Change of Basis (Optional)

In Euclidean Space

Suppose $\mathbb{V}$ is a vector space. A **basis** for $\mathbb{V}$ provides us with a *coordinate system* for the space; a way of describing the "locations" of vectors. Different coordinate systems are useful in different contexts.

For example, consider $\mathbb{R}^2$. The "standard" basis for $\mathbb{R}^2$ is $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$, where

$$\begin{aligned} \mathbf{e}_1 &= (1,0), \\ \text{and } \mathbf{e}_2 &= (0,1). \end{aligned}$$

If $\mathbf{x} \in \mathbb{R}^2$, and $\mathbf{x} = (x_1, x_2)$, then the "coordinates" of $\mathbf{x}$ with respect to $\mathcal{E}$ are the numbers x_1 and x_2, because

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

Intuitively, if we think of $\mathbf{e}_1$ as pointing "east", and $\mathbf{e}_2$ as pointing "north", this says,

> In order to get to the point $\mathbf{x}$, first walk x_1 steps *east*, and then walk x_2 steps *north*.

This coordinate system works well if you are living in a city such as Toronto, where the streets run east-west and north-south (see **Part (A)** of Figure 1 on the following page). However, in Montréal, the streets actually run at angles; "Montréal north" actually points *northwest*, so the streets run northwest-to-southeast and southwest-to-northeast. In Montréal, coordinates given in terms of the basis $\mathcal{E}$ are almost useless (see **Part (B)** of Figure 1 on the next page).

In Montréal, therefor, it makes sense to use a different basis. For example, we might use the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$\begin{aligned} \mathbf{b}_1 &= (1,1) \quad \text{points } \textit{northeast}, \\ \text{and } \mathbf{b}_2 &= (-1,1), \text{ points } \textit{northwest}. \end{aligned}$$

(see **Part (C)** of Figure 1 on the following page)
Thus, if $\mathbf{x}$ had coordinates $\vec{x}^{\mathcal{B}} = (x_1^{\mathcal{B}}, x_2^{\mathcal{B}})$ with respect to $\mathcal{B}$, this means

$$\mathbf{x} = x_1^{\mathcal{B}}\mathbf{b}_1 + x_2^{\mathcal{B}}\mathbf{b}_2.$$

In other words:

