Symmetry Groupoids and Patterns of Synchrony in Coupled Cell Networks

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Abstract

A coupled cell system is a network of dynamical systems, or 'cells', coupled together. Such systems can be represented schematically by a directed graph whose nodes correspond to cells and whose edges represent couplings. A symmetry of a coupled cell system is a permutation of the cells that preserves all internal dynamics and all couplings. Symmetry can lead to patterns of synchronized cells, rotating waves, multirhythms, and synchronized chaos. We ask whether symmetry is the only mechanism that can create such states in a coupled cell system, and show that it is not.

The key idea is to replace the symmetry group by the symmetry groupoid, which encodes information about the input sets of cells. (The *input set* of a cell consists of that cell and all cells connected to that cell.) The admissible vector fields for a given graph — the dynamical systems with the corresponding internal dynamics and couplings — are precisely those that are equivariant under the symmetry groupoid. A pattern of synchrony is 'robust' if it arises for all admissible vector fields. The first main result shows that robust patterns of synchrony (invariance of 'polydiagonal' subspaces under all admissible vector fields) is equivalent to the combinatorial condition that an equivalence relation on cells is 'balanced'. The second main result shows that admissible vector fields restricted to polydiagonal subspaces are themselves admissible vector fields for a new coupled cell network, the 'quotient network'. The existence of quotient networks has surprising implications for synchronous dynamics in coupled cell systems.

In a final section we develop the local bifurcation theory of coupled cell systems, by analogy with symmetric bifurcation theory, using the concept of an 'interior symmetry', which is closely related to the groupoid structure. We prove analogs of the Equivariant Branching Lemma for steady-state bifurcation, and the Equivariant Hopf Theorem for bifurcation to time-periodic states for 'synchrony-breaking' bifurcations. We end by suggesting analogous contexts in which symmetry groupoids can be defined and may prove useful.

1 Introduction

We use the term *cell* to indicate a system of ODEs. A *coupled cell system* is a set of cells with coupling, that is, a dynamical system whose variables correspond to cells, such that the output of certain cells affects the time-evolution of other cells. The salient feature of a coupled cell system is that the output from each cell is considered to be significant in its own right. A coupled cell system is not merely a system of ODEs, but a system of ODEs equipped with canonical observables — the individual cells (see [9]). From a mathematical point of view these output signals can be compared, and this observation leads to a variety of notions of 'synchrony'. For surveys, see Boccaletti *et al.* [2] and Wang [15].

In this paper we discuss the *architecture* of a coupled cell system: which cells influence which, which cells are 'identical', and which couplings are 'identical'. We focus on how the system architecture leads naturally to synchrony. To do this, we must define carefully when two cells or two couplings are 'identical' or 'equivalent'. Indeed, the main point of this paper is to provide a general mathematical foundation for these ideas. This foundation uses the algebraic structure of groupoids, see Brandt [1], Higgins [11], and greatly generalizes the uses of symmetry in coupled cell systems that we have explored previously [8, 9]. We mention that coupled cell systems are used to model a variety of physically interesting systems. For examples, see [9] and references therein. In this section we illustrate the issues through several examples.

Two-Cell Systems

We begin with the simplest system of two identical cells (with coordinates x_1 and x_2 in $\mathbf{R}^k$). Without making any specific assumption of the form of the 'internal dynamics' of each cell or the form of the 'coupling between cells', the differential equations for the coupled system have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \\ \dot{x}_2 &= f(x_2, x_1) \end{aligned} \tag{1.1}$$

that is, the same function f governs the dynamics of both cells. There are three issues that we discuss concerning system (1.1): the graph (diagram, network) associated to a coupled cell system, symmetry, and synchrony.

Informally, the 'network' of a coupled cell system is a finite directed graph whose nodes represent cells and whose edges represent couplings. Nodes are labeled to indicate 'equivalent' cells, which have the same phase space and the same internal dynamic. Edges are labeled to indicate 'equivalent' couplings. The graph associated to system (1.1) is given in Figure 1. We think of this graph as representing a pair of systems of differential equations in the following way. The two cells are indicated by identical symbols — so they have the same state variables. That is, the coordinates x_1 of cell 1 and x_2 of cell 2 lie in the same phase space $\mathbf{R}^k$. Since we can interchange cells 1 and 2 without changing the graph, we assume that the same is true for the system of differential equations and that they must have the form (1.1). Note that for this interchange to work, the arrow $1 \to 2$ must be the same as the arrow $2 \to 1$.

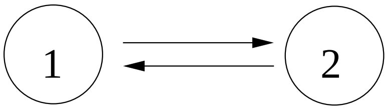

Figure 1: A two-cell network.

The discussion in the previous paragraph can be summarized by: the permutation $\sigma(x_1, x_2) = (x_2, x_1)$ is a symmetry of the system (1.1). Indeed, more is true: every system of differential equations on $\mathbf{R}^k \times \mathbf{R}^k$ that is equivariant with respect to σ has the form (1.1). That is, abstractly the study of pairs of identical cells that are identically coupled is the same as the study of σ-equivariant systems. Two consequences follow from this remark. First, synchrony in two-cell systems (solutions such that $x_1(t) = x_2(t)$ for all time t) is a robust phenomenon and should not be viewed as surprising. Second, time-periodic solutions can exhibit a kind of generalized synchrony, in which the two cells oscillate a half-period out of phase.

The first remark can be restated: the diagonal subspace $V = \{x_1 = x_2\} \subset \mathbf{R}^k \times \mathbf{R}^k$ is flow-invariant for every system (1.1). This remark can be verified in two ways. By inspection restrict (1.1) to V, obtaining

$$\begin{array}{rcl} \dot{x}_1 & = & f(x_1, x_1) \\ \dot{x}_1 & = & f(x_1, x_1) \end{array}$$

It follows that if the initial conditions for a solution satisfy $x_1(0) = x_2(0)$, then $x_1(t) = x_2(t)$ for all time t, and V is flow-invariant. Alternately, we can observe that V is the fixed-point subspace $\mathrm{Fix}(\sigma)$, and fixed-point subspaces are well-known to be flow-invariant.

The second remark is related to general theorems about spatio-temporal symmetries of time-periodic solutions to symmetric systems of ODEs. The H/K Theorem ([4, 8]) implies the existence of functions f having time-periodic solutions of period T satisfying

$$x_2(t) = x_1(t + T/2) \tag{1.2}$$

as long as the phase space of each cell has dimension $k \geq 2$. Indeed, in this case, such solutions can be found by Hopf bifurcation. (Note that when $k = 1$, nonconstant periodic solutions satisfying (1.2) must intersect the diagonal V, and hence be in V for all time: a contradiction.)

A Three-Cell Network

Consider the three-cell network illustrated in Figure 2. The systems of differential equations

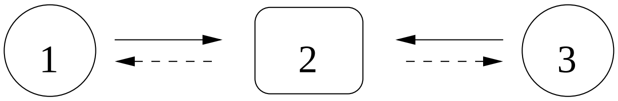

Figure 2: A three-cell network with transposition symmetry.

corresponding to this network have the form

$$\begin{array}{rcl} \dot{x}_1 & = & f(x_1, x_2) \\ \dot{x}_2 & = & g(x_2, x_1, x_3) \\ \dot{x}_3 & = & f(x_3, x_2) \end{array} \tag{1.3}$$

where $g(x_2, x_1, x_3) = g(x_2, x_3, x_1)$, $x_1, x_3 \in \mathbf{R}^k$, and $x_2 \in \mathbf{R}^\ell$. Note that all such systems are equivariant with respect to the permutation $\tau(x_1, x_2, x_3) = (x_3, x_2, x_1)$, and that synchronous solutions (where $x_1(t) = x_3(t)$ for all time t) occur robustly because the 'polydiagonal' subspace $W = \{x : x_1 = x_3\}$ is flow-invariant for (1.3).

There are two differences between the three-cell network in Figure 2 and the two-cell network in Figure 1. First, not all τ-equivariant systems on $\mathbf{R}^k \times \mathbf{R}^\ell \times \mathbf{R}^k$ have the form (1.3), since in the general τ-equivariant system f can depend nontrivially on both x_1 and x_3. So there can be additional structure in coupled cell systems that does not correspond directly to symmetry. Second, the half-period out of phase time-periodic solutions satisfy

$$x_3(t) = x_1(t + T/2) \quad \text{and} \quad x_2(t) = x_2(t + T/2) \tag{1.4}$$

In particular, the oscillations in cell 2 are forced by symmetry to occur at twice the frequency of those in cells 1 and 3. So multirhythms [8] can be forced by the architecture of coupled cell networks.

Another Three-Cell Network

We now show that robust synchrony is possible in networks that have no symmetry. Consider the three-cell network in Figure 3. This network has no symmetry, but the network structure forces the 'polydiagonal' subspace $Y = \{x : x_1 = x_2\}$ to be flow-invariant. To verify this point observe that the coupled cell systems associated with this network have the form

$$\begin{array}{rcl} \dot{x}_1 & = & f(x_1, x_2, x_3) \\ \dot{x}_2 & = & f(x_2, x_1, x_3) \\ \dot{x}_3 & = & g(x_3, x_1) \end{array} \tag{1.5}$$

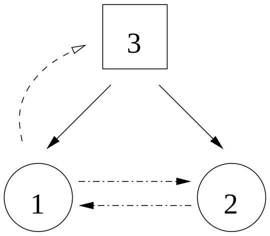

Figure 3: A three-cell network without symmetry.

where $x_1, x_2 \in \mathbf{R}^k$ and $x_3 \in \mathbf{R}^\ell$. Restricting the first two equations to Y yields

$$\begin{array}{rcl} \dot{x}_1 & = & f(x_1, x_1, x_3) \\ \dot{x}_1 & = & f(x_1, x_1, x_3) \end{array}$$

implying that Y is a flow-invariant subspace.

There is a precise sense in which cells 1 and 2 are identical within this network, and it is this observation that will enable us to prove the flow-invariance of subspaces like Y in a more abstract (and general) setting. Define the 'input set' of a cell j to be the cell j and all cells i that connect to cell j. Also include the arrows from cells i to j.

We can now explain why Y is flow-invariant, in terms of a permutation that acts on the network. This permutation is not a symmetry of the whole network, but it preserves enough structure to create a flow-invariant subspace. The key property is that the input sets of cells 1 and 2 are isomorphic via the permutation σ that maps $(1\ 2\ 3) \rightarrow (2\ 1\ 3)$.

If the system (1.5) were equivariant with respect to σ, then the fixed-point space of σ would be flow-invariant by [10, 8]. Moreover, the fixed-point space of σ is Y. However, (1.5) is *not* equivariant with respect to σ. Indeed, if we apply σ, then the equation transforms into

$$\begin{array}{rcl} \dot{x}_2 & = & f(x_2, x_1, x_3) \\ \dot{x}_1 & = & f(x_1, x_2, x_3) \\ \dot{x}_3 & = & g(x_3, x_2) \end{array} \tag{1.6}$$

The first two equations are the same as in (1.5), but the third equation is not. However, the third *is* the same on the space Y, where $x_2 = x_1$. So the restriction of the equations to Y is σ-equivariant, and this is enough to make Y flow-invariant.

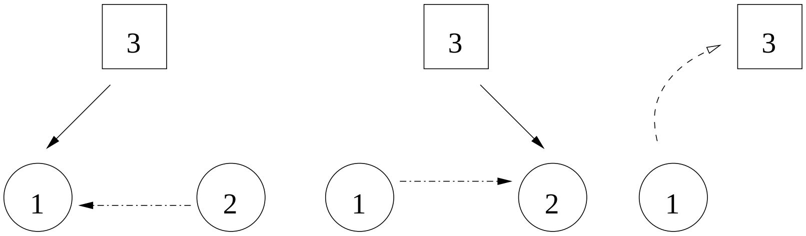

Figure 4: Input sets for three-cell network without symmetry.

Traveling Waves in a Seven-Cell Network

Consider the seven-cell linear network in Figure 5. The corresponding differential equations have the general form

$$\begin{array}{lllllllll} \dot{x}_1 & = & B(x_1) & \dot{x}_2 & = & A(x_2,x_1) & \dot{x}_3 & = & A(x_3,x_2) \\ \dot{x}_4 & = & A(x_4,x_3) & \dot{x}_5 & = & A(x_5,x_4) & \dot{x}_6 & = & A(x_6,x_5) \\ & & & \dot{x}_7 & = & A(x_7,x_6) & & & \end{array} \tag{1.7}$$

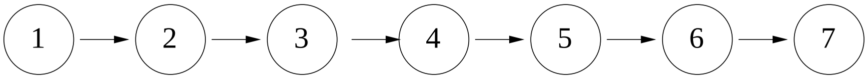

Figure 5: Seven-cell linear network.

It does not seem to be a simple matter to determine whether traveling waves are present in this network. If the cell phase spaces are all 1-dimensional, there are no nontrivial time-periodic states, so no traveling waves. With higher-dimensional phase spaces, special assumptions are needed to produce traveling waves. However, if we introduce back coupling from cell 3 to cell 1, as shown in Figure 6, traveling waves can typically be expected, even in the 1-dimensional case, as explained below. This is curious, because informally Figure 6 would normally be considered as being less regular in form than Figure 5. So the issue of 'regular form' for a coupled cell network is fairly subtle. The key feature here is that all input sets for cells in the network in Figure 6 are isomorphic, whereas as this is not true for the cells in Figure 5. It is this additional 'symmetry' on the groupoid level that makes traveling waves typical. Indeed, Figure 6 has many groupoid symmetries (42 in all).

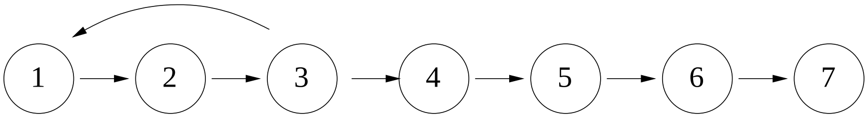

Figure 6: Seven-cell linear network with back connection.

We discuss why traveling wave solutions arise, in two ways. First, the assumption that all of the cells and arrows in Figure 6 are identical implies that the first equation in (1.7) is now

$$\dot{x}_1 = A(x_1, x_3)$$

If we set

$$\begin{array}{l} x_7 = x_4 = x_1 = y_1 \\ x_6 = x_3 = y_3 \\ x_5 = x_2 = y_2 \end{array} \tag{1.8}$$

then the system of seven equations reduces to a three-equation system

$$\begin{array}{lll} \dot{y}_1 & = & A(y_1,y_3) \\ \dot{y}_2 & = & A(y_2,y_1) \\ \dot{y}_3 & = & A(y_3,y_2) \end{array} \tag{1.9}$$

which is the general form associated with the directed ring of coupled cells in Figure 7. It is not hard to show using Hopf bifurcation that the system (1.9) can support a discrete rotating wave $y(t)$, where

$$y_2(t) = y_1\left(t - \frac{T}{3}\right) \quad y_3(t) = y_2\left(t - \frac{T}{3}\right)$$

and y_1 is periodic of period T. This solution yields a *traveling wave* solution for the network in Figure 5. See a sample simulation in Figure 8. (The number 7 is not significant here: the same ideas work for any chain containing 3 or more cells and with feedback from any cell other than the first.)

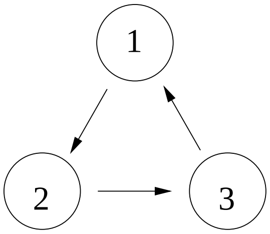

Figure 7: Three-cell directed ring: quotient of the network in Figure 6

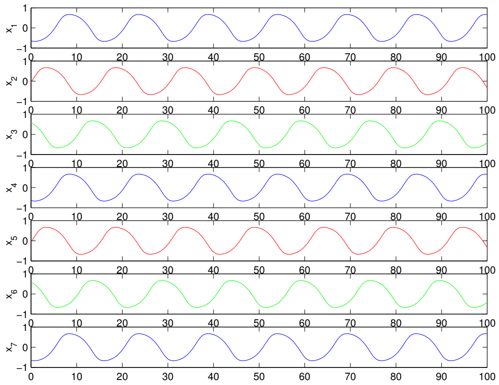

Figure 8: Traveling wave solution in seven-cell chain.

More importantly, the three-cell ring in Figure 7 is a quotient network of the one in Figure 6, where the quotient map β takes x_1, x_4, x_7 to y_1; x_2, x_5 to y_2; and x_3, x_6 to y_3. We define 'quotient' in Section 8, but the key point is that solutions for the three-cell ring naturally 'lift' to solutions for the seven-cell network, via (1.8). The crucial features here are that β induces an isomorphism from each input set in the first network to an input set in the second network, and every coupled cell system of differential equations in the quotient lifts to a coupled cell system in the first network.

