Determinants

Determinants measure **volume** in N-dimensional space.

Volumes of Boxes

In one dimension:

Let $\mathbb{I}^1$ denote the **unit interval** in $\mathbb{R}$:

$$\mathbb{I}^1 \ = \ \{x \ ; \ 0 \leq x \leq 1\}$$

The *length* of $\mathbb{I}^1$ is 1.

If $\mathbb{L} = \{x \ ; \ 0 \leq x \leq b\}$ is any closed interval in $\mathbb{R}$, then $\ell_{ength}[\mathbb{L}] = b$.

In two dimensions:

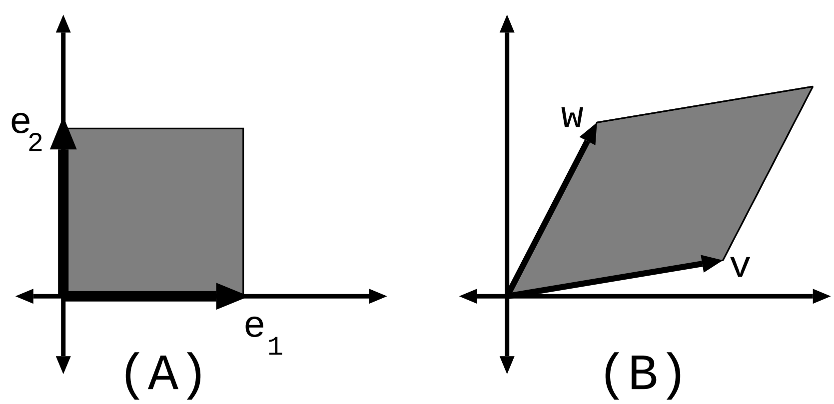

Figure 1: Parallelogram in $\mathbb{R}^2$

Let $\mathbb{I}^2$ be the **unit box** in $\mathbb{R}^N$:

$$\mathbb{I}^N \ = \ \{(x_1, x_2) \ ; \ 0 \leq x_1, x_2 \leq 1\}$$

$\mathbb{I}^2$ has one corner at 0, and 2 edges extend from 0, along the vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$. We say $\mathbb{I}^2$ is the **parallelogram spanned by** $\{\mathbf{e}_1, \mathbf{e}_2\}$. (see **Part (A)** of Figure 1)

$\mathbb{I}^2$ has an *area* of 1.

Suppose we "flip" the two edges of $\mathbb{I}^3$; metaphorically speaking, we now have the parallelogram "spanned by" $\{\mathbf{e}_2, \mathbf{e}_1\}$; one with opposite "orientation". To reflect the reversal of orientation, we say this new parallelepiped has "oriented area" of "-1".

Suppose $\mathbb{P}$ is a **parallelogram** on $\mathbb{R}^2$, with one corner at zero, and edges along the vectors (v_1, v_2) and (w_1, w_2) (see **Part (B)** of Figure 1 on the page before) Then the **oriented area** of $\mathbb{P}$ is given by the formula:

$$\mathbf{Area}^{\pm}[\mathbb{P}] \quad = \quad v_1 w_2 - v_2 w_1$$

The **(unsigned) area** of $\mathbb{P}$ is therefor:

$$\mathbf{Area}[\mathbb{P}] \quad = \quad \left| v_1 w_2 - v_2 w_1 \right|$$

(check this).

In three dimensions:

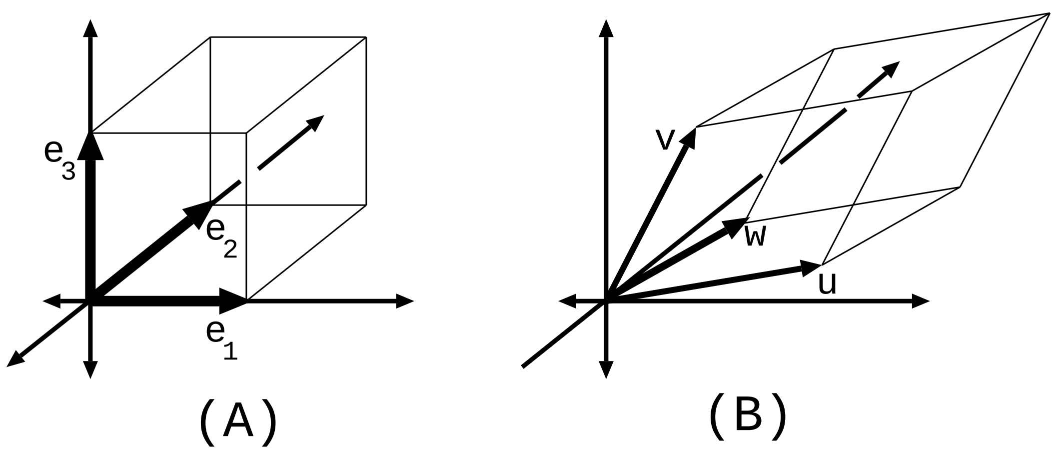

Figure 2: Parallelepipeds in $\mathbb{R}^3$

Let $\mathbb{I}^3$ be the **unit box** in $\mathbb{R}^N$:

$$\mathbb{I}^3 \;=\; \{(x_1, x_2, x_3) \;;\; 0 \leq x_1, x_2, x_3 \leq 1\}$$

$\mathbb{I}^3$ has one corner at 0, and 3 edges extend from 0, along the vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$ and $\mathbf{e}_3 = (0,0,1)$. We say $\mathbb{I}^3$ is the **parallelepiped spanned by** $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. (see **Part (A)** of Figure 2)

$\mathbb{I}^3$ has a *volume* of 1.

Suppose we "flip" dimensions 1 and 2 of $\mathbb{I}^3$; metaphorically speaking, we now have the parallelepiped "spanned by" $\{\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3\}$; one with opposite "orientation". To reflect the reversal of orientation, we say this new parallelepiped has an "oriented volume" of "-1".

