Infinite horizon, nondiscounted integenerational social choice under uncertainty
or
Additive representation of separable preferences over infinite products

Informal Microeconomic Theory Seminar
Université de Montréal

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Problem 1. Can we represent nondiscounted, time-separable, infinite-horizon intertemporal preferences, using an additive utility function?
In intergenerational social choice (e.g. environmental policy), discounting is ethically indefensible. But nondiscounted utility sums over an infinite number of future periods are generally infinite or undefined.

Problem 2. Can we formalize the 'Principle of Insufficient Reason' (i.e. uniform probability distribution) when there are infinitely many possible states of nature? (Important for choice under uncertainty/ambiguity.)

Problem 3. Can we axiomatize the utilitarian social welfare function when the population size is variable?

Fact: A continuous, separable preference order on $\mathbb{R}^N$ (for $3 \leq N < \infty$) can be represented using an additive utility function.
Problem 4. Can we extend this to the case when N is (uncountably) infinite? Can we eliminate the topological conditions?

Goal. We will use nonstandard analysis and the theory of linearly ordered abelian groups to answer these questions.

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Let $\mathcal{X}^{\mathcal{I}}$ be the set of all $\mathcal{I}$-indexed sequences of elements from $\mathcal{X}$.
An element $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ assigns an outcome x_i to each i in $\mathcal{I}$.
The space $\mathcal{X}^{\mathcal{I}}$ has at least three interpretations:

(i) *Intertemporal choice.* $\mathcal{I} :=$ an infinite sequence of moments in time (e.g. $\mathcal{I} = \mathbb{N}$ or $\mathcal{I} = \mathbb{R}_+$). $\mathcal{X} :=$ the set of possible outcomes which could happen at each moment. Thus, $\mathbf{x} =$ a history where outcome x_i happens at time i (e.g. a consumption stream).

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(iii) *Variable population social choice.* $\mathcal{I} :=$ infinite set of 'potential people'. $\mathcal{X} :=$ set of possible personal outcomes available to each person, including an outcome o ('nonexistence'). If $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $x_i = o$ for all but finitely many coordinates, then $\mathbf{x}$ represents a finite (but arbitrarily large) population.

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One can also combine these interpretations:

(iv) *Variable population intertemporal social choice under uncertainty.* (Example: anthropogenic climate change, nuclear waste disposal, etc.)

Let $\mathcal{T}$ represent a time-stream (e.g. $\mathcal{T} := \mathbb{N}$).

Let $\mathcal{S}$ be a set of possible 'states of nature'.

Let $\mathcal{P}$ be a set of 'possible people'.

Suppose at least one of $\mathcal{T}$, $\mathcal{S}$, or $\mathcal{P}$ is infinite, and let $\mathcal{I} := \mathcal{T} \times \mathcal{S} \times \mathcal{P}$.

Let $\mathcal{X}$ be a space of personal outcomes, including a 'nonexistence' outcome o.

Then an element $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ represents a policy which will assign personal outcome $x_{t,s,p}$ to person p at time t, if the state of nature s occurs (for every $t \in \mathcal{T}$, $s \in \mathcal{S}$, and $p \in \mathcal{P}$).

One can also combine these interpretations:

(iv) *Variable population intertemporal social choice under uncertainty.* (Example: anthropogenic climate change, nuclear waste disposal, etc.)

Let $\mathcal{T}$ represent a time-stream (e.g. $\mathcal{T} := \mathbb{N}$).

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Permutation invariance

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For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, if $\pi \in \Pi_{\mathrm{fin}}$, then $\mathcal{I}(\mathbf{x}, \pi(\mathbf{x})) \subseteq \mathcal{I}(\pi)$, so $d(\mathbf{x}, \pi(\mathbf{x})) < \infty$.

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* To make outcome x 'twice as likely' as outcome y, map twice as many states to x.

A *permutation* is a function $\pi : \mathcal{I} \longrightarrow \mathcal{I}$ that is one-to-one & onto (bijective).

Let $\mathcal{I}(\pi) := \{i \in \mathcal{I};\ \pi(i) \neq i\}$. We say π is *finitary* if $\mathcal{I}(\pi)$ is finite.

