Estimating the Spectral Measure of a Multivariate Stable Distribution via Spherical Harmonic Analysis

Marcus Pivato* and Luis Seco†

October 2, 2000

Abstract

A new method is developed for estimating the spectral measure of a multivariate stable probability measure, by representing the measure as a sum of spherical harmonics.

Introduction:

Stable probability distributions are the natural generalizations of the normal distribution, and share with it two key properties:

- **Stability:** The normal distribution is **stable** in the sense that, if $\mathbf{X}$ and $\mathbf{Y}$ are independent random variables, with identical normal distributions, then $\mathbf{X}+\mathbf{Y}$ is also normal, and

$$\frac{1}{2^{1/2}}(\mathbf{X}+\mathbf{Y}) \quad \underset{distr}{\cong} \quad \mathbf{X} \quad \underset{distr}{\cong} \quad \mathbf{Y}$$

  In a similar fashion, if $\mathbf{X}$ and $\mathbf{Y}$ are independent, identically distributed (i.i.d) stable random variables, then $\mathbf{X}+\mathbf{Y}$ is also stable, and its distribution is the same as $\mathbf{X}$ and $\mathbf{Y}$ when renormalized by $2^{-1/\alpha}$. The **stability exponent** α ranges from 0 to 2 . When $\alpha = 2$, we have the familiar normal distribution.

*Department of Mathematics, University of Toronto, **email:** `pivato@math.toronto.edu`

†RiskLab and Department of Mathematics, University of Toronto, **email:** `seco@math.toronto.edu`

- **Renormalization Limit:** The Central Limit Theorem says that the normal distribution is the natural limiting distribution of a suitably renormalized infinite sum of independent random variables with finite variance. If $\mathbf{X}_1, \mathbf{X}_2, \ldots$ is a sequence of such variables, then the random variables

$$\frac{1}{N^{1/2}} \sum_{n=1}^{N} \mathbf{X}_n$$

converge, in density, to a normal distribution. Similarly, if $\{\mathbf{Y}_k\}_{k=1}^{\infty}$ are independent random variables whose distributions decay according to a power law with exponent $-1-\alpha$, then the random variables

$$\frac{1}{N^{1/\alpha}} \sum_{n=1}^{N} \mathbf{Y}_n$$

converge, in distribution, to an α-stable distribution

Thus, stable distributions model random aggregations of many small, independent perturbations. For example, stable distributions model the motions of Markovian stochastic processes whose increments exhibit power laws. Stable distributions arise with surprising frequency in certain systems, especially those involving many independent interacting units with sensitive dependencies between them. They have appeared in mathematical finance [28],[14],[13], [29],[42],[15], [37],[18],[19],[11],[30],[**?**], Internet traffic statistics [**?**], [32],[31], [52], and arise in mathematical models of random scalar fields [53], [22], radar [34], and signal processing [40], [8], [9], telecommunications [**?**], and even the power distribution of ocean waves [**?**].

For further examples, see [53], [41], or [17]. The definitive reference on univariate stable distributions is [53]; the definitive reference on multivariate distributions and stable processes is [41]. Other recent references are [**?**], [**?**], [**?**], and a forthcoming book by Nolan [**?**]; slightly older references are [17] and [1].

Although one-dimensional stable distributions are well-understood, there are many open questions in the multivariate regime. The simplicity of the multivariate Gaussian universe does not extend to non-Gaussian multivariate stable distributions. An N-dimensional Gaussian distribution is completely determined by its $N \times N$ covariance matrix, which transforms nicely under linear changes of coordinates. In particular, by orthogonally diagonalizing the matrix, we can find an orthonormal basis for $\mathbb{R}^N$, relative to which the multivariate normal variable is revealed as a sum of independent univariate normal variables —this is *Principle Component Analysis.*

For a general multivariate stable distribution, however, the situation is much more complex. Since the marginals do not have finite variance, it does not make sense to define a "covariance matrix" in the usual way; none of the integrals would converge. Various modified notions of "covariance" have been proposed (see, for example, [41]), but these do not transform in any simple way under changes of coordinates. In particular, there is nothing analogous to a "principle components analysis". Instead, the correlation structure of a stable distribution on $\mathbb{R}^D$ is determined by an arbitrary measure, Γ, on the sphere $\mathbb{S}^{D-1} = \left\{\vec{x} \in \mathbb{R}^D \ ; \ \|\vec{x}\| = 1\right\}$, called the **spectral measure**:

Theorem 1: *Let $\alpha \in [0, 2)$, and let ρ be an α-stable probability measure on $\mathbb{R}^D$, with center $\vec{\mu} \in \mathbb{R}^D$. Then ρ has Fourier Transform:*

$$\chi[\vec{\xi}] \quad = \quad \exp(\Phi[\vec{\xi}])$$

where Φ (the "log Fourier transform") is given:

$$\Phi[\vec{\xi}] \quad = \quad \left\langle \vec{\mu}, \vec{\xi} \right\rangle \mathbf{i} \quad + \quad \int_{\mathbb{S}^{D-1}} \eta^{(\alpha)} \left\langle \vec{\xi}, \mathbf{s} \right\rangle \ d\Gamma[\mathbf{s}] \tag{1}$$

$$\textit{where} \ \ \eta^{(\alpha)}(\theta) \quad := \quad -|\theta|^\alpha \ - \ \mathcal{B}_\alpha \cdot \theta^{\langle \alpha \rangle} \mathbf{i}, \tag{2}$$

$$\textit{with} \ \ \theta^{\langle \alpha \rangle} \quad := \quad \begin{cases} \mathbf{sign}(\theta) \cdot |\theta|^\alpha & \textit{if} \quad \alpha \neq 1 \\ \theta \cdot \log|\theta| & \textit{if} \quad \alpha = 1 \end{cases}$$

$$\textit{and} \ \ \mathcal{B}_\alpha \quad := \quad \begin{cases} \tan\left(\frac{\pi\alpha}{2}\right) & \textit{if} \quad \alpha \neq 1 \\ -\frac{2}{\pi} & \textit{if} \quad \alpha = 1 \end{cases}$$

and where Γ is some nonnegative Borel measure on $\mathbb{S}^{D-1}$.

Proof: See [41], §2.3, p.65, or [26].

□ `[Theorem 1]`

Γ is called the **spectral measure** of the distribution[1], and is essentially an "infinite-dimensional" data-structure, so it is clear that, in general, no $N \times N$ matrix can possibly be adequate for representing it. A "principle components" type decomposition is only valid when the spectral measure consists of $2D$ antipodaly positioned atoms.

Estimating Γ is much difficult than estimating a covariance matrix. Whereas the terms of a covariance matrix can be directly computed by estimating the correlation between coordinates, Γ is only indirectly visible; the image of Γ under a sort of "spherical convolution" appears in the *logarithm* of the characteristic function of the distribution; there is no more direct way to observe it.

In this paper, we develop an method for estimating Γ from the log-characteristic function Φ. Assume for simplicity that the distribution is centered at the origin, and let the **spherical log-characteristic function** be the function $\mathbf{g} : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ determined by restricting Φ to the sphere. Then, for all $\vec{\xi} \in \mathbb{S}^{D-1}$, we have

$$\mathbf{g}[\vec{\xi}] \quad = \quad \int_{\mathbb{S}^{D-1}} \eta^{(\alpha)} \left\langle \vec{\xi}, \mathbf{s} \right\rangle \; d\Gamma[\mathbf{s}] \qquad (3)$$

The characteristic function of a distribution is easy to estimate from empirical data, and thus, we assume we have a good estimate of $\mathbf{g}$ on some suitably fine mesh over $\mathbb{S}^{D-1}$. Hence, the problem is to recover Γ from $\mathbf{g}$.

Abusing notation, we might rewrite equation (3) as "$\mathbf{g} \;=\; \eta^{(\alpha)} * \Gamma$". If $D = 2$ or $D = 4$, then $\mathbb{S}^{D-1}$ is a topological group, and this "convolution" can be interpreted literally, via the formula:

$$\eta^{(\alpha)} * \Gamma(\vec{\xi}) \;=\; \int_{\mathbb{S}^{D-1}} \eta^{(\alpha)}(\vec{\xi}.\mathbf{s}^{-1}) \; d\Gamma[\mathbf{s}].$$

In other dimensions, however, $\mathbb{S}^{D-1}$ is not a topological group, and therefore, convolution *per se* is not well-defined. We must instead think of $\mathbb{S}^{D-1}$ as a homogeneous manifold under the action of $\mathbb{SO}^{D}[\mathbb{R}]$, and define a kind of "convolution" in terms of this group action.