Question: If $\mathbb{P}$ is a **parallelepiped** in $\mathbb{R}^3$, with one corner at zero, spanned by the vectors (u_1, u_2, u_3), (v_1, v_2, v_3) and (w_1, w_2, w_3), what is the (oriented) **volume** of $\mathbb{P}$? (see **Part (B)** of Figure 2 on the preceding page)

In N dimensions:

Let $\mathbb{I}^N$ be the **unit box** in $\mathbb{R}^N$:

$$\mathbb{I}^N \;=\; \{(x_1, \ldots, x_N) \; ; \; 0 \leq x_1, \ldots, x_N \leq 1\}$$

$\mathbb{I}^N$ has one "corner" at 0, and N "edges", all of length 1, come out of this corner, extending in the directions of the "standard basis" vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$.

Intuitively, we might say that this box has an **N-dimensional volume** of 1.

Question: If $\mathbb{P}$ is a "**hyperparallelepiped**" in $\mathbb{R}^N$, with one corner at zero, and edges along the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_N$, what is the **volume** of $\mathbb{P}$?

Determinants of Matrices

The way to compute the oriented volume of an N-dimensional parallelepiped is with **determinants**.

Definition 1: *Determinant of a Matrix*

Let $\boxed{\mathsf{A}} = \begin{bmatrix} \uparrow & \uparrow & \ldots & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_N \\ \downarrow & \downarrow & \ldots & \downarrow \end{bmatrix}$. The **determinant** of $\boxed{\mathsf{A}}$ is the **oriented volume** of the N-dimensional parallelepiped $\mathbb{P}$, having one corner at zero, and edges along the vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N\}$. The determinant is therefor a *real number*, denoted $\det \left[\!\left[\boxed{\mathsf{A}} \right]\!\right]$.

Examples 2:

- If $\boxed{\mathrm{A}} = [a]$ is a 1×1 matrix, then $\det \left[\!\left[\boxed{\mathrm{A}} \right]\!\right] \;=\; a$.

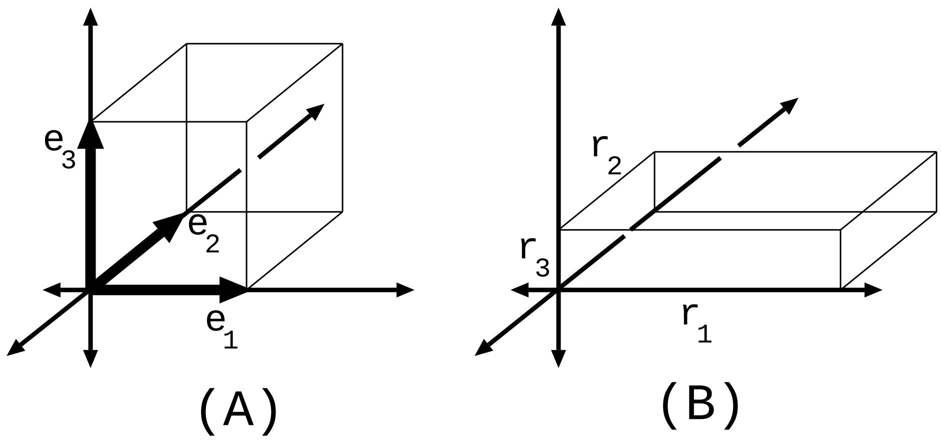

Figure 3: Boxes in $\mathbb{R}^3$

- If $\boxed{\mathrm{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 matrix, then

$$\det\left[\boxed{\mathrm{A}}\right] \;=\; a_{11} \cdot a_{22} \;-\; a_{21} \cdot a_{12}.$$

- $\det \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \;=\; 1$, because this is just the volume of a $1 \times 1 \times 1 \times \ldots \times 1$ **cube** in $\mathbb{R}^N$. (see **Part (A)** of Figure 3)

- $\det \begin{bmatrix} r_1 & 0 & 0 & \dots & 0 \\ 0 & r_2 & 0 & \dots & 0 \\ 0 & 0 & r_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & r_N \end{bmatrix} \;=\; r_1 \cdot r_2 \cdot \ldots \cdot r_N$, because this is just the volume of a $r_1 \times r_2 \times r_3 \times \ldots \times r_N$ **box** in $\mathbb{R}^N$. (see **Part (B)** of Figure 3)

Properties of Determinants

We can immediately deduce that $\det\left[\boxed{\mathrm{A}}\right]$ must have certain properties.

Let $\boxed{\mathrm{A}} = \begin{bmatrix} \uparrow & \uparrow & \dots & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_N \\ \downarrow & \downarrow & \dots & \downarrow \end{bmatrix}$.

- If $r \in \mathbb{R}$, then

$$\det\begin{bmatrix} \uparrow & \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & r.\mathbf{a}_k & \dots & \mathbf{a}_N \\ \downarrow & \downarrow & & \downarrow & & \downarrow \end{bmatrix} = r.\det\left[\!\left[\boxed{\mathrm{A}}\right]\!\right]$$

because multiplying $\mathbf{a}_k$ by r simply corresponds to multiplying the "thickness" in the direction of $\mathbf{a}_k$ of the parallelepiped by a factor of r.

- Thus, $\det\left[r.\boxed{\mathrm{A}}\right] = r^N \det\left[\!\left[\boxed{\mathrm{A}}\right]\!\right]$.
- If we *switch* two adjacent columns, we *reverse* the orientation of the parallilepiped, and therefor, we **negate** the sign:

$$\det\begin{bmatrix} \uparrow & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_j & \dots & \mathbf{a}_k & \dots & \mathbf{a}_N \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \end{bmatrix}$$

$$= \quad -1\cdot\det\begin{bmatrix} \uparrow & \uparrow & & \uparrow & \rightsquigarrow & \uparrow & & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_k & \dots & \mathbf{a}_j & \dots & \mathbf{a}_N \\ \downarrow & \downarrow & & \downarrow & \hookleftarrow & \downarrow & & \downarrow \end{bmatrix}$$

- If two distinct columns of $\boxed{\mathrm{A}}$ are the **same**, then the corresponding parallelepiped is "flat" in $\mathbb{R}^N$, only occupying an $(N-1)$-dimensional space. Thus, $\det\left[\!\left[\boxed{\mathrm{A}}\right]\!\right] = 0$.
- Also, if any columns of $\boxed{\mathrm{A}}$ is **zero**, then the corresponding parallelepiped again only occupies $(N-1)$ dimensions, so, $\det\left[\!\left[\boxed{\mathrm{A}}\right]\!\right] = 0$.

