Cellular Automata vs. Quasisturmian Shifts

Marcus Pivato
Trent University

http://xaravve.trentu.ca/pivato/Research/qca.pdf

This research supported by NSERC Canada.

Let $\mathbf{T} := \mathbb{T}^1 = \mathbb{R}/\mathbb{Z} \cong [0,1)$ be circle group. Fix $\mathtt{a} \in \mathbf{T}$ (irrational).

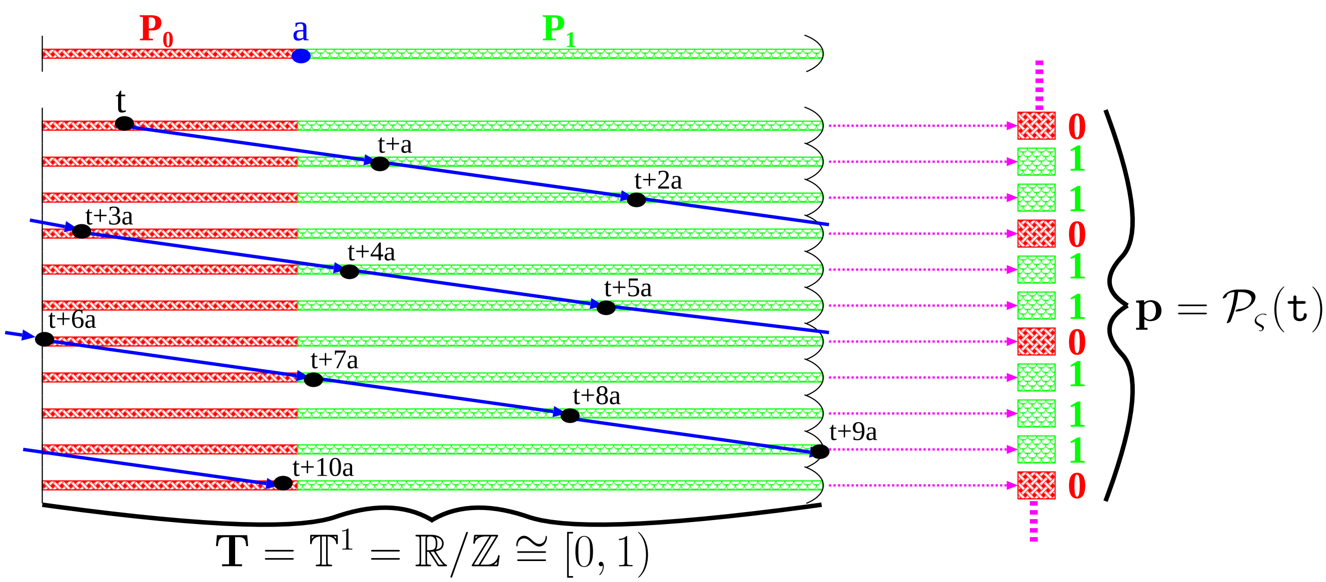
Define **irrational rotation:** $\varsigma : \mathbf{T} \longrightarrow \mathbf{T}$ by $\varsigma(\mathtt{t}) := \mathtt{t} + \mathtt{a}$, $\forall\, \mathtt{t} \in \mathbf{T}$.

If $\mathbf{P}_0 := (0, \mathtt{a})$ & $\mathbf{P}_1 := (\mathtt{a}, 1)$, then $\mathcal{P} := \{\mathbf{P}_0, \mathbf{P}_1\}$ is a partition of $\mathbf{T}$.

Let $\mathcal{A} := \{0, 1\}$. For $\forall\, \mathtt{t} \in \mathbf{T}$, the **$\mathcal{P}$-itinerary** of $\mathtt{t}$ is the $\mathcal{A}$-sequence

$$\mathcal{P}_\varsigma(\mathtt{t}) := [\ldots, p_{-1}, p_0, p_1, p_2, \ldots] \quad \text{where} \quad p_\ell := \begin{cases} 0 & \text{if} \quad \varsigma^\ell(\mathtt{t}) \in \mathbf{P}_0 \\ 1 & \text{if} \quad \varsigma^\ell(\mathtt{t}) \in \mathbf{P}_1 \end{cases}$$

$\mathbf{p} := \mathcal{P}_\varsigma(\mathtt{t})$ is a **Sturmian sequence** [Morse & Hedlund (1940)]



We get function $\mathcal{P}_\varsigma : \widetilde{\mathbf{T}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ (where $\widetilde{\mathbf{T}} \subset \mathbf{T}$ dense G_δ) such that

$$\mathcal{P}_\varsigma \circ \varsigma \quad = \quad \sigma \circ \mathcal{P}_\varsigma.$$

Let $\mathfrak{P} := \text{closure}\left[\mathcal{P}_\varsigma(\widetilde{\mathbf{T}})\right] \subset \mathcal{A}^{\mathbb{Z}}$. Then $\mathfrak{P}$ is a **Sturmian shift**.

