Variable population voting rules

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Introduction

- Suppose society must collectively choose from some set $\mathcal{X}$ of alternatives, using a voting rule F.
- Suppose we split the voters into two subgroups, and each subgroup, using rule F, selects the alternative x.
- Then the combined group, using F, should also select alternative x.
- We say the rule F satisfies *reinforcement* if it has this property.
- Smith (1973) and Young (1974,1975) showed that 'scoring rules' (e.g. Borda rule) are the only preference aggregation rules which satisfy reinforcement and are anonymous and neutral (i.e. invariant under relabeling of the voters and/or alternatives).
- This led to axiomatic characterizations of the Borda rule, Kemeny rule, and plurality rule, by Young & Levenglick, Nitzan & Rubinstein, Richelson, Morkelyunas, and others, as the only rules satisfying reinforcement and certain other axioms.

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- I will extend Myerson's result, by considering infinite signal sets, and removing his hypotheses of universal domain and overwhelming majority.
- I will do this by considering scoring rules where the scores can range over a *linearly ordered abelian group*, instead of the real numbers.
- I will also characterize *balance rules* (another class introduced by Myerson (1995)), the *formally utilitarian* voting rule, and the *range voting* rule.

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- Let $\mathcal{V}$ be the (finite or infinite) set of 'signals' or 'messages' which could be sent by each voter.
- Let $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$ and $\mathbb{Z} := \{\pm n; \ \ n \in \mathbb{N}\}$.
- For any $\mathbf{n} \in \mathbb{Z}^{\mathcal{V}}$, let $\|\mathbf{n}\| := \sum_{v \in \mathcal{V}} |n_v|$.
- Let $\mathbb{N}^{\langle \mathcal{V} \rangle} := \{\mathbf{n} \in \mathbb{N}^{\mathcal{V}}; \ \ \|\mathbf{n}\| < \infty\}$.
- If $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, then $\mathbf{n}$ represents an anonymous profile of voters: for each $v \in \mathcal{V}$, we interpret n_v as the number of voters sending the signal v, while $\|\mathbf{n}\|$ is the (finite) size of the whole population.
- A *domain* is any collection of profiles $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ such that $\mathbf{0} \in \mathcal{D}$. (The set $\mathbb{N}^{\langle \mathcal{V} \rangle}$ itself is the *universal domain*.)
- Let $\mathcal{X}$ be a (finite or infinite) set of social alternatives.
- A *(variable population, anonymous) voting rule* is a correspondence $F : \mathcal{D} \rightrightarrows \mathcal{X}$ such that $F(\mathbf{0}) = \mathcal{X}$.
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Loags and scoring rules

A *linearly ordered abelian group* (loag) is a triple $(\mathcal{R}, +, >)$, where $(\mathcal{R}, +)$ is an abelian group, and $>$ is a complete, antisymmetric, transitive binary relation on $\mathcal{R}$ such that, for all $r, s \in \mathcal{R}$, if $r > 0$, then $r + s > s$.

Examples: (a) The set $\mathbb{R}$ of real numbers is a loag, with the standard ordering and addition operator. So is $\mathbb{Z}$.
(b) For any $n \in \mathbb{N}$, the space $\mathbb{R}^n$ is a loag under vector addition and the lexicographic order.

For any $\mathbf{s} = (s_v)_{v \in \mathcal{V}} \in \mathcal{R}^{\mathcal{V}}$, define group homomorphism $\mathbf{s} : \mathbb{Z}^{\langle \mathcal{V} \rangle} \longrightarrow \mathcal{R}$ by setting $\mathbf{s}(\mathbf{d}) := \sum_{v \in \mathcal{V}} s_v \, d_v$, for all $\mathbf{d} \in \mathbb{Z}^{\langle \mathcal{V} \rangle}$. (Well-defined since $\|\mathbf{d}\| < \infty$.)

An *$\mathcal{R}$-valued score system* is an $\mathcal{X}$-indexed set $\mathrm{S} := \{\mathbf{s}^x\}_{x \in \mathcal{X}} \subset \mathcal{R}^{\mathcal{V}}$.
Fix a domain $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$. Define the S-*scoring rule* $F_{\mathrm{S}} : \mathcal{D} \rightrightarrows \mathcal{X}$ as follows:

$$F_{\mathrm{S}}(\mathbf{d}) \quad := \quad \operatorname*{argmax}_{x \in \mathcal{X}} \ \mathbf{s}^x(\mathbf{d}), \qquad \text{for all } \mathbf{d} \in \mathcal{D}.$$

Idea: $\mathbf{s}^x(\mathbf{d})$ is the 'score' which alternative x receives from profile $\mathbf{d}$; a voter who sends signal v contributes s_v^x 'points' to this score.
The alternative with the highest score wins.