We seek to isolate the abstract structural features that are responsible for the behavior described in the above examples, and to place the discussion in a rigorous, formal context. We structure the paper as follows. Coupled cell networks are rigorously defined in terms of nodes and arrows in Section 2. The key concept, the groupoid structure of a coupled cell network, is defined in Section 3 in terms of input sets. The phase space and admissible vector fields associated to a coupled cell network are discussed in Section 4. Basically, the intuitive ideas presented in this introduction are formalized as equivariance with respect to the symmetry groupoid of the network. Section 5 describes an extended example, which motivates the rest of the paper. In Section 6 we introduce three different notions of robust synchrony: flow-invariant subspaces, fixed-point subspaces of subgroupoids, and balanced equivalence relations. We prove that these notions are all equivalent. Quotient maps and quotient networks, which constitute a fourth equivalent notion for synchrony, are discussed in Section 8. Quotient networks are an especially useful concept because they illuminate the generic dynamics of vector fields restricted to synchronous invariant subspaces, which can include phase-locked states and synchronized chaos. Examples illustrating these points are discussed in Section 7. The relationship between the dynamics on a synchronous subspace and the induced dynamics on the quotient network is discussed in Section 9. Section 10 initiates the 'synchrony-breaking' bifurcation theory of not necessarily symmetric coupled cell systems by introducing the notion of 'interior symmetry' and applies it to prove analogs of the Equivariant Branching Lemma and the Equivariant Hopf Theorem [10, 8].

2 Coupled Cell Networks

We begin by formally defining a coupled cell network.

Definition 2.1 A *coupled cell network* G consists of:

(a) A finite set $\mathcal{C} = \{1, \ldots, N\}$ of *nodes* or *cells*.

(b) A finite set of ordered pairs $\mathcal{E} \subseteq \mathcal{C} \times \mathcal{C}$ of *directed edges* or *arrows*.

Each edge (c, d) has a *tail* c and a *head* d.

(c) An equivalence relation $\sim_C$ on cells in $\mathcal{C}$.

The *type* or *cell label* of cell c is the $\sim_C$-equivalence class $[c]_C$ of c.

(d) An equivalence relation $\sim_E$ on edges in $\mathcal{E}$.

The *type* or *coupling label* of edge e is the $\sim_E$-equivalence class $[e]_E$ of e.

An edge (c, c) is an *internal edge*; a cell is *active* if it has an internal edge. We assume that every cell is active, that is,

$$\Delta_{\mathcal{C}} = \{(c, c) : c \in \mathcal{C}\} \subset \mathcal{E}. \tag{2.1}$$

In addition, we require the following *compatibility conditions*:

(e) Equivalent edges have equivalent tails and heads. That is,

if $(i, c) \sim_E (j, d)$ then $i \sim_C j$ and $c \sim_C d$.

(f) Internal edges are equivalent if their tails are equivalent. Internal edges and non-internal edges are never equivalent. That is, for all $c, d, d' \in \mathcal{C}$

$$(c,c) \sim_E (d,d') \iff d = d' \text{ and } d \sim_C c$$

Formally, the coupled cell network G is the quadruple $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$. ◇

We represent a coupled cell network G by a diagram constructed as follows.

1) For each $\sim_C$-equivalence class of cells choose a distinct *node symbol* $\bigcirc, \square, \triangle$, and so on.

2) For each $\sim_E$-equivalence class of non-internal edges, choose a distinct *arrow* $\rightarrow, \Rightarrow, \rightsquigarrow$, and so on.

The compatibility conditions in Definition 2.1 state that arrows between distinct cells can be identical only when the nodes at the heads are identical and the nodes at the tails are identical, and that node symbols can be interpreted as arrows from a cell to itself.

The above definition is essentially the standard concept of a directed graph (or digraph) in graph theory (see for example Tutte [14], Wilson [17]), modified to incorporate labeling of nodes and edges. We assume that the graph is finite because this makes the associated dynamical systems (discussed in Section 4) finite-dimensional. However, most of the theory generalizes to infinite graphs. The assumption that all cells are active can be removed, at the expense of notational complications, but the details are routine and we do not treat this case here.

Example 2.2 Suppose that the network G is defined by:

$$\begin{array}{lll} \mathcal{C} & = & \{1,2,3,4\} \\ \mathcal{E} & = & \{(1,2),(1,3),(2,4),(3,1),(3,4)(4,1)\} \\ \sim_C & \text{has equivalence classes} & \{1\},\{2,3\},\{4\} \\ \sim_E & \text{has equivalence classes} & \{(1,2),(1,3)\},\{(2,4),(3,4)\},\{(3,1)\},\{(4,1)\} \end{array}$$

and all nodes are active. Then the diagram of G has the form shown in Figure 9 for the given choices of symbols. ◇

3 Input Sets and Groupoids

In this section we define the basic algebraic structure of a coupled cell network — its symmetry groupoid. Some preliminary concepts are required.

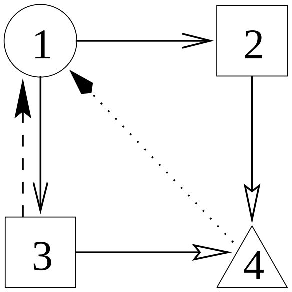

Figure 9: Example of the diagram of a coupled cell network.

Input Sets

As discussed in more detail in Section 4 the variables that appear in a given component f_c of the vector fields f associated to a coupled cell network depend only on those cells that are linked to cell c by an arrow. This observation is abstracted as:

Definition 3.1 The *input set* $I(c)$ of a cell c is

$$I(c) = \{i \in \mathcal{C} : (i, c) \in \mathcal{E}\}$$

Since all cells are active, $c \in I(c)$. We call c the *base cell* of $I(c)$. $\diamond$

Two cells are considered to be *identical within the network* if they have isomorphic input sets. We call such cells ‘input equivalent’.

Example 3.2 We return to Example 2.2 whose diagram is shown in Figure 9. The input sets are shown in Figure 10. There are three $\sim_I$-equivalence classes: $\{1\}, \{2, 3\}$, and $\{4\}$. The isomorphism between $I(2)$ and $I(3)$ is the bijection $\tau : \{2, 1\} \to \{3, 1\}$ for which $\tau(2) = 3$ and $\tau(1) = 1$. $\diamond$

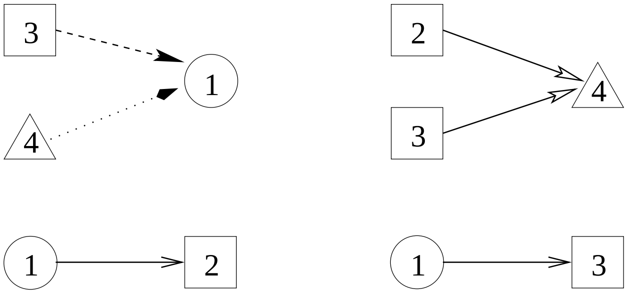

Figure 10: Input sets for Figure 9.

Definition 3.3 The relation $\sim_I$ of *input equivalence* on $\mathcal{C}$ is defined by $c \sim_I d$ if and only if there exists a base cell preserving bijection

$$\beta : I(c) \to I(d) \tag{3.1}$$

(by which we mean that $\beta(c) = d$), such that for all $i \in I(c)$

$$(i, c) \sim_E (\beta(i), d) \tag{3.2}$$

Any such bijection β is called an *input isomorphism* from cell c to cell d. The set $B(c, d)$ denotes the collection of all input isomorphisms from cell c to cell d. ◇

On setting $i = c$ in (3.2), we see that c and d have the same type ($c \sim_C d$) if they are input equivalent ($c \sim_I d$). The converse is easily seen to be false. Moreover, $B(c, d)$ is empty unless $c \sim_I d$.

The Symmetry Groupoid

We now introduce the central concept of this paper, the 'symmetry groupoid' of a coupled cell network. The symmetry groupoid is a generalization of the symmetry group of a symmetric network. It includes not just symmetries of the whole network, but symmetries between particular subgraphs — namely, the input sets.

Definition 3.4 The *symmetry groupoid* of a coupled cell network G is the disjoint union

$$\mathcal{B}_G = \dot{\bigcup}_{c,d \in \mathcal{C}} B(c, d)$$

◇

The term 'groupoid' was introduced by Brandt [1] and is developed at length in Higgins [11]. The term refers to an algebraic structure that is similar to a group, with the exception that products of elements may not always be defined. Different authors formalize groupoids in slightly different (but mostly equivalent) ways. Essentially, a groupoid must satisfy three conditions:

1) The product operation is associative in the sense that whenever one of $\alpha(\beta\gamma)$ and $(\alpha\beta)\gamma$ is defined, then so is the other, and they are equal.

2) There are distinguished elements ε_j that act as identity elements, in the sense that $\varepsilon_j \alpha = \alpha$ and $\alpha \varepsilon_j = \alpha$ whenever these are defined. (Here the indices j correspond to the 'objects' of the groupoid, which in our case are the cells.)

3) Every element α has an inverse α^{-1}, in the sense that both $\alpha\alpha^{-1}$ and $\alpha^{-1}\alpha$ are identities.

In the case of $\mathcal{B}_G$, the groupoid structure is captured by the following.

1) We define the product of $\beta_1 \in B(c,d)$ and $\beta_2 \in B(c',d')$ if and only if $c' = d$, and then we set $\beta_2\beta_1 = \beta_2 \circ \beta_1 \in B(c,d')$ where $\circ$ denotes composition of maps. Composition is of course associative when it is defined.

2) The identity elements $\mathrm{id}_{I(c)}$ for $c \in \mathcal{C}$ are the groupoid identity elements.

3) For inverses, observe that $\beta \in B(c,d)$ if and only if $\beta^{-1} \in B(d,c)$.

It follows in particular that $B(c,c)$ is a group, the *vertex group* corresponding to c. Vertex groups are important in groupoid theory, and play a key role in this paper.

Remark 3.5 The term 'disjoint union' in Definition 3.4 is used in a technical sense. The sets $B(c,d)$ for different pairs (c,d) are not necessarily disjoint. For example, if $\mathcal{C} = \{1,2\}$ where cells 1 and 2 are inequivalent, each coupled to the other by inequivalent arrows, then $B(1,1)$ and $B(2,2)$ both consist of the identity map on $\{1,2\}$. It is convenient to *make* them disjoint. One way to do this is to replace each bijection $\beta \in B(c,d)$ by the triple (β,c,d). Then β defines the permutation, c is an index specifying its 'domain', and d is an index specifying its 'range'. As far as the groupoid structure is concerned, the product $\beta_2\beta_1$ is defined only when $c' = d$. This occurs when the 'range' of β_1 is equal to the 'domain' of β_2, in the sense just specified. However, the set-theoretic ranges and domains of the corresponding bijections may permit the composition of β_1 and β_2 as *functions* in cases where we do not wish to permit them to be multiplied in the groupoid.

The point here is that we are not dealing merely with bijections on sets, but with base-point preserving bijections on based sets. Composition must respect the base points as well as the sets.

For simplicity, we use β to denote an input isomorphism, rather than the cumbersome (β,c,d), because the appropriate c,d are usually obvious. ◇

Example 3.6 Again, we return to Example 2.2. The non-empty sets $B(c,d)$ are:

$B(1,1)$: The identity map on $\{1,3,4\}$.

$B(2,2)$: The identity map on $\{2,1\}$.

$B(3,3)$: The identity map on $\{3,1\}$.

$B(4,4)$: The identity map on $\{4,2,3\}$, and the permutation σ on $\{4,2,3\}$ for which $\sigma(2) = 3, \sigma(3) = 2, \sigma(4) = 4$.

$B(2,3)$: The map $\tau : \{2,1\} \to \{3,1\}$ for which $\tau(1) = 1, \tau(2) = 3$.

$B(3,2)$: The inverse τ^{-1} of τ. ◇

Subgroupoids and Connected Components

For the basics of groupoids see Brandt [1], Brown [3], Higgins [11], and MacLane [12]. For applications see Weinstein [16]. Groupoids combine several features of groups with features of graphs, and we discuss one of each now. The group-theoretic notion is that of a subgroupoid; the graph-theoretic one is that of a connected component.

A subset $\mathcal{S} \subset \mathcal{B}_G$ is a *subgroupoid* if $\mathcal{S}$ is closed under products (when defined) and taking inverses.

The *connected components* of the groupoid $\mathcal{B}_G$ are in one-to-one correspondence with $\sim_I$-equivalence classes on $\mathcal{C}$. Specifically, let A be a $\sim_I$-equivalence class. Then the subgroupoid

$$\mathcal{S}(A) = \dot{\bigcup}_{c,d \in A} B(c,d) \tag{3.3}$$

is a *connected component* of $\mathcal{B}_G$. Moreover:

Lemma 3.7 *The groupoid $\mathcal{B}_G$ is the disjoint union of its connected components. That is,*

$$\mathcal{B}_G = \dot{\bigcup}_A \mathcal{S}(A)$$

where A runs through the $\sim_I$-equivalence classes in $\mathcal{C}$. Moreover, if A, A' are two distinct such classes, and $\beta \in \mathcal{S}(A), \beta' \in \mathcal{S}(A')$, then the product $\beta\beta'$ is not defined. □

We say that two cells $c, d \in \mathcal{C}$ are in the *same connected component* of $\mathcal{B}_G$ if and only if $c \sim_I d$. The mental image here is that associated with any groupoid there is a graph, whose elements are the vertices of the groupoid and whose (directed) edges are the groupoid elements. It is the connected components of this graph that are being described. See Higgins [11] chapter 3.

If c, d belong to the same connected component, then the vertex groups $B(c,c)$ and $B(d,d)$ are *conjugate*, in the sense that there exists $\gamma \in B(c,d)$ such that

$$B(c,c) = \gamma^{-1} B(d,d) \gamma$$

In particular, $B(c,c)$ and $B(d,d)$ are isomorphic groups.

Structure of $B(c,d)$

For later use, we determine the general structure of the sets $B(c,d)$.

1) If $c \not\sim_I d$ then $B(c,d) = \emptyset$.

2) If $c = d$ then we define an equivalence relation $\equiv_c$ on $I(c)$ by

$$j_1 \equiv_c j_2 \iff (j_1, c) \sim_E (j_2, c)$$

for $j_1, j_2 \in I(c)$. Let the $\equiv_c$-equivalence classes of $I(c)$ be $K_0, \ldots, K_r$, for $r = r(c)$, so that

$$I(c) = K_0 \,\dot{\cup} \cdots \dot{\cup} K_r \tag{3.4}$$

We may choose $K_0 = \{c\}$ by Definition 2.1(f). Let

$$k_s = |K_s| \qquad (0 \leq s \leq r)$$

Then $B(c,c)$ is a group, given by

$$B(c,c) = \mathbf{S}_{k_1} \times \cdots \times \mathbf{S}_{k_r}$$

where each $\mathbf{S}_{k_s}$ comprises all permutations of K_s, extended by the identity on $I(c) \setminus K_s$.

3) If $c \sim_I d$ and $c \neq d$, define $\equiv_d$ on $I(d)$ in the same way. Let $\beta \in B(c,d)$, and let

$$L_s = \beta(K_s) \qquad (0 \leq s \leq r(c))$$

Then β and β^{-1} preserve $\sim_E$, so the $\equiv_d$-equivalence classes of $I(d)$ are the $L_s(0 \leq s \leq r(c))$, and $r(d) = r(c)$.

Choose a fixed but arbitrary $\beta_0 \in B(c,d)$, having the above property. Then

$$B(c,d) = B(d,d)\beta_0 = \beta_0 B(c,c)$$

Conversely, any $\beta_0 : I(c) \to I(d)$ such that $L_s = \beta_0(K_s)$ for $0 \leq s \leq r(c)$ lies in $B(c,d)$.

4 Vector Fields on a Coupled Cell Network

We now define the class $\mathcal{F}_G^P$ of vector fields corresponding to a given coupled cell network G. This class consists of all vector fields that are 'compatible' with the labeled graph structure, or equivalently are 'symmetric' under the groupoid $\mathcal{B}_G$. It also depends on a choice of 'total phase space' P, which we assume is fixed throughout the subsequent discussion. For example, in the two-cell system (1.1) we have $P = \mathbf{R}^k \times \mathbf{R}^k$, which depends on the choice of k.

For each cell in $\mathcal{C}$ define a *cell phase space* P_c. This must be a smooth manifold of dimension ≥ 1, which for simplicity we assume is a nonzero finite-dimensional real vector space. We require

$$c \sim_C d \quad \Longrightarrow \quad P_c = P_d$$

and we employ the same coordinate systems on P_c and P_d. Only these identifications are canonical. If $P_c = P_d$ or P_c is isomorphic to P_d when $c \not\sim_C d$, then the identification of P_c and P_d will be deemed accidental, and will have no significance for our present purposes. In other words, the relation $c \sim_C d$ means that cells c and d have the same phase space, but not that they have isomorphic dynamics.

Define the corresponding *total phase space* to be

$$P = \prod_{c \in \mathcal{C}} P_c$$

and employ the coordinate system

$$x = (x_c)_{c \in \mathcal{C}}$$

on P.

The *cell projection* corresponding to cell c is the natural projection

$$\pi_c : P \to P_c$$

More generally, suppose that $\mathcal{D}$ is any subset of $\mathcal{C}$. Define

$$P_{\mathcal{D}} = \prod_{c \in \mathcal{D}} P_c$$

and let

$$\pi_{\mathcal{D}} : P \to P_{\mathcal{D}}$$

be the natural projection. Further, write

$$x_{\mathcal{D}} = \pi_{\mathcal{D}}(x)$$

and suppress braces when $\mathcal{D}$ is a singleton. That is, $\pi_c(x) = x_c = x_{\{c\}}$.