Let $\Pi_{\rm fin}$ be the group of all finitary permutations of $\mathcal{I}$.

For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, if $\pi \in \Pi_{\rm fin}$, then $\mathcal{I}(\mathbf{x}, \pi(\mathbf{x})) \subseteq \mathcal{I}(\pi)$, so $d(\mathbf{x}, \pi(\mathbf{x})) < \infty$.

Thus, a finitary preorder $(\succeq)$ can compare $\mathbf{x}$ to $\pi(\mathbf{x})$.

Say that $(\succeq)$ is $\Pi_{\rm fin}$-*invariant* if $\mathbf{x} \approx \pi(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $\pi \in \Pi_{\rm fin}$.

- In interpretation (i) (*Intertemporal choice*), $\Pi_{\rm fin}$-invariance means there are *no time preferences*: the near and far future are equally important.
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Permutation invariance

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Separable preferences

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The preorder $(\succeq)$ is *separable* if the following holds: for any $\mathcal{J} \subset \mathcal{I}$, with $\mathcal{K} := \mathcal{I} \setminus \mathcal{J}$, and for every $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}' \in \mathcal{X}^{\mathcal{I}}$ such that

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(In Savage's risky decision theory, separability is called the *sure thing principle* or Axiom P2. In social choice, separability is a special case of the axiom of *independence of* (or *elimination of*) *indifferent individuals*, which in turn is a special case of the *Extended Pareto* axiom.)

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Separable preferences

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An *abelian group* is a set $\mathcal{R}$ equipped with a binary operator "$+$" with the following properties:

- There is an *identity* element $0 \in \mathcal{R}$ such that $0 + r = r$ for all $r \in \mathcal{R}$.
- For every $r \in \mathcal{R}$, there is an *inverse* $-r \in \mathcal{R}$ such that $r + (-r) = 0$.
- $+$ is *commutative*: $r + s = s + r$ for all $r, s \in \mathcal{R}$.
- $+$ is *associative*: $r + (s + t) = (r + s) + t$ for all $r, s, t \in \mathcal{R}$.

Example: The set $\mathbb{R}$ of real numbers is an abelian group under addition. So is the set $\mathbb{Z}$ of integers, and the set $\mathbb{Q}$ of rational numbers. For any $N \in \mathbb{N}$, the space $\mathbb{R}^N$ is an abelian group under vector addition.

A *linear order* on $\mathcal{R}$ is a transitive binary relation $(>)$ such that, for all $r, s \in \mathcal{R}$:

- either $r > s$ or $s > r$, but not both ($(>)$ is *complete* & *antisymmetric*).
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For example: the standard order on $\mathbb{R}$, $\mathbb{Z}$, or $\mathbb{Q}$ is a linear order. Also, the *lexicographical order* $(\gg)$ on $\mathbb{R}^N$ is a linear order. (For any $\mathbf{r}, \mathbf{s} \in \mathbb{R}^N$, we have $\mathbf{r} \gg \mathbf{s}$ if there is some $n \in [1 \ldots N]$ such that $r_m = s_m$ for all $m < n$, while $r_n > s_n$.)

Linearly ordered abelian groups

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In fact, *Hahn's Embedding Theorem* says that any linearly ordered abelian group can be represented as an ordered subgroup of a lexicographically ordered vector space $\mathbb{R}^{\Omega}$ (where Ω could be infinite).

Heuristically, a linearly ordered group $(\mathcal{R}, +, >)$ is a 'measurement scale'. For example, a function $u : \mathcal{X} \longrightarrow \mathcal{R}$ can be treated as a *cardinal utility function*: we can meaningfully make statements like "$u(x) + u(y) > u(z)$".

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $u(x_i) - u(y_i) = 0$ for all $i \in \mathcal{I} \setminus \mathcal{I}(\mathbf{x}, \mathbf{y})$. Thus, if $d(\mathbf{x}, \mathbf{y}) < \infty$, then

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Linearly ordered abelian groups; the additive preorder

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(i) In intertemporal choice, $\sum_{i\in\mathcal{I}} \Big(u(x_i) - u(y_i)\Big)$ is the nondiscounted sum of future u-utility differences between histories $\mathbf{x}$ and $\mathbf{y}$.