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Loags and scoring rules

A *linearly ordered abelian group* (loag) is a triple $(\mathcal{R}, +, >)$, where $(\mathcal{R}, +)$ is an abelian group, and $>$ is a complete, antisymmetric, transitive binary relation on $\mathcal{R}$ such that, for all $r, s \in \mathcal{R}$, if $r > 0$, then $r + s > s$.
Examples: (a) The set $\mathbb{R}$ of real numbers is a loag, with the standard ordering and addition operator. So is $\mathbb{Z}$.
(b) For any $n \in \mathbb{N}$, the space $\mathbb{R}^n$ is a loag under vector addition and the lexicographic order.

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Let $(\mathcal{R}, +, >)$ be a linearly ordered abelian group.
Let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ be a domain of profiles.
An $\mathcal{R}$*-valued balance system* is an $\mathcal{X}^2$-indexed collection $\mathrm{B} := \{\mathbf{b}^{x,y}\}_{x,y \in \mathcal{X}} \subset \mathcal{R}^{\mathcal{V}}$ such that $\mathbf{b}^{x,y} = -\mathbf{b}^{y,x}$ for all $x, y \in \mathcal{X}$ (hence $\mathbf{b}^{x,x} = 0$ for all $x \in \mathcal{X}$), and such that,

$$\max_{x \in \mathcal{X}} \ \min_{y \in \mathcal{X}} \ \mathbf{b}^{x,y}(\mathbf{d}) \quad \geq \quad 0, \qquad \text{for all } \mathbf{d} \in \mathcal{D}. \qquad (*)$$

Define the *balance rule* $F_{\mathrm{B}} : \mathcal{D} \rightrightarrows \mathcal{X}$ as follows:
For all $\mathbf{d} \in \mathcal{D}$ and $x \in \mathcal{X}$, $\Big(x \in F_{\mathrm{B}}(\mathbf{d})\Big) \iff \Big(\mathbf{b}^{x,y}(\mathbf{d}) \geq 0 \text{ for all } y \in \mathcal{X}.\Big)$
(Condition $(*)$ just means that $F_{\mathrm{B}}(\mathbf{d}) \neq \emptyset$ for all $\mathbf{d} \in \mathcal{D}$.)

Example:
Let $\mathrm{S} = \{\mathbf{s}^x\}_{x \in \mathcal{X}}$ be an $\mathcal{R}$-valued score system on $(\mathcal{X}, \mathcal{V})$.
For all $x, y \in \mathcal{X}$, define $\nabla^{x,y}\mathrm{S} := \mathbf{s}^x - \mathbf{s}^y \in \mathcal{R}^{\mathcal{V}}$, to obtain a balance system $\nabla\mathrm{S} := \{\nabla^{x,y}\mathrm{S}\}_{x,y \in \mathcal{X}}$. Then $F_{\nabla\mathrm{S}}(\mathbf{n}) = F_{\mathrm{S}}(\mathbf{n})$ for all $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$.
Thus, every scoring rule is a balance rule.

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Balance rules

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Example. Let $\mathcal{V}$ be the set of all preference orders over $\mathcal{X}$.
For all $x, y \in \mathcal{X}$ and $v \in \mathcal{V}$, define

$$b_v^{x,y} := \begin{cases} 1 & \text{if } v \text{ prefers } x \text{ to } y; \\ -1 & \text{if } v \text{ prefers } y \text{ to } x; \\ 0 & \text{if } v \text{ is indifferent between } x \text{ and } y. \end{cases}$$

Then F_{B} is the *Condorcet rule*: for any $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$, we have $x \in F(\mathbf{n})$ if and only if x is a *Condorcet winner* in the profile $\mathbf{n}$ (i.e. for any other $y \in \mathcal{X}$, at least as many voters strictly prefer x over y as the number who strictly prefer y over x).
Unfortunately, $F(\mathbf{n}) = \emptyset$ for some $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ (the 'Condorcet paradox').
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The Condorcet balance rules

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Then F_{B} is the *Condorcet rule*: for any $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, we have $x \in F(\mathbf{n})$ if and only if x is a *Condorcet winner* in the profile $\mathbf{n}$ (i.e. for any other $y \in \mathcal{X}$, at least as many voters strictly prefer x over y as the number who strictly prefer y over x).
Unfortunately, $F(\mathbf{n}) = \emptyset$ for some $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ (the 'Condorcet paradox').
Let $\mathcal{D} \subset \mathbb{N}^{\langle \mathcal{V} \rangle}$ be the set of all profiles having a Condorcet winner.
Then $F_{\mathrm{B}} : \mathcal{D} \rightrightarrows \mathcal{X}$ is a balance rule.