Finally, suppose that $\mathcal{D}_1, \mathcal{D}_2$ are subsets of $\mathcal{C}$, and that there is a bijection $\beta : \mathcal{D}_1 \to \mathcal{D}_2$ such that $\beta(d) \sim_C d$ for all $d \in \mathcal{D}_1$. Define the *pullback map*

$$\beta^* : P_{\mathcal{D}_2} \to P_{\mathcal{D}_1}$$

by

$$(\beta^*(z))_j = z_{\beta(j)} \qquad \forall j \in \mathcal{D}_1, z \in P_{\mathcal{D}_2} \tag{4.1}$$

By direct calculation it is easy to verify three simple properties of the pullback:

$$\begin{array}{rcl} (\beta\gamma)^* & = & \gamma^*\beta^* \\ \mathrm{id}^* & = & \mathrm{id} \\ (\gamma^{-1})^* & = & (\gamma^*)^{-1} \end{array} \tag{4.2}$$

Note the reversed order in the first of these equations.

We use pullback maps to relate different components of the vector field associated with a given coupled cell network. Specifically, the class of vector fields that is encoded by a coupled cell network will be defined using the following concept:

Definition 4.1 A vector field $f : P \to P$ is *$\mathcal{B}_G$-equivariant* or *G-admissible* if:

(a) For all $c \in \mathcal{C}$ the component $f_c(x)$ depends only on $x_{I(c)}$; that is, there exists $\hat{f}_c : P_{I(c)} \to P_c$ such that

$$f_c(x) = \hat{f}_c(x_{I(c)}) \tag{4.3}$$

(b) For all $c, d \in \mathcal{C}$ and $\beta \in B(c, d)$ (so that in particular $d = \beta(c)$)

$$\hat{f}_d(x_{I(d)}) = \hat{f}_c(\beta^*(x_{I(d)})) \qquad \forall x \in P \tag{4.4}$$

For brevity, we write this condition as

$$f_{\beta(c)}(x) = f_c(\beta^*(x)) \qquad \forall x \in P \tag{4.5}$$

However, when using (4.5) it is necessary to check that $f_d(x)$ depends only on $x_{I(d)}$. Otherwise, $\beta^*(x)$ is not defined. ◇

We call (a) the *domain condition* and (b) the *equivariance condition* on f.

Remark 4.2 If β belongs to the vertex group $B(c,c)$ then (4.5) implies that

$$f_c(\beta^*(x)) = f_c(x) \qquad \forall x \in P \tag{4.6}$$

That is, f_c is $B(c,c)$-*invariant*. It is easy to check that this property is the same as the usual property of invariance under a group, provided we consider $B(c,c)$ as acting on $P_{I(c)}$. ◇

Definition 4.3 For a given choice of the P_c we define the class $\mathcal{F}_G^P$ to consist of all G-admissible vector fields on P. ◇

These are the most general vector fields on P that are consistent with the coupled cell network.

Example 4.4 We describe $\mathcal{F}_G^P$ for the diagram of Figure 9. There are three cell types ($\bigcirc, \square, \triangle$) and we choose three corresponding phase spaces U, V, W. Then the state variable is $x = (x_1, x_2, x_3, x_4)$ where $x_1 \in U$; $x_2, x_3 \in V$; $x_4 \in W$. There are four arrow types. We claim that the G-admissible vector fields f are those of the form:

$$\begin{array}{rcll} f_1(x) & = & A(x_1, x_3, x_4) & \text{where } A : U \times V \times W \to U \\ f_2(x) & = & B(x_2, x_1) & \text{where } B : V \times U \to V \\ f_3(x) & = & B(x_3, x_1) & \\ f_4(x) & = & C(x_4, x_2, x_3) & \text{where } C : W \times V \times V \to U \end{array}$$

and C is symmetric in x_2, x_3.

To prove this, we consider the equivariance condition (4.5) for all the bijections β listed in Example 3.6. There are two nontrivial cases: $B(2,3)$ and $B(4,4)$. First, suppose that $c = 2, d = 3$ and consider the bijection $\tau : I(2) \to I(3)$ for which $\tau(2) = 3, \tau(1) = 1$. Suppose that we define the function $B : P_{I(2)} \to P_2$ by

$$B(x_2, x_1) = f_2(x)$$

so that $B = \hat{f}_2$. Then $f_3(x) = \hat{f}_3(x_3, x_1)$ and we wish to express this in terms of B.

It is easy to work out the pullback of τ. If we write the elements of $P_{I(3)}$ in the form $x = (x_3, x_1)$, then $y = \tau^*(x)$ takes the form $y = (y_2, y_1) \in P_{I(2)}$ where

$$\begin{array}{rcl} y_2 & = & (x)_{\tau(2)} = x_3 \\ y_1 & = & (x)_{\tau(1)} = x_1 \end{array}$$

Then

$$\tau^*(x_3, x_1) = (x_3, x_1)$$

and condition (4.5) tells us that

$$\hat{f}_3(x) = B(x_3, x_1)$$

as claimed. (The pullback τ^* is not the identity, because its range and domain are different. It is an *identification*.)

Similarly, if we consider $\sigma \in B(4,4)$ then we have a function C defined by $f_4(x) = C(x_4, x_2, x_3)$. Now the pullback $\sigma^* : P_{I(4)} \to P_{I(4)}$ acts as

$$\sigma^*(x_4, x_2, x_3) = (x_4, x_3, x_2)$$

and condition (4.5) tells us that

$$C(x_4, x_2, x_3) = \hat{f}_4(x) = C(x_4, x_3, x_2)$$

so that C is symmetric in x_2, x_3. ◇

Here and from now on we adopt the convention that x_c is the first variable listed in the argument of $\hat{f}_c$. We can show that $\hat{f}_c$ is symmetric in some subset of variables by putting a bar over that set, so that here

$$f_4(x) = C(x_4, \overline{x_2, x_3})$$

(To do this, we have to order the variables suitably, and in some cases this cannot be done consistently. The use of a bar is convenient for the purposes of this paper.) Note that the network G is *not* symmetric under the 2-cycle (2 3), because the arrow from cell 3 to cell 1 does not correspond to an arrow from cell 2 to cell 1.

Admissible Vector Fields

The proofs of the main theorems of this paper rely on the construction of certain special G-admissible vector fields. In this subsection we describe these constructions.

As motivation, consider Example 4.4. Here, the most general G-admissible vector field is specified by three functions A, B, C. These functions can be assigned independently of each other. There is one such function for each $\sim_I$-equivalence class of cells, that is, each connected component of $\mathcal{B}_G$. If $c \in \mathcal{C}$ then f_c is $B(c,c)$-invariant; if $d \sim_I c$ then f_d is uniquely defined by f_c through the condition of $\mathcal{B}_G$-equivariance. We now give a formal statement of these properties and show that they are valid in general.

The main point is that Lemma 3.7 implies that $\mathcal{B}_G$-equivariance imposes conditions relating components f_c, f_d of f when c, d lie in the same connected component of $\mathcal{B}_G$, but not otherwise. We can therefore construct G-admissible vector fields g on P whose components g_c are zero for all c outside a fixed $\sim_I$-equivalence class. We will prove that such vector fields span $\mathcal{F}_G^P$.

Let $Q \subseteq \mathcal{C}$ be a $\sim_I$-equivalence class. Define

$$\mathcal{F}_G^P(Q) = \{f \in \mathcal{F}_G^P : f_s(x) = 0 \quad \forall\, s \notin Q\} \tag{4.7}$$

Vector fields in $\mathcal{F}_G^P(Q)$ are *supported on* Q. The subset $\mathcal{F}_G^P(Q)$ is a linear subspace of $\mathcal{F}_G^P$.

The key constraint on a vector field in $\mathcal{F}_G^P(Q)$ is $B(q,q)$-equivariance for some fixed but arbitrary $q \in Q$. In fact:

Lemma 4.5 *Given a $\sim_I$-equivalence class $Q \subseteq \mathcal{C}$, let $q \in Q$ and let $g_q : P_{I(q)} \to P_q$ be any $B(q,q)$-invariant mapping. Then g_q extends uniquely to a vector field in $\mathcal{F}_G^P(Q)$.*

Proof For any $r \in Q$, choose $\beta_0 \in B(q,r)$ (which exists since $r \sim_I q$) so that $\beta_0(q) = r$. Equivariance forces us to define

$$g_r(y) = g_q(\beta_0^*(y)) \qquad\qquad \forall y \in P_{I(r)} \tag{4.8}$$

so the extension to $r \in Q$ is unique if it exists. It is easy to show that g_r does not depend on the choice of β_0. Finally, if $r \notin Q$ we define $g_r(x) = 0$.

We have now extended g_q to a vector field g on the whole of P. We claim that $g \in \mathcal{F}_G^P(Q)$. Clearly, the components g_r of g with $r \notin Q$ vanish. It is therefore sufficient to show that if $\gamma \in B(r,s)$ and $z \in P_{I(s)}$ then

$$g_s(z) = g_r(\gamma^*(z)) \tag{4.9}$$

The component g_s is defined by choosing $\beta_1 \in B(q,s)$ and setting

$$g_s(y) = g_q(\beta_1^*(y)) \qquad\qquad \forall y \in P_{I(s)} \tag{4.10}$$

To establish (4.9), let

$$\delta = \beta_1^{-1}\gamma\beta_0 \in B(q,q)$$

so that

$$\gamma = \beta_1\delta\beta_0^{-1}$$

Then, using (4.2), we compute:

$$\begin{aligned} g_r(\gamma^*(z)) &= g_r((\beta_1\delta\beta_0^{-1})^*(z)) \\ &= g_r((\beta_0^*)^{-1}\delta^*\beta_1^*(z)) \\ &= g_q(\beta_0^*(\beta_0^*)^{-1}\delta^*\beta_1^*(z)) \\ &= g_q((\delta^*(\beta_1^*(z)) \\ &= g_q(\beta_1^*(z)) \\ &= g_s(z) \end{aligned}$$

(where $\delta^*(\beta_1^*(z)) = \beta_1^*(z)$ because $\beta_1^*(z) \in I(q)$ and g_q is $B(q,q)$-invariant). This calculation proves (4.9). $\square$

The importance of such vector fields g stems from:

Proposition 4.6

$$\mathcal{F}_G^P = \bigoplus_Q \mathcal{F}_G^P(Q)$$

where Q *runs over the* $\sim_I$*-equivalence classes of* G*.*

Proof Suppose that $f \in \mathcal{F}_G^P$, so that f is $\mathcal{B}_G$-equivariant. Let Q be a $\sim_I$-equivalence class, and pick $q \in Q$. Define $g \in \mathcal{F}_G^P(Q)$ by setting

$$g_q(x) = f_q(x) \qquad \forall x \in P$$

which is $B(q,q)$-invariant since f is $\mathcal{B}_G$-equivariant. For the same reason,

$$g_r(x) = f_r(x) \qquad \forall x \in P, r \in Q$$

where g_r is defined as in Lemma 4.5. Recall that $g_s(x) = 0$ for all $s \notin Q$. Repeating this construction for all $\sim_I$-equivalence classes Q we see that

$$\mathcal{F}_G^P = \sum \mathcal{F}_G^P(Q)$$

But the definition of $\mathcal{F}_G^P(Q)$ shows that

$$\mathcal{F}_G^P(Q) \cap \sum_{R \neq Q} \mathcal{F}_G^P(R) = \{0\}$$

(where R ranges over $\sim_I$-equivalence classes other than Q), so the sum is direct. □

5 Patterns of Synchrony: Example

There are many kinds of synchrony in coupled cell systems: for surveys see Boccaletti *et al.* [2] and Wang [15]. Most notions of synchrony depend on specific dynamics of cells and couplings. Some notions are model-independent; that is, they are valid for any vector field consistent with the given cell architecture. It is important to distinguish model-independent properties from model-dependent ones; otherwise model-independent features of a system may inadvertently be used as evidence for specific equations.

We now approach the central issue of this paper: conditions under which certain cells in a coupled cell network can synchronize *as a consequence of the network architecture*. Because the theoretical issues are somewhat abstract, we first discuss a motivating example.

Example 5.1 Consider the ten-cell network G_1 of Figure 11. There are two cell types. Cells 0 and 1 have type ○, cells 2, 3, 4, 5, 6, 7, 8, 9 have type □. There are three arrow types $\longrightarrow, \Longrightarrow, - - \rhd$. The shading on the nodes divides $\mathcal{C}$ into three classes

$$\{0,1\}, \{2,3,6,8\}, \{4,5,7,9\} \tag{5.1}$$

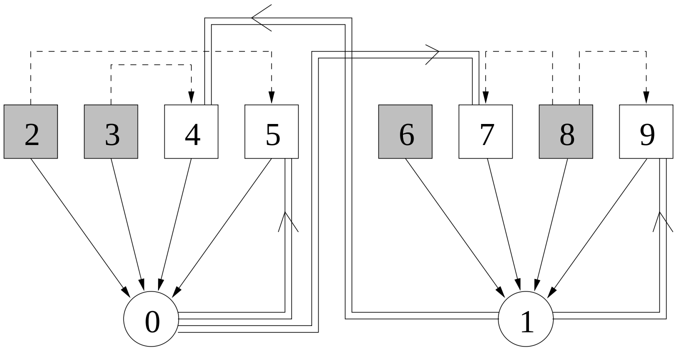

Figure 11: A 10-cell system. The shading indicates a possible pattern of synchronous cells.

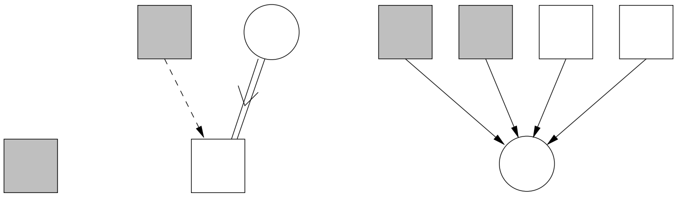

Figure 12: Input types for the 10-cell system, including shading.

There are three distinct input types, illustrated in Figure 12. Cells 2, 3, 6, 8 have no inputs except themselves. Cells 4, 5, 7, 9 have two inputs: one of type ◯, the other of type □ and in the class $\{2, 3, 6, 8\}$. Cells 0, 1 have four inputs, all of type □; of these, two are in the class $\{2, 3, 6, 8\}$ and the other two are in the class $\{4, 5, 7, 9\}$.

With appropriate choice of phase spaces, a vector field $f \in \mathcal{F}^P_{G_1}$ takes the form:

$$\begin{array}{rcl} f_0 & = & A(x_0, \overline{x_2, x_3, x_4, x_5}) \\ f_1 & = & A(x_1, \overline{x_6, x_7, x_8, x_9}) \\ f_2 & = & B(x_2) \\ f_3 & = & B(x_3) \\ f_4 & = & C(x_4, x_1, x_3) \end{array} \qquad \begin{array}{rcl} f_5 & = & C(x_5, x_0, x_2) \\ f_6 & = & B(x_6) \\ f_7 & = & C(x_7, x_0, x_8) \\ f_8 & = & B(x_8) \\ f_9 & = & C(x_9, x_1, x_8) \end{array} \tag{5.2}$$

Consider the space

$$Y = \{(u, u, v, v, w, w, v, w, v, w)\}$$

determined by making entries constant on the classes (5.1). On Y the vector field f restricts to:

$$\begin{array}{rcl} g_0 & = & A(u, \overline{v, v, w, w}) \\ g_1 & = & A(u, \overline{v, w, v, w}) \\ g_2 & = & B(v) \\ g_3 & = & B(v) \\ g_4 & = & C(w, u, v) \end{array} \qquad \begin{array}{rcl} g_5 & = & C(w, u, v) \\ g_6 & = & B(v) \\ g_7 & = & C(w, u, v) \\ g_8 & = & B(v) \\ g_9 & = & C(w, u, v) \end{array} \tag{5.3}$$

By symmetry g_0, g_1 are identical. Bearing this in mind, we see that Y is flow-invariant for f.

Identifying elements of Y with triples (u, v, w) we obtain an induced vector field $\overline{f}$ of the form

$$\begin{array}{rcl} \overline{f}_0 & = & A'(u, \overline{v, w}) \\ \overline{f}_1 & = & B(v) \\ \overline{f}_3 & = & C(w, u, v) \end{array}$$

where

$$A'(u, \overline{v, w}) = A(u, \overline{v, v, w, w})$$

This is the class of admissible vector fields for the simpler coupled cell network G_2 shown in Figure 13. Here cells v, w have the same *type*, but we have shaded cell v to show which equivalence class it corresponds to. ◇

The coupled cell network G_2 is an example of a quotient network. What structure in G_1 makes Y flow-invariant for all $f \in \mathcal{F}^P_{G_1}$, and permits this reduction to G_2 on Y? The key feature is how the three classes (5.1) relate to input isomorphisms. In Section 6 we develop the theory of flow-invariant subspaces, and in Section 8 we develop the general theory of such reductions.

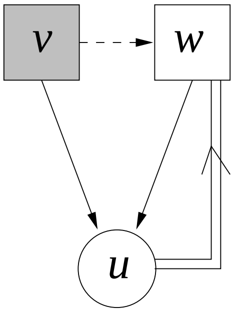

Figure 13: Quotient network G_2 of the 10-cell system G_1. Shading relates cells to those in G_1.