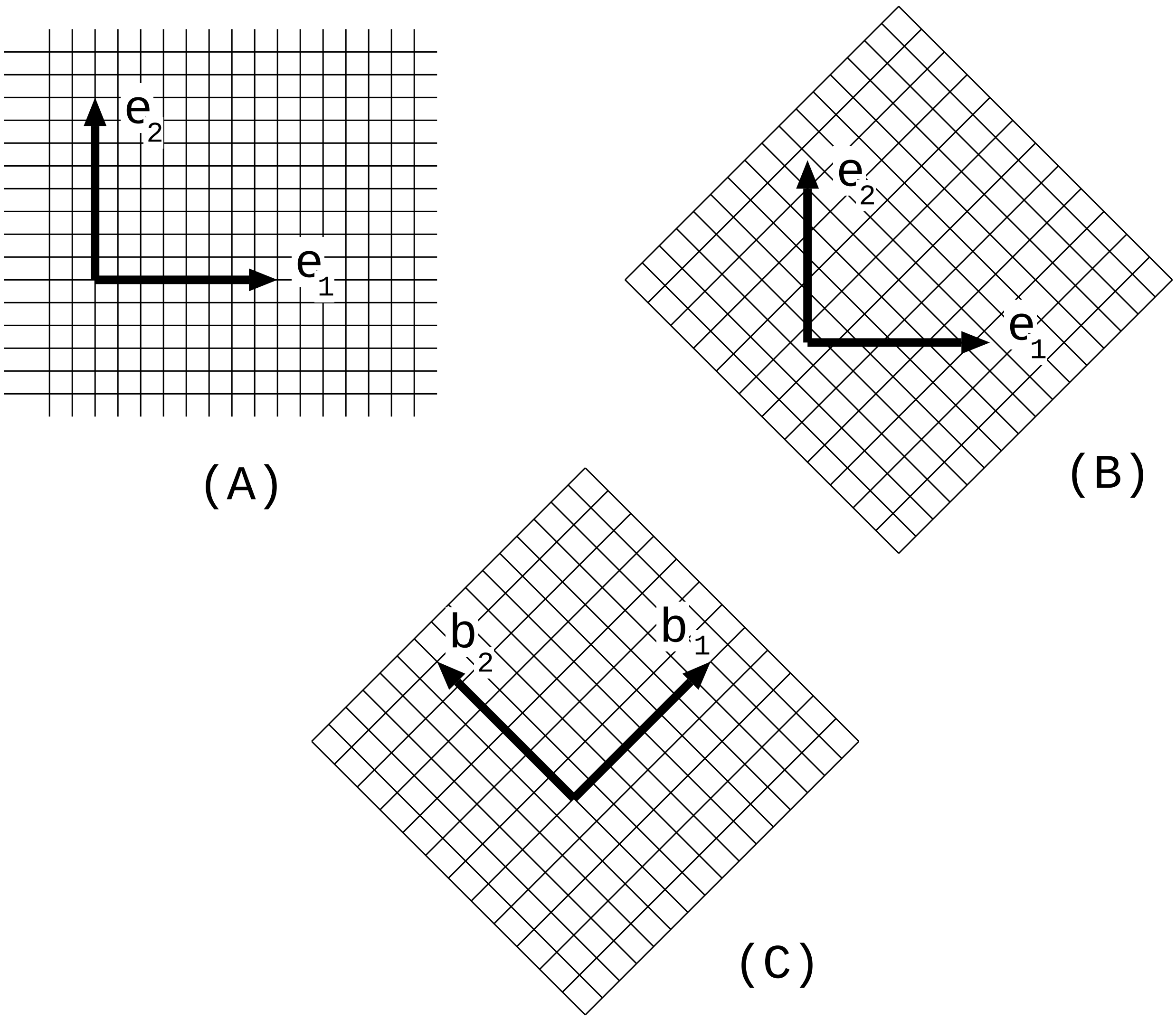

Figure 1: Coordinate systems in Toronto and Montréal

> In order to get to the point $\mathbf{x}$, first walk $x_1^{\mathcal{B}}$ steps *northeast*, and then walk $x_2^{\mathcal{B}}$ steps *northwest*.

This would make it easy to find $\mathbf{x}$ by walking along Montréal's tilted streets But suppose a person from Toronto (not knowing better) provided you with the coordinates of a point with respect to the north-south-east-west basies $\mathcal{E}$. How could you convert them to coordinates with respect to the basis $\mathcal{B}$? This is the problem of **change of basis**.

Example 1: Consider the point $\mathbf{x} = 1\mathbf{e}_1 + 2\mathbf{e}_2$. Thus, with respect to the basis $\mathcal{E}$, $\mathbf{x}$ has coordinates

$$\vec{x} = (1, 2)$$

How can we compute the coordinates of $\mathbf{x}$ with respect to $\mathcal{B}$?
Loosely speaking, we use the following procedure:

1. Compute the coordinates of $\mathbf{e}_1$ with respect to $\mathcal{B}$. (**Part C** of Figure 2 on the next page):

$$\mathbf{e}_1 = \frac{1}{2}\mathbf{b}_1 - \frac{1}{2}\mathbf{b}_2$$

2. Compute the coordinates of $\mathbf{e}_2$ with respect to $\mathcal{B}$ (**Part D** of Figure 2 on the following page)

$$\mathbf{e}_2 = \frac{1}{2}\mathbf{b}_1 + \frac{1}{2}\mathbf{b}_2$$

3. Compute the coordinates of $\mathbf{x}$ with respect to $\mathcal{E}$ (**Part E** of Figure 2 on the next page)

$$\mathbf{x}_2 = 1\mathbf{e}_1 + 2\mathbf{e}_2$$

4. Now, substitute in the expressions for $\mathbf{e}_1$ and $\mathbf{e}_2$ in terms of the basis $\mathcal{B}$ (**Part F** of Figure 2 on the following page) and simplify:

$$\begin{aligned} \mathbf{x}_2 &= 1\mathbf{e}_1 + 2\mathbf{e}_2 \\ &= 1\left(\frac{1}{2}\mathbf{b}_1 - \frac{1}{2}\mathbf{b}_2\right) \quad + \quad 2\left(\frac{1}{2}\mathbf{b}_1 + \frac{1}{2}\mathbf{b}_2\right) \\ &= \frac{3}{2}\mathbf{b}_1 + \frac{1}{2}\mathbf{b}_2 \end{aligned}$$

(**Part G** of Figure 2 on the next page)

The formal generalization of this procedure is the following theorem:

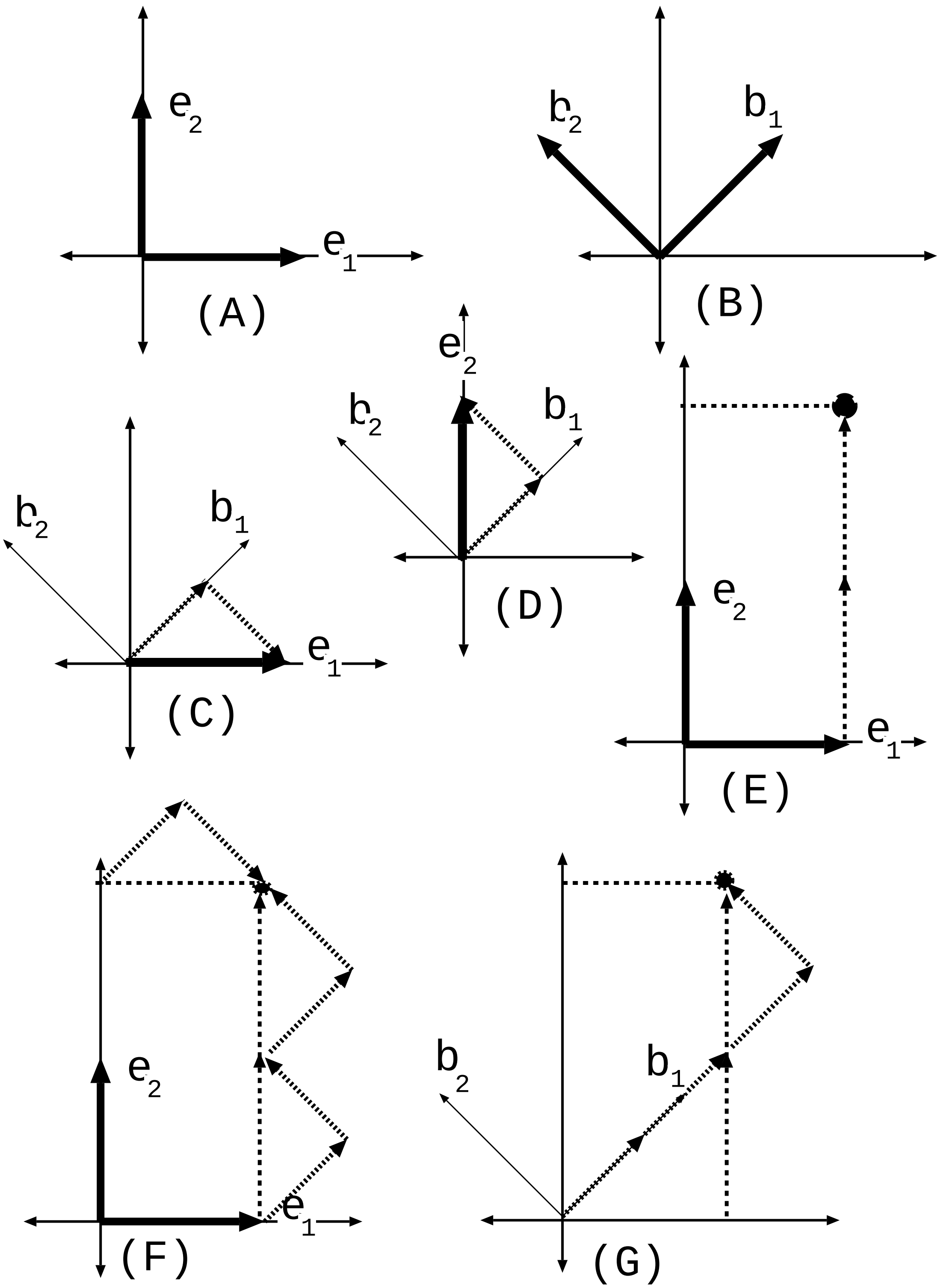

Figure 2: Changing from one coordinate system to another

Proposition 2: *Change of Basis in $\mathbb{R}^N$*

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_N\}$ is some **basis** for $\mathbb{R}^N$, and $\vec{x} \in \mathbb{R}^N$. To find the **coordinates** for $\vec{x}$ in the basis $\mathcal{B}$, do the following:

Let $\boxed{\mathsf{B}} = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_N \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix}$. $\mathcal{B}$ is a basis, so $\boxed{\mathsf{B}}$ is **invertible**. Let $\boxed{\mathsf{A}} = \boxed{\mathsf{B}}^{-1}$. Then:

1. The nth **column** of $\boxed{\mathsf{A}}$ is the *coordinate N-tuple*[1] of $\mathbf{e}_n$, with respect to $\mathcal{B}$.

2. Suppose $\vec{x} = (x_1, \ldots, x_N)$. Let $\vec{y} = \boxed{\mathsf{A}} \cdot \vec{x}$; with $\vec{y} = (y_1, \ldots, y_N)$. Then $(y_1, \ldots, y_N)$ is the coordinate N-tuple of $\vec{x}$ with respect to $\mathcal{B}$.