The Condorcet balance rules

Recall. The *balance rule* $F_{\mathrm{B}} : \mathcal{D} \rightrightarrows \mathcal{X}$ is defined as follows:
For all $\mathbf{d} \in \mathcal{D}$ and $x \in \mathcal{X}$, $\Big(x \in F_{\mathrm{B}}(\mathbf{d})\Big) \iff \Big(\mathbf{b}^{x,y}(\mathbf{d}) \geq 0$ for all $y \in \mathcal{X}.\Big)$

Example. Let $\mathcal{V}$ be the set of all preference orders over $\mathcal{X}$.
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Let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$. A rule $F : \mathcal{D} \rightrightarrows \mathcal{X}$ satisfies *reinforcement* if the following is true: for any $\mathbf{n}, \mathbf{m} \in \mathcal{D}$, if $F(\mathbf{n}) \cap F(\mathbf{m}) \neq \emptyset$, then

$$\mathbf{n} + \mathbf{m} \in \mathcal{D}, \qquad \text{and} \qquad F(\mathbf{n} + \mathbf{m}) \quad = \quad F(\mathbf{n}) \cap F(\mathbf{m}).$$

Idea: the profile $(\mathbf{n} + \mathbf{m})$ represents a union of two disjoint sub-populations, represented by profiles $\mathbf{n}$ and $\mathbf{m}$.
Reinforcement says: if $x \in \mathcal{X}$ and both $\mathbf{n}$ and $\mathbf{m}$ endorse x (i.e. if $x \in F(\mathbf{n})$ and $x \in F(\mathbf{m})$), then we should have $x \in F(\mathbf{n} + \mathbf{m})$.
Furthermore, in this case, $F(\mathbf{n} + \mathbf{m})$ should consist of *only* those $x \in \mathcal{X}$ which receive this joint endorsement.

We now come to our first result:

Theorem 1. *Let $\mathcal{X}$ and $\mathcal{V}$ be arbitrary sets, let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ be any domain, and let $F : \mathcal{D} \rightrightarrows \mathcal{X}$ be a voting rule.*
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Neutrality

Not every balance rule is a scoring rule, even when $\mathcal{D} = \mathbb{N}^{\langle \mathcal{V} \rangle}$.

We need additional hypotheses to characterize scoring rules.

Let $\Pi_{\mathcal{V}}$ be the group of all permutations of $\mathcal{V}$. For any $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ and $\pi \in \Pi_{\mathcal{V}}$, let $\pi(\mathbf{n}) := \mathbf{m}$, where $m_v := n_{\pi^{-1}(v)}$ for all $v \in \mathcal{V}$.

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A voting rule $F : \mathcal{D} \rightrightarrows \mathcal{X}$ is *neutral* if there exists a group homomorphism $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ (the *neutralizer*) such that, for all $\pi \in \Pi_{\mathcal{X}}$, if $\widetilde{\pi} := \nu(\pi)$, then the domain $\mathcal{D}$ is $\widetilde{\pi}$-invariant, and $F(\widetilde{\pi}(\mathbf{d})) = \pi(F(\mathbf{d}))$ for all $\mathbf{d} \in \mathcal{D}$.

Idea: Every alternative in $\mathcal{X}$ is treated equally: for any $x, y \in \mathcal{X}$, and every profile $\mathbf{d} \in \mathcal{D}$ with $x \in F(\mathbf{d})$, there is a permutation $\mathbf{d}'$ of $\mathbf{d}$ with $y \in F(\mathbf{d}')$.

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A score system $\mathsf{S} = \{\mathbf{s}^x\}_{x \in \mathcal{X}}$ is *ν-neutral* if, for all $\pi \in \Pi_{\mathcal{X}}$ and $x, y \in \mathcal{X}$, if $\pi(y) = x$ and $\widetilde{\pi} := \nu(\pi)$, then $\mathbf{s}^x \widetilde{\pi} = \mathbf{s}^y$.

Example: All the scoring rules mentioned above have neutral score systems, with the obvious neutralizers.

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We will now extend this result.

A domain $\mathcal{D} \subseteq \mathbb{N}^{\langle\mathcal{V}\rangle}$ is a *cone* if $\mathbf{d}_1 + \mathbf{d}_2 \in \mathcal{D}$ whenever $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{D}$, and also, $\mathbf{d} \in \mathcal{D}$ whenever $n\,\mathbf{d} \in \mathcal{D}$ for some $n \in \mathbb{N}$.

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A domain $\mathcal{D} \subseteq \mathbb{N}^{\langle\mathcal{V}\rangle}$ is a *cone* if $\mathbf{d}_1 + \mathbf{d}_2 \in \mathcal{D}$ whenever $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{D}$, and also, $\mathbf{d} \in \mathcal{D}$ whenever $n\,\mathbf{d} \in \mathcal{D}$ for some $n \in \mathbb{N}$.

Example: (a) The universal domain $\mathbb{N}^{\langle\mathcal{V}\rangle}$ itself is a cone.

(b) If $f : \mathbb{R}^{\langle\mathcal{V}\rangle} \longrightarrow \mathbb{R}$ is a linear function, then the sets $\{\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle};\ \ f(\mathbf{n}) \geq 0\}$ and $\{\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle};\ \ f(\mathbf{n}) = 0\}$ are cones.

(c) The intersection of any collection of cones is a cone.

Here is our second main result:

Theorem 2. *Let $\mathcal{X}$ be a finite set, let $\mathcal{V}$ be any set, let $\mathcal{D} \subseteq \mathbb{N}^{\langle\mathcal{V}\rangle}$ be a cone, and let $F : \mathcal{D} \rightrightarrows \mathcal{X}$ be any voting rule.*

Then F is neutral and satisfies reinforcement if and only if F is a scoring rule with a neutral score system.