6 Patterns of Synchrony: Theory

We now isolate the abstract features of Example 5.1 that make Y flow-invariant. The classes (5.1) can be represented as the equivalence classes corresponding to an equivalence relation. The properties of this equivalence relation, relative to the symmetry groupoid of the network, turn out to control the existence of the flow-invariant subspace Y and the quotient network G_2. We begin by considering the equivalence relation.

Let $G = (\mathcal{C}, \mathcal{E}, \sim_C, \sim_E)$ be a coupled cell network. Choose a total phase space P, and let $\bowtie$ be an equivalence relation on $\mathcal{C}$, partitioning the cells into equivalence classes. We assume that $\bowtie$ is a refinement of $\sim_{\mathcal{C}}$; that is, if $c \bowtie d$, then c and d have the same cell labels. It follows that the *polydiagonal subspace*

$$\Delta_{\bowtie} = \{x \in P : x_c = x_d \text{ whenever } c \bowtie d \quad \forall c, d \in \mathcal{C}\}$$

is well defined since x_c and x_d lie in the same space $P_c = P_d$. The polydiagonal $\Delta_{\bowtie}$ is a linear subspace of P.

For instance, in Example 5.1 we can define $\bowtie$ to have equivalence classes (5.1), in which case $\Delta_{\bowtie} = Y$.

Definition 6.1 A trajectory $x(t)$ of $f \in \mathcal{F}_G^P$ is $\bowtie$-*polysynchronous* if its components are constant on $\bowtie$-equivalence classes. That is,

$$c \bowtie d \Longrightarrow x_c(t) = x_d(t) \qquad \forall t \in \mathbf{R}$$

or $x(t) \in \Delta_{\bowtie}$ for all $t \in \mathbf{R}$. $\Diamond$

Polysynchronous states are *patterns of synchrony*. Trivially, any trajectory is polysynchronous with respect to the relation of equality (which partitions $\mathcal{C}$ into its individual cells). Only nontrivial polysynchrony is interesting.

Robust Polysynchrony

Definition 6.2 Let $\bowtie$ be an equivalence relation on $\mathcal{C}$. Then $\bowtie$ is *robustly polysynchronous* if $\Delta_{\bowtie}$ is invariant under every vector field $f \in \mathcal{F}_G^P$. That is,

$$f(\Delta_{\bowtie}) \subseteq \Delta_{\bowtie} \quad \forall f \in \mathcal{F}_G^P$$

Equivalently, if $x(t)$ is a trajectory of any $f \in \mathcal{F}_G^P$, with initial condition $x(0) \in \Delta_{\bowtie}$, then $x(t) \in \Delta_{\bowtie}$ for all $t \in \mathbf{R}$. $\Diamond$

We now find necessary and sufficient conditions on $\bowtie$ to ensure that $\bowtie$ is robustly polysynchronous. We begin by showing that robust polysynchrony can occur only between cells that have isomorphic input sets. This is intuitively clear because these are the only cells that involve the 'same' function in the corresponding components of admissible vector fields, and the proof bears out this intuition.

Lemma 6.3 *If $\bowtie$ is robustly polysynchronous, then $\bowtie$ refines $\sim_I$. That is, for all $c, d \in \mathcal{C}$*

$$c \bowtie d \Longrightarrow c \sim_I d$$

Proof By the discussion immediately preceding Lemma 3.7 we need to show that if $c \bowtie d$, then c and d are in the same connected component Q of $\mathcal{B}_G$. Suppose they are not; then we will show that $\Delta_{\bowtie}$ is not flow-invariant. Choose $x(0) \in \Delta_{\bowtie}$ so that $x_c(0) = x_d(0) \neq 0$ and choose $f \in \mathcal{F}_G^P(Q)$ where $d \notin Q$. Let $x(t)$ be the solution to the differential equation f.

Since $f \in \mathcal{F}_G^P(Q)$ implies that f_d vanishes, $x_d(t) = x_d(0)$ for all t. If we can choose f so that $f_c(x(0)) \neq 0$, then $x_c(t) \neq x_c(0)$ for small t, so $\Delta_{\bowtie}$ is not flow-invariant and we are finished.

It remains to choose such an f. By Lemma 4.5 we need only find a $B(c,c)$-invariant mapping $g_c : P_{I(c)} \to P_c$ such that $g_c(x(0)) \neq 0$, since such an invariant mapping extends to a vector field in $\mathcal{F}_G^P(Q)$. For example, we may take $g_c(x) = x_c(0) \neq 0$, which is $B(c,c)$-invariant. □

Balanced Equivalence Relations

In order to motivate our characterization of robustly synchronous equivalence relations $\bowtie$, we repeat the analysis of Example 5.1 with a slightly different equivalence relation. Suppose that we partition the ten cells into the classes

$$\{0,1\}, \{2,3,8\}, \{4,5,6,7,9\}$$

so that the color of cell 6 changes from gray to white. Now the associated polydiagonal is

$$Y' = \{(u,u,v,v,w,w,w,w,v,w)\}$$

The general vector field (5.2) remains unchanged, but its restriction (5.3) to Y changes in just one component: now

$$g_1 = A(u, \overline{w,w,v,w})$$

This is no longer the same as g_0, so no reduction to the three-cell network is possible.

What is the source of this difference? The symmetry property of A (that is, its $B(1,1)$-invariance) implies that the order of the v's and w's does not matter, but there are three occurrences of w in g_1 and only two occurrences in g_0. Similarly there is one occurrence of v in g_1 but there are two occurrences in g_0. This difference in 'multiplicity' makes g_1 differ from g_0, and so destroys the possibility of Y' being an invariant subspace.

This and similar examples lead to the following concept:

Definition 6.4 An equivalence relation $\bowtie$ on $\mathcal{C}$ is *balanced* if for all $c, d \in \mathcal{C}$ with $c \bowtie d$ and $c \neq d$, there exists $\gamma \in B(c,d)$ such that $i \bowtie \gamma(i)$ for all $i \in I(c)$. ◇

In particular, $B(c,d) \neq \emptyset$ implies $c \sim_I d$. Therefore, balanced equivalence relations refine $\sim_I$.

The equivalence relation for Example 5.1 is balanced; the modified equivalence relation is not balanced. It turns out that this is the crucial distinction when it comes to constructing a quotient network: see Theorem 6.5 below.

There is a relatively simple graphical way to test whether a given equivalence relation $\bowtie$ is balanced. Color the cells in a network so that two cells have the same color precisely when they are in the same $\bowtie$-equivalence class. Then $\bowtie$ is balanced if and only if every pair of $\bowtie$-equivalent cells is connected by a color-preserving groupoid element.

For example, consider the seven-cell network in Figure 14. Let $\bowtie$ be the equivalence relation with equivalence classes

$$\{1,4,7\} \quad \{2,5\} \quad \{3,6\}$$

as indicated by colors in Figure 14. Observe that the light gray cells have input sets 'white to light gray', the white cells have input sets 'dark gray to white', and the dark gray cells have input sets 'light gray to dark gray'. So $\bowtie$ is a balanced equivalence relation, since all cells in the same equivalence class have identically colored input sets.

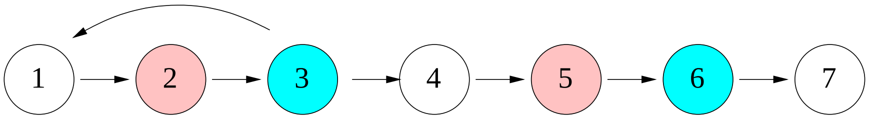

Figure 14: Seven-cell linear network with $\bowtie$-equivalence classes indicated by color.

The Main Theorem on Polysynchrony

An examination of these examples leads to the following general result:

Theorem 6.5 *Let $\bowtie$ be an equivalence relation on a coupled cell network. Then $\bowtie$ is robustly polysynchronous if and only if $\bowtie$ is balanced.*

Proof If $\bowtie$ is balanced, then Definition 6.4 implies that $\Delta_\bowtie$ is invariant under any admissible vector field, that is, $\bowtie$ is robustly polysynchronous. This is obvious in the 'color' interpretation: as we have seen, $\bowtie$ is balanced if and only if every pair of cells of the same color are related by a color-preserving input isomorphism. This implies that if $f \in \mathcal{F}_G^P$ and $c \bowtie d$, then $f_c(x) = f_d(x)$ for all $x \in \Delta_\bowtie$. That is, $\Delta_\bowtie$ is flow-invariant for f.

To prove the converse, suppose that $\bowtie$ is robustly polysynchronous. Then every $f \in \mathcal{F}_G^P$ maps $\Delta_\bowtie$ to $\Delta_\bowtie$. We wish to prove that $\bowtie$ is balanced; that is, if $c \bowtie d$ and $c \neq d$ then there exists $\gamma \in B(c,d)$ such that $i \bowtie \gamma(i)$ for all $i \in I(c)$. Since $\bowtie$ refines $\sim_I$ (Lemma 6.3), the set $B(c,d)$ is nonempty.

Define $K_0, \ldots, K_r$ as in (3.4), so that there is a partition

$$I(c) = K_0 \dot\cup \cdots \dot\cup K_r$$

where i, i' belong to the same K_s if and only if $(i,c) \sim_E (i',c)$. Similarly, there is a partition

$$I(d) = L_0 \dot\cup \cdots \dot\cup L_r$$

with the corresponding property. We may choose the numbering so that

$$i \in K_s, j \in L_s \Longrightarrow (i,c) \sim_E (j,d) \qquad 1 \leq s \leq r \tag{6.1}$$

As before, we may take $K_0 = \{c\}, L_0 = \{d\}$. (Because $c \sim_I d$, the sets K_s and L_s have the same cardinality for $0 \leq s \leq r$, and the same r occurs for $I(c)$ and $I(d)$.)

Suppose that we can prove that for any s with $0 \leq s \leq r$ and any $\bowtie$-equivalence class $U \subseteq \mathcal{C}$:

$$|U \cap K_s| = |U \cap L_s| \tag{6.2}$$

Then we can define a bijection $\gamma : I(c) \to I(d)$ such that

$$\gamma(U \cap K_s) = U \cap L_s \qquad (0 \leq s \leq r) \tag{6.3}$$

for all U. By (6.1), $\gamma \in B(c,d)$. Moreover, (6.3) implies that $\gamma(i) \bowtie i$ for all $i \in I(c)$. For we may take U such that $i \in U$, and then $\gamma(i) \in U$ as well.

Thus it remains to prove the cardinality condition (6.2). To do so, we introduce a $\mathcal{B}_G$-equivariant map h, which depends on s, and apply it to an element $y \in \Delta_{\bowtie}$ that depends on U, as follows.

Let $M : P_i \to P_c$ be a nonzero linear map where $i \in K_s$. Let $h_c : P_{I(c)} \to P_c$ be defined by

$$h_c(x) = M\left(\sum_{i \in K_s} x_i\right) \tag{6.4}$$

which is $B(c,c)$-invariant, since K_s is a $B(c,c)$-orbit. By Lemma 4.5 we may then define, for all other $c' \in \mathcal{C}$:

$$h_{c'}(x) = h_c(\beta^*(x)) = \sum_{j \in \beta(K_s)} M(x_j)$$

where β is some (hence any) element of $B(c,c')$ and $c' \sim_I c$, and

$$h_{c'}(x) = 0$$

otherwise. Moreover, the resulting h is $\mathcal{B}_G$-equivariant. Since $\bowtie$ is polysynchronous, h maps $\Delta_{\bowtie}$ to itself.

Next, define $y \in P$ by

$$y_j = \begin{cases} v & j \in U \\ 0 & j \notin U \end{cases}$$

for some fixed $v \in P_a$ for which $M(v) \neq 0$, where $a \in U$.

We are assuming that $c \sim_I d$ so that $B(c,d) \neq \emptyset$. Let $\beta \in B(c,d)$, which implies that $\beta(K_s) = L_s$. Clearly

$$h_d(x) = \sum_{j \in L_s} M(x_j) \qquad \forall x \in \Delta_{\bowtie} \tag{6.5}$$

Since h preserves $\Delta_{\bowtie}$,

$$h_c(x) = h_d(x) \qquad \forall x \in \Delta_{\bowtie} \tag{6.6}$$

Apply (6.5),(6.6) to y:

$$\begin{aligned} h_c(y) &= |U \cap K_s| M(v) \\ h_d(y) &= |U \cap L_s| M(v) \end{aligned}$$

By (6.6), since $M(v) \neq 0$, we deduce that

$$|U \cap K_s| = |U \cap L_s|$$

for all U and all $0 \leq s \leq r$. But this is (6.2), so $\bowtie$ is balanced. $\square$

7 Dynamics on Polysynchronous Subspaces

As illustrated in Example 5.1 the restriction of a coupled cell vector field to a polysynchronous subspace has itself a special structure. The restriction is an admissible vector field for an associated 'quotient' coupled cell network. In this section we construct the quotient network for a given polysynchronous subspace and illustrate some of the implications for the dynamics of the restriction. We begin with an example.

Example 7.1 Consider the five-cell network illustrated in Figure 15 (left). All cells are identical, so a phase space for this network has the form $P = (\mathbf{R}^k)^5$ for some k. Since all cells are also identical within the network (that is, have isomorphic input sets) the diagonal (x, x, x, x, x) is polysynchronous.

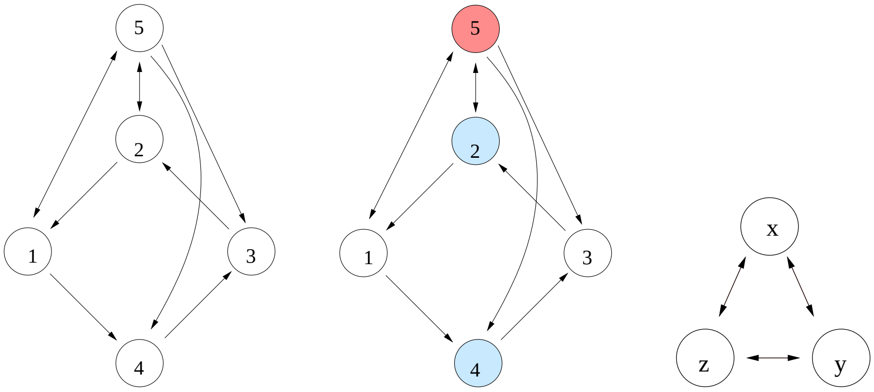

Figure 15: A five-cell identical cell network with a balanced relation leading to a quotient three-cell bidirectional ring.

There is, however, a more interesting $3k$-dimensional polysynchronous subspace $\Delta_{\bowtie}$ associated to the balanced relation illustrated in Figure 15 (center). That subspace is

$$\Delta_{\bowtie} = \{(x, y, x, y, z) : x, y, z \in \mathbf{R}^k\}$$

Next we discuss the structure of the restriction of an admissible coupled cell vector field to $\Delta_{\bowtie}$. The general admissible vector field has the form

$$\begin{array}{rcl} \dot{x}_1 & = & f(x_1, \overline{x_2, x_5}) \\ \dot{x}_2 & = & f(x_2, \overline{x_3, x_5}) \\ \dot{x}_3 & = & f(x_3, \overline{x_4, x_5}) \\ \dot{x}_4 & = & f(x_4, \overline{x_5, x_5}) \\ \dot{x}_5 & = & f(x_5, \overline{x_1, x_2}) \end{array} \qquad (7.1)$$

where $f : (\mathbf{R}^k)^3 \to \mathbf{R}^k$ is symmetric in the last two arguments. The restriction of (7.1) to $\Delta_{\bowtie}$ has the form

$$\begin{array}{rcl} \dot{x} & = & f(x, \overline{y, z}) \\ \dot{y} & = & f(y, \overline{z, x}) \\ \dot{z} & = & f(z, \overline{x, y}) \end{array} \tag{7.2}$$

Observe that (7.2) is the general vector field associated to the three-cell bidirectional ring illustrated in Figure 15 (right). We will show that there is a general construction that leads to this three-cell quotient; but first we discuss some implications for the dynamics of the five-cell system.

Observe also that the restriction (7.2) has $\mathbf{D}_3$ symmetry and is, in fact, the general $\mathbf{D}_3$-equivariant vector field on $(\mathbf{R}^k)^3$. So it is possible for a quotient network to have symmetry even when the original network has none. It is known that when $k \geq 2$ such vector fields can support discrete rotating waves and solutions where two cells are out of phase while the third cell has twice the frequency of the other two [10, 8]. These solutions are also solutions to the original five-cell system. Typical simulations are shown in Figure 16. The middle and right simulations are obtained just by changing initial conditions.

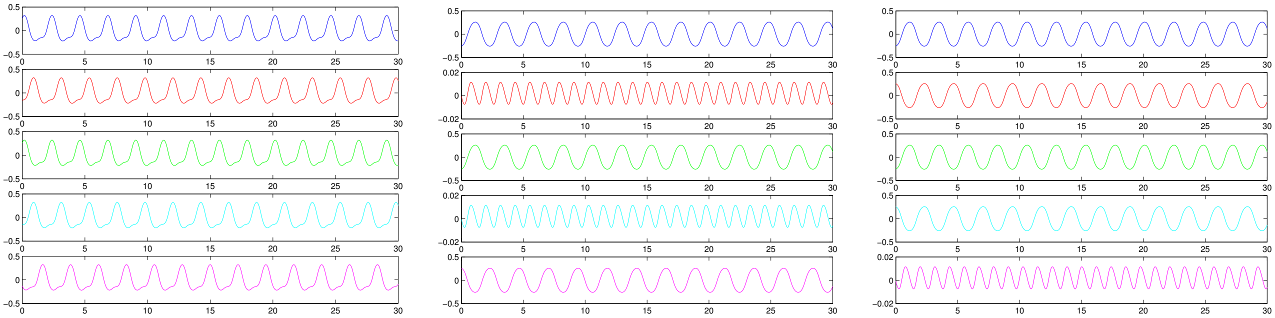

Figure 16: Simulations in five-cell identical cell network in Figure 15. (Left) rotating wave; (middle) double frequency in cells 2 and 4; (right) double frequency in cell 5.