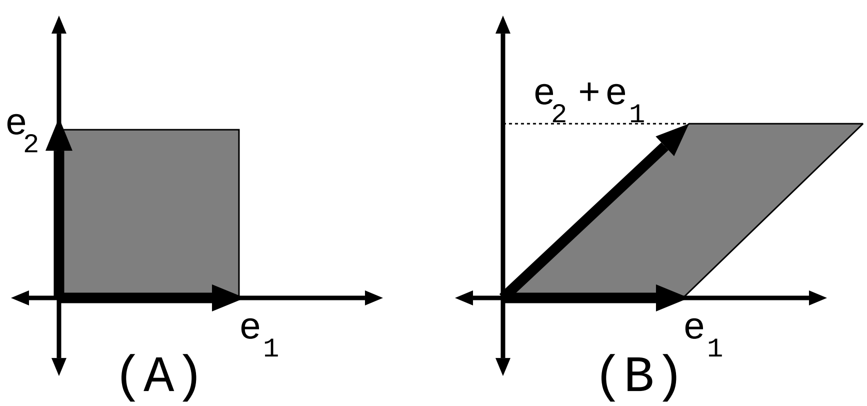

Figure 4: Adding $\mathbf{e}_1$ to $\mathbf{e}_2$ doesn't change the area of the parallellogram

- If we **add** a multiple of one column to another, then the determinant is unchanged: for any k and j, and any real number $r \in \mathbb{R}$:

$$\det\begin{bmatrix} \uparrow & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & (\mathbf{a}_j + r.\mathbf{a}_k) & \dots & \mathbf{a}_k & \dots & \mathbf{a}_N \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \end{bmatrix}$$
$$= \det\begin{bmatrix} \uparrow & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_j & \dots & \mathbf{a}_k & \dots & \mathbf{a}_N \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \end{bmatrix}$$

(This is easiest to see in $\mathbb{R}^2$: Imagine beginning with the unit square, and adding $\mathbf{e}_1$ to $\mathbf{e}_2$. See Figure **??** on page ??. Convince yourself that the resulting parallelogram has the same area as the square you started with).

- If $\mathbf{b}$ is **linearly independent** of $\mathbf{a}_1, \dots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \dots, \mathbf{a}_N$, then

$$\det\begin{bmatrix} \uparrow & \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{b} + \mathbf{a}_j & \dots & \mathbf{a}_N \\ \downarrow & \downarrow & & \downarrow & & \downarrow \end{bmatrix}$$
$$= \det\begin{bmatrix} \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_{j-1} & \mathbf{a}_j & \mathbf{a}_{j+1} & \dots & \mathbf{a}_N \\ \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix} +$$
$$\det\begin{bmatrix} \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow & & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_{j-1} & \mathbf{b} & \mathbf{a}_{j+1} & \dots & \mathbf{a}_N \\ \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

Column Reduction Algorithm For Computing Determinants

This suggests the following algorithm for computing the determinant of $\boxed{\mathrm{A}}$: *column*-reduce $\boxed{\mathrm{A}}$ to **echelon form**, and keep track of the **column operations** you perform, and their effect on the determinant.

Suppose that we column-reduce $\boxed{\mathrm{A}}$ via the sequence:

$$\boxed{\mathrm{A}} = \boxed{A_0} \to \boxed{A_1} \to \boxed{A_2} \to \dots \boxed{A_M} = \boxed{\mathrm{E}}$$

where $\boxed{\mathrm{E}}$ is the echelon form. Then

$$\det\left[\boxed{\mathrm{A}}\right] = c_1 \cdot c_2 \cdot \dots \cdot c_M \cdot \det\left[\boxed{\mathrm{E}}\right],$$

where:

- If $\boxed{A_{j-1}} \to \boxed{A_j}$ involves *exchanging* two columns, then $c_j = -1$.

- If $\boxed{A_{j-1}} \rightarrow \boxed{A_j}$ involves *adding* a multiple of one column to another, then $c_j = 1$.

- If $\boxed{A_{j-1}} \rightarrow \boxed{A_j}$ involves *multiplying* a column by a constant r, then $c_j = \frac{1}{r}$.

If $\boxed{\mathrm{E}}$ is *not* the identity matrix, then it *must* have a zero column. Thus, $\det\left[\boxed{\mathrm{E}}\right] = 0$, and thus, $\det\left[\boxed{\mathrm{A}}\right] = 0$.

Otherwise, $\boxed{\mathrm{E}}$ *is* the identity matrix. Thus, $\det\left[\boxed{\mathrm{E}}\right] = 1$, and thus, $\det\left[\boxed{\mathrm{A}}\right] = c_1 \cdot \ldots \cdot c_M$.

Example 3: Let $\boxed{\mathrm{A}} = \begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 3 \\ 1 & 0 & -1 \end{bmatrix}$. Then $\boxed{\mathrm{A}}$ has column-reduction:

$$\begin{bmatrix} 3 & 2 & 4 \\ 4 & 3 & 3 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{-C_2 \mapsto C_1} \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 3 \\ 1 & 0 & -1 \end{bmatrix} \quad (c_1 = 1)$$

$$\xrightarrow{-2C_1 \mapsto C_2} \begin{bmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 1 & -2 & -1 \end{bmatrix} \quad (c_2 = 1)$$

$$\xrightarrow{-4C_1 \mapsto C_3} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & -2 & -5 \end{bmatrix} \quad (c_3 = 1)$$

$$\xrightarrow[C_2 \mapsto C_3]{-C_2 \mapsto C_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -2 & -7 \end{bmatrix} \quad (c_4 = 1)$$

$$\xrightarrow{\frac{-1}{7} \times C_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad (c_5 = -7)$$

$$\xrightarrow[2C_3 \mapsto C_2]{-3C_3 \mapsto C_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (c_6 = 1)$$

Thus, $\det\left[\boxed{\mathrm{A}}\right] = -7$.