Overwhelming majority

A voting rule F satisfies *overwhelming majority* (OWM) if, for any $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, there exists some $M \in \mathbb{N}$ such that, for all $m > M$, we have $F(m\mathbf{n} + \mathbf{n}') \subseteq F(\mathbf{n})$.

Idea: If one sub-population of voters (represented by $m\,\mathbf{n}$) is much larger than another sub-population (represented by $\mathbf{n}'$), then the choice of the combined population should be determined by the choice of the larger sub-population —except that the smaller sub-population may act as a 'tie-breaker' in some cases.

Recall: Myerson (1995) showed that, if the voting rule in Theorem 2 satisfies overwhelming majority, then not only is it a scoring rule, but the score system is *real-valued*.

Our next result makes an analogous statement for the balance rule in Theorem 1, *without* assuming neutrality.

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A voting rule $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ satisfies the *tie condition* (TC) if, for all distinct $x, y \in \mathcal{X}$:

(TC1) There exists some $\mathbf{n} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ with $F(\mathbf{n}) = \{x, y\}$; and

(TC2) For any finite $\mathcal{W} \subseteq \mathcal{V}$, there exists some $\mathbf{m} \in \mathbb{N}^{\langle\mathcal{V}\rangle}$ such that $m_w > 0$ for all $w \in \mathcal{W}$, and $F(\mathbf{m}) \supseteq \{x, y\}$.

Example: Any nontrivial neutral balance rule satisfies TC.

Proposition 3. *Let $F : \mathbb{N}^{\langle\mathcal{V}\rangle} \rightrightarrows \mathcal{X}$ be a balance rule satisfying* TC. *F satisfies overwhelming majority if and only if $F = F_{\mathrm{B}}$ for some real-valued balance system* B.

Proof sketch. Overwhelming majority and TC imply that the loag $\mathcal{R}$ is Archimedean.

Then Hölder's theorem implies that $\mathcal{R}$ is isomorphic to a subgroup of $\mathbb{R}$. □

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Let $\mathcal{V}$ and $\mathcal{W}$ be two sets of 'signals', and let $\alpha : \mathcal{W} \longrightarrow \mathcal{V}$ ('translation'). Define $\alpha_* : \mathbb{N}^{\langle \mathcal{W} \rangle} \longrightarrow \mathbb{N}^{\langle \mathcal{V} \rangle}$ as follows: for any $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{W} \rangle}$, and any $v \in \mathcal{V}$, $\alpha_*(\mathbf{n})_v := \sum\{n_w;\ \ w \in \mathcal{W} \text{ and } \alpha(w) = v\}$.

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Given two voting rules $F : \mathbb{N}^{\langle \mathcal{V} \rangle} \rightrightarrows \mathcal{X}$ and $G : \mathbb{N}^{\langle \mathcal{W} \rangle} \rightrightarrows \mathcal{X}$, we say that F is *at least as expressive as* G if there is a some 'translation' function $\alpha : \mathcal{W} \longrightarrow \mathcal{V}$ such that, for all $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{W} \rangle}$, $F(\alpha_*(\mathbf{n})) = G(\mathbf{n})$.

Idea: for any $w \in \mathcal{W}$, voting for w in the rule G is effectively equivalent to voting for $\alpha(w)$ in F. Thus, the voters can express any profile of opinions via F which they could have expressed via G.

The rule F is the *most expressive* member of some class of rules if it is at least as expressive as every other element of that class.

Proposition 4. *Let $\mathcal{X}$ be a finite set.*
Formally utilitarian voting is the most expressive $\mathcal{X}$-valued voting rule which satisfies reinforcement, neutrality, and overwhelming majority.

Range voting

For any $v \in \mathcal{V}$, define $\mathbf{1}^v \in \mathbb{N}^{\langle \mathcal{V} \rangle}$ by $(\mathbf{1}^v)_v := 1$, whereas $(\mathbf{1}^v)_w := 0$ for all $w \in \mathcal{V} \setminus \{v\}$.

A voting rule $F : \mathbb{N}^{\langle \mathcal{V} \rangle} \rightrightarrows \mathcal{X}$ *admits minority overrides* if, for any $\mathbf{n} \in \mathbb{N}^{\langle \mathcal{V} \rangle}$, there is some $v \in \mathcal{V}$ such that $F(\mathbf{n} + \mathbf{1}^v) \neq F(\mathbf{n})$.

Idea: Regardless of the size of the population and the weight of existing public opinion, a single voter can always cast a vote which changes the outcome. (Example: Formally utilitarian voting).

Such 'overrides' not only generate political instability; they are arguably undemocratic. It might be better if F did *not* admit minority overrides.

If $\mathcal{V}$ is finite, then any rule satisfying overwhelming majority will not admit minority overrides.

However, we will be interested in the case when $\mathcal{V}$ is infinite.