The eigenfunctions of the Laplacian operator on $\mathbb{S}^{D-1}$ are called **spherical harmonics**, and form an orthonormal basis for $\mathbf{L}^2(\mathbb{S}^{D-1})$,

[1]This terminology is standard, but somewhat unfortunate, since Γ is unrelated to any one of half a dozen other "spectra" and "spectral measures" currently existent in mathematics. Perhaps it would be more appropriate to call Γ a **Feldheim measure**, since Feldheim [16] was the first to define it.

analogous to the Fourier basis for $\mathbf{L}^2(\mathbb{S}^1)$ from classical harmonic analysis. The expression of a function on $\mathbb{S}^{D-1}$ in terms of this basis is called its **spherical Fourier transform**. A function $f \in \mathbf{L}^2(\mathbb{S}^{D-1})$ is called **zonal** if it is rotationally invariant around a particular coordinate axis —for example, $\eta^{(\alpha)}$ is zonal. There is a way of convolving arbitrary functions by zonal functions, and, just as in classical harmonic analysis, convolution of a function f by η translates into componentwise multiplication of their respective Fourier transforms. Thus, to deconvolve f and η, it suffices divide the Fourier transform of $\eta * f$ by that of η.

The advantage of this approach is twofold: First, it provides a natural *continuous* representation of the spectral measure, obviating the need to approximate it with a sum of atoms. Second, it is computationally faster. The computations involved are still expensive: numerically integrating on a sphere using a mesh of density $\epsilon \sim 1/N$ is a computation of order $\mathcal{O}\left(N^{(D-1)}\right)$, and computing a convolution of two functions is thus a computation of order $\mathcal{O}\left(N^{2(D-1)}\right)$. However, there is no need to explicitly compute a matrix inverse first, because a closed-form expression exists for the elements of the orthonormal basis.

Organization of this Paper: In §1, we summarize previous work on this problem. In §2, we develop some background material, treating $\mathbb{S}^{D-1}$ as homogeneous manifold under the action of $\mathbb{SO}^D[\mathbb{R}]$, and reviewing zonal functions, the eigenfunctions of the Laplacian, and a suitable notion of convolution, and provide explicit formulae for the spherical harmonics. In §3, we define the spherical Fourier transform and show how to compute "deconvolution" using this transform. In §4, we characterize the rate of convergence of the spherical Fourier series as an estimate of the spectral measure, and relate this to convergence of the underlying stable distribution.

1 Summary of previous Work:

Early on, Press [36] developed an estimation scheme for multivariate stable distributions, through a straightforward generalization of his one-dimensional method. Press's method, however, only works for "pseudo-Gaussian" distributions, with log-characteristic functions of the form:

$$\Phi_{\mathbf{X}}(\vec{\xi}) \quad = \quad \left\langle \vec{\xi}, \vec{\mu} \right\rangle \mathbf{i} \; + \; \left\langle \vec{\xi}, \Omega\vec{\xi} \right\rangle^{\alpha/2}$$

where Ω is some symmetric, positive semidefinite "covariance matrix". If Ω has unit eigenvectors $\vec{\omega}_1, \ldots, \vec{\omega}_D$, with eigenvalues $\lambda_1, \ldots \lambda_D$ (ie. as a covariance matrix, we have "principle components" $\lambda_1\vec{\omega}_1$, ..., $\lambda_1\vec{\omega}_1$), then the spectral measure of this distribution is symmetric and atomic, with atoms at each of $\pm\vec{\omega}_1, \ldots, \pm\vec{\omega}_D$, with masses $\lambda_1, \ldots \lambda_D$ —in other words:

$$\Gamma = \sum_{d=1}^{D} \lambda_d \left(\delta_{\vec{\omega}_d} + \delta_{-\vec{\omega}_d}\right)$$

Press proposes to solve for the components of the matrix Ω by empirically estimating the log characteristic function at some collection of frequencies $\{\vec{\xi}_1, \ldots, \vec{\xi}_N\}$, where $N = D(D+1)/2$, and then solving a system of N linear equations. He claims that his method will generalize to a *sum* of pseudo-Gaussians:

$$\Phi_{\mathbf{X}}(\vec{\xi}) \quad = \quad \left\langle \vec{\xi}, \vec{\mu} \right\rangle \mathbf{i} \; + \; \sum_{m=1}^{M} \left\langle \vec{\xi}, \Omega_m\vec{\xi} \right\rangle^{\alpha/2}$$

(where $\Omega_1, \ldots, \Omega_M$ are linearly independent, symmetric, positive semidefinite matrices). However, in this case, one no longer ends up with a system of linear equations, so it is not clear that the method is tractable. In any event, Press's method only applies to multivariate distributions with particularly simple atomic spectral measures, which furthermore must be symmetrically distributed. Empirical evidence (see, for example, [4]) suggests that the stable distributions found in financial data are significantly skewed; symmetry is not a reasonable assumption.

Cheng, Rachev and Xin [44],[5] develop a more sophisticated method, by expressing a stable random vector in polar coordinates, and then examining the order statistics of the radial component, as a function of the angular component. They utilize the theorem of Araujo and Giné (Corollary 6.20(b), Chapter 3, p. 152 of [1]) stating that the radial distribution decays most slowly in those angular directions with the heaviest concentration of spectral mass; these differences in decay rate are then used to estimate the density distribution of the spectral measure.

Nolan, Panorska, and McCulloch [25], [24], develop a method based upon a discrete approximation of the spectral measure. If the spectral measure is treated as a sum of a finite number of atoms,

$$\Gamma \;=\; \sum_{\mathbf{a}\in\mathcal{A}} \gamma_{\mathbf{a}}\delta_{\mathbf{a}},$$

then, for any fixed $\vec{\xi} \in \mathbb{S}^{D-1}$, the function $\eta^{(\alpha)}_{\vec{\xi}}(\mathbf{s}) \;=\; \eta^{(\alpha)}\left\langle \vec{\xi}, \mathbf{s}\right\rangle$ of Theorem 1 can be restricted to a function $\eta^{(\alpha)}_{\vec{\xi}} : \mathcal{A} \longrightarrow \mathbb{C}$. The set of all discrete measures supported on $\mathcal{A}$ is a finite-dimensional vector space, which we can identify with $\mathbb{C}^{\mathcal{A}}$, and $\eta^{(\alpha)}_{\vec{\xi}}$ is just a linear functional on this vector space. If $\Xi \subset \mathbb{S}^{D-1}$ is some finite set, then we can define a linear map:

$$F : \mathbb{C}^{\mathcal{A}} \longrightarrow \mathbb{C}^{\Xi}$$

where, for each $\vec{\xi} \in \Xi$,

$$F(\Gamma)_{\vec{\xi}} \;=\; \mathbf{g}(\vec{\xi}) \;=\; \int_{\mathbb{S}^{D-1}} \eta^{(\alpha)}_{\vec{\xi}}\; d\Gamma$$

The method of Nolan *et al.* then comes down to *inverting* this linear transformation to recover Γ from an empirical estimate of $\mathbf{g}$. They explicitly implemented their method in the two-dimensional case (ie. when the spectral measure lives on a circle), and tested it against a variety of distributions. They found that it worked fairly well for a variety of measures on the circle, and consistently outperformed the method of Chen *et al.* The methods of Chen *et al.* and Nolan *et al.* are also discussed in §5 of [33].

Finally, Hurd *et al.* [**?**] develop...

2 $\mathbb{S}^{D-1}$ as a Homogeneous Riemannian Manifold: Zonal functions, Laplacians, and Convolution

[The development of background material here loosely follows the discussion in [47] chapter 3, section 3. A more friendly approach is [35].
]

$\mathbb{S}^{D-1}$ is a compact Riemannian manifold, and $\mathbb{G} = \mathbb{SO}^D[\mathbb{R}]$ is a (nonabelian) compact Lie group, acting transitively and isometrically on $\mathbb{S}^{D-1}$ by rotations. We will develop a version of harmonic analysis on $\mathbb{S}^{D-1}$ as a homogeneous Riemannian manifold (this theory is actually applicable to any homogeneous Riemannian manifold; it may be helpful to keep this in mind).

Let $\mathcal{L}^{\mathrm{bsg}}$ be the canonical volume measure induced on $\mathbb{S}^{D-1}$ by its Riemann structure. For example, on $\mathbb{S}^2$, $\mathcal{L}^{\mathrm{bsg}}$ is the usual "surface area" measure. $\mathbb{S}^{D-1}$ is compact, so $\mathcal{L}^{\mathrm{bsg}}$ is finite —assume $\mathcal{L}^{\mathrm{bsg}}$ is normalized to have total mass 1. Let $\mathbf{L}^2(\mathbb{S}^{D-1}) = \mathbf{L}^2(\mathbb{S}^{D-1}, \mathcal{L}^{\mathrm{bsg}}; \mathbb{C})$. The action of $\mathbb{G}$ on $\mathbb{S}^{D-1}$ induces a linear $\mathbb{G}$-action on $\mathbf{L}^2(\mathbb{S}^{D-1})$ in the obvious way: if $\phi \in \mathbf{L}^2(\mathbb{S}^{D-1})$ and $g \in \mathbb{G}$, then $g.\phi : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ is defined: $g.\phi(m) = \phi(g.m)$.