Example 4: *Upper Triangular Matrices*
Suppose $\boxed{\mathrm{A}}$ is an **upper triangular matrix:**

$$\boxed{\text{A}} = \begin{bmatrix} a_1 & * & * & * & \ldots & * \\ 0 & a_2 & * & * & \ldots & * \\ 0 & 0 & a_3 & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & a_N \end{bmatrix}$$

then det $\left[\boxed{\text{A}}\right] = a_1 \cdot a_2 \ldots a_N$.

Remark 5: *Second Definition of Determinant*
The previous algorithm shows that the determinant is in fact *uniquely defined* as the function $f : \mathcal{M}_{N\times N} \longrightarrow \mathbb{R}$ satisfying the following five **axioms**:

1. If $\boxed{\text{B}}$ is obtained from $\boxed{\text{A}}$ by **adding** a multiple of one column to another, then $f\left(\boxed{\text{B}}\right) = f\left(\boxed{\text{A}}\right)$.

2. If $\boxed{\text{B}}$ is obtained from $\boxed{\text{A}}$ by **multiplying** a column by r, then $f\left(\boxed{\text{B}}\right) = r.f\left(\boxed{\text{A}}\right)$.

3. If $\boxed{\text{B}}$ is obtained from $\boxed{\text{A}}$ by **switching** two columns, then $f\left(\boxed{\text{B}}\right) = -f\left(\boxed{\text{A}}\right)$.

4. If $\boxed{\text{A}}$ has a **zero column**, then $f\left(\boxed{\text{A}}\right) = 0$

5. $f\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} = 1.$

If f is any function satisfying these five axioms, then we can use the previous algorithm to compute f...... and we will end up with the determinant! Hence, f must *be* the determinant.

Sometimes the determinant is *defined* as the function satisfying these five axioms.

The Laplace Expansion

The "Laplace Expansion" is a recursive formula for computing the determinant of a matrix in terms of the determinants of "submatrices".

Definition 6: *Submatrix, Minor, Cofactor*

Let $\boxed{\mathsf{A}}$ be an $N \times N$ matrix. Let $i, j \in [1...N]$.

The (i,j)th **submatrix** of $\boxed{\mathsf{A}}$ is the $(N-1)\times(N-1)$ matrix $\boxed{A_{[i,j]}}$ obtained by *deleting* the ith row and the jth column of $\boxed{\mathsf{A}}$. In other words,

$$\text{If } \boxed{\mathsf{A}} = \left[\begin{array}{ccc|c|ccc}
a_{11} & \cdots & a_{1,k-1} & a_{1,k} & a_{1,k+1} & \cdots & a_{1,N} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{j-1,1} & \cdots & a_{j-1,k-1} & a_{j-1,k} & a_{j-1,k+1} & \cdots & a_{j-1,N} \\
\hline
a_{j,1} & \cdots & a_{j,k-1} & a_{j,k} & a_{j,k+1} & \cdots & a_{j,N} \\
\hline
a_{j+1,1} & \cdots & a_{j+1,k-1} & a_{j+1,k} & a_{j+1,k+1} & \cdots & a_{j+1,N} \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{N,1} & \cdots & a_{N,k-1} & a_{N,k} & a_{N,k+1} & \cdots & a_{N,N}
\end{array}\right],$$

$$\text{then } \boxed{A_{[i,j]}} = \left[\begin{array}{ccc|ccc}
a_{11} & \cdots & a_{1,k-1} & a_{1,j+1} & \cdots & a_{1,N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{j-1,1} & \cdots & a_{j-1,k-1} & a_{j-1,j+1} & \cdots & a_{j-1,N} \\
\hline
a_{j+1,1} & \cdots & a_{j+1,k-1} & a_{j+1,j+1} & \cdots & a_{j+1,N} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{N,1} & \cdots & a_{N,k-1} & a_{N,j+1} & \cdots & a_{N,N}
\end{array}\right],$$

The (i,j)th **minor** of $\boxed{\mathsf{A}}$ is the **determinant** of the (i,j)th submatrix:

$$M_{ij}\left(\boxed{\mathsf{A}}\right) = \det\left(\boxed{A_{[i,j]}}\right)$$

The (i,j)th **cofactor** of $\boxed{\mathsf{A}}$ is the (i,j)th minor, subjected to a *sign change*:

$$C_{i,j}\left(\boxed{\mathsf{A}}\right) = (-1)^{(i+j)} M_{ij}\left(\boxed{\mathsf{A}}\right)$$

The factor $(-1)^{(i+j)}$ is called the **sign** of the (i,j)th position.

Example 7: If $\boxed{\mathsf{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then

$$\begin{aligned} \boxed{A_{[2,3]}} &= \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} \\ M_{2,3}\left(\boxed{\mathrm{A}}\right) &= 8-14 \;=\; -6 \\ C_{2,3}\left(\boxed{\mathrm{A}}\right) &= (-1)^{2+3} M_{2,3}\left(\boxed{\mathrm{A}}\right) \;=\; (-1)(-6) \;=\; 6. \end{aligned}$$

Example 8: If $\boxed{\mathrm{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$\begin{aligned} \boxed{A_{[1,1]}} &= [a_{22}]\,, & \boxed{A_{[2,1]}} &= [a_{12}]\,; \\ M_{1,1}\left(\boxed{\mathrm{A}}\right) &= a_{22}, & M_{1,2}\left(\boxed{\mathrm{A}}\right) &= a_{12}; \\ C_{1,1}\left(\boxed{\mathrm{A}}\right) &= +a_{22}, & C_{1,2}\left(\boxed{\mathrm{A}}\right) &= -a_{12}; \end{aligned}$$

Proposition 9: *The Laplace Expansion*

If $\boxed{\mathrm{A}}$ is an $N \times N$ matrix, then we can compute the determinant of $\boxed{\mathrm{A}}$ via the following recursive formula:

For any fixed $j \in [1...N]$,

$$\det\left(\boxed{\mathrm{A}}\right) = \sum_{n=1}^{N} a_{n,j} C_{n,j}$$

This formula is called the **Laplace Expansion** (along the **jth column**) for the determinant of $\boxed{\mathrm{A}}$