Facts:
- Minimal: every point of $\mathfrak{P}$ has dense σ-orbit.
- The smallest complexity of any nonperiodic subshift.
- $\#(\mathfrak{P}\text{-words of length } n) \;=\; n+1$. Thus, $h_{\text{top}}(\mathfrak{P}) = 0$.

Let $K \geq 1$. Let $\mathbf{T} := \mathbb{T}^K = \mathbb{R}^K/\mathbb{Z}^K \cong [0,1)^K$ be the K-torus.

Fix $\mathtt{a} \in \mathbf{T}$ (monothetic). Define **irrational rotation:** $\varsigma : \mathbf{T} \longrightarrow \mathbf{T}$ by

$$\varsigma(\mathtt{t}) \quad := \quad \mathtt{t} + \mathtt{a}, \quad \forall\, \mathtt{t} \in \mathbf{T}.$$

$\mathcal{A}$:= finite alphabet. An $\mathcal{A}$-indexed **open partition** of $\mathbf{T}$ is collection $\mathcal{P} = \{\mathbf{P}_a\}_{a\in\mathcal{A}}$ of open subsets of $\mathbf{T}$, so that:

If $\mathbf{P}_* := \bigsqcup_{a\in\mathcal{A}} \mathbf{P}_a$, then:

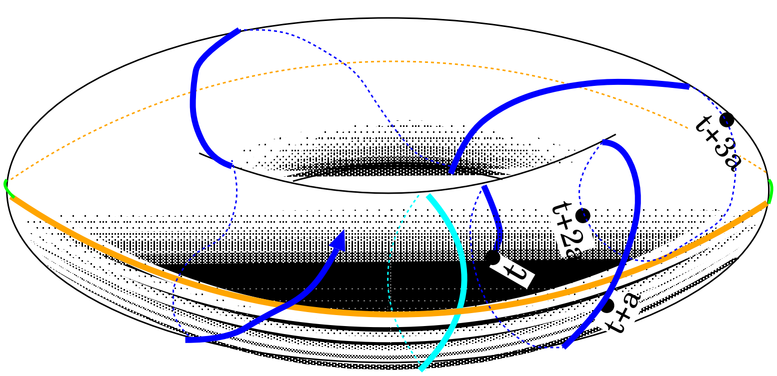
- $\lambda[\mathbf{P}_*] = 1$. ($\lambda$:= Lebesgue measure on $\mathbf{T}$.)
- $\mathbf{P}_*$ is dense.

Define measurable function $\mathcal{P} : \mathbf{T} \longrightarrow \mathcal{A}$ by $\mathcal{P}^{-1}\{a\} = \mathbf{P}_a$, $\forall\, a \in \mathcal{A}$.

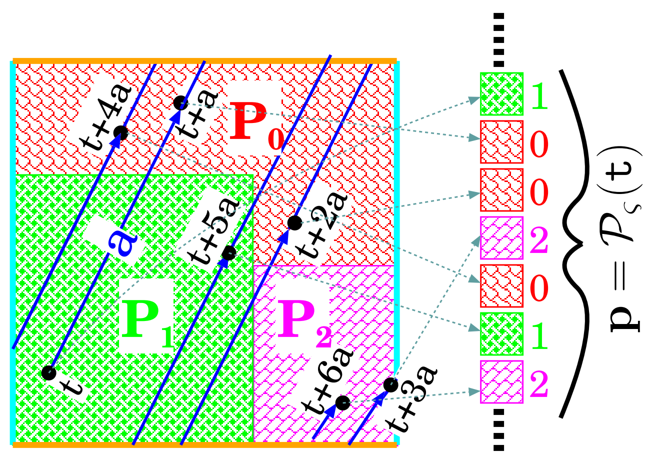
For any $\mathtt{t} \in \mathbf{T}$, the **itinerary** of $\mathtt{t}$ is the sequence

$$\mathcal{P}_\varsigma(\mathtt{t}) := [\ldots, p_{-1}, p_0, p_1, p_2, \ldots] \text{ where } p_\ell := \mathcal{P}\left(\varsigma^\ell(\mathtt{t})\right), \text{ for all } \ell \in \mathbb{Z}.$$