Let $\mathbf{C}^\infty(\mathbb{S}^{D-1})$ be the space of smooth, complex-valued functions on $\mathbb{S}^{D-1}$. $\mathcal{L}^{\mathrm{bsg}}$ is finite, so $\mathbf{C}^\infty(\mathbb{S}^{D-1})$ is a linear subspace of $\mathbf{L}^2(\mathbb{S}^{D-1})$ (though not a closed subspace). $\mathbb{G}$ acts smoothly on $\mathbb{S}^{D-1}$, so $\mathbf{C}^\infty(\mathbb{S}^{D-1})$ is $\mathbb{G}$-invariant. We consider the restricted action of $\mathbb{G}$ on $\mathbf{C}^\infty(\mathbb{S}^{D-1})$.

Let $\triangle : \mathbf{C}^\infty(\mathbb{S}^{D-1}) \longrightarrow \mathbf{C}^\infty(\mathbb{S}^{D-1})$ is the Laplacian operator.

Theorem 2: *(The Laplacian on $\mathbb{S}^D$) [45]*

First consider the case $D = 1$. Endow the circle $\mathbb{S}^1$ with the angular coordinate system $\theta \in (0, 2\pi)$, so that any point on $\mathbb{S}^1_* = \mathbb{S}^1 - \{(1,0)\}$ has coordinates

$$(\cos(\theta),\ \sin(\theta))$$

If $f : \mathbb{S}^1_* \longrightarrow \mathbb{C}$, then, relative to this coordinate system, we have:

$$\triangle_{\mathbb{S}^1} f = \frac{\partial^2 f}{\partial \theta^2}.$$

More generally, define $\mathbb{S}^D_* = \mathbb{S}^D \setminus \left(\mathbb{R}^{D-1} \times [0,\infty) \times \{0\}\right)$, and then define the diffeomorphism

$$\begin{array}{rcl} \mathbb{S}^{D-1}{}_* \times (0,\pi) & \longrightarrow & \mathbb{S}^D{}_* \\ (\mathbf{s}, \phi) & \mapsto & [\cos(\phi);\ \ \sin(\phi)\cdot \mathbf{s}] \end{array}$$

Then we have the following inductive formula:

$$\triangle_{\mathbb{S}^D} f \quad = \quad \frac{\partial^2 f}{\partial \phi^2} \; + \; (D-1)\cot(\phi)\frac{\partial f}{\partial \phi} \; + \; \frac{1}{\sin(\phi)^2}\triangle_{\mathbb{S}^{D-1}} f. \qquad (4)$$

□

$\triangle$ commutes with the isometric $\mathbb{G}$ action: for all $g \in \mathbb{G}$,

$$\triangle(g.\phi) \;=\; g.(\triangle\phi)$$

Let $\Lambda := \{\lambda \in \mathbb{C} \,;\, -\lambda \text{ is an eigenvalue of } \triangle\}$, and for each $\lambda \in \Lambda$, let $\mathbb{V}_\lambda := \left\{\phi \in \mathbf{C}^\infty(\mathbb{S}^{D-1}) \,;\, \triangle\phi = -\lambda\phi\right\}$ be the corresponding eigenspace. Thus, $\mathbb{V}_\lambda$ is a $\mathbb{G}$-invariant subspace.

The eigenfunctions of the Laplacian on $\mathbb{S}^{D-1}$ are called **spherical harmonics**. Further information on spherical harmonics can be found in chapter 4, section 3 of [47]; chapter II of [3]; chapters 3 and 5 of [21]; chapters 7 and 8 of [27]; §11 and §12 of [45]; and also in [38], [39], [20], [2], [6], [43], [46], and [50].

Let $e = (1, 0, \ldots, 0) \in \mathbb{S}^{D-1}$, and define

$$\mathbb{G}_e \;=\; \{g \in \mathbb{G} \,;\, g.e \;=\; e\},$$

the set of all orthogonal transformations of $\mathbb{R}^D$ fixing the e-axis. In other words, $\mathbb{G}_e$ is the set of all "rotations" of the remaining $(D-1)$ dimensions about this axis; hence, there is a natural isomorphism $\mathbb{G}_e \cong \mathbb{SO}^{D-1}[\mathbb{R}]$. $\mathbb{G}_e$ is thus a connected, compact subgroup of $\mathbb{G}$. The action of $\mathbb{G}$ upon $\mathbf{C}^\infty(\mathbb{S}^{D-1})$ restricts to an action of $\mathbb{G}_e$, and the spaces $\mathbb{V}_\lambda$ remain invariant under this new action.

Definition 3: *Zonal function*

A function $\zeta \in \mathbf{C}^\infty(\mathbb{S}^{D-1})$ is called **zonal** *(relative to $\mathbb{G}$ and the fixed point $e \in \mathbb{S}^{D-1}$) if it is invariant under the action of $\mathbb{G}_e$. Formally, for any $\mathbb{G}_e$-invariant subspace $\mathbb{V} \subset \mathbf{C}^\infty(\mathbb{S}^{D-1})$, define*

$$\mathcal{Z}_e(\mathbb{V}) \;:=\; \{\zeta \in \mathbb{V} \,;\, \forall g \in \mathbb{G}_e, \;\; g.\zeta \;=\; \zeta\}$$

Thus, the zonal elements of $\mathbf{C}^\infty(\mathbb{S}^{D-1})$ are smooth functions rotationally invariant about the e axis. Clearly, any such function must be of the form

$$\zeta(\mathbf{x}) \;=\; \zeta_1(x_1)$$

where $\zeta_1 : [-1,1] \longrightarrow \mathbb{C}$, and where $\mathbf{x} = (x_1, x_2, \ldots, x_D)$ is any element of $\mathbb{S}^{D-1}$.

Proposition 4:

1. *If* $\mathbb{V} \subset \mathbf{C}(\mathbb{S}^{D-1})$ *is a nontrivial* $\mathbb{G}$*-invariant subspace, then* $\mathcal{Z}_e(\mathbb{V})$ *is nontrivial.*
2. *If* $\dim(\mathcal{Z}_e(\mathbb{V})) = 1$*, then* $\mathbb{V}$ *is an irreducible* $\mathbb{G}$*-module.*

Proof:

Proof of Part 1:

Claim 1: $\mathbb{V}$ *contains an element* ϕ *such that* $\phi(e) \neq 0$.

Proof: Since $\mathbb{V}$ is nontrivial, there is some $\psi \in \mathbb{V}$ which is nonzero *somewhere* –say $\psi(x) \neq 0$. Since $\mathbb{G}$ acts transitively on $\mathbb{S}^{D-1}$, find $g \in \mathbb{G}$ so that $g.e = x$. Thus, if $\phi = g.\psi$, then $\phi(e) = \psi(g.e) = \psi(x) \neq 0$. Since $\mathbb{V}$ is $\mathbb{G}$-invariant, $\phi \in \mathbb{V}$ is the element we seek. .... □ [Claim 1]

Now, $\mathbb{G}_e$ is a closed subgroup of the compact group $\mathbb{G}$; thus, $\mathbb{G}_e$ is compact, so it has a finite Haar measure $\mathcal{H}^{aar}$. Define

$$\zeta \ := \ \int_{\mathbb{G}_e} g.\phi \ d\mathcal{H}^{aar}[g]$$

Since $\mathcal{H}^{aar}$ is finite, this integral is well-defined. Since $\mathbb{V}$ is a closed, $\mathbb{G}$-invariant subspace, the element ζ is in $\mathbb{V}$. Furthermore, since $\zeta(e) = \phi(e)$, and $\phi(e) \neq 0$, we conclude that ζ is nontrivial. Finally, note that ζ is $\mathbb{G}_e$-invariant by construction —in other words, it is zonal.

Proof of Part 2: Suppose $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$, where $\mathbb{V}_1, \mathbb{V}_2$ are $\mathbb{G}$-invariant. Then by **Part 1**, we can construct linearly independent zonal functions $\zeta_1 \in \mathcal{Z}_e(\mathbb{V}_1)$ and $\zeta_2 \in \mathcal{Z}_e(\mathbb{V}_2)$. Since $\zeta_1, \zeta_2 \in \mathcal{Z}_e(\mathbb{V})$, this contradicts the hypothesis that $\dim(\mathcal{Z}_e(\mathbb{V})) = 1$.

