplicity** of each eigenvalue is simply the number of times it appears in this list. The **spectrum** of $\boxed{\text{A}}$ is the set $\{a_1, \ldots, a_N\}$ (where each element only appears once in the set, even if it has multiplicity greater than 1).

(Of course, exactly the same result is true for *lower* diagonal matrices.)

Example 17: If $\boxed{\text{A}} = \begin{bmatrix} 3 & -1 & 1/6 \\ 0 & 3 & \sqrt{2} \\ 0 & 0 & -4 \end{bmatrix}$, then the **spectrum** of $\boxed{\text{A}}$ is the set $\{3, -4\}$. The eigenvalue 3 has **multiplicity** 2, while the eigenvalue -4 has **multiplicity** 1.

Example 18: *Block-Diagonal Matrix*

Suppose $\boxed{\text{A}}$ has "block decomposition" so that the "diagonal blocks" are all *square* and nonzero, and all other blocks are *zero*. In other words

$$\boxed{\text{A}} \quad = \quad \left[\begin{array}{|c|c|c|c|c|} \hline A_1 & 0 & 0 & \ldots & 0 \\ \hline 0 & A_2 & 0 & \ldots & 0 \\ \hline 0 & 0 & A_3 & \ldots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \ldots & A_N \\ \hline \end{array}\right],$$

where $\boxed{A_1}$, $\boxed{A_2}$, $\ldots$, $\boxed{A_N}$ are *square* matrices (although possible of different sizes). Then

$$c_{\boxed{A}}(x) \quad = \quad c_{\boxed{A_1}}(x) \cdot c_{\boxed{A_2}}(x) \cdot \ldots \cdot c_{\boxed{A_N}}(x)$$

Proof: Exercise.

□

Example 19:

$$\text{If } \boxed{\text{A}} \quad = \quad \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 5 & 7 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sqrt{5} & -5 \\ 0 & 0 & 0 & \pi & 23 \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix} \quad = \quad \left[\begin{array}{|cc|ccc|} \hline \multicolumn{2}{|c|}{\text{B}} & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ \hline 0 & 0 & & & \\ 0 & 0 & & \text{C} & \\ 0 & 0 & & & \\ \hline \end{array}\right],$$

$$\text{where } \boxed{\text{B}} \quad = \quad \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} \qquad \text{and } \boxed{\text{C}} \quad = \quad \begin{bmatrix} 1 & \sqrt{5} & -5 \\ 0 & \pi & 23 \\ 0 & 0 & \sqrt{2} \end{bmatrix},$$

$$\begin{aligned} \text{then } c_{\boxed{A}}(x) &= c_{\boxed{B}}(x) \cdot c_{\boxed{C}}(x) \\ &= \Big[(x-1)\cdot(x-7) \ - \ 5\cdot 3\Big] \cdot \Big[(x-1)(x-\pi)(x-\sqrt{2})\Big] \end{aligned}$$

Example 20: *Companion Matrices, Rational Canonical Form*

$$\text{If } \boxed{\text{A}} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & a_0 \\ 1 & 0 & 0 & \ldots & 0 & a_1 \\ 0 & 1 & 0 & \ldots & 0 & a_2 \\ 0 & 0 & 1 & \ldots & 0 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & a_N \end{bmatrix},$$

$$\text{then } c_{\boxed{A}}(x) = x^{N+1} + a_N x^N + a_{N-1} x^{N-1} + \ldots + a_1 x + a_0.$$

Proof: Exercise.

□

The matrix $\boxed{\text{A}}$ is called the **companion matrix** of the polynomial $p(x) = x^{N+1} + a_N x^N + a_{N-1} x^{N-1} + \ldots + a_1 x + a_0$.

Suppose $p_1(x),\ p_2(x),\ \ldots,\ p_K(x)$ are polynomials, with companion matrices $\boxed{A_1}$, $\boxed{A_2}$, $\ldots$, $\boxed{A_K}$, and

$$\boxed{\text{A}} = \begin{bmatrix} A_1 & 0 & 0 & \ldots & 0 \\ 0 & A_2 & 0 & \ldots & 0 \\ 0 & 0 & A_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & A_K \end{bmatrix}$$

then $c_{\boxed{A}} = p_1(x) \cdot p_2(x) \cdot \ldots \cdot p_K(x)$.

A matrix of this type is said to be in **rational canonical form**.

(Although it is beyond the scope of this course, a powerful theorem in linear algebra says that *every* matrix is similar to a unique matrix in rational canonical form.)

Example 21: *Jordan Matrix*
Let $\boxed{\mathrm{A}}$ be an $N \times N$ matrix of the form:

$$\boxed{\mathrm{A}} = \begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \lambda \end{bmatrix}.$$

Then $c_{\boxed{A}}(x) = (x - \lambda)^N.$

Proof: Exercise.

A matrix of this type is called a **Jordan Matrix**. Note that the only **eigenvalue** of this matrix is λ, and λ has **multiplicity** N. However, if $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is the linear transformation associated with multiplication by $\boxed{\mathrm{A}}$, then the λ-eigenspace of f associated with λ is only *one*-dimensional, *not* N-dimensional, as you might expect.

This shows that the *eigenspace-dimension* of an eigenvalue is not necessarily the same as the *multiplicity* of that eigenvalue.