Example 10: Let $\boxed{\mathrm{A}} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, and choose $j = 3$. Then

$$\begin{aligned} \det\left[\boxed{\mathrm{A}}\right] &= a_{1,3} \cdot C_{1,3}\left(\boxed{\mathrm{A}}\right) \;+\; a_{2,3} \cdot C_{2,3}\left(\boxed{\mathrm{A}}\right) \;+\; a_{2,3} \cdot C_{2,3}\left(\boxed{\mathrm{A}}\right) \\ &= 3 \cdot (-1)^{1+3} \det \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} \;+\; 6 \cdot (-1)^{2+3} \det \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} \\ &\quad + \; 9 \cdot (-1)^{3+3} \det \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &= 3(+1)(-3) \quad + \quad 6(-1)(-6) \quad + \quad 9(+1)(-3) \\ &= -9+36-27 \\ &= 0 \end{aligned}$$

Example 11: If $\boxed{\text{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$\begin{aligned} \det \boxed{\text{A}} &= a_{11} \cdot (-1)^{1+1} \det \boxed{A_{[1,1]}} \quad + \quad a_{21} \cdot (-1)^{2+1} \det \boxed{A_{[2,1]}} \\ &= a_{11} \cdot (-1)^2 a_{22} \quad + \quad a_{21} \cdot (-1)^3 a_{12} \\ &= a_{11}a_{22} \quad - \quad a_{21}a_{12}, \end{aligned}$$

the familiar formula.

Example 12: If $\boxed{\text{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\begin{aligned} &\det\left(\boxed{\text{A}}\right) \\ &= a_{11} \cdot (-1)^{1+1} \det \boxed{A_{[1,1]}} \quad + \quad a_{21} \cdot (-1)^{2+1} \det \boxed{A_{[2,1]}} \\ &\qquad + \quad a_{31} \cdot (-1)^{3+1} \det \boxed{A_{[3,1]}} \\ &= a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \quad - \quad a_{21} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix} \\ &\qquad + \quad a_{31} \det \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix} \\ &= a_{11}\left(a_{22}a_{32} - a_{23}a_{33}\right) \quad - \quad a_{21}\left(a_{12}a_{32} - a_{13}a_{33}\right) \\ &\qquad + \quad a_{31}\left(a_{12}a_{22} - a_{23}a_{33}\right) \\ &= a_{11}a_{22}a_{32} - a_{11}a_{23}a_{33} - a_{21}a_{12}a_{32} + a_{21}a_{13}a_{33} \\ &\qquad + a_{31}a_{12}a_{22} - a_{31}a_{23}a_{33} \end{aligned}$$

This formula can be used to compute 3×3 determinants quickly.... if you can remember it.

Proof of "Laplace Expansion" (sketch): Fix j. Let $f : \mathcal{M}_{N\times N} \longrightarrow \mathbb{R}$ be the function defined by the Laplace Expansion along the j**th column**:

$$f\left(\boxed{\text{A}}\right) = \sum_{n=1}^{N} a_{n,j}(-1)^{(n+j)} \det \boxed{A_{[i,j]}}$$

It is straightforward to **check** that this function satisfies the **five axioms** listed in the *Second Definition of Determinant.* Thus, it must be *equal* to the determinant.

□ [Laplace Expansion]

The Laplace Expansion also gives us a slick method of computing the inverse.

Definition 13: *Cofactor Matrix, Adjoint-Cofactor Matrix*

If $\boxed{\mathsf{A}}$ is an $N \times N$ matrix, then the **cofactor matrix** of $\boxed{\mathsf{A}}$ is the matrix whose (i,j)th entry is the (i,j)th cofactor of $\boxed{\mathsf{A}}$:

$$\mathbf{cof}\left(\boxed{\mathsf{A}}\right) = \begin{bmatrix} C_{11}\left(\boxed{\mathsf{A}}\right) & \dots & C_{1N}\left(\boxed{\mathsf{A}}\right) \\ \vdots & \ddots & \vdots \\ C_{N1}\left(\boxed{\mathsf{A}}\right) & \dots & C_{NN}\left(\boxed{\mathsf{A}}\right) \end{bmatrix}$$

The **adjoint-cofactor matrix** is the *transpose* of the cofactor matrix:

$$\mathbf{adj}\left(\boxed{\mathsf{A}}\right) = \begin{bmatrix} C_{11}\left(\boxed{\mathsf{A}}\right) & \dots & C_{N1}\left(\boxed{\mathsf{A}}\right) \\ \vdots & \ddots & \vdots \\ C_{1N}\left(\boxed{\mathsf{A}}\right) & \dots & C_{NN}\left(\boxed{\mathsf{A}}\right) \end{bmatrix}$$

(this is often called the **adjoint matrix**, but this terminology is ambiguous; in other parts of linear algebra, the "adjoint matrix" means something completely different).

Theorem 14: *Adjoint-Cofactor formula for Matrix Inverse*
If $\boxed{\mathsf{A}}$ is a square matrix with **adjoint-cofactor matrix adj** $\left(\boxed{\mathsf{A}}\right)$, then

$$\mathbf{adj}\left(\boxed{\mathsf{A}}\right) \cdot \boxed{\mathsf{A}} = \det\left(\boxed{\mathsf{A}}\right) \cdot \boxed{\mathbf{Id}_M}$$

As a consequence, if $\boxed{\mathsf{A}}$ is **invertible**, then its inverse is given:

$$\boxed{\mathsf{A}}^{-1} = \frac{1}{\det\left(\boxed{\mathsf{A}}\right)} \cdot \mathbf{adj}\left(\boxed{\mathsf{A}}\right)$$

Proof: The second assertion follows immediately from the first. The first assertion is an immediate consequence of the Laplace Expansion. The (i,j)th entry of the product matrix $\mathbf{adj}\left(\boxed{A}\right)\cdot\boxed{A}$, is the **dot product** of the jth column of $\boxed{A}$ and the ith row of $\mathbf{adj}\left(\boxed{A}\right)$:

$$\sum_{n=1}^{N} a_{nj} C_{ni}\left(\boxed{A}\right).$$

- If $j = i$, then this is just the **Laplace Expansion** (along the jth column) for the determinant of $\boxed{A}$. Hence, the **diagonal** entries of $\mathbf{adj}\left(\boxed{A}\right)\cdot\boxed{A}$ are all equal to $\det\boxed{A}$.
- If $j \neq i$, then it is the Laplace expansion for the determinant of a matrix whose ith column and jth column are *identical* (check this); hence the determinant must be zero. Hence, the **off-diagonal** entries of $\mathbf{adj}\left(\boxed{A}\right)\cdot\boxed{A}$ are all equal to 0.