It is also possible for the restricted system to exhibit symmetric chaos, as illustrated in the five-cell simulations in Figures 17 and 18. $\diamond$

Perhaps the simplest example of a network that has no symmetry, but does have a quotient network with symmetry, is the three-cell network in Figure 19. This network is part of the same family of networks as the seven-cell network described in Section 1 (Figure 6), and again in Section 6 (Figure 14).

Construction of the Natural Quotient Network

Let $\bowtie$ be a balanced equivalence relation on a coupled cell network $G = (\mathcal{C}, \sim_C, \mathcal{E}, \sim_E)$. In a series of steps we construct the quotient network $G_{\bowtie}$ corresponding to the polysynchronous subspace $\Delta_{\bowtie}$. To do this we need to define the cells and edges of the quotient network and

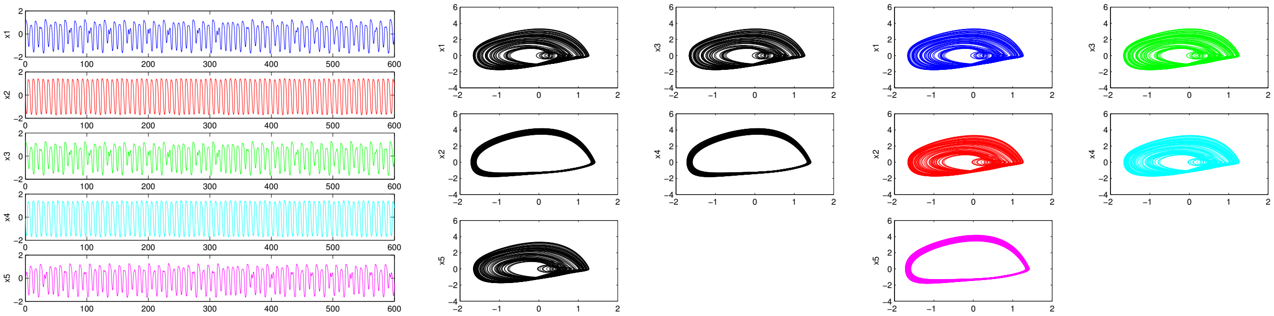

Figure 17: Simulations in five-cell identical cell network in Figure 15. (Left) time series for chaotic attractor with $\mathbf{Z}_2$ symmetry; (middle) phase plane with cells 1,3,5 and cells 2,4 exhibiting symmetry on average; (right) double frequency in cell 5.

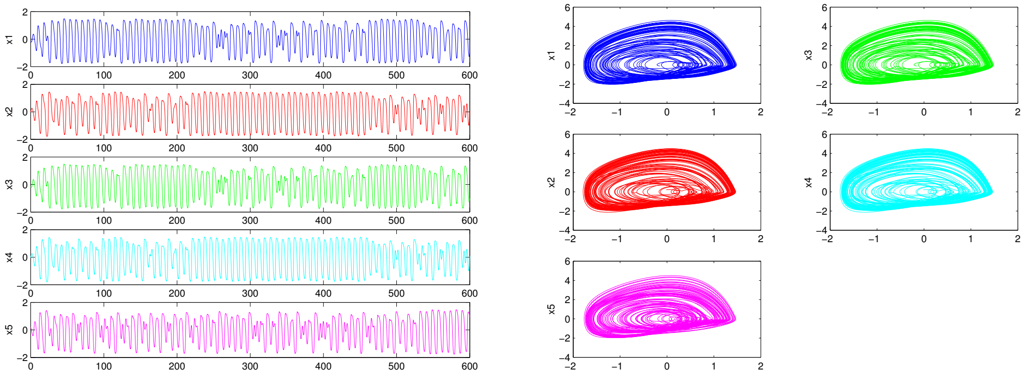

Figure 18: Simulations in five-cell identical cell network in Figure 15. (Left) time series for chaotic attractor with $\mathbf{Z}_3$ symmetry; (right) phase planes with all cells exhibiting symmetry on average.

the equivalence relations on them; that is, we must define $\mathcal{C}_{\bowtie}, \sim_{C_{\bowtie}}, \mathcal{E}_{\bowtie}, \sim_{E_{\bowtie}}$. Most steps are straightforward, but those related to edge-equivalence are more complicated.

(A) Let $\overline{c}$ denote the $\bowtie$-equivalence class of $c \in \mathcal{C}$. The cells in $\mathcal{C}_{\bowtie}$ are the $\bowtie$-equivalence classes in $\mathcal{C}$; that is,

$$\mathcal{C}_{\bowtie} = \{\overline{c} : c \in \mathcal{C}\}$$

Thus we obtain $\mathcal{C}_{\bowtie}$ by forming the *quotient* of $\mathcal{C}$ by $\bowtie$, that is, $\mathcal{C}_{\bowtie} = \mathcal{C}/\bowtie$.

(B) Define

$$\overline{c} \sim_{C_{\bowtie}} \overline{d} \iff c \sim_C d$$

This is well-defined since $\bowtie$ refines $\sim_C$.

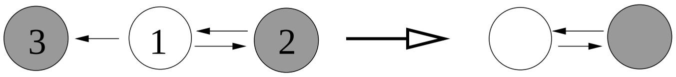

Figure 19: A three-cell network with no symmetry having a quotient two-cell network with $\mathbf{Z}_2$ symmetry.

(C) The edges in the quotient network are the projection of edges in the original network that do not link distinct but $\bowtie$-equivalent cells. That is,

$$\mathcal{E}_{\bowtie} = \{(\bar{i}, \bar{c}) : (i, c) \in \mathcal{E}, i \not\bowtie c\} \cup \{(\bar{c}, \bar{c}) : c \in \mathcal{C}\}$$

(D) We now define $\sim_{E_{\bowtie}}$. Suppose that $(j, d) \in \mathcal{E}_{\bowtie}$, and let $c \in \mathcal{C}$ satisfy $\bar{c} = d$. Define

$$\Omega_c(j) = \{i \in I(c) : \bar{i} = j\} \tag{7.3}$$

Now let $(j_1, d_1), (j_2, d_2) \in \mathcal{E}_{\bowtie}$. We say that

$$(j_1, d_1) \sim_{E_{\bowtie}} (j_2, d_2)$$

if and only if for some $c_1, c_2 \in \mathcal{C}$ with $\overline{c_1} = d_1, \overline{c_2} = d_2$ there exists $\gamma \in B(c_1, c_2)$ such that

$$\gamma(\Omega_{c_1}(j_1)) = \Omega_{c_2}(j_2) \tag{7.4}$$

Remark 7.2 We interrupt our discussion to provide a word picture of the construction of edge equivalence $\sim_{E_{\bowtie}}$. Suppose that the cells of $\mathcal{C}$ are colored by $\bowtie$-equivalence classes, as discussed previously. So every cell in $\mathcal{C}_{\bowtie}$ can be identified with a unique color. The set $\Omega_c(j)$ consists of those cells in the input set $I(c)$ having color j. The edges (j_1, d_1) and (j_2, d_2) are $\sim_{E_{\bowtie}}$-equivalent if there is an input equivalence of $I(c_1)$ to $I(c_2)$ that maps cells of color j_1 to cells of color j_2. In particular, the number of cells in $I(c_1)$ of color j_1 must equal the number of cells in $I(c_2)$ of color j_2.

This completes the construction of $G_{\bowtie} = (\mathcal{C}_{\bowtie}, \mathcal{E}_{\bowtie}, \sim_{C_{\bowtie}}, \sim_{E_{\bowtie}})$, except for one final technical remark. As stated, the definition of $\sim_{E_{\bowtie}}$ appears to depend on the choice of c_1, c_2 in (D). In fact, it does not:

Lemma 7.3 *Suppose that $\bowtie$ is balanced. Let $c_1, c_2, c_1', c_2' \in \mathcal{C}$, where $c_1 \bowtie c_1'$ and $c_2 \bowtie c_2'$. Let $j_1, j_2 \in \mathcal{C}_{\bowtie}$. Suppose that there exists $\beta \in B(c_1, c_2)$ such that*

$$\beta(\Omega_{c_1}(j_1)) = \Omega_{c_2}(j_2)$$

Then there exists $\beta' \in B(c_1', c_2')$ such that

$$\beta'(\Omega_{c_1'}(j_1)) = \Omega_{c_2'}(j_2)$$

Proof Since $\bowtie$ is balanced, there exists (for $k = 1, 2$) an element $\gamma_k \in B(c_k, c'_k)$ such that $\gamma_k(i) \bowtie i$ for all $i \in I(c_k)$. Therefore

$$\gamma_k(\Omega_{c_k}(j_k)) = \Omega_{c'_k}(j_k)$$

Clearly $\beta' = \gamma_2 \beta \gamma_1^{-1}$ is an input isomorphism and by construction $\beta'(\Omega_{c'_1}(j_1)) = \Omega_{c'_2}(j_2)$. $\square$

Lemma 7.3 implies that if (D) holds for some choice of c_1, c_2 satisfying the required conditions, then it holds for any choice of c_1, c_2.

Finally, we show that $G_\bowtie$ is a coupled cell network. To do so, we must verify the compatibility conditions in Definition 2.1(e,f).

(E) If $(j_1, d_1) \sim_{E_\bowtie} (j_2, d_2)$ then $j_1 \sim_{C_\bowtie} j_2$ and $d_1 \sim_{C_\bowtie} d_2$.

Choose $c_1, c_2 \in \mathcal{C}$ such that $\overline{c_1} = d_1$ and $\overline{c_2} = d_2$. The definition of $\sim_{E_\bowtie}$ implies there exists $\gamma \in B(c_1, c_2)$ such that $\gamma(\Omega_{c_1}(j_1)) = \Omega_{c_2}(j_2)$. Since γ is an input isomorphism, it preserves cell type, so $c_1 \sim_C c_2$. But now the definition of $\sim_{C_\bowtie}$ shows that $d_1 \sim_{C_\bowtie} d_2$. Next choose any $i \in \Omega_{c_1}(j_1)$. Then $\gamma(i) \in \Omega_{c_2}(j_2)$, and $i \sim_C \gamma(i)$. Therefore $j_1 \sim_{C_\bowtie} j_2$.

(F) Internal edges are never equivalent to non-internal ones; that is,

$$(j_1, j_1) \sim_{E_\bowtie} (j_2, d_2) \iff j_2 = d_2 \text{ and } j_2 \sim_{C_\bowtie} j_1$$

for all $j_1, j_2, d_2 \in \mathcal{C}_\bowtie$.

We prove $\Longrightarrow$. Assume that $(j_1, j_1) \sim_{E_\bowtie} (j_2, d_2)$ and choose $c_1, c_2 \in \mathcal{C}$ such that $\overline{c_1} = j_1$ and $\overline{c_2} = d_2$. The definition of $\sim_{E_\bowtie}$ implies that there exists $\gamma \in B(c_1, c_2)$ satisfying (7.4). Therefore $\gamma(c_1) = c_2$, so $c_1 \sim_C c_2$. Moreover, $\Omega_{c_1}(j_1) = \{c_1\}$, so $\gamma(\Omega_{c_1}(j_1)) = \gamma(\{c_1\}) = \{c_2\}$. Hence $\Omega_{c_2}(j_2) = \{c_2\}$, so $j_2 = \bar{c}_2 = d_2$.

The converse is obtained by direct calculation.

It remains to prove that the restriction of each G-admissible vector field to $\Delta_\bowtie$ is a $G_\bowtie$-admissible vector field. This result follows from Theorem 9.2, whose proof uses 'quotient maps', which are introduced in Section 8.

Remark 7.4 *On the symmetry groupoid of the natural quotient.*

It is reasonable to ask for a characterization of the symmetry groupoid of the natural quotient $G/\bowtie$ in terms of the symmetry groupoid of G and its relation to $\bowtie$.

Define

$$\begin{aligned} \Sigma^\bowtie(c, d) &= \{\sigma \in B(c, d) : \sigma(i) \bowtie i \qquad \forall i \in I(c)\} \\ \mathrm{T}^\bowtie(c, d) &= \{\tau \in B(c, d) : i \bowtie j \Longrightarrow \tau(i) \bowtie \tau(j) \qquad \forall i, j \in I(c)\} \end{aligned}$$

Then define two subgroupoids of $\mathcal{B}_G$ by:

$$\begin{aligned} \Sigma^\bowtie &= \dot{\bigcup}_{c,d \in \mathcal{C}} \Sigma^\bowtie(c, d) \\ \mathrm{T}^\bowtie &= \dot{\bigcup}_{c,d \in \mathcal{C}} \mathrm{T}^\bowtie(c, d) \end{aligned}$$

Note that $\mathrm{T}^{\bowtie}$ consists precisely of the $\bowtie$-compatible elements of $\mathcal{B}_G$. It follows that $\mathcal{B}_{G/\bowtie}$ consists precisely of the bijections induced on $\mathcal{C}/\bowtie$ by the subgroupoid $\mathrm{T}^{\bowtie}$ of $\mathcal{B}_G$.

Moreover, the elements of $\Sigma^{\bowtie}$ act as the identity on $\mathcal{C}/\bowtie$. In fact, they form the isotropy subgroupoid of any generic element of the polydiagonal $\Delta_{\bowtie}$ (that is, an element $x \in \Delta_{\bowtie}$ such that $x_i = x_j \Leftrightarrow i \bowtie j$). By analogy with the group-symmetric case, we expect $\mathcal{B}_{G/\bowtie}$ to be equal to the quotient groupoid $\mathrm{T}^{\bowtie}/\Sigma^{\bowtie}$. Moreover, $\mathrm{T}^{\bowtie}$ ought to be the 'normalizer groupoid' of $\Sigma^{\bowtie}$ in $\mathcal{B}_G$.

Dias and Stewart [5] prove the above statements. We omit the proofs here because they involve technicalities about quotient groupoids that would take us too far afield. ◇

Remark 7.5 *On the lifting of $G_{\bowtie}$ admissible vector fields.*

In symmetric dynamics the issue of 'hidden symmetry' arises. Here, the restriction of an equivariant vector field onto the fixed-point space of a subgroup Σ is always equivariant under the normalizer of Σ, but sometimes it obeys extra constraints. See [10, 8]. The next example shows that the same issue arises in the groupoid context. In particular, vector fields that are admissible with respect to the quotient network $G_{\bowtie}$ don't always lift to vector fields that are admissible with respect to the original coupled cell network G.

Consider the four-cell network in Figure 20 (left). The equivalence relation $\bowtie$ indicated by color is balanced, and induces a quotient map ϕ to the three-cell network in Figure 20 (right).

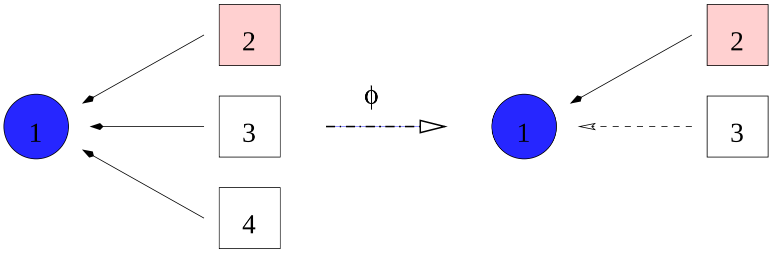

Figure 20: A four-cell example.

Admissible vector fields of the four-cell network have the form

$$\begin{array}{rcl} \dot{x}_1 & = & A(x_1, \overline{x_2, x_3, x_4}) \\ \dot{x}_2 & = & B(x_2) \\ \dot{x}_3 & = & B(x_3) \\ \dot{x}_4 & = & B(x_4) \end{array} \tag{7.5}$$

Admissible vector fields of the three-cell network have the form

$$\begin{array}{rcl} \dot{u}_1 & = & f(u_1, u_2, u_3) \\ \dot{u}_2 & = & g(u_2) \\ \dot{u}_3 & = & g(u_3) \end{array} \tag{7.6}$$

If we identify (x_1, x_2, x_3, x_4) with (u_1, u_2, u_3, u_3) we induce a vector field from (7.5), and we thereby obtain one of the form

$$\begin{array}{rcl} \dot{u}_1 & = & A(u_1, \overline{u_2, u_3, u_3}) \\ \dot{u}_2 & = & B(u_2) \\ \dot{u}_3 & = & B(u_3) \end{array} \tag{7.7}$$

which is admissible by (7.6).