□ [Proposition 4]

The isometric action of $\mathbb{G}_e$ on $\mathbb{S}^{D-1}$ induces a linear, isometric action upon the tangent space $\mathsf{T}_e\mathbb{S}^{D-1}$. If $\vec{v} \in \mathsf{T}_e\mathbb{S}^{D-1}$ is the derivative of a path $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathbb{S}^{D-1}$ with $\gamma(0) = e$, then $g.\vec{v}$ is the derivative of the path $(g.\gamma) : (-\epsilon, \epsilon) \longrightarrow \mathbb{S}^{D-1}$. The action of of $\mathbb{G}_e$ on $\mathbb{S}^{D-1}$ is **rank one**, meaning that $\mathbb{G}_e$ acts transitively on the set of unit tangent vectors $\mathsf{T}_e\mathbb{S}^{D-1}$. For any $r > 0$, let $\mathbb{B}(e, r)$ be the ball of radius r about e in $\mathbb{S}^{D-1}$, relative to the intrinsic Riemannian metric. The following is clear:

Lemma 5: *For all r, $\mathbb{G}_e$ acts transitively on $\partial\mathbb{B}(e,r)$ in $\mathbb{S}^{D-1}$.*

Proposition 6: *If $\mathbb{S}^{D-1}$ is of rank one, then each eigenspace $\mathbb{V}_\lambda$ of $\triangle$ is an* **irreducible** *$\mathbb{G}$-module.*

Proof: By Proposition 4, it suffices to show that $\dim[\mathcal{Z}_e(\mathbb{V}_\lambda)] = 1$. So, suppose that $\zeta_1, \zeta_2 \in \mathcal{Z}_e(\mathbb{V}_\lambda)$ are linearly independent. Since they are zonal, $\zeta_1(u)$ and $\zeta_2(u)$ are functions only of the distance from u to e. So, for some $u \in \mathbb{S}^{D-1}$ with $\mathbf{dist}[u,e] = r$, define $z_1 := \zeta_1(u)$ and $z_2 := \zeta_2(u)$, and let $\zeta := z_2\zeta_1 - z_1\zeta_2$. Thus, ζ is also zonal. We aim to show that ζ is the zero function; thus, ζ_1 and ζ_2 are just scalar multiples of one another.

Now, by construction, $\zeta(u) = 0$, and thus, $\zeta \equiv 0$ on $\partial\mathbb{B}(e;r)$. At the same time, however, ζ is a linear combination of two elements of $\mathbb{V}_\lambda$; hence, it is also in $\mathbb{V}_\lambda$ —ie. ζ is a $(-\lambda)$-eigenfunctions of $\triangle$. Fix λ, and let r get small. If r is made small enough, then the Dirichlet boundary condition $\zeta_{|\partial\mathbb{B}(e;r)} \equiv 0$ forces the smallest eigenvalue of $\triangle$ to be larger in absolute value than λ, creating a contradiction.

$\square$ `[Proposition 6]`

One consequence of this irreducibility is

Theorem 7: *(Schur's Lemma) [48]*
If $\mathbb{V}$ is complex Banach space and an irreducible $\mathbb{G}$-module, and $\phi : \mathbb{V} \longrightarrow \mathbb{V}$ is a continuous, complex-linear map that commutes with the $\mathbb{G}$-action, then ϕ must be multiplication by a scalar. $\square$

Now consider the D-torus $\mathbb{T}^D$, equipped with the standard equivariant metric. The eigenfunctions of the Laplacian on are the periodic functions of the form $\mathcal{E}_{\mathbf{n}}(\mathbf{x}) = \exp(2\pi\mathbf{i} \cdot \langle \mathbf{n}, \mathbf{x} \rangle)$, with $\mathbf{n} \in \widehat{\mathbb{T}^D} \cong \mathbb{Z}^D$, where $\mathbf{x} \in [0,1)^D$ and $[0,1)^D$ is identified with $\mathbb{T}^D$ in the obvious way. These eigenfunctions form an orthonormal basis for $\mathbf{L}^2(\mathbb{T}^D)$. The same is true for arbitrary homogeneous Riemannian manifolds, and in particular, for the sphere:

Theorem 8:

- *If $\lambda_1 \neq \lambda_2$, then the eigenspaces $\mathbb{V}_{\lambda_1}$ and $\mathbb{V}_{\lambda_2}$ are orthogonal as subsets of $\mathbf{L}^2(\mathbb{S}^{D-1})$.*

- *The eigenspaces of $\triangle$ span $\mathbf{L}^2(\mathbb{S}^{D-1})$. In other words:*

$$\mathbf{L}^2(\mathbb{S}^{D-1}) \;=\; \bigoplus_{\lambda\in\Lambda} \mathbb{V}_\lambda$$

Proof: See for example [51], chapter 6, p. 255; or [7], Theorem 3.21, p. 156. Or treat $\triangle$ as an elliptic differential operator, and use [12], §6.5, Theorem 1, p. 335. Alternately, employ the Spectral Theorem for unbounded self-adjoint operators (see [10], chapter X, section 4, p. 319).

□ `[Theorem 8]`

Definition 9: *Equivariant Function*

*If $\eta : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$, then say that η is a $\mathbb{G}$-***equivariant** *if, for all $m, n \in \mathbb{S}^{D-1}$ and $g \in \mathbb{G}$,*

$$\eta(g.m, g.n) \;=\; \eta(m,n)$$

Since $\mathbb{G}$ acts isometrically and transitively on $\mathbb{S}^{D-1}$, this is equivalent to saying that $\eta(\mathbf{x},\mathbf{y})$ is a function only of the distance $\mathbf{dist}\,[\mathbf{x},\mathbf{y}]$.

For instance, if the function $\eta^{(\alpha)} : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ defined by equation (2) is $\mathbb{G}$-equivariant.

$\mathbb{G}$-equivariant functions are interesting because we can define a sort of **convolution** with them.

Definition 10: *Convolution*

If η is $\mathbb{G}$-equivariant, $\phi : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$, and both are $\mathcal{L}^{bsg}$-integrable, then define $\eta * \phi : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ by

$$(\eta * \phi)(\mathbf{s}) \;=\; \int_{\mathbb{S}^{D-1}} \eta(\mathbf{s},\theta)\phi(\theta)\; d\mathcal{L}^{bsg}[\theta]$$

For example, if Γ is a measure on $\mathbb{S}^{D-1}$, with Radon-Nikodym derivative $\gamma : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$, then $\eta * \gamma : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ is defined

$$\eta * \gamma(\mathbf{s}) \;=\; \int_{\mathbb{S}^{D-1}} \eta(\mathbf{s},\theta)\gamma(\theta)\; d\mathcal{L}^{bsg}[\theta] \;=\; \int_{\mathbb{S}^{D-1}} \eta(\mathbf{s},\theta)\; d\Gamma[\theta]$$

In particular, if Γ is a spectral measure and $\eta = \eta^{(\alpha)}$, then this formula is identical to equation (3). In other words,

$$\eta^{(\alpha)} * \gamma \quad = \quad \mathbf{g}$$

where $\mathbf{g}$ is the spherical log-characteristic function.

Recall again the case of $\mathbb{T}^D$ The eigenfunctions of the Laplacian, $\{\mathcal{E}_\mathbf{n}\ ;\ \mathbf{n} \in \mathbb{Z}^D\}$, are well-behaved under convolution: classical harmonic analysis tells us that

$$\left(\sum_{\mathbf{n}\in\mathbb{Z}^D} a_\mathbf{n}\mathcal{E}_\mathbf{n}(\mathbf{x})\right) * \left(\sum_{\mathbf{n}\in\mathbb{Z}^D} b_\mathbf{n}\mathcal{E}_\mathbf{n}(\mathbf{x})\right) \quad = \quad \sum_{\mathbf{n}\in\mathbb{Z}^D} (a_\mathbf{n} \cdot b_\mathbf{n})\, \mathcal{E}_\mathbf{n}(\mathbf{x})$$

It turns out that this phenomenon generalizes to arbitrary homogeneous Riemannian manifolds.

Proposition 11: *(Convolution and Eigenfunctions)*
Let $\eta : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ be $\mathbb{G}$-equivariant. Fix $\lambda \in \Lambda$ and $\zeta \in \mathcal{Z}_e(\mathbb{V}_\lambda)$, and define $A_\lambda \in \mathbb{C}$ by:

$$A_\lambda \quad := \quad \frac{(\eta * \zeta)(e)}{\zeta(e)}.$$

Then, for any $\phi \in \mathbb{V}_\lambda$, $\quad \eta * \phi \ = \ A_\lambda \cdot \phi$.

Proof: Let $T_\eta : \mathcal{C}^\infty(\mathbb{S}^{D-1}) \longrightarrow \mathcal{C}^\infty(\mathbb{S}^{D-1})$ be defined: $T_\eta(\phi) = \eta * \phi$.