Proposition 5. *Let $\mathcal{X}$ be a finite set.*
Range voting is the most expressive $\mathcal{X}$-valued voting rule which satisfies reinforcement, neutrality, overwhelming majority, and does not admit minority overrides.

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Despite Propositions 3-5, OWM is not always normatively compelling. In some cases, a non-real-valued scoring system may be more appropriate.

Example. Let $\mathcal{X} \subsetneq \mathcal{X}_1 \times \mathcal{X}_2$, where $\mathcal{X}_1$ is a space of alternatives in one 'policy dimension', while $\mathcal{X}_2$ is a space of alternatives in another dimension.

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Problem. If we apply the scoring rules $F_{{}_1\mathcal{S}} : \mathbb{N}^{\langle \mathcal{V}_1 \rangle} \rightrightarrows \mathcal{X}_1$ and $F_{{}_2\mathcal{S}} : \mathbb{N}^{\langle \mathcal{V}_2 \rangle} \rightrightarrows \mathcal{X}_2$ separately, then we may end up selecting an element of $(\mathcal{X}_1 \times \mathcal{X}_2) \setminus \mathcal{X}$ (infeasible).

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Nonsolution. Combine ${}_1\mathrm{S}$ and ${}_2\mathrm{S}$ into a $\mathbb{R}$-valued score system S on $(\mathcal{X}, \mathcal{V}_1 \times \mathcal{V}_2)$ by defining $s^{x_1,x_2}_{v_1,v_2} := {}_1s^{x_1}_{v_1} + {}_2s^{x_2}_{v_2}$ for all $(x_1, x_2) \in \mathcal{X}$ and $(v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2$.

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Flaw. This does not respect the lexicographical priority of $\mathcal{X}_1$ over $\mathcal{X}_2$.

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Solution. Let $\mathcal{R} := \mathbb{R}^2$ with the vector addition operation '$+$' and the lexicographical ordering '$\succ$' (i.e. $(r_1, r_2) \succ (s_1, s_2)$ if and only if either $r_1 > s_1$, or $r_1 = s_1$ and $r_2 > s_2$).

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Solution. Let $\mathcal{R} := \mathbb{R}^2$ with the vector addition operation '$+$' and the lexicographical ordering '$\succ$' (i.e. $(r_1, r_2) \succ (s_1, s_2)$ if and only if either $r_1 > s_1$, or $r_1 = s_1$ and $r_2 > s_2$).
Define $\mathcal{R}$-valued score system S on $(\mathcal{X}, \mathcal{V}_1 \times \mathcal{V}_2)$ by setting $s_{v_1, v_2}^{x_1, x_2} := ({}_1s_{v_1}^{x_1}, {}_2s_{v_2}^{x_2})$ for all $(x_1, x_2) \in \mathcal{X}$ and $(v_1, v_2) \in \mathcal{V}_1 \times \mathcal{V}_2$.

Proof sketches

Homogeneous partial orders

Let $(\mathcal{R},+)$ be an abelian group, and let $(\succeq)$ be a binary relation on $\mathcal{R}$.
$(\succeq)$ is *homogeneous* if, for all $r,s\in\mathcal{R}$, we have $(r\succeq s) \iff (r-s\succeq 0)$.
Thus, $(\mathcal{R},+,\succ)$ is a loag iff $(\succ)$ is a homogeneous *linear* order on $\mathcal{R}$.
The *positive conoid* of $(\succeq)$ is the set $\mathcal{P}_{\succeq} := \{r\in\mathcal{R};\ r\succeq 0\}$.
The conoid $\mathcal{P}_{\succeq}$ completely encodes the relation $(\succeq)$: for any $r,s\in\mathcal{R}$,

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The proof of Theorem 1 uses the following result:

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Proof sketch. (Fuchs, 1950 or Everett, 1950) Similar to 'classic' Szpilrajn lemma.

1. Show that the set of homogeneous partial orders extending $(\succ)$ satisfies the Ascending Chain Condition.
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Proof sketch. (Fuchs, 1950 or Everett, 1950) Similar to 'classic' Szpilrajn lemma.

1. Show that the set of homogeneous partial orders extending $(\succ)$ satisfies the Ascending Chain Condition.
2. Then use Zorn's Lemma to obtain maximal extension.
3. This maximal extension must be a homogeneous linear order. $\square$

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For all $x \in \mathcal{X}$, let $\mathcal{C}_x := \{\mathbf{d} \in \mathcal{D};\ x \in F(\mathbf{d})\}$.
Then $\mathcal{C}_x$ is a preorder conoid in the abelian group $\mathbb{Z}^{\langle \mathcal{V} \rangle}$.
(Proof: We have $\mathbf{0} \in \mathcal{C}_x$ because $\mathbf{0} \in \mathcal{D}$ and $F(\mathbf{0}) = \mathcal{X}$ by definition. Meanwhile, $\mathcal{C}_x$ is closed under addition because F satisfies reinforcement).