Proof:

$$\begin{aligned} \sum_{n=1}^{N} y_n \mathbf{b}_n &= y_1 \begin{bmatrix} \uparrow \\ \mathbf{b}_1 \\ \downarrow \end{bmatrix} + y_2 \begin{bmatrix} \uparrow \\ \mathbf{b}_2 \\ \downarrow \end{bmatrix} + \ldots + y_N \begin{bmatrix} \uparrow \\ \mathbf{b}_N \\ \downarrow \end{bmatrix} \\ &= \boxed{\mathrm{B}} \cdot \vec{y} \\ &= \boxed{\mathrm{B}} \cdot \boxed{\mathrm{B}}^{-1} \cdot \vec{x} \\ &= \vec{x} \end{aligned}$$

In particular, the coordinate N-tuple of $\mathbf{e}_n$ with respect to $\mathcal{E}$ is, of course $\left(\underbrace{0, \ldots, 0}_{n}, 1, 0, \ldots, 0 \right)$, so the coordinate N-tuple of $\mathbf{e}_n$ with respect to $\mathcal{B}$ is

$$\begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_N \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad = \quad \mathbf{a}_n, \quad \text{the } n\text{th column of } \boxed{\mathrm{A}}.$$

□ [Proposition 2]

[1] An N-**tuple** is an ordered sequence of N numbers —in other words, it is an element in $\mathbb{R}^N$. The term "N-tuple" is a sort of mathematician's generalization of the words "couple", "triple", "quadruple", "quintuple", etc...

Definition 3: *Change-of-Basis Matrix*

The matrix $\boxed{\text{A}}$ in the previous theorem is called the **change-of-basis matrix** from the (standard) basis $\mathcal{E}$ to the basis $\mathcal{B}$.