Note that not every $G_{\bowtie}$-admissible vector field (7.6) can be extended to a G-admissible vector field. Compare the linear terms in A (namely, $\alpha u_1 + \beta(u_2 + 2u_3)$) with the linear terms in f in (7.6) (namely, $\alpha u_1 + \beta u_2 + \gamma u_3$). This is a groupoid analog of hidden symmetry, and raises similar issues. Dias and Stewart [5] give a complete groupoid-theoretic characterization of the cases when every $G_{\bowtie}$-admissible vector field extends to a G-admissible vector field. $\diamond$

8 Quotient Maps

In this section we give a formal definition of a quotient map $\phi : G_1 \to G_2$, where G_1 and G_2 are coupled cell networks. The definition is purely graph-theoretic.

Definition 8.1 Let $G_i = (\mathcal{C}_i, \mathcal{E}_i, \sim_{C_i}, \sim_{E_i})$ be coupled cell networks. The map

$$\phi : \mathcal{C}_1 \to \mathcal{C}_2$$

is a *quotient map* if

(a) *Cells lift*: ϕ is surjective.

(b) *Input arrows lift*: If $(i, c) \in \mathcal{E}_1$, then $(\phi(i), \phi(c)) \in \mathcal{E}_2$. Conversely, if $(j, d) \in \mathcal{E}_2$ and $c \in \mathcal{C}_1$ such that $\phi(c) = d$, then there exists $i \in \mathcal{C}_1$ such that $\phi(i) = j$ and $(i, c) \in \mathcal{E}_1$.

(c) *Input isomorphisms lift*: Let $d, d' \in \mathcal{C}_2$ and $\beta_2 \in B(d, d')$. Choose $c, c' \in \mathcal{C}_1$ such that $\phi(c) = d$ and $\phi(c') = d'$. Then there exists $\beta_1 \in B(c, c')$ such that

$$\beta_2(\phi(i)) = \phi(\beta_1(i)) \tag{8.1}$$

for all $i \in I(c)$. $\diamond$

There are several observations that follow directly from the definition of a quotient map $\phi : \mathcal{C}_1 \to \mathcal{C}_2$. Define the equivalence relation $\bowtie_\phi$ on $\mathcal{C}_1$ by

$$c \bowtie_\phi c' \quad \Longleftrightarrow \quad \phi(c) = \phi(c') \tag{8.2}$$

That is, any two cells in $\mathcal{C}_1$ that project by ϕ onto the same cell in $\mathcal{C}_2$ have the same color (that is, are $\bowtie_\phi$-equivalent).

Lemma 8.2 *Let $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ be a quotient map. Then:*

(a) If $\phi(c) = d$ then
$$\phi(I(c)) = I(d) \tag{8.3}$$

(b) For every $c, c' \in \mathcal{C}_1$ such that $\phi(c) = \phi(c')$ there is an input isomorphism $\beta \in B(c, c')$ such that
$$\phi(i) = \phi(\beta(i)) \tag{8.4}$$
for all $i \in I(c)$.

(c) The equivalence relation $\bowtie_\phi$ is balanced.

Proof Part (a) follows directly from Definition 8.1(b). Part (b) follows from Definition 8.1(c) by setting $d = d'$ and $\beta_2 = \mathrm{id}$ on $I(d)$. The existence of $\beta \in B(c, c')$ in Part (b) implies that $c \sim_{I_1} c'$ and hence $c \sim_{C_1} c'$. Using (8.2), identity (8.4) is equivalent to $i \bowtie_\phi \beta(i)$ for all $i \in I(c)$, which is the definition of 'balanced' in Definition 6.4. Thus Part (c) holds. $\square$

Quotient Networks are Examples of Quotient Maps

Theorem 8.3 *Assume that $\bowtie$ is a balanced equivalence relation on $\mathcal{C}$ and let $\mathcal{C}_\bowtie$ be the natural coupled cell network whose cells are the equivalence classes of $\bowtie$. Let $\overline{c}$ denote the $\bowtie$-equivalence class of the cell $c \in \mathcal{C}$. Then the map $\phi : \mathcal{C} \to \mathcal{C}_\bowtie$ defined by $c \mapsto \overline{c}$ is a quotient map.*

Proof We verify that $\phi : \mathcal{C} \to \mathcal{C}_\bowtie$ is a quotient map; that is, we verify Definition 8.1 (a-c).

(a) Cells lift since ϕ is onto by construction.

(b) Input arrows lift by definition. See part (C) in the construction of the network $\mathcal{C}_\bowtie$.

(c) We show that input isomorphisms lift. Recall that Definition 8.1(c) states: For every $d, d' \in \mathcal{C}_\bowtie$, $c, c' \in \mathcal{C}_1$ such that $\overline{c} = d, \overline{c'} = d'$, and $\beta_2 \in B(d, d')$, there exists $\beta_1 \in B(c, c')$ such that $\beta_2(\overline{i}) = \overline{\beta_1(i)}$ for all $i \in I(c)$. So we must construct the input isomorphism $\beta_1 : I(c) \to I(c')$.

We first show that input sets lift; that is, $\overline{I(c)} = I(\overline{c})$ for each $c \in \mathcal{C}$. Suppose that $(j, \overline{c}) \in I(\overline{c})$. Since input arrows lift, there exist $i', c' \in \mathcal{C}$ such that $\overline{i'} = j$, $\overline{c'} = \overline{c}$, and $(i', c') \in \mathcal{E}$. By construction of $\mathcal{C}_\bowtie$, c' and c are $\bowtie$-equivalent. Since $\bowtie$ is balanced, there exists $\gamma \in B(c', c)$ for which $i = \gamma(i') \bowtie i'$. It follows that $\gamma(i', c') = (i, c) \in \mathcal{E}$ and that $\overline{i} = j$. Thus each input arrow in $I(\overline{c})$ lifts to an input arrow in $I(c)$.

Next we show that β_1 exists. The set $\Omega_c(j)$, defined in (7.3), consists of cells in $I(c)$ that are $\bowtie$-equivalent and project onto the node $j \in \mathcal{C}_{\bowtie}$. Therefore, we can choose a finite set J of j such that

$$I(c) = \dot{\bigcup}_{j\in J} \Omega_c(j)$$

Since input sets lift, the existence of β_2 implies that

$$I(c') = \dot{\bigcup}_{j\in J} \Omega_{c'}(\beta_2(j))$$

We construct the permutation β_1 by finding bijections

$$\beta_1|_{\Omega_c(j)} : \Omega_c(j) \to \Omega_{c'}(\beta_2(j))$$

for all $j \in J$, and letting β_1 be their union. The existence of $\beta_2 \in B(d, d')$ implies that $(j, d) \sim_{E_{\bowtie}} (\beta_2(j), d')$. Recall from (7.4) that the definition of $\sim_{E_{\bowtie}}$ implies that there exists $\gamma \in B(c, c')$ such that $\gamma(\Omega_c(j)) = \Omega_c(\beta_2(j))$. Thus $\beta_2(\bar{i}) = \overline{\gamma(i)}$ for all $i \in \Omega_c(j)$. Now set $\beta_1|_{\Omega_c(j)} = \gamma$. □

Non-Uniqueness and Universality

Theorem 8.4 *Assume that $\bowtie$ is a balanced equivalence relation on $\mathcal{C}$ and let $G_{\bowtie}$ be the associated natural quotient network with quotient map ϕ. Then the pair $(G_{\bowtie}, \phi)$ is universal. That is, if G' is a coupled cell network with a quotient map ϕ', then there is a quotient map $\xi : G_{\bowtie} \to G'$ such that $\phi'(\overline{c}) = \xi(\phi(\overline{c})) \quad \forall \overline{c} \in \mathcal{C}_{\bowtie}$.*

In this situation we say that (G', ϕ') *factors through* $(G_{\bowtie}, \phi)$. Note that with the definition of ϕ given in Theorem 8.3, $c \bowtie d$ if and only if $c \bowtie_{\phi} d$.

First, we give an example to show that quotient networks need not be unique. Then we prove Theorem 8.4, which shows that the natural quotient is universal. That is, all other quotient networks are quotients of $G_{\bowtie}$ of a rather trivial kind: distinct cells remain distinct.

Example 8.5 Figure 21 shows three coupled cell networks. The network G is the 7-cell chain of Figure 6. The network $G_{\bowtie}$ is the 3-cell ring of Figure 7 in which all three arrows are equivalent. The network G' is another 3-cell ring, in which the arrows are *not* equivalent. It is easy to see that there exist three quotient maps $\phi : G \to G_{\bowtie}$, $\phi' : G \to G'$, and $\xi : G_{\bowtie} \to G'$, shown by the coloring of the figure. Moreover,

$$\phi'(c) = \xi(\phi(c)) \qquad \forall c \in \mathcal{C}_{\bowtie}$$

Clearly ϕ and ϕ' induce the same equivalence relation on G; that is, $\bowtie_{\phi} = \bowtie_{\phi'}$. However, $G_{\bowtie}$ and G' are not isomorphic.

In fact, there are three other quotient networks with the same equivalence relation. Namely, form a 3-cell ring and define two arrows to be equivalent but the third different. These three networks can be inserted between $G_{\bowtie}$ and G'. ◇

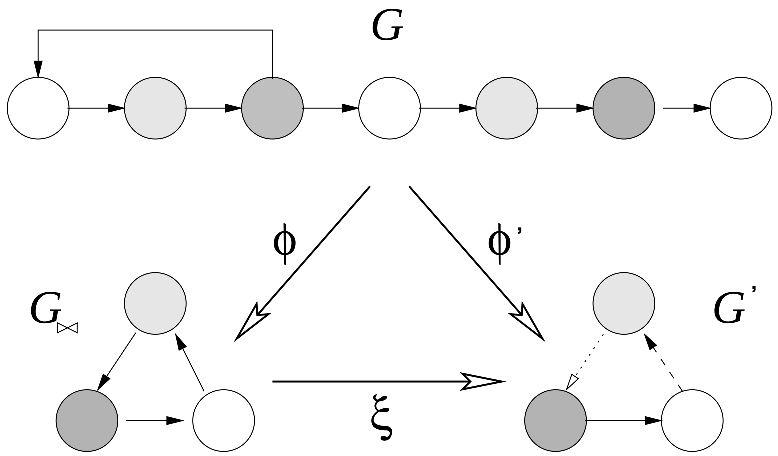

Figure 21: Two distinct quotients with the same equivalence relation.

The essential point now is that Example 8.5 exhibits the only way in which uniqueness fails. The natural quotient $G_{\bowtie}$ defined above is the one in which as many arrows as possible are edge-equivalent. All other quotients are obtained from the natural one by employing the same cells, and refining $\sim_E$.

It will be helpful to introduce the following concept:

Definition 8.6 Let $\gamma : J \to K$ be a bijection between subsets $J, K \subset \mathcal{C}$, and let $\bowtie$ be an equivalence relation on $\mathcal{C}$. Say that γ is $\bowtie$-*compatible* if for all $j_1, j_2 \in J$

$$j_1 \bowtie j_2 \Leftrightarrow \gamma(j_1) \bowtie \gamma(j_2) \tag{8.5}$$

$\diamond$

Essentially, the point here is that γ *permutes* $\bowtie$-equivalence classes. Note that in the definition of 'balanced' we have the stronger condition $i \bowtie \gamma(i)$, in which γ *fixes* $\bowtie$-equivalence classes.

Such maps arise for the following reason. Suppose that $\phi : G \to G'$ is any quotient map of coupled cell networks, and let $\beta \in \mathcal{B}_{G'}$ be an input isomorphism. The definition of 'quotient' requires there to exist a lift $\tilde{\beta} \in \mathcal{B}_G$. The definition of 'lift' clearly implies:

$$\tilde{\beta} \text{ is } \bowtie_\phi \text{-compatible} \tag{8.6}$$

with $J = I(c), K = I(d)$ whence $\tilde{\beta} \in B(c, d)$.

Proof of Theorem 8.4 Suppose that G is a coupled cell network, and $\bowtie$ is a balanced equivalence relation on $\mathcal{C}$. Let $(G_{\bowtie}, \phi)$ be the natural quotient by $\bowtie$, so that $\bowtie_\phi = \bowtie$. Let (G', ϕ') be any quotient network with $\bowtie_{\phi'} = \bowtie$. We claim that ϕ' factors through ϕ.

Define $\xi : G_{\bowtie} \to G'$ as follows. Let $\bar{c} \in \mathcal{C}_{\bowtie}$ be a cell of $G_{\bowtie}$. Define

$$\xi(\bar{c}) = \phi'(c)$$

The map ξ is well-defined because $\bowtie_{\phi'} = \bowtie = \bowtie_{\phi}$. It is a bijection $\xi : \mathcal{C}_{\bowtie} \to \mathcal{C}'$.

We claim that ξ is a quotient map. The defining properties are obvious, except for the condition that input isomorphisms lift from G' to $G_{\bowtie}$. Suppose that $\beta \in \mathcal{B}_{G'}$. Then β lifts from G' to G, yielding an input isomorphism $\tilde{\beta} \in \mathcal{B}_G$. By (8.6), $\tilde{\beta}$ is $\bowtie$-compatible. Therefore it induces a bijection γ on $\mathcal{C}_{\bowtie}$ defined by

$$\gamma(\overline{c}) = \tilde{\beta}(c)$$

The definition of edge-equivalence in the construction of $G_{\bowtie}$ implies that γ is an input isomorphism in $G_{\bowtie}$. Therefore every $\beta \in \mathcal{B}_{G'}$ lifts to some $\gamma \in \mathcal{B}_{G_{\bowtie}}$. □

Several other properties follow directly from this proof. To state them, we need:

Definition 8.7 Let $\xi : G \to G'$ be bijective on cells. Then G' is an *edge-refinement* of G if

$$\xi(i,c) \sim_{E'} \xi(j,d) \quad \Rightarrow \quad (i,c) \sim_E (j,d)$$

◇

Corollary 8.8 *(a) Every quotient network corresponding to a given balanced equivalence relation $\bowtie$ is an edge-refinement of the natural quotient $G_{\bowtie}$.*

(b) Conversely, every edge-refinement of $G_{\bowtie}$ is a quotient network corresponding to $\bowtie$.

(c) Let G', G'' be edge-refinements of $G_{\bowtie}$. Then G'' is an edge-refinement of G' if and only if $\mathcal{B}_{G'} \supseteq \mathcal{B}_{G''}$.

(d) The condition $\mathcal{B}_{G'} \supseteq \mathcal{B}_{G''}$ is equivalent to $\mathcal{F}^P_{G'} \subseteq \mathcal{F}^P_{G''}$ for any choice of phase space P on cells, where cells in G', G'' are identified if they correspond to the same $\bowtie$-class of cells in G.

The above corollary establishes that the phenomena described in Example 8.5 are typical of the general case.

9 Induced Vector Fields are Admissible

Now we come to the second main theorem of this paper. We show that any quotient map $\phi : G_1 \to G_2$ converts G_1-admissible vector fields into G_2-admissible vector fields in a natural way.

The basic idea is the following. Let Δ_ϕ denote the polydiagonal subspace corresponding to the equivalence relation $\bowtie_\phi$ (previously denoted $\Delta_{\bowtie_\phi}$). We claim that the space of G_1-admissible vector fields restricted to Δ_ϕ can be naturally identified with a subspace of the space of G_2-admissible vector fields. The main consequence of this observation is that interesting dynamics (rotating waves, symmetric chaos) in this subspace for the cell system

G_2 corresponds to the same dynamics in the cell system G_1, in which $\bowtie$-equivalent cells are synchronous.

We first choose cell phase spaces P_c for $c \in \mathcal{C}_1$. Then $\phi(c) \in \mathcal{C}_2$, and we let the corresponding cell phase space be $\overline{P}_{\phi(c)} = P_c$. The space $\overline{P}_{\phi(c)}$ is well-defined since quotient maps preserve the relation $\sim_c$.

Choose a set of representatives $\mathcal{R}$ for the map ϕ. That is, $\mathcal{R} \subseteq \mathcal{C}_1$ and for each $d \in \mathcal{C}_2$ there exists a unique $c \in \mathcal{R}$ such that $\phi(c) = d$. Thus the set of all $\phi(c)$ runs through the elements of $\mathcal{C}_2$ without duplication when c runs through $\mathcal{R}$. Then define

$$\overline{P} = \prod_{c \in \mathcal{R}} \overline{P}_{\phi(c)} = \prod_{c \in \mathcal{R}} P_c$$

If $x = (x_c)_{c \in \mathcal{C}_1}$ defines coordinates on P, we can consider $y = (y_{\phi(c)})_{\phi(c) \in \mathcal{C}_2}$ as defining coordinates on $\overline{P}$. Moreover, for each $c \in \mathcal{C}_1$ there exists a unique $r \in \mathcal{R}$ such that $\phi(c) = \phi(r)$, and then $y_{\phi(c)}$ is identified with $y_{\phi(r)}$.

In Section 8 we introduced the notion of a quotient map between coupled cell networks. The key property that we wish to ensure is that a quotient map $\phi : G_1 \to G_2$ induces a natural mapping $\hat{\phi} : \mathcal{F}_{G_1}^{P} \to \mathcal{F}_{G_2}^{\overline{P}}$ where $\overline{P}$ is obtained by identifying the relevant factors of P.

Quotients Preserve Admissibility

We now establish an important property of quotient maps: they induce admissible vector fields.