For any $x, y \in \mathcal{X}$, let $\mathcal{P}_{x,y} := \{\mathbf{c}_x - \mathbf{c}_y;\ \mathbf{c}_x \in \mathcal{C}_x \text{ and } \mathbf{c}_y \in \mathcal{C}_y\}$.
$\mathcal{P}_{x,y}$ is a preorder conoid (because $\mathcal{C}_x$ and $\mathcal{C}_y$ are). Note: $\mathcal{P}_{y,x} = -\mathcal{P}_{x,y}$.

Let $(\succeq)$ be the homogenous preorder on $\mathbb{Z}^{\langle \mathcal{V} \rangle}$ defined by $\mathcal{P}_{x,y}$.
Let $\mathcal{O}_{x,y} := \{\mathbf{z} \in \mathbb{Z}^{\langle \mathcal{V} \rangle};\ \mathbf{z} \approx \mathbf{0}\}$. Let $\mathcal{R}_{x,y} := \mathbb{Z}^{\langle \mathcal{V} \rangle}/\mathcal{O}_{x,y}$.
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Let $\mathbf{b}^{x,y} : \mathbb{Z}^{\langle \mathcal{V} \rangle} \longrightarrow \mathcal{R}_{x,y}$ be the quotient map (i.e. $\mathbf{b}^{x,y}(\mathbf{z}) := \mathbf{z} + \mathcal{O}_{x,y}$).
Note: $\mathcal{O}_{y,x} = \mathcal{O}_{x,y}$, so $\mathcal{R}_{y,x} = \mathcal{R}_{x,y}$ as groups, and $\mathbf{b}^{y,x} = \mathbf{b}^{x,y}$.
But $(\underset{x,y}{>})$ is the negative ordering to $(\underset{y,x}{>})$ (i.e. $r \underset{x,y}{>} r' \Leftrightarrow -r \underset{y,x}{>} -r'$).
Thus, WOLOG, redefine $(\underset{y,x}{>})$ to be identical with $(\underset{x,y}{>})$, and redefine $\mathbf{b}^{y,x}$ to be $-\mathbf{b}^{x,y}$ (or vice versa) Finally, let $\mathbf{b}^{x,x} := 0$ for all $x \in \mathcal{X}$.

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(b) Furthermore, if $y \notin F(\mathbf{d})$, then $\mathbf{b}^{x,y}(\mathbf{d}) > 0$ (hence, $y \notin F_{\mathrm{B}}(\mathbf{d})$).

Now, for all $\mathbf{d} \in \mathcal{D}$, Claim 1(a) implies that $F(\mathbf{d}) \subseteq F_{\mathrm{B}}(\mathbf{d})$.
Meanwhile, Claim 1(b) implies that $F(\mathbf{d}) \supseteq F_{\mathrm{B}}(\mathbf{d})$. Thus $F = F_{\mathrm{B}}$.

Now we must construct a single loag $\mathcal{R}$ and a collection of functions $\widetilde{\mathbf{b}}_{x,y} : \mathcal{V} \longrightarrow \mathcal{R}$ (for all $x, y \in \mathcal{X}$) such that $F_{\mathrm{B}} = F_{\widetilde{\mathrm{B}}}$.
Let $\mathcal{R} := \prod_{x,y \in \mathcal{X}} \mathcal{R}_{x,y}$ with 'Pareto' order it gets from orders on factors.
Extend this to a homog. linear order on $\mathcal{R}$ using Homog. Szpilrajn.

Theorem 1. *Let $\mathcal{X}$ and $\mathcal{V}$ be any sets, let $\mathcal{D} \subseteq \mathbb{N}^{\langle\mathcal{V}\rangle}$ be any domain. A voting rule $F : \mathcal{D} \rightrightarrows \mathcal{X}$ satisfies reinforcement $\Leftrightarrow$ F is a balance rule.*

Proof sketch. "$\Longrightarrow$" $\forall\, x, y \in \mathcal{X}$, $\mathcal{C}_x := \{\mathbf{d} \in \mathcal{D};\ x \in F(\mathbf{d})\}$ and $\mathcal{P}_{x,y} := \{\mathbf{c}_x - \mathbf{c}_y;\ \mathbf{c}_x \in \mathcal{C}_x \text{ and } \mathbf{c}_y \in \mathcal{C}_y\}$. Let $(\succeq) :=$ homog. preorder on $\mathbb{Z}^{\langle\mathcal{V}\rangle}$ defined by $\mathcal{P}_{x,y}$. Let $\mathcal{O}_{x,y} := \{\mathbf{z} \in \mathbb{Z}^{\langle\mathcal{V}\rangle};\ \mathbf{z} \approx \mathbf{0}\}$. Let $\mathcal{R}_{x,y} := \mathbb{Z}^{\langle\mathcal{V}\rangle}/\mathcal{O}_{x,y}$. Then $(\succeq)$ projects to homog. partial order $(\succ)$ on $\mathcal{R}_{x,y}$. Homog. Szpilrajn extends $(\succ)$ to homog. linear order $(\underset{x,y}{>})$ on $\mathcal{R}_{x,y}$.
Let $\mathbf{b}^{x,y} : \mathbb{Z}^{\langle\mathcal{V}\rangle} \longrightarrow \mathcal{R}_{x,y}$ be the quotient map.
WOLOG, assume $\mathcal{R}_{x,y} = \mathcal{R}_{y,x}$ and $\mathbf{b}_{x,y} = -\mathbf{b}_{y,x}$, while $\mathbf{b}_{x,x} = 0$.