□ [Theorem 14]

Corollary 15: *Cramer's Rule*
If $\boxed{A}$ is an invertible $N \times N$ matrix, and $\mathbf{Y}$ is an N-dimensionl vector, then the solution to the system of linear equations:

$$\boxed{A}\cdot\mathbf{X} = \mathbf{Y}$$

is $\mathbf{X} = [x_1, x_2, \ldots, x_N]$, where

$$x_1 = \frac{\det\boxed{A_1}}{\det\boxed{A}}; \qquad x_2 = \frac{\det\boxed{A_2}}{\det\boxed{A}}; \qquad \ldots \quad x_N = \frac{\det\boxed{A_N}}{\det\boxed{A}};$$

where, for each n, $\boxed{A_n}$ is the matrix obtained by replacing the n**th column** of $\boxed{A}$ by $\mathbf{Y}$.

□

Products and Inverses

Theorem 16: *Product and Inverse Formulae for Determinants* Let $\boxed{A}$ be an $N \times N$ matrix.

1. $\left(\boxed{A} \text{ is } \textbf{invertible}\right) \iff \left(\det\left(\boxed{A}\right) \neq 0\right)$.

2. if $\boxed{A}$ *is* invertible, then $\det\left(\boxed{A}^{-1}\right) = \det\left(\boxed{A}\right)^{-1}$.

3. If $\boxed{\mathsf{B}}$ is another $N \times N$ matrice then

$$\det\left(\boxed{\mathsf{A}} \cdot \boxed{\mathsf{B}}\right) \;=\; \det\left(\boxed{\mathsf{A}}\right) \cdot \det\left(\boxed{\mathsf{B}}\right)$$

Proof: Let $\boxed{\mathrm{E}}$ be the **column reduced echelon form** of $\boxed{\mathrm{A}}$. Recall that we can make $\boxed{\mathrm{A}}$ into $\boxed{\mathrm{E}}$ by applying a sequence of **elementary column operations** to $\boxed{\mathrm{A}}$, which is equivalent to multiplying $\boxed{\mathrm{A}}$ on the right by a sequence of **elementary column operation matrices**. In other words:

$$\boxed{\mathrm{E}} \;=\; \boxed{\mathrm{A}} \cdot \boxed{X_1} \cdot \boxed{X_2} \cdot \ldots \cdot \boxed{X_K}$$

Where $\boxed{X_k}$ are all invertible matrices defined as follows:

- If the kth step in the reduction involves **multiplying** column n by r, then

$$\boxed{X_k} \;=\; \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & r & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 1 \\ & & & & [n] & & \end{bmatrix}.$$

  Thus, $\det\left(\boxed{X_k}\right) = r$.

- If the kth step in the reduction involves **switching** columns i and j, then

$$\boxed{X_k} \;=\; \begin{bmatrix} \uparrow & & \uparrow & \rightsquigarrow & \uparrow & & \uparrow \\ \mathbf{e}_1 & \ldots & \mathbf{e}_j & \ldots & \mathbf{e}_i & \ldots & \mathbf{e}_N \\ \downarrow & & \downarrow & \hookleftarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \\ 0 & 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \\ 0 & 0 & 1 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 1 & \ldots & 0 & [i] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & \ldots & 0 & \ldots & 0 & [j] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 1 & \\ & & & & [i] & & [j] & & & \end{bmatrix}.$$

Thus, $\det\left(\boxed{X_1}\right) = (-1)$.

- If the kth step in the reduction involves **adding** r times column i to column j, then

$$\boxed{X_k} = \left[\begin{array}{ccccccc} \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathbf{e}_1 & \ldots & (\mathbf{e}_j + r.\mathbf{e}_i) & \ldots & \mathbf{e}_i & \ldots & \mathbf{e}_N \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \end{array}\right]$$

$$= \left[\begin{array}{cccccccccc} 1 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \\ 0 & 1 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \\ 0 & 0 & 1 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & 1 & \ldots & 0 & \ldots & 0 & [i] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & r & \ldots & 1 & \ldots & 0 & [j] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 1 & \\ & & & & [i] & & [j] & & & \end{array}\right].$$

Thus, $\det\left(\boxed{X_1}\right) = 1$.

If you look at the column-reduction algorithm for computing determinants, you will see that it says:

$$\det\left(\boxed{\mathrm{A}}\right) = \frac{1}{\det\left(\boxed{X_1}\right)} \cdot \frac{1}{\det\left(\boxed{X_2}\right)} \cdot \ldots \cdot \frac{1}{\det\left(\boxed{X_J}\right)} \det\left(\boxed{\mathrm{E}}\right)$$

Proof of Part 1"$\Longrightarrow$":

There are two cases:

- If $\boxed{\mathrm{A}}$ is **invertible**, then the **column-reduced echelon form** of $\boxed{\mathrm{A}}$ must be the identity; in other words, $\boxed{\mathrm{E}} = \boxed{\mathbf{Id}}$, so $\det\left(\boxed{\mathrm{E}}\right) = 1$, and therefor,

$$\det\left(\boxed{\mathrm{A}}\right) = \frac{1}{\det\left(\boxed{X_1}\right)} \cdot \frac{1}{\det\left(\boxed{X_2}\right)} \cdot \ldots \cdot \frac{1}{\det\left(\boxed{X_J}\right)}$$

- If $\boxed{\mathrm{A}}$ is *not* invertible, then the **column-reduced echelon form** of $\boxed{\mathrm{A}}$ must contain a **zero column** (check this). But if $\boxed{\mathrm{E}}$ has a zero column, then $\det\left(\boxed{\mathrm{E}}\right) = 0$, and therefor,

$$\det\left(\boxed{A}\right) \;=\; \frac{1}{\det\left(\boxed{X_1}\right)}\cdot\ldots\cdot\frac{1}{\det\left(\boxed{X_J}\right)}\cdot 0 \;=\; 0.$$

Proof of Part 1 "$\Longleftarrow$": If $\det\left(\boxed{A}\right) \neq 0$, then this means that $\det\left(\boxed{E}\right)$ must be nonzero, which means that $\boxed{E} = \boxed{\mathbf{Id}}$. Thus, since

$$\boxed{\mathbf{Id}} \;=\; \boxed{E} \;=\; \boxed{A}\cdot\boxed{X_1}\cdot\boxed{X_2}\cdot\ldots\cdot\boxed{X_K},$$

we conclude that $\boxed{A}$ is invertible, and $\boxed{A}^{-1} = \boxed{X_1}\cdot\boxed{X_2}\cdot\ldots\cdot\boxed{X_K}$.