Claim 1: *The operator T_η commutes with the $\mathbb{G}$-action: for all $g \in \mathbb{G}$, $T_\eta[g.\phi] = g.T_\eta[\phi]$.*

Proof: For any $m \in \mathbb{S}^{D-1}$,

$$\begin{aligned}
T_\eta[g.\phi](m) \quad &= \quad [\eta * (g.\phi)]\,(m) \\
&= \quad \int_{\mathbb{S}^{D-1}} \eta(m,n)\phi(g.n)\ d\mathcal{L}^{\text{leg}}[n] \\
&=_{(1)} \quad \int_{\mathbb{S}^{D-1}} \eta(m,g^{-1}.n')\phi(n')\ d\mathcal{L}^{\text{leg}}[n'] \\
&=_{(2)} \quad \int_{\mathbb{S}^{D-1}} \eta(g.m,n')\phi(n')\ d\mathcal{L}^{\text{leg}}[n'] \\
&= \quad (\eta * \phi)(g.m) \\
&= \quad g.(\eta * \phi)(m).
\end{aligned}$$

(1) where $n' := g.n$

(2) Because η is $\mathbb{G}$-equivariant. .... $\Box$ `[Claim 1]`

Claim 2: *T_η commutes with $\triangle$.*

Proof: For each $y \in \mathbb{S}^{D-1}$, define $\eta_y : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ by $\eta_y(x) = \eta(y,x) = \eta(x,y)$. Thus,

$$\begin{aligned}
(\eta * \phi)(x) &= \int_{\mathbb{S}^{D-1}} \eta(x,y) \cdot \phi(y) \; d\mathcal{L}^{\text{bsg}}[y] \\
&= \int_{\mathbb{S}^{D-1}} \phi(y) \cdot \eta_y(x) \; d\mathcal{L}^{\text{bsg}}[y] \\
\text{Hence,} \quad \triangle\, (\eta * \phi)(x) &= \triangle \int_{\mathbb{S}^{D-1}} \phi(y) \cdot \eta_y(x) \; d\mathcal{L}^{\text{bsg}}[y] \\
&= \int_{\mathbb{S}^{D-1}} \phi(y) \cdot \triangle\eta_y(x) \; d\mathcal{L}^{\text{bsg}}[y] \qquad (*)
\end{aligned}$$

because $\triangle$ is a linear operator.

Claim 2.1: $\triangle\eta_y(x) = \triangle\eta_x(y)$.

Proof: Find some $g \in \mathbb{G}$ so that $g.x = y$ and $g.y = x$. Thus for any $m \in \mathbb{S}^{D-1}$,

$$\begin{aligned}
\eta_x(m) &= \eta(x,m) = \eta(g.x,\; g.m) = \eta(y,\; g.m) \\
&= \eta_y(g.m) = (g.\eta_y)(m)
\end{aligned}$$

$$\begin{aligned}
\text{In other words,} \quad \eta_x &= (g.\eta_y). \\
\text{Thus,} \quad \triangle\, \eta_x &= \triangle(g.\eta_y) = g.(\triangle\eta_y). \\
\text{In particular,} \quad \triangle\, \eta_x(y) &= g.(\triangle\eta_y)(y) = \triangle\eta_y(g.y) \\
&= \triangle\eta_y(x).
\end{aligned}$$

.... $\Box$ `[Claim 2.1]`

Hence, we can rewrite expression $(*)$ as:

$$\int_{\mathbb{S}^{D-1}} \phi(y) \cdot \triangle\eta_x(y) \; d\mathcal{L}^{\text{bsg}}[y]$$

But $\mathbb{S}^{D-1}$ is a manifold without boundary, so $\triangle$ is self-adjoint (see, for example [51], chapter 6). Hence,

$$\begin{aligned}
\int_{\mathbb{S}^{D-1}} \phi(y) \cdot \triangle \eta_x(y) \; d\mathcal{L}^{bsg}[y] &= \int_{\mathbb{S}^{D-1}} \triangle\phi(y) \cdot \eta_x(y) \; d\mathcal{L}^{bsg}[y] \\
&= \int_{\mathbb{S}^{D-1}} \eta(x,y) \cdot \triangle\phi(y) \; d\mathcal{L}^{bsg}[y] \\
&= \eta * (\triangle\phi)(x)
\end{aligned}$$

.... □ [Claim 2]

It follows from **Claim 2** that T_η must leave invariant all eigenspaces of $\triangle$; in other words, for all $\lambda \in \Lambda$, $\mathbb{V}_\lambda$ is invariant under T_η. But by **Claim 1**, the restricted map

$$\left(T_\eta\right)_| : \mathbb{V}_\lambda \longrightarrow \mathbb{V}_\lambda$$

is then an isomorphism of linear $\mathbb{G}$-modules. Since $\mathbb{G}$ acts **irreducibly** on $\mathbb{V}_\lambda$ (by Proposition 6), it follows from Schur's Lemma that T_η must act on $\mathbb{V}_\lambda$ by scalar multiplication: thus, there is some $A_\lambda \in \mathbb{C}$ so that, for all $\phi \in \mathbb{V}_\lambda$,

$$T_\eta(\phi) \;=\; A_\lambda \cdot \phi$$

....in other words, $\eta * \phi = A_\lambda \cdot \phi$. In particular, if $\zeta \in \mathcal{Z}_e(\mathbb{V}_\lambda)$, then $\eta * \zeta \;=\; A_\lambda \cdot \zeta$; hence we must have $A_\lambda = \frac{\eta*\zeta(e)}{\zeta(e)}$.

□ [Proposition 11]

Corollary 12: *Let $\zeta \in \mathcal{Z}_e(\mathbb{V}_\lambda)$ be a zonal eigenfunction, normalized so that $\|\zeta\|_2 = 1$. Define $Z : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ by*

$$Z(x,y) \;=\; \zeta(g_x^{-1}.y)$$

where $g_x \in \mathbb{G}$ is any element so that $g_x.e = x$. Then Z is well-defined, independent of the choice of g_x, and is $\mathbb{G}$-equivariant. If we then define $\mathbb{P}_\lambda : \mathbf{L}^2(\mathbb{S}^{D-1}) \longrightarrow \mathbf{L}^2(\mathbb{S}^{D-1})$ by

$$\mathbb{P}_\lambda(\phi) \;=\; \zeta(e) \cdot (Z * \phi)$$

then $\mathbb{P}_\lambda$ is the **orthogonal projection** *from $\mathbf{L}^2(\mathbb{S}^{D-1})$ onto the eigenspace $\mathbb{V}_\lambda$.*

Proof:

Proof of "Well Defined": If $g_1, g_2 \in \mathbb{G}$ so that $g_1.e = g_2.e = x$, then $g_1^{-1}.g_2.e = e$; thus, $g_1^{-1}.g_2 \in \mathbb{G}_e$. Thus, since ζ is zonal about e,

$$\zeta(g_2^{-1}.y) \;=\; \zeta(g_1^{-1}.g_2.g_2^{-1}.y) \;=\; \zeta(g_1^{-1}.y)$$

Proof of "Equivariant": Let $x, y \in \mathbb{S}^{D-1}$, and $h \in \mathbb{G}$. Note that we can pick $g_{(h.x)} = h.g_x$. Thus,

$$\begin{aligned} Z(h.x,\ h.y) &= \zeta(g_{(h.x)}^{-1}.h.y) \;=\; \zeta((h.g_x)^{-1}.h.y) \;=\; \zeta(g_x^{-1}.h^{-1}.h.y) \\ &= \zeta(g_x^{-1}.y) \;=\; Z(x,\ y). \end{aligned}$$

Proof of "Orthogonal Projection": Since $\mathbb{P}_\lambda$ is defined by a convolution integral, it is clearly a linear operator. It then suffices to show that $\mathbb{P}_\lambda$ fixes $\mathbb{V}_\lambda$, and annihilates $\mathbb{V}_\lambda^\perp$.