(ii) In risky choice, $\sum_{i\in\mathcal{I}} \Big(u(x_i) - u(y_i)\Big)$ is the difference between the expected u-utility of lottery $\mathbf{x}$ and that of lottery $\mathbf{y}$ (assuming a uniform probability distribution on $\mathcal{I}(\mathbf{x}, \mathbf{y})$).

(iii) In social choice, ($\underset{u}{\succeq}$) is a generalized utilitarian social welfare order.

(iv) In risky intertemporal social choice, ($\underset{u}{\succeq}$) is the nondiscounted intertemporal Harsanyi utilitarian social welfare order.

We now come to our first main result.

Additive preorder

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(b) *Furthermore, $\mathcal{R}$ and u can be built with a universal property: if $(\mathcal{R}', +, >)$ is another linearly ordered abelian group, and $(\succeq)$ is also the additive preorder defined by some function $u' : \mathcal{X} \longrightarrow \mathcal{R}'$, then there exists $r' \in \mathcal{R}'$ and an order-preserving group homomorphism $\psi : \mathcal{R} \longrightarrow \mathcal{R}'$ such that $u'(x) = \psi[u(x)] + r'$ for all $x \in \mathcal{X}$.*

Proof sketch. Fix $o \in \mathcal{X}$, and let $\mathbf{o} \in \mathcal{X}^{\mathcal{I}}$ be the element with o in every coordinate. Let $\mathcal{A}$ be the free abelian group generated by $\mathcal{X} \setminus \{o\}$. (An element of $\mathcal{A}$ has the form "$J_1 x_1 + \cdots + J_n x_n$", where $J_1, \ldots, J_n \in \mathbb{Z}$ and $x_1, \ldots, x_n \in \mathcal{X} \setminus \{o\}$.)
If $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, and $d(\mathbf{x}, \mathbf{o}) < \infty$, then $\mathbf{x}$ defines an element $a_{\mathbf{x}} \in \mathcal{A}$ in the obvious way. (Example: if $\mathbf{x} = (x, x, x, y, y, o, o, o, \ldots)$, then $a_{\mathbf{x}} = 3\,x + 2\,y$). The preorder $(\succeq)$ then induces a preorder $(\underset{*}{\succeq})$ on $\mathcal{A}$.
Let $\mathcal{C}_0 := \{a \in \mathcal{A};\ a \underset{*}{\approx} 0\}$; then $\mathcal{C}_0$ is a subgroup of $\mathcal{A}$.
Let $\mathcal{R} := \mathcal{A}/\mathcal{C}_0$; then $\mathcal{R}$ is a linearly ordered abelian group. Define $u : \mathcal{X} \longrightarrow \mathcal{R}$ by treating $\mathcal{X}$ as a subset of $\mathcal{A}$, and applying quotient map.

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Theorem 1 applies to choices between alternatives which differ at only finitely many $\mathcal{I}$-coordinates. However, it is insufficient for choice problems which implicate infinitely many coordinates.

To fix this, we will use methods from nonstandard analysis.

Let $\mathcal{R}$ be a linearly ordered abelian group. One can construct a larger linearly ordered group ${}^*\mathcal{R}$ by supplementing $\mathcal{R}$ with a rich collection of 'infinite' and 'infinitesimal' elements with their own well-defined arithmetic. (Formally, ${}^*\mathcal{R}$ is an ultrapower of $\mathcal{R}$; more details later.) For example, if $\mathcal{R}$ is the additive group $\mathbb{R}$ of real numbers, then ${}^*\mathbb{R}$ is the additive group of *hyperreal* numbers.

For any function $u : \mathcal{X} \longrightarrow \mathcal{R}$ and any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, it is possible to evaluate the 'sum' ${}^*\sum_{i \in \mathcal{I}} u(x_i)$ as an element of ${}^*\mathcal{R}$ in a unique and well-defined way.

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$\left({}^{*}\!\underset{u}{\succeq} \right)$ is a *complete*, Π_{fin}-invariant, separable preorder on $\mathcal{X}^{\mathcal{I}}$, whose finitary part is the additive preorder $\left(\underset{u}{\succeq} \right)$.