Proof of Part 2: First suppose $\det\left(\boxed{A}\right) = 0$. Thus, by **Part 1**, $\boxed{A}$ is not invertible. Hence, $\boxed{A}\cdot\boxed{B}$ is *also* not invertible, hence $\det\left(\boxed{A}\cdot\boxed{B}\right) = 0$, again by **Part 1**.

Likewise, if $\det\left(\boxed{B}\right) = 0$, then $\det\left(\boxed{A}\cdot\boxed{B}\right) = 0$.

Hence, assume that $\det\left(\boxed{A}\right)$ and $\det\left(\boxed{B}\right)$ are both nonzero —in other words, that both $\boxed{A}$ and $\boxed{B}$ are invertible, and their **column-reduced echelon form** is the identity matrix.

Consider the argument concerning the column-reduction of $\boxed{A}$; we can apply the same argument to $\boxed{B}$. The column-reduced echelon form of $\boxed{B}$ is $\boxed{\mathbf{Id}}$, and is obtained by multiplying by **elementary column operation matrices:**

$$\boxed{\mathbf{Id}} \;=\; \boxed{B}\cdot\boxed{Y_1}\cdot\boxed{Y_2}\cdot\ldots\cdot\boxed{Y_J}$$

Thus,

$$\det\left(\boxed{B}\right) \;=\; \frac{1}{\det\left(\boxed{Y_1}\right)}\cdot\frac{1}{\det\left(\boxed{Y_2}\right)}\cdot\ldots\cdot\frac{1}{\det\left(\boxed{Y_J}\right)}$$

Now consider $\boxed{A}\cdot\boxed{B}$. Note that:

$$\begin{aligned}\boxed{A}\cdot\boxed{B}\cdot\left(\boxed{Y_1}\cdot\ldots\cdot\boxed{Y_J}\right)\left(\boxed{X_1}\cdot\ldots\cdot\boxed{X_K}\right) &= \boxed{A}\cdot\boxed{\mathbf{Id}}\cdot\left(\boxed{X_1}\cdot\ldots\cdot\boxed{X_K}\right)\\ &= \boxed{A}\cdot\left(\boxed{X_1}\cdot\ldots\cdot\boxed{X_K}\right)\\ &= \boxed{\mathbf{Id}}\end{aligned}$$

Hence, we can column-reduce $\boxed{A} \cdot \boxed{B}$ to echelon form by subjecting it to this sequence of *elementary column operations.* Hence, by the same reasoning used earlier to prove the *Column reduction formula* for computing the determinant,

$$\det\left(\boxed{A} \cdot \boxed{B}\right) = \left(\frac{1}{\det\left(\boxed{Y_1}\right)} \cdot \ldots \cdot \frac{1}{\det\left(\boxed{Y_J}\right)}\right) \cdot \left(\frac{1}{\det\left(\boxed{X_1}\right)} \cdot \ldots \cdot \frac{1}{\det\left(\boxed{Y_J}\right)}\right)$$
$$= \det\boxed{B} \cdot \det\boxed{A}.$$

Proof of Part 2: If $\boxed{A}$ is invertible, then, by **Part 3**,

$$\det\left(\boxed{A}\right) \cdot \det\left(\boxed{A}^{-1}\right) = \det\left(\boxed{A} \cdot \boxed{A}^{-1}\right) = \det\left(\boxed{\mathbf{Id}_M}\right) = 1.$$

□ `[Theorem 16]`

The Amazing Transposition Property

So far, everything we have said about determinants has been in terms of the **columns** of the matrix. Amazingly, all the same things are true if we define determinants in terms of the **rows** of the matrix.

Theorem 17: *Transposition Formula*
Let $\boxed{\mathsf{A}}$ be a square matrix, and $\boxed{\mathsf{A}}^t$ be its **transpose**. Then

$$\det\boxed{\mathsf{A}} = \det\left(\boxed{\mathsf{A}}^t\right)$$

Proof: First suppose that $\boxed{A}$ is **invertible**. Then the **column reduced echelon form** of $\boxed{A}$ is the identity matrix $\boxed{\mathbf{Id}_M}$. Suppose that this is achieved via the sequence of **elementary column operation matrices:**

$$\boxed{\mathbf{Id}_M} = \boxed{A} \cdot \boxed{X_1} \cdot \boxed{X_2} \cdot \ldots \cdot \boxed{X_K}$$

Thus, $$\boxed{X_K}^{-1} \cdot \ldots \cdot \boxed{X_2}^{-1} \cdot \boxed{X_1}^{-1} = \boxed{A}$$

In other words, if $\boxed{Y_k} = \boxed{X_k}^{-1}$ for all k, then

$$\boxed{\mathrm{A}} = \boxed{Y_K} \cdot \ldots \cdot \boxed{Y_2} \cdot \boxed{Y_1}$$

$$\text{Therefore, } \boxed{\mathrm{A}}^t = \boxed{Y_1}^t \cdot \boxed{Y_2}^t \cdot \ldots \cdot \boxed{Y_K}^t$$