If $\phi \in \mathbb{V}_\lambda$, then by Proposition 11,

$$Z * \phi \;=\; \frac{(Z * \zeta)(e)}{\zeta(e)} \cdot \phi \qquad\qquad \text{thus,} \quad \mathbb{P}_\lambda(\phi) \;=\; (Z * \zeta)(e) \cdot \phi,$$

so it suffices to show that $(Z * \zeta)(e) = 1$. But:

$$\begin{aligned} Z * \zeta(e) &= \int_{\mathbb{S}^{D-1}} Z(e, y)\zeta(y)\ d\mathcal{L}^{\text{bsg}}[y] \\ &= \int_{\mathbb{S}^{D-1}} \zeta(g_e^{-1}.y) \cdot \zeta(y)\ d\mathcal{L}^{\text{bsg}}[y] \\ &= \int_{\mathbb{S}^{D-1}} \zeta(y) \cdot \zeta(y)\ d\mathcal{L}^{\text{bsg}}[y] \qquad (\text{since } g_e = \mathbf{Id}) \\ &= \|\zeta\|_2^2 \\ &= 1 \qquad\qquad (\text{by hypothesis}) \end{aligned}$$

On the other hand, if $\phi \in \mathbb{V}_\lambda^\perp$, then for all $\mathbf{s} \in \mathbb{S}^{D-1}$,

$$
\begin{aligned}
Z * \phi(\mathbf{s}) &= \int_{\mathbb{S}^{D-1}} \zeta(g_{\mathbf{s}}^{-1}.y) \cdot \phi(y)\ d\mathcal{L}^{\mathsf{bsg}}[y] \\
&= \int_{\mathbb{S}^{D-1}} \left(g_{\mathbf{s}}^{-1}.\zeta\right)(y) \cdot \phi(y)\ d\mathcal{L}^{\mathsf{bsg}}[y] \\
&= \left\langle g_{\mathbf{s}}^{-1}.\zeta, \quad \phi \right\rangle \\
&= 0
\end{aligned}
$$

since $g_{\mathbf{s}}^{-1}.\zeta \in \mathbb{V}_\lambda \perp \phi$.

□ [Corollary 12]

Proposition 13: *(Zonal Eigenfunctions of $\triangle$ on $\mathbb{S}^{D-1}$)* *The eigenvalues of $\triangle$ on $\mathbb{S}^{D-1}$ are all of the form*

$$
\lambda_N = N \cdot (N + D - 2).
$$

for some $N \in \mathbb{N}$. Let ζ_N be a corresponding eigenfunction, and assume that ζ_N is zonal (relative to $\mathbb{SO}^D[\mathbb{R}]$ and e).

Case $D = 2$: *Modulo multiplication by some normalizing constant,*

$$
\zeta_N(\theta) = \cos(N \cdot \theta)
$$

where we use the coordinate system $(0, 2\pi) \ni \theta \mapsto (\cos(\theta),\ \sin(\theta)) \in \mathbb{S}^1$. If we write ζ_N in terms of Cartesian coordinates $\mathbf{x} = (x_1, x_2)$ on $\mathbb{R}^2$, we get the **Čebyšev polynomials:**

$$
\zeta_N(\mathbf{x}) = 2^{(N-1)} x^N + \sum_{n=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^n 2^{(N-1-2n)} \frac{N}{n} \binom{N-n-1}{n-1} x_1^{(N-2n)}. \tag{5}
$$

Case $D = 3$: *Modulo multiplication by some constant, ζ_N is a* **Legendre Polynomial:**

$$
\zeta_N(\mathbf{x}) = \sum_{n=0}^{\lfloor N/2 \rfloor} (-1)^n 2^{N-2n} \frac{\Gamma\left[\frac{1}{2} + N - n\right]}{\Gamma\left[\frac{1}{2}\right] \cdot n! \cdot (N-2n)!} \cdot x^{N-2n}
$$

General Case: *Assume that ζ_N is of unit norm. Then ζ_N is a normalized* **Gegenbauer polynomial***:*

$$\begin{aligned}
\zeta_N(\mathbf{x}) &= \frac{1}{K_N^{(\nu)}} \, C_N^{(\nu)}(x_1) \\
\textit{where } \; C_N^{(\nu)}(x) &= \sum_{n=0}^{\lfloor N/2 \rfloor} (-1)^n 2^{N-2n} \cdot c_{N;n}^{(\nu)} \cdot x^{N-2n} \\
\textit{with } \; c_{N;n}^{(\nu)} &= \frac{\Gamma\left(\nu + (N-n)\right)}{\Gamma\left(\nu\right) \cdot n! \cdot (N-2n)!} \\
\textit{and where } \; \left(K_N^{(\nu)}\right)^2 &= \int_{\mathbb{S}^{D-1}} \left| C_N^{(\nu)}(x_1) \right|^2 d\mathbf{x} \\
&= \frac{2 \cdot \pi^{(D-1)/2}}{\Gamma(\nu)^2} \cdot \sum_{k=0}^{2 \cdot \lfloor N/2 \rfloor} (-1)^k \cdot 2^{2N-2k} \cdot \frac{\Gamma\left(N-k+\frac{1}{2}\right)}{\Gamma\left(N-k+\frac{D}{2}\right)} \\
&\qquad \cdot \left(\sum_{n=0}^{k} c_{N;n}^{(\nu)} c_{N;(k-n)}^{(\nu)} \right),
\end{aligned}$$

where $\nu := \frac{D-2}{2}$.

Proof:

Proof of Characterization of Eigenvalues: See, for example, [51], chapter 6, [47], chapter 3, or [35].

Proof of Case $D = 2$: It is clear from the definition of the Laplacian on $\mathbb{S}^1$ that the function ζ_N is an eigenfunction of $\triangle\mathbb{S}^1$. The subgroup of $\mathbb{SO}^2[\mathbb{R}]$ fixing e is just the two-element group of maps $(x_1, x_2) \mapsto (x_1, \pm x_2)$; since the function ζ_N is symmetric relative to the x_2 variable, it is zonal relative to these maps.

The formula (5) is then just a standard trigonometric identity, where we identify $x_1 = \cos(\theta)$; see, for example [23], §1.33 #3, p. 27.

Proof of Case $D = 3$: This is just the Gegenbauer polynomial when $D = 3$. For a direct proof, see, for example [3], Theorem 1, §2.1, p. 90, where there is unfortunately an error in the definition of the Legendre functions —see [45], §1, p.2, for a correct definition).

Proof of General Case: This is just a big computation. See [47] or [35].

$\square$ [Proposition 13]

3 Spherical Fourier Series

Theorem 14: *(Spherical Fourier Analysis)*

For all $n \in \mathbb{N}$, let $\zeta_n : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ be the zonal harmonic polynomials defined by Proposition 13, and then define $\mathcal{Z}_n : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ by

$$\mathcal{Z}_n(\mathbf{x}, \mathbf{y}) \;=\; \zeta_n(e) \cdot \zeta_n(\langle \mathbf{x}, \mathbf{y} \rangle)$$

Then $\mathcal{Z}_n$ is rotationally equivariant.

Now, suppose $\gamma \in \mathbf{L}^2(\mathbb{S}^{D-1};\ \mathbb{C})$. If we define $\gamma_n \;:=\; \mathcal{Z}_n * \gamma$ then $\gamma_n \in \mathbb{V}_{(\lambda_n)}$, and γ has the orthogonal decomposition:

$$\gamma \;=\; \sum_{n=1}^{\infty} \gamma_n. \tag{6}$$

Proof: This follows from Theorem 8, and Corollary 12, using the zonal functions provided by Proposition 13.

$\square$ [Theorem 14]

Corollary 15: *((De)convolution on Spheres)*

Suppose $\eta : \mathbb{S}^{D-1} \times \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ is rotationally equivariant, and suppose that $\mathbf{g} := \eta * \gamma$. If, for all $n \in \mathbb{N}$, ζ_n and $\mathcal{Z}_n$ are as in Theorem 14, and we define

$$\mathbf{g}_n \;:=\; \mathcal{Z}_n * \mathbf{g} \quad \text{and} \quad A_n \;:=\; \frac{(\eta * \zeta_n)(\mathbf{e}_1)}{\zeta_n(\mathbf{e}_1)}$$

then $\mathbf{g}_n = A_n \cdot \gamma_n$.

Conversely, suppose that γ is unknown, but we know η and $\mathbf{g}$. We can reconstruct γ via the formula:

$$\gamma \;=\; \sum_{n=1}^{\infty} \frac{1}{A_n} \mathbf{g}_n$$

Proof: This follows from the previous theorem, and Proposition 11.