Also, $\left({}^{*}\!\underset{u}{\succeq} \right)$ satisfies a weak continuity condition called $\mathfrak{U}\mathfrak{F}$-*continuity*. (*Roughly:* if $\mathbf{x}_{\mathcal{J}} \underset{u}{\succeq} \mathbf{y}_{\mathcal{J}}$ for 'almost all' finite subsets $\mathcal{J} \subseteq \mathcal{I}$, then $\mathbf{x} \underset{u}{\succeq} \mathbf{y}$. Precise definition given later).

Theorem 2. *Let $(\succeq)$ be a preorder on $\mathcal{X}^{\mathcal{I}}$. Then*

(a) *$(\succeq)$ is Π_{fin}-invariant, separable and $\mathfrak{U}\mathfrak{F}$-continuous if and only if there exists some linearly ordered abelian group $(\mathcal{R},+,>)$ and some function $u : \mathcal{X} \longrightarrow \mathcal{R}$ such that $(\succeq) = \left({}^{*}\!\underset{u}{\succeq} \right)$.*

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Hyperadditive preorders

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Also, $(\overset{*}{\underset{u}{\succeq}})$ satisfies a weak continuity condition called $\mathfrak{UF}$-*continuity*. (*Roughly:* if $\mathbf{x}_{\mathcal{J}} \underset{u}{\succeq} \mathbf{y}_{\mathcal{J}}$ for 'almost all' finite subsets $\mathcal{J} \subseteq \mathcal{I}$, then $\mathbf{x} \underset{u}{\succeq} \mathbf{y}$. Precise definition given later).

Theorem 2. *Let $(\succeq)$ be a preorder on $\mathcal{X}^{\mathcal{I}}$. Then*

(a) *$(\succeq)$ is Π_{fin}-invariant, separable and $\mathfrak{UF}$-continuous if and only if there exists some linearly ordered abelian group $(\mathcal{R},+,>)$ and some function $u:\mathcal{X}\longrightarrow\mathcal{R}$ such that $(\succeq) = (\overset{*}{\underset{u}{\succeq}})$.*

(b) *$\mathcal{R}$ and u can be built with same universal property as in Theorem 1.*

In what sense does u represents individual preferences in Theorems 1 and 2?

Let $x \in \mathcal{X}$, let $i \in \mathcal{I}$, and let $\mathbf{z}_{-i} \in \mathcal{X}^{\mathcal{I}\setminus\{i\}}$. Let $(x, \mathbf{z}_{-i})$ be the element of $\mathcal{X}^{\mathcal{I}}$ which has x in the ith coordinate and $\mathbf{z}_{-i}$ in the other coordinates.

Let $(\succeq)$ be a separable, Π_{fin}-invariant, finitary preorder on $\mathcal{X}^{\mathcal{I}}$.

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Proposition 3. *Let $(\mathcal{R}, +, >)$ be a linearly ordered group, let $u : \mathcal{X} \longrightarrow \mathcal{R}$, and let $(\succeq)$ be the (hyper)additive preorder on $\mathcal{X}^{\mathcal{I}}$ defined by u. Then:*

(a) *u is an ordinal utility function for $(\underset{1}{\succeq})$:* $\quad (x \underset{1}{\succeq} y) \Leftrightarrow (u(x) \geq u(y))$.

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Strong Pareto/dominance property

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In Theorems 1 and 2, when is u real-valued? (Equivalent: when is $\mathcal{R} \subseteq \mathbb{R}$?)

Fix $o \in \mathcal{X}$. Define $\mathbf{o} \in \mathcal{X}^{\mathcal{I}}$ by $o_i := o$ for all $i \in \mathcal{I}$.