Example 4: Consider the "Montréal basis" $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$\begin{aligned} \mathbf{b}_1 &= (1,1) \quad \text{and} \quad \mathbf{b}_2 = (-1,1). \\ \text{Thus, } \boxed{\text{B}} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \\ \text{so } \boxed{\text{B}}^{-1} &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

Suppose $\vec{x} = (1,2)$, and let $\vec{y} = \boxed{\text{B}}^{-1}\vec{x} = \left(\frac{3}{2}, \frac{1}{2}\right)$. Thus,

- $\mathbf{x}$ has coordinates $(1,2)$ with respect to the "standard" basis $\mathcal{E}$.
- $\mathbf{x}$ has coordinates $\left(\frac{3}{2}, \frac{1}{2}\right)$ with respect to the Montréal basis $\mathcal{B}$.

Example 5:

$$\begin{aligned} \text{Suppose } \mathbf{b}_1 &= \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \quad \text{and} \quad \mathbf{b}_1 = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right), \\ \text{Then } \boxed{\text{B}} &= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}. \\ \text{Thus, } \boxed{\text{A}} &= \boxed{\text{B}}^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}. \end{aligned}$$

Thus, if $\vec{x} = (1,2)$, then

$$\vec{y} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} + 1 \\ \frac{-1}{2} + \sqrt{3} \end{bmatrix}$$

In an abstract vector space

The same idea will work in an abstract vector space. We will use the following convention: if $\mathbb{V}$ is a vector space, and $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_N\}$ is a basis for $\mathbb{V}$, and $\mathbf{x}$ is a point in $\mathbb{V}$, then we will write the **coordinates** of $\mathbf{x}$ with respect to $\mathcal{B}$ as the *N-tuple* $\vec{x} = (x_1, \ldots, x_N)$. This means that

$$\mathbf{x} \;=\; x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \ldots x_N\mathbf{b}_N.$$

Note: $\vec{x}$ is an N-tuple of real numbers, but $\mathbf{x}$ *is not necessarily an N-tuple.* (For example $\mathbf{x}$ might be a matrix, a polynomial, or a function). Also, notice that the coordinate n-tuple $\vec{x}$ *depends on the choice of basis.*

In $\mathbb{R}^N$, there is a natural basis to work with, the "standard" basis $\mathcal{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$. We say that $\mathcal{E}$ is a **canonical** basis for $\mathbb{R}^N$. In a general vector space, however, there may not be any *canonical* basis; any basis is just as good or bad as any other, and the choice is essentially arbitrary.

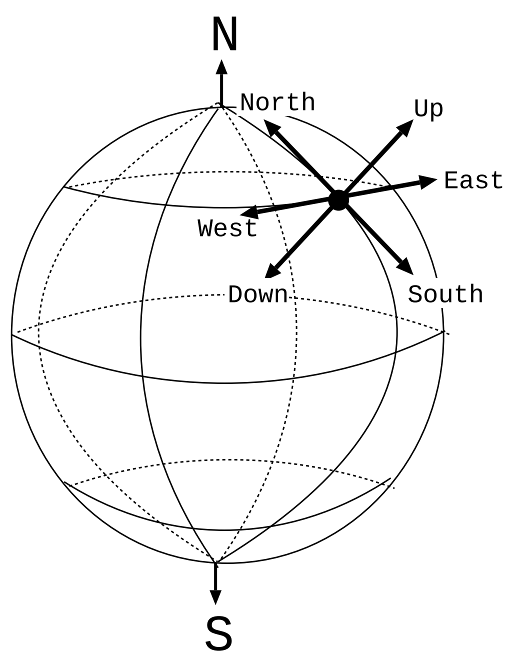

Figure 3: The canonical coordinate system on the Earth's surface

This situation may be compared to the difference between navigation on the surface of the Earth and navigation in outer space. On the surface of the earth, there is a "natural" coordinate system to work with. The motion of the sun across the sky defines directions we call "east" and "west". The Earth's magnetic field defines directions "north" and "south", while the Earth's gravitational field defines "up" and "down". (See Figure **??** on page ??. We can assign a basis element to each of “east”, “north” and “up”, and we get a “natural” (or *canonical*) basis for navigation near Earth’s surface. Two different cultures, on opposite sides of the planet, which never had any contact with one another, would both naturally arive at the same coordinate system.