Claim. *Let $\mathbf{d} \in \mathcal{D}$ and let $x \in F(\mathbf{d})$. Then:*
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Neutral perfect balance systems

Let $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ be a group homomorphism. Let B be a balance system.

- Say B is *ν-neutral* if, for all $x, y, x', y' \in \mathcal{X}$ and $\pi \in \Pi_{\mathcal{X}}$, if $x' := \pi^{-1}(x)$ and $y' := \pi^{-1}(y)$ and $\widetilde{\pi} = \nu(\pi)$, then $\mathbf{b}^{x,y}\widetilde{\pi} = \mathbf{b}^{x',y'}$.
- Say B is *perfect* on the domain $\mathcal{D}$ if, for any $\mathbf{d} \in \mathcal{D}$, any $x \in F_{\mathrm{B}}(\mathbf{d})$ and any $y \in \mathcal{X} \setminus F_{\mathrm{B}}(\mathbf{d})$, we have $\mathbf{b}^{x,y}(\mathbf{d}) > 0$.

Example: ∇S is perfect. But the Condorcet balance system is not perfect.

The next result is also important to prove Theorem 2.

Proposition B. *Let $\mathcal{X}$ be a finite set, let $\mathcal{V}$ be any set, let $\mathcal{D} \subseteq \mathbb{N}^{\langle \mathcal{V} \rangle}$ be a domain, and let $F : \mathcal{D} \rightrightarrows \mathcal{X}$ be a balance rule.*
Let $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ be a group homomorphism. Then F is ν-neutral if and only if $F = F_{\mathrm{B}}$ for some ν-neutral perfect balance system B.

Proof sketch. Start with any balance system $\widetilde{\mathrm{B}}$ for F.
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Then $F_{\widetilde{\mathrm{B}}} = F_{\mathrm{B}}$, and F_{B} is ν-neutral by construction.

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Then $F_{\widetilde{\mathrm{B}}} = F_{\mathrm{B}}$, and F_{B} is ν-neutral by construction. $\square$

Proof sketch for Theorem 2

If F satisfies reinforcement, then Theorem 1 says F is a balance rule.

If $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{V}}$ is a homomorphism, and F is also ν-neutral, then Proposition B says F has a ν-neutral, perfect balance system B.

We must show that B satisfies condition $(*)$ in Lemma A.

For simplicity, suppose $|\mathcal{X}| = 3$, and let $\mathcal{X} = \{x, y, z\}$.

Find $\pi \in \Pi_{\mathcal{X}}$ with $\pi(x) = z$, $\pi(y) = x$, and $\pi(z) = y$. Thus, $\pi^3 = \mathrm{Id}_{\mathcal{X}}$.

Let $\widetilde{\pi} := \nu(\pi) \in \Pi_{\mathcal{V}}$. Thus, $\widetilde{\pi}^3 = \mathrm{Id}_{\mathcal{V}}$.

Let $\mathbf{d} \in \mathcal{D}$, and let $\widetilde{\mathbf{d}} := \mathbf{d} + \widetilde{\pi}(\mathbf{d}) + \widetilde{\pi}^2(\mathbf{d})$. Then $\widetilde{\pi}(\widetilde{\mathbf{d}}) = \widetilde{\mathbf{d}}$.

Thus, neutrality says $\pi[F(\widetilde{\mathbf{d}})] = F(\widetilde{\mathbf{d}})$. But $F(\widetilde{\mathbf{d}}) \neq \emptyset$. Thus, $F(\widetilde{\mathbf{d}}) = \mathcal{X}$.

Thus, $\mathbf{b}^{x,y}(\widetilde{\mathbf{d}}) = 0$ (because $x, y \in F_{\mathrm{B}}(\widetilde{\mathbf{d}})$). But then

$$
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&= \left(\mathbf{b}^{x,y} + \mathbf{b}^{\pi^{-1}(x),\pi^{-1}(y)}\widetilde{\pi} + \mathbf{b}^{\pi^{-2}(x),\pi^{-2}(y)}\right)(\mathbf{d}) \quad \text{(by neutrality of B)} \\
&= \left(\mathbf{b}^{x,y} + \mathbf{b}^{y,z} + \mathbf{b}^{z,x}\right)(\mathbf{d}) \qquad \text{(by the definition of } \pi\text{)}.
\end{aligned}
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Thus, $\mathbf{b}^{x,y}(\mathbf{d}) + \mathbf{b}^{y,z}(\mathbf{d}) = -\mathbf{b}^{z,x}(\mathbf{d}) = \mathbf{b}^{x,z}(\mathbf{d})$.