For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ with $d(\mathbf{x}, \mathbf{o}) < \infty$, and any $N \in \mathbb{N}$, define $\mathbf{x}^N$ as follows.

$$\begin{array}{llllllll} \mathbf{x}: & \ldots & x_1x_2x_3x_4x_5 & o\,o\,o\,o\,o & o\,o\,o\,o\,o & o\,o\,o\,o\,o & o\,o\,o\,o\,o & o\,o\,o\,o\,o \quad \ldots \\ \mathbf{x}^4: & \ldots & \underbrace{x_1x_2x_3x_4x_5}_{\mathcal{J}_1} & \underbrace{x_1x_2x_3x_4x_5}_{\mathcal{J}_2} & \underbrace{x_1x_2x_3x_4x_5}_{\mathcal{J}_3} & \underbrace{x_1x_2x_3x_4x_5}_{\mathcal{J}_4} & o\,o\,o\,o\,o & o\,o\,o\,o\,o \quad \ldots \end{array}$$

(1) Find disjoint $\mathcal{J}_1, \ldots, \mathcal{J}_N \subset \mathcal{I}$ with $|\mathcal{J}_n| = d(\mathbf{x}, \mathbf{o})$ for all $n \in [1...N]$.
(2) Let $\beta_n : \mathcal{J}_n \longrightarrow \mathcal{I}(\mathbf{x}, \mathbf{o})$ be bijections for $n \in [1 \ldots N]$.
(3) Define $x_j^N := x_{\beta_n(j)}$ for all $n \in [1 \ldots N]$ and $j \in \mathcal{J}_n$.
(4) Define $x_i^N := o$ for all $i \in \mathcal{I} \setminus \mathcal{J}_1 \sqcup \cdots \sqcup \mathcal{J}_N$.

($\succeq$) is *Archimedean* if and only if: for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ with $d(\mathbf{x}, \mathbf{o}) < \infty$, $d(\mathbf{y}, \mathbf{o}) < \infty$, and $\mathbf{x} \succ \mathbf{o}$, there exists some $N \in \mathbb{N}$ such that $\mathbf{x}^N \succeq \mathbf{y}$.
(This definition is independent of the choice of o, because ($\succeq$) is separable.)

Proposition 4. *Let ($\succeq$) be a strictly finitary preorder on $\mathcal{X}^{\mathcal{I}}$.*

($\succeq$) is $\Pi_{\rm fin}$-invariant, separable, and Archimedean if and only if there exists some $u : \mathcal{X} \longrightarrow \mathbb{R}$ such that ($\succeq$) is the additive preorder defined by u.

Archimedean preferences and real-valued utility

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Formal definition of ${}^*\mathcal{R}$: ultrafilters

Let $\mathcal{F}$:= {all finite subsets of $\mathcal{I}$}.

Let $\mathfrak{P}$:= the power set of $\mathcal{F}$.

A *free ultrafilter* is a subset $\mathfrak{U}\mathfrak{F} \subset \mathfrak{P}$ (i.e. a family of collections of finite subsets of $\mathcal{I}$) with the following properties:

- **(F0)** No finite subset of $\mathcal{F}$ is an element of $\mathfrak{U}\mathfrak{F}$. (Hence, $\emptyset \notin \mathfrak{U}\mathfrak{F}$.)
- **(F1)** If $\mathcal{D}, \mathcal{E} \in \mathfrak{U}\mathfrak{F}$, then $\mathcal{D} \cap \mathcal{E} \in \mathfrak{U}\mathfrak{F}$.
- **(F2)** For any $\mathcal{E} \in \mathfrak{U}\mathfrak{F}$ and $\mathcal{P} \in \mathfrak{P}$, if $\mathcal{E} \subseteq \mathcal{P}$, then $\mathcal{P} \in \mathfrak{U}\mathfrak{F}$ also.
- **(UF)** For any $\mathcal{P} \in \mathfrak{P}$, either $\mathcal{P} \in \mathfrak{U}\mathfrak{F}$ or $\mathcal{P}^{\complement} \in \mathfrak{U}\mathfrak{F}$ (but not both).

Idea: Elements of $\mathfrak{U}\mathfrak{F}$ are 'large' collections of finite subsets of $\mathcal{I}$; if $\mathcal{G} \in \mathfrak{U}\mathfrak{F}$ and a certain statement holds for all $\mathcal{J} \in \mathcal{G}$, then this statement holds for 'almost all' finite subsets $\mathcal{J} \subseteq \mathcal{I}$. (In particular, axioms (F0) and (UF) imply that $\mathcal{F} \in \mathfrak{U}\mathfrak{F}$.)

The existence of a free ultrafilter on $\mathcal{F}$ follows from Zorn's Lemma.