However, in deep space, there is no sun, no magnetic field, and no gravity. There is no natural direction to call “up” or “down”, or “north” or “south”. You can *pick* a triple of directions, and arbitrarily *define* them to be “east”, “north”, and “up”, but in a sense, any choice is as good as any other. If your friend picks a different direction system, you can’t argue that yours is somehow “better” than hers.

Because of this, in general, we must simply pick some basis at random, and work within it. Often, it will be necessary to change from one basis to another.

Proposition 6: *Change of Basis in finite-dimensional vector space*

Let $\mathbb{V}$ be a vector space, and suppose $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ is a **basis** for $\mathbb{V}$. Suppose $\mathbf{x}$ is a point in $\mathbb{V}$, and has coordinate N-tuple $\vec{x} = (x_1, \ldots, x_N)$ with respect to $\mathcal{A}$.

Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_N\}$ is *another* basis for $\mathbb{V}$.

1. To find the **coordinates** for $\mathbf{x}$ in the basis $\mathcal{B}$, do the following:
   - For each $n \in [1..N]$, suppose the *coordinate N-tuple* of $\mathbf{a}_n$ with respect to $\mathcal{B}$ is $\vec{a}_n = (a_{1n}, a_{2n}, \ldots, a_{nn})$
   - Let $\boxed{\mathsf{A}} = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_N \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix}$.
   - Let $\vec{x}^{\mathcal{B}} = \boxed{\mathsf{A}} \cdot \vec{x}$. Then $\vec{x}^{\mathcal{B}} = (x_1^{\mathcal{B}}, \ldots, x_N^{\mathcal{B}})$ is the coordinate N-tuple of $\mathbf{x}$ with respect to $\mathcal{B}$.
2. Sometimes, it is easier to find the coordinates of $\mathcal{A}$ with respect to $\mathcal{B}$ than the other way around. In this case, do the following:
   - For each $n \in [1..N]$, suppose the *coordinate N-tuple* of $\mathbf{b}_n$ with respect to $\mathcal{A}$ is $\vec{b}_n = (b_{1,n}, b_{2,n}, \ldots, b_{N,n})$
   - Let $\boxed{\mathsf{B}} = \begin{bmatrix} \uparrow & \uparrow & \cdots & \uparrow \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_N \\ \downarrow & \downarrow & \cdots & \downarrow \end{bmatrix}$.
   - Let $\boxed{\mathsf{A}} = \boxed{\mathsf{B}}^{-1}$. Then $\boxed{\mathsf{A}}$ is the same matrix as described in **Part 1**
   - Thus, if $\vec{x}^{\mathcal{B}} = \boxed{\mathsf{A}} \cdot \vec{x}$, then $\vec{x}^{\mathcal{B}} = (x_1^{\mathcal{B}}, \ldots, x_N^{\mathcal{B}})$ is the coordinate N-tuple of $\mathbf{x}$ with respect to $\mathcal{B}$.

Proof: Exercise.

□ [Proposition 6]

Remark 7: Notice that the second algorithm provided here is exactly the same as the change-of-basis algorithm for $\mathbb{R}^D$, when $\mathcal{A} = \mathcal{E}$. This should give you a hint how to prove **Part 2**. The proof of **Part 1** then follows.

Definition 8: *Change-of-Basis Matrix*

The matrix $\boxed{\mathrm{A}}$ in the previous theorem is called the **change-of-basis matrix** from the basis $\mathcal{A}$ to the basis $\mathcal{B}$.

Remark 9: To *change back* from the basis $\mathcal{B}$ to the basis $\mathcal{A}$, we simply reverse the previous procedure. Thus, if $\boxed{\mathrm{A}}$ is the change-of-basis matrix from $\mathcal{A}$ to $\mathcal{B}$, then $\boxed{\mathrm{A}}^{-1}$ is the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{A}$. (Exercise: Convince yourself of this.)