$\square$ [Corollary 15]

Definition 16: *Spherical Fourier Coefficients*

If $\gamma \in \mathbf{L}^2(\mathbb{S}^{D-1})$, then the **spherical Fourier Coefficients** *of γ are the functions $\gamma_n := \mathcal{Z}_n * \gamma$, for $n \in \mathbb{N}$. (Notice that these "coefficients" are themselves functions, not numbers). The* **spherical Fourier series** *for γ is then the orthogonal decomposition*

$$\gamma = \sum_{n=1}^{\infty} \gamma_n.$$

Example 17: *(Spherical Fourier series on $\mathbb{S}^1$)*

Let for $N \in \mathbb{N}$, let $\zeta_N : \mathbb{S}^1 \longrightarrow \mathbb{C}$ be as in **Part 1** of Proposition 13:

$$\zeta_N(\theta) \quad = \quad \cos(N\theta) \quad = \quad \frac{1}{2}\left(\mathcal{E}_N(\theta) + \mathcal{E}_{(-N)}(\theta)\right)$$

where we identify $\mathbb{S}^1 \cong [0, 2\pi)$, and define $\mathcal{E}_K(\theta) := \exp(K\theta \cdot \mathbf{i})$. Let $\mathcal{Z}_N : \mathbb{S}^1 \times \mathbb{S}^1 \longrightarrow \mathbb{C}$ be defined from ζ_N as in Theorem 15. Then, for any $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{R}$,

$$\begin{aligned}
\gamma_N &= \mathcal{Z}_N * \gamma \\
&=_{(1)} \gamma * \zeta_N \;=\; \frac{1}{2}\left(\gamma * \mathcal{E}_N \;+\; \gamma * \mathcal{E}_{(-N)}\right) \\
&=_{(2)} \frac{1}{2}\left(\widehat{\gamma}(N) \cdot \mathcal{E}_N \;+\; \widehat{f}(-N) \cdot \mathcal{E}_{(-N)}\right) \\
&=_{(3)} \frac{1}{2}\left(\widehat{\gamma}(N) \cdot \mathcal{E}_N \;+\; \overline{\widehat{\gamma}(N) \cdot \mathcal{E}_N}\right) \\
&= \mathbf{re}\left[\widehat{\gamma}(N) \cdot \mathcal{E}_N\right].
\end{aligned}$$

(1) where the convolution is now meant in the "usual" sense on the group $\mathbb{S}^1 = \mathbb{T}^1$.

(2) here, $\widehat{\gamma}$ is the (classical) Fourier transform of γ as a function on the circle.

(3) because γ is real-valued.

Now, if we write $\widehat{\gamma}(N) = r_N \exp(\phi_N \cdot \mathbf{i})$, where $r_N \in [0, \infty)$ and $\phi_N \in [0, 2\pi)$, then, for any $\theta \in \mathbb{S}^1 \cong [0, 2\pi)$, we have:

$$
\begin{aligned}
\gamma_N(\theta) &= \mathbf{re}\left[r_N \cdot \exp(\phi_N \mathbf{i}) \cdot \mathcal{E}_N(\theta)\right] \\
&= r_N \cdot \mathbf{re}\left[\exp(\phi_N \mathbf{i}) \cdot \exp((N \cdot \theta \cdot \mathbf{i})\right] \\
&= r_N \cdot \mathbf{re}\left[\exp\left(N \cdot \left(\theta + \frac{\phi_N}{N}\right) \cdot \mathbf{i}\right)\right] \\
&= r_N \cdot \mathbf{re}\left[\mathcal{E}_N\left(\theta + \frac{\phi_N}{N}\right)\right] \\
&= r_N \cdot \zeta_N\left(\theta + \frac{\phi_N}{N}\right).
\end{aligned}
$$

In other words, convolving ζ_N by γ is equivalent to multiplying the magnitude of ζ_N by r_N, and rotating the phase by ϕ_N/N.

4 Asymptotic Decay and Convergence Rates

In classical harmonic analysis, the infinitesimal properties of a function f are in many ways reflected in the asymptotic behaviour of its Fourier transform, and vice versa. Generally, the smoother f is, the more rapidly $\widehat{f}$ decays near infinity. Conversely, if f is very "jaggy", undifferentiable, or discontinuous, then $\widehat{f}$ decays slowly or not at all near infinity, reflecting a concentration of the "energy" of f in high frequency Fourier components.

Hence, when approximating f by a partial Fourier sum, the more jaggy f is, the more slowly the sum converges, and the more terms we must include to ensure ourselves of a good approximation.

A similar phenomenon manifests when approximating a functions $\gamma : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$ by a spherical Fourier series. By relating the decay rate of the spherical Fourier series to the smoothness of γ, we will be able to estimate the error introduced by approximating γ with a partial spherical Fourier sum.

Say that a sequence of functions $[\gamma_n|_{n=1}^{\infty}]$ is of **order** less than or equal to $\mathcal{O}(n^{-\alpha})$ if

$$
\lim_{n \to \infty} \frac{\|\gamma_n\|_2}{n^\alpha} < \infty
$$

(with the limit possibly zero).

Theorem 18: *Let $\gamma : \mathbb{S}^{D-1} \longrightarrow \mathbb{C}$, and suppose that γ is continuously $2M$-differentiable. Then the sequence $[\gamma_n|_{n=1}^{\infty}]$ is of order less than or equal to $\mathcal{O}(n^{-(2M+1)})$*

Proof: First suppose that γ is twice continuously differentiable. Thus, using formula (4) on page 9, we can apply $\triangle_{\mathbb{S}^{D-1}}$ to γ. Let $\alpha = \triangle_{\mathbb{S}^{D-1}}\gamma$. Since α is a continuous function, it is in $\mathbf{L}^2(\mathbb{S}^{D-1})$, and we can compute the spherical Fourier coefficients $\alpha_n = \mathcal{Z}_n * \alpha$, for all n, and conclude:

$$\alpha \;=\; \sum_{n=1}^{\infty} \alpha_n.$$

In particular, since this sum converges absolutely in $\mathbf{L}^2(\mathbb{S}^{D-1})$, we know that the sequence $[\alpha_n|_{n=1}^{\infty}]$ is of order less than $\mathcal{O}(n^{-1})$. By construction, we know that $\gamma_n = \mathcal{Z}_n * \gamma$ is an eigenfunction of $\triangle_{\mathbb{S}^{D-1}}$, with eigenvalue $\lambda_n = n(n+D-2)$. By Claim 2 of Proposition 11, the Laplacian operator commutes with convolution operators. Thus,

$$\begin{aligned}
n(n+D-2)\gamma_n &= \triangle_{\mathbb{S}^{D-1}}\gamma_n \\
&= \triangle_{\mathbb{S}^{D-1}}(\mathcal{Z}_n * \gamma) \\
&= \mathcal{Z}_n * (\triangle_{\mathbb{S}^{D-1}}\gamma) \\
&= \mathcal{Z}_n * \alpha \\
&= \alpha_n
\end{aligned}$$

Since this is true for all n, we conclude that $[\gamma_n|_{n=1}^{\infty}]$ is of order less than or equal to $\mathcal{O}\left(\frac{1}{n(n+D-2)}\right) \cdot \mathcal{O}(n^{-1}) \;=\; \mathcal{O}(n^{-3})$.

Proceed inductively to prove the general case.

□ [Theorem 18]

Conclusion

By expressing the log characteristic function $\mathbf{g}$ of equation (3) as a spherical Fourier series via Theorem 14, and then applying the "deconvolution" formula provided by Corollary 15, we can reconstruct a spherical Fourier series for the spectral measure Γ. Of course, for practical purposes, we can only ever compute a finite number of terms of this series. The degree of approximation error introduced by finitely truncating the spherical Fourier series is determined by the asymptotic decay rate of the coefficients. As with classical Fourier series, this decay rate is a function of the "smoothness" of Γ (Theorem 18). For an extremely singular Γ, unsurprisingly, the coefficients may decay in size slowly, so the series will take a long time to converge to a good approximation.

The advantage of this approach is that, once we have expressed $\mathbf{g}$ in terms of its spherical Fourier series, computing Γ is extremely straightforward; we need only divide the Fourier series of $\mathbf{g}$ by the constants A_n of Corollary 15. Computation of the Fourier series, in turn, involves convolution with Gegenbauer polynomials. A closed-form expression for these polynomials is given (Theorem 13), and the convolution can be computed by numerical integration over $\mathbb{S}^{D-1}$, a task with complexity $\mathcal{O}\left(N^{2(D-1)}\right)$, to be contrasted with the $\mathcal{O}\left(N^{3(D-1)} + N^{2(D-1)}\right)$ required by an explicit matrix-inversion approach (where $N \sim 1/\epsilon$ reflects a precision of ϵ in our approximation).

Through a linear combination of spherical harmonics, we can explicitly represent Γ as a *continuous* object on $\mathbb{S}^{D-1}$, rather than as a sum of atoms. Of course, if Γ in reality *was* discrete, this representation might be misleading, and a discrete representation might actually be preferable. However, in many cases Γ is absolutely continuous, relative to the Lebesgue measure —for example, if the stable distribution is *sub-Gaussian* (see [41], §2.5). In these cases, an explicitly continuous representation may be preferable to avoid introducing anomalous asymptotic behaviour to the distribution.