Suppose that $\phi : G_1 \to G_2$ is a quotient map. There is an injective map $\alpha : \overline{P} \to P$ defined by

$$\alpha(y)_c = y_{\phi(c)} \qquad \forall c \in \mathcal{C}_1, y \in \overline{P} \tag{9.1}$$

Note that $\Delta_\phi = \alpha(\overline{P})$, so $\alpha : \overline{P} \to \Delta_\phi$ is a bijection. Replacing y by $\alpha^{-1}x$, for $x \in \Delta_\phi$, equation (9.1) becomes:

$$(\alpha^{-1}x)_{\phi(c)} = x_c \qquad \forall c \in \mathcal{C}_1 \tag{9.2}$$

Definition 9.1 Since $f \in \mathcal{F}_{G_1}^{P}$ leaves Δ_ϕ invariant, we can define a vector field $\overline{f}$ on $\overline{P}$, the *induced vector field*, by restricting f to Δ_ϕ and projecting the result onto $\overline{P}$ by α^{-1}. More precisely,

$$\overline{f}(y) = \alpha^{-1}(f(\alpha(y))) \qquad \forall y \in \overline{P} \tag{9.3}$$

We will also denote $\overline{f}$ by $\hat{\phi}(f)$. ◇

The main result of this section is:

Theorem 9.2 *For any $f \in \mathcal{F}_{G_1}^{P}$, the induced vector field $\overline{f}$ lies in $\mathcal{F}_{G_2}^{\overline{P}}$.*

Proof Since $f \in \mathcal{F}^P_{G_1}$ is G_1-admissible, it satisfies the two conditions of Definition 4.1: the domain condition (4.3) and the equivariance condition (4.5).

The domain condition states: For all $c \in \mathcal{C}_1$ there exists a function $\hat{f}_c : P_{I(c)} \to P_c$ such that

$$f_c(x) = \hat{f}_c(x_{I(c)}) \tag{9.4}$$

Equivalently, f_c depends only on variables from cells in $I(c)$.

$\mathcal{B}_{G_1}$-equivariance states: For all $c, c' \in \mathcal{C}_1$ and for all $\gamma \in B(c, c')$ we have

$$\hat{f}_{c'}(x) = \hat{f}_c(\gamma^*(x)) \qquad \forall x \in P_{I(c')} \tag{9.5}$$

where

$$(\gamma^*(x))_i = x_{\gamma(i)} \qquad \forall i \in I(c) \tag{9.6}$$

and is undefined elsewhere.

We must verify Definition 4.1. That is, we must show that:

(a) The vector field $\overline{f}$ satisfies the domain condition for G_2.

(b) The vector field $\overline{f}$ satisfies the equivariance condition for G_2.

To prove (a), let $d \in \mathcal{C}_2$, and suppose that $\phi(c) = d$. We must show that $\overline{f}_d(y)$ depends only on $y_{I(d)}$, for $y \in \overline{P}$. We have:

$$\begin{array}{rcll} \overline{f}_d(y) & = & \overline{f}_{\phi(c)}(y) & \\ & = & (\overline{f}(y))_{\phi(c)} & \\ & = & (\alpha^{-1}(f(\alpha(y))))_{\phi(c)} & \text{by (9.3)} \\ & = & (f(\alpha(y)))_c & \text{by (9.2)} \\ & = & f_c(\alpha(y)) & \text{by definition} \end{array}$$

By (9.4), the value of $f_c(\alpha(y))$ depends only on $(\alpha(y))_{I(c)}$. But by (9.1):

$$(\alpha(y))_{I(c)} = y_{\phi(I(c))} = y_{I(d)}$$

since $\phi(I(c)) = I(d)$ by (8.3). This proves (a).

We interrupt this proof to verify a lemma.

Lemma 9.3 *Let $d, d' \in \mathcal{C}_2$ and let $\beta \in B(d, d')$. Choose $c, c' \in \mathcal{C}_1$ such that $\phi(c) = d, \phi(c') = d'$. Suppose that β lifts to $\tilde{\beta} \in B(c, c')$. Then for all $y \in \overline{P}$ we have*

$$\tilde{\beta}^*(\alpha(y)) = \alpha(\beta^*(y)) \tag{9.7}$$

Proof For all $i \in I(c)$ we have

$$\begin{array}{rcll} (\tilde{\beta}^*(\alpha(y)))_i & = & (\alpha(y))_{\tilde{\beta}(i)} & \text{by (9.6)} \\ & = & y_{\phi(\tilde{\beta}(i))} & \text{by (9.1)} \\ & = & y_{\beta(\phi(i))} & \text{by (8.1)} \\ & = & (\beta^*(y))_{\phi(i)} & \text{by (9.6)} \\ & = & (\alpha(\beta^*(y)))_i & \text{by (9.1)} \end{array}$$

which proves (9.7). $\square$

Now we return to the proof of Theorem 9.2. To prove (b) we must show that for all $d, d' \in \mathcal{C}_2$ and $\beta \in B(d, d')$,

$$\overline{f}_{d'}(y) = \overline{f}_d(\beta^*(y)) \qquad \forall y \in P_{I(d')} \tag{9.8}$$

where by definition

$$(\beta^*(y))_i = y_{\beta(i)}$$

Choose $c, c' \in \mathcal{C}_1$ such that $\phi(c) = d, \phi(c') = d'$. Use Definition 8.1(c) to lift $\beta : I(d) \to I(d')$ to $\tilde{\beta} : I(c) \to I(c')$. That is, by (8.1):

$$\beta(\phi(i)) = \phi(\tilde{\beta}(i)) \qquad \forall i \in I(c)$$

We know that

$$f_{c'}(x) = f_c(\tilde{\beta}^*(x)) \qquad \forall x \in P$$

Therefore, setting $x = \alpha(y)$, we have

$$f_{c'}(\alpha(y)) = f_c(\tilde{\beta}^*(\alpha(y))) \qquad \forall y \in \overline{P} \tag{9.9}$$

By Lemma 9.3,

$$f_{c'}(\alpha(y)) = f_c(\alpha(\beta^*(y))) \tag{9.10}$$

Now, by definition (9.3), $\overline{f}(y) = \alpha^{-1}(f(\alpha(y)))$. Therefore

$$\begin{array}{rcll} \overline{f}_{d'}(y) & = & (\alpha^{-1}(f(\alpha(y))))_{d'} & \\ & = & (\alpha^{-1}(f(\alpha(y))))_{\phi(c')} & \\ & = & (f(\alpha(y)))_{c'} & \text{by (9.2)} \\ & = & f_{c'}(\alpha(y)) & \end{array}$$

Similarly,

$$\begin{array}{rcll} \overline{f}_d(\beta^*(y)) & = & (\overline{f}(\beta^*(y)))_d & \\ & = & (\alpha^{-1}(f(\alpha(\beta^*(y)))))_d & \\ & = & (f(\alpha(\beta^*(y))))_c & \text{by (9.2)} \\ & = & f_c(\alpha(\beta^*(y))) & \\ & = & f_c(\tilde{\beta}^*(\alpha(y))) & \text{by (9.7)} \end{array}$$

and the result follows from (9.9). $\square$

Theorem 9.2 is valid for all of the quotients In Example 8.5. However, it is clear that $\mathcal{B}_{G_{\bowtie}}$ is a proper subset of $\mathcal{B}_{G'}$. Therefore $\mathrm{im}(\hat{\phi}) = \mathrm{im}(\hat{\phi}') \subseteq \mathcal{B}_{G_{\bowtie}} \subset \mathcal{B}_{G'}$, so we gain more information about induced vector fields $\overline{f}$ and their lifts f if we work with $(G_{\bowtie}, \phi)$ rather than (G', ϕ').

Note that Example 7.5 is the natural quotient, so $\hat{\phi}$ need not be surjective when ϕ is natural. It is never surjective when ϕ is not the natural quotient map.

10 Interior Symmetries and Local Bifurcation Theory

Finally, we discuss some special types of local bifurcation in coupled cell systems. In symmetric dynamics, there are two main local bifurcation theorems. The Equivariant Branching Lemma ([10] Chapter XIII Section 3) proves the existence of certain branches of symmetry-breaking steady-state solutions; the Equivariant Hopf Theorem ([10] Chapter XVI Section 4) proves the existence of certain branches of spatio-temporal symmetry-breaking time-periodic solutions. In this section we prove that both of these theorems generalize to coupled cell systems in a groupoid-equivariant setting. The analogue of the Equivariant Branching Lemma is a natural generalization of the symmetric case, but the analogue of the Equivariant Hopf Theorem has novel features. In particular, instead of proving the existence of states with certain spatio-temporal symmetries, we prove the existence of states that on certain subsets of cells are superpositions of synchronous states with states having spatio-temporal symmetries.

The main concept involved in both bifurcation theorems is the notion of an 'interior symmetry'. Roughly, this is a permutation of cells that preserves a certain amount of input structure. Interior symmetries are related to the symmetry groupoid, but in general do not belong to it.

Recall that in general a *symmetry* γ of a differential equation

$$\dot{x} = f(x)$$

with phase space $X = \mathbf{R}^k$ is a linear map $x \mapsto \gamma x$ on X that commutes with f; that is, γ satisfies

$$f(\gamma x) = \gamma f(x) \qquad \forall x \in X$$

Let x_0 be an equilibrium that is fixed by γ. Then the Jacobian matrix of f at x_0 commutes with γ; that is,

$$(df)_{x_0}\gamma = \gamma(df)_{x_0}$$

This commutativity can force $(df)_{x_0}$ to have multiple eigenvalues [10] and is the source of many of the complications in equivariant bifurcation theory. However, it can also help resolve those complications. Similar problems arise in the context of interior symmetries, and again the structure that causes them can sometimes be used to resolve them.

Throughout this section we work with a coupled cell network G on a set $\mathcal{C}$ of N nodes, and use the same notation as before. In particular, f is a G-admissible vector field on a phase space P. The symmetries that concern us here are permutations of the cells.

First, we relate the symmetry group of a coupled cell network to its symmetry groupoid. Suppose that a permutation σ is a symmetry of the coupled cell system. Then it is easy to show that for each cell j,

$$\sigma|_{I(j)} : I(j) \to I(\sigma(j)) \quad \text{is an input isomorphism.} \tag{10.1}$$

Moreover, it is easy to see that the converse is also true. That is, if σ satisfies (10.1) for every cell j, then σ is a symmetry. The proof is a direct application of $\mathcal{B}_G$-equivariance.

We now introduce an important concept for the local bifurcation theory of coupled cell networks:

Definition 10.1 Let $\mathcal{S} \subseteq \mathcal{C}$ be a subset of cells, and let σ be a permutation of $\mathcal{C}$ that is the identity on the complement $\mathcal{C} \setminus \mathcal{S}$ of $\mathcal{S}$. Then σ is an *interior symmetry* on $\mathcal{S}$ if σ satisfies (10.1) for every $j \in \mathcal{S}$.

The *interior symmetry group* $\Sigma_{\mathcal{S}}$ is the set of all interior symmetries on $\mathcal{S}$. It is obviously a group. ◇

The interior symmetry group $\Sigma_{\mathcal{C}}$ on the whole network $\mathcal{C}$ is the usual symmetry group Γ of the entire coupled cell system.

The simplest example of a network with a nontrivial interior symmetry is the three-cell network of Figure 3, discussed in the introduction. Because of the (dotted) arrow from cell 1 to cell 3, the permutation $(1\ 2) \in \mathbf{S}_3$ is not a group symmetry of the network, but it is an interior symmetry on the subset $\mathcal{S} = \{1, 2\}$.

Groupoid equivariance of the cell system implies that

$$f_{\mathcal{S}}(\sigma x_{\mathcal{S}}, x_{\mathcal{C} \setminus \mathcal{S}}) = \sigma f_{\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{C} \setminus \mathcal{S}}) \tag{10.2}$$

where $f_{\mathcal{S}}$ is the cell system vector field on the cells in $\mathcal{S}$ and $\sigma \in \Sigma_{\mathcal{S}}$.

Suppose that $\mathrm{T} \subset \Sigma_{\mathcal{S}}$ is a subgroup. Then

$$\mathrm{Fix}(\mathrm{T}) = \{(x_{\mathcal{S}}, x_{\mathcal{C} \setminus \mathcal{S}}) \in P : \delta x_{\mathcal{S}} = x_{\mathcal{S}} \quad \forall \delta \in \mathrm{T}\} \tag{10.3}$$

Proposition 10.2 *Let* T *be a subgroup of* $\Sigma_{\mathcal{S}}$*, and let* f *be a* G*-admissible vector field. Then the subspace* $\mathrm{Fix}(\mathrm{T})$ *is flow-invariant for* f*.*

Proof The set consisting of $\delta|_{I(j)}$ for all $j \in \mathcal{S}$ and $\delta \in \mathrm{T}$, together with the identity elements in $I(j)$ for all $j \notin \mathcal{S}$, is a subgroupoid of $\mathcal{B}_G$ whose fixed-point subspace is $\mathrm{Fix}(\mathrm{T})$. Now apply (10.2) and (10.3). **CHANGE PROOF — NO SUBGROUPOIDS** □

Linear Theory

We discuss steady-state and Hopf bifurcations from equilibria in $\mathrm{Fix}(\Sigma_{\mathcal{S}})$ that break interior symmetry.

The action of the group $\Sigma_{\mathcal{S}}$ decomposes $\mathcal{S}$ as

$$\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_k$$

where each $\mathcal{S}_j$ is an orbit of the action. Let

$$W = \{x \in P : x_j = 0 \quad \forall j \in \mathcal{C} \setminus \mathcal{S} \quad \text{and} \quad \sum_{i \in \mathcal{S}_\ell} x_i = 0 \quad \text{for } 1 \leq \ell \leq k\} \tag{10.4}$$

Note that W is a $\Sigma_{\mathcal{S}}$-invariant subspace.

We can write the state space P as

$$P = W \oplus \mathrm{Fix}(\Sigma_{\mathcal{S}}) \tag{10.5}$$

In particular, (10.4) implies that vectors in W, when written in coupled cell coordinates, have zero components on all cells not in $\mathcal{S}$.

Bifurcation theory concerns changes in solutions of an ODE as parameters are varied, so we introduce an explicit bifurcation parameter $\lambda \in \mathbf{R}$. We assume that f (hence also its components f_c) depend explicitly on λ, and that the ODE

$$\dot{x} = f(x, \lambda) \tag{10.6}$$

has a 'trivial' equilibrium x_0. In the present context, we may assume without loss of generality that

$$f(x_0, \lambda) \equiv 0$$

and that the bifurcation occurs at $\lambda = 0$. Let $L = (df)_{x_0}$. Proposition 10.2 implies that $\mathrm{Fix}(\Sigma_{\mathcal{S}})$ is invariant under L, so that L has the block form

$$L = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \tag{10.7}$$

with respect to the decomposition (10.5); that is, $A : W \to W$ and $B : \mathrm{Fix}(\Sigma_{\mathcal{S}}) \to \mathrm{Fix}(\Sigma_{\mathcal{S}})$. Thus the eigenvalues of L are the eigenvalues of A, together with those of B.

Local bifurcation (steady-state or Hopf) occurs when some eigenvalue of L has zero real part. That eigenvalue is either associated with A or with B, and it is the former case that concerns us here. We say that f undergoes a bifurcation at x_0 that *breaks interior symmetry* if A has an eigenvalue with zero real part. In this case, steady-state bifurcation occurs when A has a zero eigenvalue, and Hopf bifurcation occurs when A has a conjugate pair of purely imaginary eigenvalues. In the Hopf case, we may assume (after rescaling time if necessary) that the purely imaginary eigenvalues of A are $\pm i$. We may also assume that the center subspace $E(A)$ is equal to $\ker A$ in steady-state bifurcation, and to the real eigenspace

$$E(A) = \{x \in P : (A^2 + 1)x = 0\}$$

in Hopf bifurcation.

The structure of L in (10.7) has several important implications:

Lemma 10.3 *(a) A commutes with the action of $\Sigma_{\mathcal{S}}$ on W.*

(b) A vector $u \in \mathrm{Fix}(\Sigma_{\mathcal{S}})$ is an eigenvector of B with eigenvalue μ if and only if u is an eigenvector of L with eigenvalue μ.

(c) If $w \in W$ is an eigenvector of A with eigenvalue μ, then there exists an eigenvector v of L with eigenvalue μ of the form

$$v = w + u$$

where $u \in \mathrm{Fix}(\Sigma_{\mathcal{S}})$.

(d) The center subspace $E(A)$ is $\Sigma_{\mathcal{S}}$-invariant.

Proof Part (a) follows from Proposition 10.2 and the chain rule. Parts (b,c) are consequences of the block form of L. Part (d) follows because part (a) implies that all eigenspaces of A are $\Sigma_{\mathcal{S}}$-invariant. □

In our bifurcation studies we shall assume:

1) Critical eigenvalues μ (0 in steady-state bifurcation and $\pm i$ in Hopf bifurcation) extend uniquely and smoothly to eigenvalues $\mu(\lambda)$ for λ near 0.

2) The *eigenvalue crossing condition*

$$\frac{d}{d\lambda}\mathrm{Re}(\mu)(0) \neq 0 \tag{10.8}$$

is valid.