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Formal definition of ${}^*\mathcal{R}$: ultrafilters

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Let $\mathcal{R}^{\mathcal{F}}$ be the set of all functions $r : \mathcal{F} \longrightarrow \mathcal{R}$.

For any $r, s \in \mathcal{R}^{\mathcal{F}}$, let $\mathcal{F}(r,s) := \{\mathcal{F} \in \mathcal{F};\ r(\mathcal{F}) \geq s(\mathcal{F})\}$.

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This defines a *complete preorder* $(\underset{\mathfrak{UF}}{\succsim})$ on $\mathcal{R}^{\mathcal{F}}$.

Let $(\underset{\mathfrak{UF}}{\approx})$ be the symmetric part of $(\underset{\mathfrak{UF}}{\succsim})$ (an equivalence relation on $\mathcal{R}^{\mathcal{F}}$).
Thus, $r \underset{\mathfrak{UF}}{\approx} s$ if they agree 'almost everywhere'. Define ${}^*\mathcal{R} := \mathcal{R}^{\mathcal{F}} / (\underset{\mathfrak{UF}}{\approx})$.
For any $r \in \mathcal{R}^{\mathcal{F}}$, let ${}^*r$ denote the equivalence class of r in ${}^*\mathcal{R}$.

Define linear order $(>)$ on ${}^*\mathcal{R}$, by $({}^*r > {}^*s) \Leftrightarrow (r \underset{\mathfrak{UF}}{\succsim} s)$, for all ${}^*r, {}^*s \in {}^*\mathcal{R}$.

$\mathcal{R}^{\mathcal{F}}$ is an abelian group under pointwise addition. Define a binary operation '$+$' on ${}^*\mathcal{R}$ by setting ${}^*r + {}^*s := {}^*(r + s)$ for all ${}^*r, {}^*s \in {}^*\mathcal{R}$.

Lemma A. $({}^*\mathcal{R}, +, >)$ *is a linearly ordered abelian group.*

${}^*\mathcal{R}$ is called an *ultrapower* of $\mathcal{R}$.

Example. If $\mathcal{R} = \mathbb{R}$, then ${}^*\mathbb{R}$ is the group of *hyperreal numbers* (the starting point of *nonstandard analysis*).

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Lemma A. $({}^*\mathcal{R}, +, >)$ *is a linearly ordered abelian group.*

${}^*\mathcal{R}$ is called an *ultrapower* of $\mathcal{R}$.

Example. If $\mathcal{R} = \mathbb{R}$, then ${}^*\mathbb{R}$ is the group of *hyperreal numbers* (the starting point of *nonstandard analysis*).

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Formal definition of $(\overset{*}{\underset{u}{\succsim}})$: hypersums [Skip to end]

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For any $\mathbf{x}, \mathbf{z} \in \mathcal{X}^{\mathcal{I}}$ and $\mathcal{J} \subseteq \mathcal{I}$, let $\mathbf{x}_{\mathcal{J}}\mathbf{z}_{\mathcal{I}\setminus\mathcal{J}}$ denote the element $\mathbf{w} \in \mathcal{X}^{\mathcal{I}}$ defined by $w_j := x_j$ for all $j \in \mathcal{J}$ and $w_i := z_i$ for all $i \in \mathcal{I} \setminus \mathcal{J}$.

Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, and let $\mathcal{G} \in \mathfrak{UF}$.

Write "$\mathbf{x} \underset{\mathcal{G}}{\succeq} \mathbf{y}$" if, for all $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$ and all $\mathcal{J} \in \mathcal{G}$, we have $\mathbf{x}_{\mathcal{J}}\mathbf{z}_{\mathcal{I}\setminus\mathcal{J}} \succeq \mathbf{y}_{\mathcal{J}}\mathbf{z}_{\mathcal{I}\setminus\mathcal{J}}$.

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The preorder $(\succeq)$ is $\mathfrak{UF}$-*continuous* if, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$:
(C1) if $\mathbf{x} \underset{\mathcal{G}}{\succeq} \mathbf{y}$ for some $\mathcal{G} \in \mathfrak{UF}$, then $\mathbf{x} \succeq \mathbf{y}$.
(C2) if $\mathbf{x} \underset{\mathcal{G}}{\succ} \mathbf{y}$ for some $\mathcal{G} \in \mathfrak{UF}$, then $\mathbf{x} \succ \mathbf{y}$.