Marcus Pivato
Department of Mathematics,
University of Toronto,
100 St. George Street,
Toronto, Ontario,
M5S 3G3
CANADA

Email: pivato@math.toronto.edu
URL: http://www.math.toronto.edu/~pivato/

Phone: (416) 535 6523
Fax: (416) 978 4107

Luis Seco
University of Toronto,
100 St. George Street,
Toronto, Ontario,
M5S 3G3
CANADA

Email: seco@math.toronto.edu
URL: www-risklab.erin.utoronto.ca/seco/index.htm
Phone: (905) 828 5392
Fax: (905) 828 5425

References

[1] A. Araujo and E. Giné. *The Central Limit Theorem for Real and Banach Valued Random Variables.* Wiley, New York, 1980.

[2] A. Erdelyi *et al.* *Higher Transcendental Functions*, volume I, II, III. McGraw-Hill, New York, 1953-1955.

[3] Audrey Terras. *Harmonic Analysis on Symmetric Spaces and Applications*, volume I. Springer-Verlag, New York, 1985.

[4] B.D. Fielitz and E.W. Smith. Asymmetric stable distributions of stock price change. *Journal of the American Statistical Association (Applications)*, 67(340):813–814, 1972.

[5] B.N. Cheng and S.T. Rachev. Multivariate stable securities in financial markets. *Mathematical Finance.*

[6] C. Müller. *Spherical Harmonics.* Number 17 in Lecture Notes in Mathematics. Springer-Verlag, New York, 1966.

[7] Isaac Chavel. *Riemannian Geometry: A modern introduction.* Cambridge UP, Cambridge, Massachussetts, 1 edition, 1993.

[8] Chrysostomos L. Nikias and A.P. Petropulu. *Higher order spectral analysis: a Nonlinear signal processing environment.* Prentice-Hall, Englewood Cliffs, New Jersey, 1993.

[9] Chrysostomos L. Nikias and M. Shao. *Signal Processing with Alpha-stable distributions and Applications*. John Wiley & Sons,, New York, 1995.

[10] John B. Conway. *A Course in Functional Analysis*. Springer-Verlag, New York, second edition, 1990.

[11] V.S. Bawa E.J. Elton and M.J. Gruber. Simple rules for optimal portfolio selection in stable paretian markets. *The Journal of Finance*, 34:1041–1047, 1979.

[12] Lawrence C. Evans. *Partial Differential Equations*. American Mathematical Society, Providence, Rhode Island, 1998.

[13] Eugene F. Fama. Mandelbrot and the stable paretian hypothesis. *Journal of Business (Chicago)*, 36:420–429, 1963.

[14] Eugene F. Fama. The distribution of daily stock prices: A test of mandelbrot's stable paretian hypothesis. *Journal of Business (Chicago)*, 38:34–105, 1965.

[15] Eugene F. Fama. Portfolio analysis in a stable paretian market. *Management Science*, 6:404–419, 1965.

[16] E. Feldheim. *Etude de la stabilité des lois de probabilitié*. PhD thesis, Thése de la Faculté des Sciences de Paris, 1937.

[17] William Feller. *An Introduction to probability theory and its applications, vol II*. John Wiley and Sons, Inc., New York, second edition, 1966.

[18] Andrea Gamba. Portfolio analysis with symmetric stable paretian returns. In Elio Canestrelli, editor, *Current Topics in Quantitative Finance*, pages 48–69. Physica-Verlag, Heidelberg, 1999.

[19] Andrea Gamba. Portfolio analysis with symmetric stable paretian returns. In *Current topics in quantitative finance*, Contributions in Management Science. Physica, Heidelberg, 1999.

[20] H. Dym and P. McKean. *Fourier Series and Integrals*. Academic Press, New York, 1972.

[21] Sigurdur Helgason. *Topics in Harmonic Analysis on Homogeneous Spaces*. Birkhäuser, Boston, Massachusetts, 1981.

[22] J. Holtsmark. Über die verbreiterung von spektrallinien. *Ann. Physik*, 4(58 (363)):577–630, 1919.

[23] I. S. Gradshteyn and I. M. Ryzhik. *Tables of Integrals, Series, and Products*. Academic Press, Toronto, corrected and enlarged edition, 1980.

[24] John P. Nolan and Anna K. Panorska. Data analysis for heavy tailed multivariate samples. *Communications in Statistics: Stochastic Models*, 13(4):687–702, 1997.

[25] J. Huston McCulloch John P. Nolan, Anna K. Panorska. Estimation of stable spectral measures. *(submitted)*, 1997.

[26] J. Kuelbs. A representation theorem for symmetric stable processes and symmetric measures on h. *Zeitschrift für Wahrscheinlichtkeitstheorie und verwalte Gebiete*, 26:259–271, 1973.

[27] N. N. Lebedev. *Special Functions and Applications*. Dover, New York, 1972.

[28] Benoit B. Mandelbrot. The variation of certain speculative prices. In *Fractals and Scaling in Finance: Discontinuities, Concentration, Risk*, New York, (orig. 1965). Springer Verlag.

[29] Rosario N. Mantegna and H. Eugene Stanley. Scaling behaviour in the dynamics of an economic index. *Nature*, 376(6):46–49, July 1995.

[30] J. Huston McCulloch. Financial applications of stable distributions. In *Statistical Methods in Finance*, volume 14 of *Handbook of Statistics*, pages 393–425. North-Holland, Amsterdam, 1996.

[31] Walter Willinger Murad S. Taqqu and A. Erramilli. A bibliographical guide to self-similar traffic and performance modelling for modern high-speed networks. In F.P.Kelly S. Zachary and I. Ziedins, editors, *Stochastic Networks: Theory and Applications*, pages 339–366. Clarendon Press, Oxford, UK, 1996.

[32] W.E. Leland Murad S. Taqqu and Walter Willinger. On the self-similar nature of ethernet traffic. *IEEE/ACM Trans. Networking*, 2:1–15, 1994.

[33] John P. Nolan. Multivariate stable distributions: approximation, estimation, simulation, and identification. *(preprint)*, 1999.

[34] P. Tsakalides and C. Nikias. Maximum likelihood localization of sources in noise modelled as a stable process. *IEEE Transactions on Signal Processing*, 43, 1995.

[35] Marcus Pivato. *Analytical Methods for Multivariate Stable Probability Distributions*. PhD thesis, University of Toronto, 2001 (forthcoming).

[36] James S. Press. Estimation in univariate and multivariate stable distributions. *Journal of the American Statistical Association (Theory and Methods)*, 67(340):842–846, 1972.

[37] James S. Press. Multivariate stable distributions. *Journal of Multivariate Analysis*, 2:444–462, 1972.

[38] R. R. Coifman and G. Wiess. Representations of compact groups and spherical harmonics. *L'enseignment Math.*, 14:123–173, 1968.

[39] R.Courant and D. Hilbert. *Methods of Mathematical Physics*, volume I. Wiley Interscience, New York, 1961.

[40] S. Cambanis and G. Miller. Linear problems in p-th order stable processes. *SIAM Journal of Applied Mathematics*, 43(1):43–69, August 1981.

[41] Gennady Samorodnitsky and Murad S. Taqqu. *Stable Non-Gaussian random processes: stochastic models with infinite variance.* Chapman and Hall, New York, 1994.

[42] P.A. Samuelson. Efficient portfolio selection for pareto-levy investments. *Journal of Financial and Quantitative Analysis*, 2:107–122, 1967.

[43] M. Sugiura. *Unitary Representations and Harmonic Analysis.* Wiley, New York, 1975.

[44] Svetlozar T. Rachev and Huang Xin. Test for association of random variables in the domain of attractiono of multivariate stable law. *Probability and Mathematical Statistics*, 14(1):125–141, 1993.

[45] Masaru Takeuchi. *Modern Spherical Functions*, volume 135 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, Rhode Island, 1994.

[46] J. D. Talman. *Special Functions.* Benjamin, New York, 1968.

[47] Michael Eugene Taylor. *Noncommutative Harmonic Analysis.* American Mathematical Society, Providence, Rhode Island, 1986.

[48] Theodor Brocker and Tammo tom Dieck. *Representations of Compact Lie Groups.* Springer-Verlag, New York, 1 edition, 1985.

[49] John P. Nolan Tomasz Byczkowski and Balram Rajput. Approximation of multidimensional stable densities. *Journal of Multivariate Analysis*, 46:13–31, 1993.

[50] N.J. Vilenkin. *Special Functions and the Theory of Group Representations*, volume 22 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, Rhode Island, 1968.

[51] Frank M. Warner. *Foundations of Differentiable Manifolds and Lie Groups*. Springer-Verlag, New York, 1983.

[52] Walter Willinger and Vern Paxson. Where mathematics meets the internet. *Notices of the Americal Mathematical Society*, 45(8):961–970, September 1998.

[53] V.M. Zolotarev. *One-dimensional Stable Distributions*. American Mathematical Society, Providence, Rhode Island, 1986.