This holds for all $\mathbf{d} \in \mathcal{D}$. Thus, condition $(*)$ holds.

Thus, Lemma A says that F is a scoring rule.

Proof sketch for Theorem 2

If F satisfies reinforcement, then Theorem 1 says F is a balance rule.
If $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{Y}}$ is a homomorphism, and F is also ν-neutral, then Proposition B says F has a ν-neutral, perfect balance system B.
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Thus, neutrality says $\pi[F(\widetilde{\mathbf{d}})] = F(\widetilde{\mathbf{d}})$. But $F(\widetilde{\mathbf{d}}) \neq \emptyset$. Thus, $F(\widetilde{\mathbf{d}}) = \mathcal{X}$.
Thus, $\mathbf{b}^{x,y}(\widetilde{\mathbf{d}}) = 0$ (because $x, y \in F_{\mathrm{B}}(\widetilde{\mathbf{d}})$). But then

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Thus, $\mathbf{b}^{x,y}(\mathbf{d}) + \mathbf{b}^{y,z}(\mathbf{d}) = -\mathbf{b}^{z,x}(\mathbf{d}) = \mathbf{b}^{x,z}(\mathbf{d})$.
This holds for all $\mathbf{d} \in \mathcal{D}$. Thus, condition $(*)$ holds.
Thus, Lemma A says that F is a scoring rule.

If F satisfies reinforcement, then Theorem 1 says F is a balance rule.
If $\nu : \Pi_{\mathcal{X}} \longrightarrow \Pi_{\mathcal{Y}}$ is a homomorphism, and F is also ν-neutral, then Proposition B says F has a ν-neutral, perfect balance system B.
We must show that B satisfies condition $(*)$ in Lemma A.
For simplicity, suppose $|\mathcal{X}| = 3$, and let $\mathcal{X} = \{x, y, z\}$.
Find $\pi \in \Pi_{\mathcal{X}}$ with $\pi(x) = z$, $\pi(y) = x$, and $\pi(z) = y$. Thus, $\pi^3 = \mathrm{Id}_{\mathcal{X}}$.
Let $\widetilde{\pi} := \nu(\pi) \in \Pi_{\mathcal{Y}}$. Thus, $\widetilde{\pi}^3 = \mathrm{Id}_{\mathcal{Y}}$.
Let $\mathbf{d} \in \mathcal{D}$, and let $\widetilde{\mathbf{d}} := \mathbf{d} + \widetilde{\pi}(\mathbf{d}) + \widetilde{\pi}^2(\mathbf{d})$. Then $\widetilde{\pi}(\widetilde{\mathbf{d}}) = \widetilde{\mathbf{d}}$.
Thus, neutrality says $\pi[F(\widetilde{\mathbf{d}})] = F(\widetilde{\mathbf{d}})$. But $F(\widetilde{\mathbf{d}}) \neq \emptyset$. Thus, $F(\widetilde{\mathbf{d}}) = \mathcal{X}$.
Thus, $\mathbf{b}^{x,y}(\widetilde{\mathbf{d}}) = 0$ (because $x, y \in F_{\mathrm{B}}(\widetilde{\mathbf{d}})$). But then

$$\begin{aligned} 0 &= \mathbf{b}^{x,y}(\widetilde{\mathbf{d}}) = \mathbf{b}^{x,y}\left(\mathbf{d} + \widetilde{\pi}(\mathbf{d}) + \widetilde{\pi}^2(\mathbf{d})\right) = \left(\mathbf{b}^{x,y} + \mathbf{b}^{x,y}\widetilde{\pi} + \mathbf{b}^{x,y}\widetilde{\pi}^2\right)(\mathbf{d}) \\ &= \left(\mathbf{b}^{x,y} + \mathbf{b}^{\pi^{-1}(x),\pi^{-1}(y)}\widetilde{\pi} + \mathbf{b}^{\pi^{-2}(x),\pi^{-2}(y)}\right)(\mathbf{d}) \quad \text{(by neutrality of B)} \\ &= \left(\mathbf{b}^{x,y} + \mathbf{b}^{y,z} + \mathbf{b}^{z,x}\right)(\mathbf{d}) \qquad \text{(by the definition of } \pi\text{)}. \end{aligned}$$

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Thus, Lemma A says that F is a scoring rule.

Other results:

- Other conditions under which a balance rule is actually a scoring rule.
- Also, examples of balance rules which are *not* scoring rules.
- Conditions under which the balance representation or scoring representation of a rule is unique up to affine transformations.

Open problems:

- Neutrality is sufficient but not necessary in Theorem 2.
  Are there normatively compelling conditions that are both necessary *and* sufficient for a scoring rule?
- Are there normatively compelling balance rules that are *not* scoring rules?
- Most of our results require $\mathcal{X}$ to be finite. Can this restriction be eliminated?

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Thank you.

These presentation slides are available at

<http://euclid.trentu.ca/pivato/Research/scoring.pdf>

The paper is available at

< http://mpra.ub.uni-muenchen.de/31896>