Now, it is easy to **check** that, for *any* **elementary column operation matrix** $\boxed{Y_k}$, we have:

$$\det\left(\boxed{Y_k}^t\right) = \det\boxed{Y_k}$$

Thus,

$$\begin{aligned}
\det\left(\boxed{\mathrm{A}}^t\right) &= \det\left(\boxed{Y_1}^t \cdot \boxed{Y_2}^t \cdot \ldots \cdot \boxed{Y_K}^t\right) \\
&= \det\left(\boxed{Y_1}^t\right) \cdot \det\left(\boxed{Y_2}^t\right) \cdot \ldots \cdot \det\left(\boxed{Y_K}^t\right) \\
&= \det\left(\boxed{Y_1}\right) \cdot \det\left(\boxed{Y_2}\right) \cdot \ldots \cdot \det\left(\boxed{Y_K}\right) \\
&= \det\left(\boxed{Y_K}\right) \cdot \ldots \cdot \det\left(\boxed{Y_2}\right) \cdot \det\left(\boxed{Y_1}\right) \\
&= \det\left(\boxed{Y_K} \cdot \ldots \cdot \boxed{Y_2} \cdot \boxed{Y_1}\right) \\
&= \det\boxed{\mathrm{A}}
\end{aligned}$$

On the other hand, if $\boxed{\mathrm{A}}$ is not invertible, then $\det\boxed{\mathrm{A}} = 0$. But if $\boxed{\mathrm{A}}$ is not invertible, then $\mathsf{rank}\left[\boxed{\mathrm{A}}\right] < N$, which means that also, $\mathsf{rank}\left[\boxed{\mathrm{A}}^t\right] < N$, which means that $\boxed{\mathrm{A}}^t$ is *also* not invertible, which means $\det\left(\boxed{\mathrm{A}}^t\right) = 0$.

□ [Theorem 17]

Corollary 18: *Columns unto Rows*

Everything we have said so far about determinants in terms of **columns** is also true in terms of **rows**. In particular:

- The determinant of matrix $\boxed{\mathsf{A}}$ is *also* the oriented volume of the parallelepiped spanned by the **row vectors** of $\boxed{\mathsf{A}}$.

- The determinant of $\boxed{\mathsf{A}}$ can *also* be computed by computing the **row** reduced echelon form of $\boxed{\mathsf{A}}$, and keeping track of the *row* operations, in exactly the same fashion as with columns.
- The determinant of $\boxed{\mathsf{A}}$ can *also* be computed by taking the Laplace Expansion along any **row** of $\boxed{\mathsf{A}}$. In other words, for any fixed $i \in [1...N]$,

$$\det \boxed{\mathsf{A}} = \sum_{n=1}^{N} a_{i,n} C_{i,n}\left(\boxed{\mathsf{A}}\right)$$

Determinants of Linear Transformations

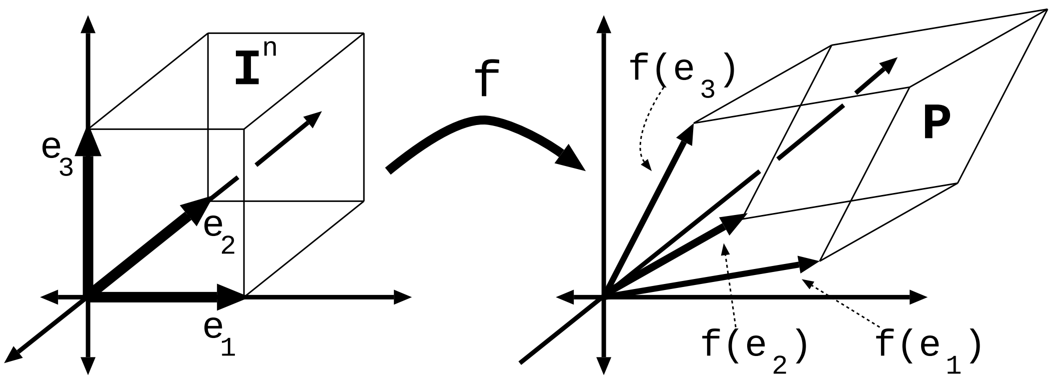

Figure 5: f maps the unit box $\mathbb{I}^N$ into the parallellepiped $\mathbb{P}$

Suppose $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a linear transformation. Let $\mathbb{P} = f(\mathbb{I}^N)$. Since f is linear, therefor $\mathbb{P}$ is a **parallelepiped** in $\mathbb{P}$.

Definition 19: *Determinant*

The **determinant** of f is the oriented volume of $\mathbb{P} = f(\mathbb{I}^N)$

Remark 20: Suppose f is equivalent to multiplication by the matrix $\begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_N \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix}$. Then $\mathbb{P}$ is the parallelepiped spanned by $\mathbf{a}_1, \ldots, \mathbf{a}_N$. Thus,

$$\det[f] = \det \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_N \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix}$$

The determinant of f measures the extent to which f "expands" "compresses" and/or "reverses" N-dimensional space.

- If $|\det(f)| > 1$, this means that $f(\mathbb{I}^N)$ is "bigger" than $\mathbb{I}^N$, so somehow, f is "stretching" space.
- If $|\det(f)| < 1$, this means that $f(\mathbb{I}^N)$ is "smaller" than $\mathbb{I}^N$, so somehow, f is "compressing" space.
- If $|\det(f)| = 1$, this means that $f(\mathbb{I}^N)$ has the same volume as $\mathbb{I}^N$; f is a **volume-preserving** deformation of space.
- If $\det(f) > 0$, this means that $f(\mathbb{I}^N)$ has the same **orientation** as $\mathbb{I}^N$; f is an **orientation-preserving** transformation of space (for example, a rotation).
- If $\det(f) > 0$, this means that $f(\mathbb{I}^N)$ has the opposite **orientation** as $\mathbb{I}^N$; f is an **orientation-reversing** transformation of space (for example, a reflection).

Example 21: Suppose $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ corresponds to multiplication by the matrix

$$\begin{bmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_N \end{bmatrix}$$

- If $|a_n| > 1$, this means that f "stretches" the nth dimension of space.
- If $|a_n| < 1$, this means that f "compresses" the nth dimension of space.
- If $a_n < 0$, this means that f "reverses" the nth dimension of space; if $a_n > 0$, then f preserves the orientation of the nth dimension.

Now, $\det[f] = a_1 \cdot a_2 \cdot \ldots \cdot a_N$. This product is **positive** either if all of $a_1, a_2, \ldots, a_N$ are positive, or if an *even* number are negative (heuristically speaking, an *even* number of orientation reversals "cancel out").

Also, $|\det[f]| = |a_1| \cdot |a_2| \cdot \ldots \cdot |a_N|$, which is larger than one only if f "stretches" space in some dimensions *more* than it "compresses" it in others.