Interior Symmetry Branching Lemma

Recall [6, 8] that an *axial* subgroup of a group action is an isotropy subgroup whose fixed-point subspace is one-dimensional. We can now prove a generalization to coupled cell systems of the Equivariant Branching Lemma:

Theorem 10.4 *Assume that* $\ker L$ *and* $\ker A$ *have the same dimension, and that* $\ker A$ *is* $\Sigma_{\mathcal{S}}$*-absolutely irreducible. Let* $\mathrm{T} \subset \Sigma_{\mathcal{S}}$ *be an axial subgroup of the action of* $\Sigma_{\mathcal{S}}$ *on* $\ker A$*, and assume the eigenvalue crossing condition (10.8). Then there exists a unique branch of equilibria, bifurcating from* $(x_0, 0)$*, with* T *symmetry.*

Proof By Proposition 10.2, $\mathrm{Fix}(\mathrm{T})$ is flow-invariant. Since T is an axial subgroup, $\ker A \cap \mathrm{Fix}(\mathrm{T})$ is one-dimensional and the bifurcation of $f|_{\mathrm{Fix}(\mathrm{T})\times\mathbf{R}}$ is a simple eigenvalue bifurcation. Moreover, since $f|_{\mathrm{Fix}(\Sigma_{\mathcal{S}})\times\mathbf{R}}$ is nonsingular at $\lambda = 0$, there is a unique branch of trivial solutions to $f = 0$ in $\mathrm{Fix}(\Sigma_{\mathcal{S}}) \times \mathbf{R}$. By the usual implicit function theorem argument, the eigenvalue crossing condition implies the existence of a unique branch of nontrivial solutions to $f = 0$ in $\mathrm{Fix}(\mathrm{T}) \times \mathbf{R}$. □

Remark 10.5 Liapunov-Schmidt reduction ([7] Chapter I Section 3) applied to the equation $f = 0$ yields a reduced equation $g = 0$, where

$$g : \ker L \times \mathbf{R} \to \ker L$$

whose solutions are in one-to-one correspondence with the solutions of $f = 0$ near the origin. Even though Theorem 10.4 proves the simultaneous existence of branches of solutions to $g = 0$ corresponding to each axial subgroup T, the reduced mapping g satisfies no obvious symmetry constraints. In particular:

1) The group $\Sigma_{\mathcal{S}}$ does not act (naturally) on $\ker L$, even though it does act on $\ker A$.

2) The bifurcating branches can be transcritical, even when the symmetry condition $N_{\Sigma_{\mathcal{S}}}(\mathrm{T})/\mathrm{T} \cong \mathbf{Z}_2$ (where $N_{\Sigma_{\mathcal{S}}}(\mathrm{T})$ is the normalizer of T in $\Sigma_{\mathcal{S}}$) would imply that generic equivariant bifurcations are pitchforks.

An example illustrating the second point is the 3-cell network of Figure 3, where we assume that each cell has one-dimensional internal dynamics. In this example, $\mathcal{S} = \{1, 2\}$ and $\Sigma_{\mathcal{S}} = \mathbf{Z}_2(1\ 2)$. Generically, the bifurcating branch is transcritical. To ensure this, it is enough to assume that the coupling from cell 1 to cell 3 is nonzero at linear level. $\diamond$

Review of the Equivariant Hopf Theorem

We now generalize the Equivariant Hopf Theorem ([10] Chapter XVI) to the context of interior symmetries of coupled cell systems. We begin this process by recalling the Equivariant Hopf Theorem.

Let $x(t)$ be a 2π-periodic solution of a system of ODEs with symmetry group Γ. A *spatio-temporal* symmetry of $x(t)$ is a pair $(\gamma, \theta) \in \Gamma \times \mathbf{S}^1$ such that

$$\gamma x(t) = x(t + \theta)$$

Here $\mathbf{S}^1$ is the circle group of phase shifts modulo the period. The group of (spatio-temporal) symmetries of a periodic solution is a subgroup $\Delta \subseteq \Gamma \times \mathbf{S}^1$. The subgroup $K = \Delta \cap \Gamma$ consists of the purely spatial symmetries of $x(t)$. It is known that K must be an isotropy subgroup of the Γ-action [10, 4, 8].

Consider now the context of Γ-equivariant vector fields $F : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ where Γ is a (compact Lie) group, so that

$$F(\gamma x, \lambda) = \gamma F(x, \lambda) \qquad \forall \gamma \in \Gamma$$

Assume that F has a Γ-invariant equilibrium x_0 for all λ, that is,

$$F(x_0, \lambda) \equiv 0$$

Suppose that x_0 undergoes a Hopf bifurcation at $\lambda = 0$. After rescaling time, we may suppose that

$$L = (d_x F)_{x_0}$$

has eigenvalues that include $\pm i$ when $\lambda = 0$. Note that L commutes with Γ since x_0 is a Γ-invariant equilibrium.

There is a natural action of $\mathbf{S}^1$ on $E(L)$ given by the exponential $\exp(sL)$, and this action commutes with Γ since L commutes with Γ. Recall [6, 8] that an isotropy subgroup $\Delta \subset \Gamma \times \mathbf{S}^1$ is $\mathbf{C}$-*axial* if

$$\dim \mathrm{Fix}_{E(L)}(\Delta) = 2$$

Suppose that the eigenvalue μ that extends i at $\lambda = 0$ crosses the imaginary axis with nonzero speed as λ varies. Then the Equivariant Hopf Theorem states that for each **C**-axial subgroup Δ, there is a branch of periodic solutions emanating from $(x_0, 0)$ with spatio-temporal symmetry group Δ. The proof uses Liapunov-Schmidt reduction in the context of loop spaces: see [10] Chapter XIV Section 4.

We now prove a generalization: for each **C**-axial subgroup Δ of the action of a group of interior symmetries on $\mathcal{C}$, there is also a branch of periodic solutions emanating from this bifurcation point, whose structure is related to Δ, but in a less straightforward way.

Hopf Theorem with Interior Symmetry

We pursue the same proof strategy to derive an analog of the Equivariant Hopf Theorem for interior symmetries. Let G be a coupled cell network, let $\mathcal{S} \subseteq \mathcal{C}$, and suppose that there is an interior symmetry group $\Sigma_{\mathcal{S}}$. Let f be a G-admissible vector field. In general, f is not $\Sigma_{\mathcal{S}}$-equivariant, and L does not commute with $\Sigma_{\mathcal{S}}$. However, the block matrix A defined in (10.7) does commute with $\Sigma_{\mathcal{S}}$, so there is a natural $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$-action on $E(A)$, where $\mathbf{S}^1$ acts by $\exp(sA)$. Let $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be a subgroup. The *spatial* symmetry group in Δ is $K = \Delta \cap \Sigma_{\mathcal{S}}$.

Definition 10.6 The subgroup $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ is *spatially* **C**-*axial* if

$$\dim \mathrm{Fix}_{E_i(A)}(\Delta) = \dim \mathrm{Fix}_{E_i(A)}(K) = 2 \tag{10.9}$$

We require the following concept:

Definition 10.7 The ODE (10.6) undergoes a *synchrony-breaking Hopf bifurcation* at x_0 when $\lambda = 0$ if:

(a) After rescaling time, the linearization $L = (df)_{x_0}$ at $\lambda = 0$ has eigenvalues $\pm i$ coming from A, as defined in Lemma 10.3(c).

(b) The eigenvalue crossing condition (10.8) holds. ◇

We use the term 'synchrony-breaking' because the equilibrium state is assumed to be in $\mathrm{Fix}(\Sigma_{\mathcal{S}})$ (and hence synchronous on each $\Sigma_{\mathcal{S}}$-orbit in $\mathcal{S}$), whereas the Hopf bifurcation critical eigenvectors (corresponding to a critical eigenvalue of A in (10.7)) are assumed to be transverse to $\mathrm{Fix}(\Sigma_{\mathcal{S}})$ (which leads to periodic solutions not in $\mathrm{Fix}(\Sigma_{\mathcal{S}})$ and hence with less synchrony).

The generalization of the Equivariant Hopf Theorem is:

Theorem 10.8 *Consider the coupled cell system (10.6). Let $\mathcal{S}$ be a subset of cells with interior symmetry group $\Sigma_{\mathcal{S}}$, and let $x_0 \in \mathrm{Fix}(\Sigma_{\mathcal{S}})$ be an equilibrium of f. Assume that a synchrony-breaking Hopf bifurcation occurs at x_0 when $\lambda = 0$. Let $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be a spatial* **C**-*axial subgroup. Then generically there exists a family of periodic solutions of (10.6), bifurcating from $(x_0, 0)$ and having period near 2π, that is synchronous on any two cells in*

$\mathcal{S}$ lying in the same K-orbit. Moreover, to lowest order in the bifurcation parameter λ, the solution $x(t)$ is of the form

$$x(t) \approx u(t) + w(t) \tag{10.10}$$

where $u(t) = e^{tL}u_0$ is synchronous on $\Sigma_{\mathcal{S}}$ group orbits of cells in $\mathcal{S}$ and $w(t) = e^{tL}v_0$ has exact Δ spatio-temporal symmetries on cells in $\mathcal{S}$.

Proof We adapt the proof of the Equivariant Hopf Theorem [10], which uses Liapunov-Schmidt reduction, to the context of a synchrony-breaking Hopf bifurcation that breaks interior symmetry. Let $\mathcal{C}_{2\pi}(P)$ be the loop space consisting of all continuous 2π-periodic functions from $\mathbf{S}^1$ into P, with the C^0 norm. The Hopf Theorem concerns periodic solutions to differential equations near a point where the Jacobian matrix has purely imaginary eigenvalues. We have rescaled time so that those purely imaginary eigenvalues are $\pm i$, so we look for periodic solutions with period near 2π. By introducing a perturbed period parameter τ we can rescale time again, from t to $s = (1 + \tau)t$, and consider 2π-periodic solutions to the equation

$$\mathcal{F}(v(s), \lambda, \tau) \equiv (1 + \tau)\frac{dv}{ds} - f(v(s), \lambda) = 0 \tag{10.11}$$

These solutions are zeros near the trivial equilibrium $(0, 0, 0)$ of the mapping

$$\mathcal{F} : \mathcal{C}^1_{2\pi}(P) \times \mathbf{R} \times \mathbf{R} \to \mathcal{C}_{2\pi}(P)$$

defined in (10.11), where $\mathcal{C}^1_{2\pi}(P)$ consists of the continuously differentiable functions in $\mathcal{C}_{2\pi}(P)$ with the C^1 norm.

The linearization of $\mathcal{F}$ about the origin is

$$\mathcal{L}(v(s)) = \frac{dv}{ds} - Lv(s)$$

and $\ker(\mathcal{L})$ consists of all functions $v(s) = \mathrm{Re}(e^{is}v)$ where v is an eigenvector of L associated to the eigenvalue i.

As is well known, the operator $\mathcal{F}$ is $\mathbf{S}^1$-equivariant with respect to the phase shift action of $\mathbf{S}^1$ on loop space. In the standard Hopf Theorem $\ker(\mathcal{L})$ is two-dimensional, and Liapunov-Schmidt reduction in the presence of symmetry leads to a reduced equation that can be solved for a unique branch of 2π-periodic solutions as long as the eigenvalue crossing condition is valid.

In the equivariant context, $\ker(\mathcal{L})$ may be high-dimensional. The proof of the Equivariant Hopf Theorem proceeds by restricting the Liapunov-Schmidt reduced equation to $\mathrm{Fix}(\Delta)$, which by the $\mathbf{C}$-axial assumption on Δ is two-dimensional, and then completing the proof as in the standard Hopf Theorem. That approach does not work in the context of interior symmetries (since the fixed-point subspace of Δ in loop space is not $\mathcal{F}$-invariant), but the fixed-point subspace of K is $\mathcal{F}$-invariant (by Proposition 10.2). Since spatial $\mathbf{C}$-axial assumes that $\dim \mathrm{Fix}(K)$ is two-dimensional the proof proceeds as in the equivariant case.

At linear level the solution of (10.6), with period near 2π, is of the form

$$v(t) = w(t) + u(t)$$

where $w(t) \in \mathrm{Fix}_W(\Delta)$ and $u(t) \in \mathrm{Fix}(\Sigma_{\mathcal{S}})$. This follows from the form of the eigenvectors in Lemma 10.3(c). In particular, $w(t)$ has spatio-temporal symmetry Δ on cells in $\mathcal{S}$, and $u(t)$ is synchronous on $\Sigma_{\mathcal{S}}$ group orbits of cells in $\mathcal{S}$. $\square$

Remark 10.9 (1) The theorem asserts no restrictions on $v_j(t)$ when $j \notin \mathcal{S}$.

(2) The structure of a Δ-symmetric wave does impose genuine restrictions. Suppose, for the sake of illustration, that Δ defines a rotating wave. That is,

$$\Delta = \langle(\zeta, \theta)\rangle \subseteq \Sigma_{\mathcal{S}} \times \mathbf{S}^1$$

where ζ is a k-cycle on $\mathcal{S}$, with $|\mathcal{S}| = k$, and $\theta = 2\pi/k$ for the period 2π case. Then on $\mathcal{S} = \{0, \ldots, k-1\}$ we have

$$w_0(t) = w_1(t + 2\pi/k) = \cdots = w_{k-1}(t + 2(k-1)\pi/k)$$

and

$$u_0(t) = u_1(t) = \cdots = u_{k-1}(t)$$

for all t. To first order in λ, the wave $v(t)$ satisfies

$$v_j(t) = w_j(t) + u_j(t)$$

If we define

$$y_j(t) = v_{j+1}(t) - v_j(t) = w_{j+1}(t) - w_j(t)$$

(subscripts modulo k) then

$$y_j(t) = y_0(t + 2\pi j/k)$$

is a rotating wave. For a general periodic state, no such relation on differences holds. $\diamond$

Example 10.10 Consider the four-cell network whose diagram is Figure 22 (left). Set $\mathcal{S} = \{1, 2, 3\}$ and observe that the interior symmetry group is $\mathbf{Z}_3$. Symmetry-breaking Hopf bifurcation that breaks $\mathbf{Z}_3$ symmetry leads to simple eigenvalues and a rotating wave. As discussed, synchrony-breaking Hopf bifurcation leads to periodic solutions that, tgo first order, are the sum of a periodic rotating wave and a periodic state that is synchronous on $\mathcal{S}$. A simulation of such a state is given in Figure 22 (right). The upper panel in the simulation shows the superimposed time series from cells 1,2,3. The lower panel shows the hidden (approximate) rotating wave obtained by superimposing the time series $x_1 - x_2$, $x_2 - x_3$, $x_3 - x_1$. $\diamond$

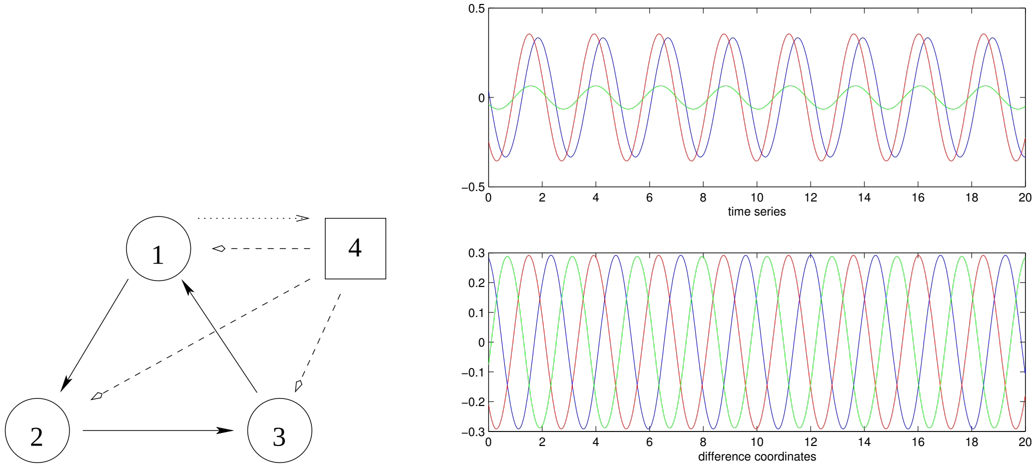

Figure 22: $\mathbf{Z}_3$ interior symmetry in a four-cell example. (Left) network; (right) simulation.

11 Final Comments

The formalism of symmetry groupoids proposed in this paper can be set up for many analogous systems that possess a network structure. Here, we have associated to each cell (node of the network) a continuous-time dynamical system defined on a manifold, and to each directed edge a coupling between such systems. We briefly consider variations on this theme.

Extra constraints can be imposed, an important one being to make the system Hamiltonian, see [9]. An analogous formalism can be introduced for discrete-time dynamics (coupled map lattices and generalizations to networks), or discrete-time discrete-space dynamics (cellular automata), and groupoid-equivariance implies constraints on the dynamics (in particular, on patterns of synchrony). If cells represent states of a stochastic process, and edges represent transitions, then the network corresponds to a Markov chain, and now the symmetry groupoid implies constraints on the stationary probability density function. We can also extend the groupoid formalism to stochastic differential equations and delay-differential equations.

The theory developed here is a preliminary step toward a formal understanding of pattern-formation in general, not necessarily symmetric, coupled cell networks. Its main focus is robust synchrony, plus a first pass at local bifurcation theory. Many other questions about the dynamics of coupled cell networks can be tackled within the groupoid framework; indeed, work is in progress on several of these. In all cases, the central role of the symmetry groupoid as a formal algebraic structure that captures the constraints imposed by the network topology is paramount.

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