Example 22: *Jordan Canonical Form*
If $\boxed{A_1}$, $\boxed{A_2}$, $\dots$, $\boxed{A_K}$ are Jordan matrices, with eigenvalues $\lambda_1, \dots, \lambda_K$, respectively, and

$$\boxed{\mathrm{A}} = \begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ 0 & A_2 & 0 & \dots & 0 \\ 0 & 0 & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A_K \end{bmatrix}$$

then $c_{\boxed{A}} = (x - \lambda_1)^{N_1} \cdot (x - \lambda_2)^{N_2} \cdot \ldots (x - \lambda_K)^{N_K}$

A matrix like $\boxed{\mathrm{A}}$ is said to be in **Jordan canonical form**.

(Although it is beyond the scope of this course, a powerful theorem in linear algebra says that *any* matrix whose characteristic polynomial completely factors is similar to a unique matrix in Jordan canonical form.)

Remark 23: The **eigenvalues** of a linear transformation $f : \mathbb{V} \longrightarrow \mathbb{V}$ depend only on f, but the **characteristic polynomial** for f seems to depend upon the **matrix representation** $\boxed{\mathrm{F}}$. If $\mathbb{V} = \mathbb{R}^N$, then there is a "natural" way to represent any linear transformation as a matrix. However, if $\mathbb{V}$ is an abstract vector space, then the matrix representation $\boxed{\mathrm{F}}$ depends upon the choice of *basis* (see the material on **Matrix Representations**) —hence, might not the characteristic polynomial also depend on the choice of basis?

In fact, it does not; if $\boxed{\mathrm{F}}$ and $\widetilde{\boxed{\mathrm{F}}}$ are two distinct matrix representations of the same transformation f, then $\boxed{\mathrm{F}}$ and $\widetilde{\boxed{\mathrm{F}}}$ have the same characteristic polynomial. (This fact will be verified in the section on **Similarity**.)

Hence, we can define the **characteristic polynomial of the linear transformation** f to be the characteristic polynomial of *any* (and hence, *every*) matrix representation of f.

Proposition 24: *Properties of the Characteristic Polynomial*
Let $\boxed{\mathsf{A}}$ be an $N \times N$ matrix. Then:

1. The **degree** of $c_{\boxed{A}}(x)$ is N; in other words,

$$c_{\boxed{A}}(x) \;=\; b_N x^N + b_{N-1} x^{N-1} + \ldots + b_1 x + b_0$$

   for some coefficients $b_N, b_{N-1}, \ldots, b_1, b_0$. Furthermore.....

2. $b_N \;=\; 1$, always.

3. $b_0 = (-1)^N \cdot \det\left(\boxed{\mathsf{A}}\right)$, always.

4. $b_{N-1} \;=\; -\text{trace}\left[\boxed{\mathsf{A}}\right]$, always.

5. Furthermore, suppose that the characteristic polynomial **completely factors**:

$$c_{\boxed{A}}(x) \;=\; (x-\lambda_1)^{N_1} \cdot (x-\lambda_2)^{N_2} \cdot \ldots \cdot (x-\lambda_K)^{N_K}.$$

$$\begin{array}{rcl} \text{Then } \; b_0 & = & (-1)^N \lambda_1^{N_1} \cdot \lambda_2^{N_2} \cdot \ldots \cdot \lambda_K^{N_K}, \\ \text{and } \; b_{N-1} & = & -N_1 \cdot \lambda_1 - \; N_2 \cdot \lambda_2 - \ldots - \; N_K \cdot \lambda_K. \end{array}$$

6. In other words if $\boxed{\mathsf{A}}$ has eigenvalues $\lambda_1, \ldots, \lambda_K$, with multiplicities $N_1, \ldots, N_K$, respectively, and $N_1 + N_2 + \ldots + N_K \;=\; N$, then

$$\begin{array}{rcl} \det\left(\boxed{\mathsf{A}}\right) & = & \lambda_1^{N_1} \cdot \lambda_2^{N_2} \cdot \ldots \lambda_K^{N_K}, \\ \text{and trace}\left[\boxed{\mathsf{A}}\right] & = & N_1 \cdot \lambda_1 \;+\; N_2 \cdot \lambda_2 \;+\; \ldots \;+\; N_k \cdot \lambda_K \end{array}$$

   (where trace $\left[\boxed{\mathsf{A}}\right] = a_{11} + a_{22} + \ldots + a_{NN}$)

Proof:

Proof of Part 3: $c_{\boxed{A}}(x) \;=\; b_N x^N + \ldots + b_1 x + b_0$, therefore $c_{\boxed{A}}(0) \;=\; b_0$. But by definition

$$\begin{aligned} c_{\boxed{A}}(0) &= \det\left(0.\boxed{\mathbf{Id}} - \boxed{\mathrm{A}}\right) \\ &= \det\left(-\boxed{\mathrm{A}}\right) \\ &= (-1)^N \det\left(\boxed{\mathrm{A}}\right) \end{aligned}$$

Proof of Part 5: In this case,

$$\begin{aligned} c_{\boxed{A}}(0) &= (0-\lambda_1)^{N_1} \cdot (0-\lambda_2)^{N_2} \cdot \ldots \cdot (0-\lambda_K)^{N_K} \\ &= (-1)^{(N_1+N_2+\ldots+N_K)} \cdot \lambda_1^{N_1} \cdot \lambda_2^{N_2} \cdot \ldots \cdot \lambda_K^{N_K} \\ &= (-1)^N \cdot \lambda_1^{N_1} \cdot \lambda_2^{N_2} \cdot \ldots \cdot \lambda_K^{N_K} \end{aligned}$$

Part 1, **Part 2** and **Part 3** are left as **exercises** (simple polynomial algebra). **Part 6** follows immediately from **Part 3**, **Part 4** and **Part 5**.

□ [Proposition 24]