Let $(\underset{\text{fin}}{\succeq})$ denote the finitary part of $(\succeq)$.
Lemma C. *Let $(\succeq)$ be a preorder on $\mathcal{X}^{\mathcal{I}}$, and let $u : \mathcal{X} \longrightarrow \mathcal{R}$. Then*

$$\Big((\underset{\text{fin}}{\succeq}) = (\underset{u}{\succeq}), \textit{ and } (\succeq) \textit{ is } \mathfrak{UF}\textit{-continuous} \Big) \iff \Big((\succeq) = (\overset{*}{\underset{u}{\succeq}}) \Big).$$

Theorem 2 follows by combining Lemmas A, B, and C.

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The preorder $(\succeq)$ is $\mathfrak{UF}$-*continuous* if, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$:

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$\mathfrak{UF}$-continuity [Skip to end]

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Problems. (a) $(\overset{*}{\underset{u}{\succsim}})$ is defined using an ultrafilter, so it is not explicitly constructable within the Zermelo-Fraenkel (ZF) axioms.

(This is unavoidable: Zame (2007) and Lauwers (2010) have shown that *any* 'reasonable' infinite-horizon intertemporal preference order is nonconstructable in ZF.)

(b) Furthermore, there are uncountably many distinct ultrafilters $\mathfrak{UF}$ satisfying the conditions of Lemma B(b); each yields a slightly different version $(\overset{*}{\underset{u}{\succsim}}{}^{\mathfrak{UF}})$ of the hyperadditive order.

This makes it hard to determine, in practice, whether $\mathbf{x} \overset{*}{\underset{u}{\succsim}} \mathbf{y}$.

Solution. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, define $\mathbf{x} \underset{w}{\succsim} \mathbf{y}$ iff there exists some finite $\mathcal{E} \subset \mathcal{I}$ such that $\sum_{j \in \mathcal{J}} u(x_j) \geq \sum_{j \in \mathcal{J}} u(y_j)$ for all finite $\mathcal{J} \subset \mathcal{I}$ with $\mathcal{E} \subseteq \mathcal{J}$.

Proposition. *Let $\mathcal{R}$ be a linearly ordered abelian group and let $u : \mathcal{X} \longrightarrow \mathcal{R}$. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $\mathbf{x} \underset{w}{\succsim} \mathbf{y}$ if and only if $\mathbf{x} \overset{*}{\underset{u}{\succsim}}{}^{\mathfrak{UF}} \mathbf{y}$ for every ultrafilter $\mathfrak{UF}$ satisfying Lemma B(b).*

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The paradoxes above may cause us to reconsider separability.

Related Literature

Lexicographical utility and probability. M. Hausner (1954); J.S. Chipman (1960,1971); P.C. Fishburn (1972-present); Fishburn and Lavalle (1990-2000).

Hyperreal utility and probability. M. Richter (1971), H.J. Skala (1974, 1975); L. Narens (1974, 1985); Blume, Brandenburger, and Dekel (1989, 1991); J.Y. Halpern (2009, 2010).

Additive separability in infinite-horizon intertemporal choice. H. Atsumi (1965); C.C. von Weizsäcker (1965); L. Lauwers (1998); Lauwers and Vallentyne (2004); Basu and Mitra (2003, 2007); Asheim and Tungodden (2004); Banerjee (2006).

Hyperreal utilitarian SWF for infinite-horizon intertemporal choice. Fleurbaey and Michel (2003; Theorem 5).

Generalized utilitarianism in variable-population social choice. Blackorby, Bossert, and Donaldson (1997)

Ultrafilters and aggregation. Kirmann and Sondermann (1972); Lauwers and van Liedekerke (1995); Zame (2007); L. Lauwers (1997, 2010).

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Merci & Thank you.

These presentation slides are available at

<http://euclid.trentu.ca/pivato/Research/separable.pdf>

The paper is available at

<http://mpra.ub.uni-muenchen.de/28262/>