$\mathbf{p} := \mathcal{P}_\varsigma(\mathtt{t})$ is a **quasisturmian sequence**



$\mathbf{T} = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z} \cong [0,1)$



Get function $\mathcal{P}_\varsigma : \widetilde{\mathbf{T}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ ($\widetilde{\mathbf{T}} \subset \mathbf{T}$ dense G_δ) such that $\mathcal{P}_\varsigma \circ \varsigma = \sigma \circ \mathcal{P}_\varsigma$.

Let $\mathfrak{P} :=$ closure $\left[\mathcal{P}_\varsigma(\widetilde{\mathbf{T}})\right] \subset \mathcal{A}^{\mathbb{Z}}$. Then $\mathfrak{P}$ is a **quasisturmian shift**.

$\mathcal{P}$ is **measurable partition** if $\{\mathbf{P}_a\}_{a\in\mathcal{A}}$ are *measurable* & $\lambda[\mathbf{P}_*] = 1$.

Let $\mu := \mathcal{P}_\varsigma(\lambda)$. Then μ is a **quasisturmian measure** on $\mathcal{A}^{\mathbb{L}}$.

Let $D \geq 1$. Let $\mathbb{L} := \mathbb{Z}^D$, a D-dimensional lattice.

Let $\tau : \mathbb{L} \longrightarrow \mathbf{T}$ be a monomorphism with dense image.

Define **rotation action** of $\mathbb{L}$ on $\mathbf{T}$:

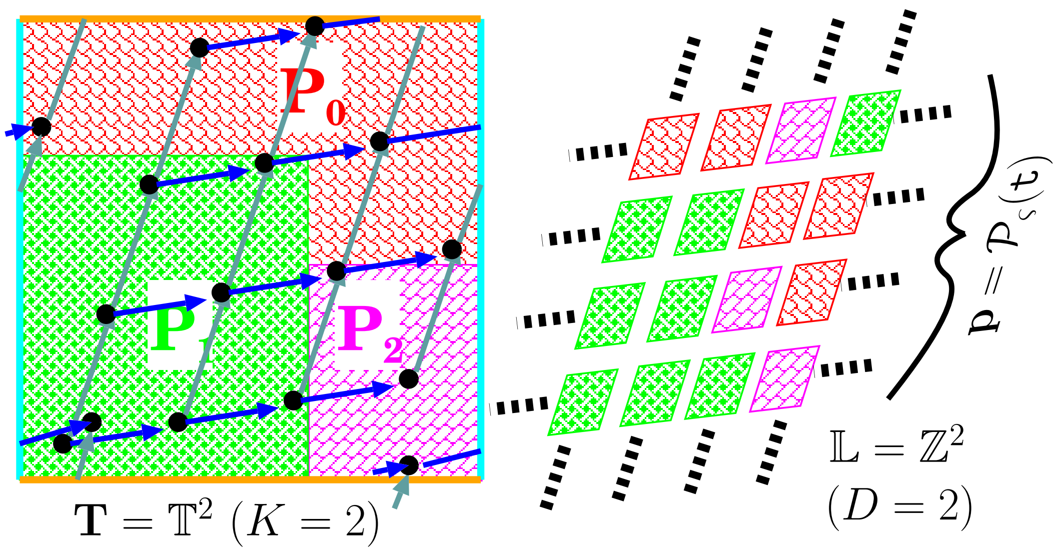
$$\text{For any } \ell \in \mathbb{L} \text{ and } \mathtt{t} \in \mathbf{T}, \quad \varsigma^{\ell}(\mathtt{t}) \quad := \quad \mathtt{t} + \tau(\ell).$$

Let $\mathcal{A}$ be a finite alphabet and let $\mathcal{P} = \{\mathbf{P}_a\}_{a \in \mathcal{A}}$ be an open partition.

For $\forall\, \mathtt{t} \in \mathbf{T}$, the **itinerary** of $\mathtt{t}$ is the D-dimensional $\mathcal{A}$-configuration:

$$\mathcal{P}_\varsigma(\mathtt{t}) \;:=\; [p_\ell]_{\ell \in \mathbb{L}} \text{ where } p_\ell \;:=\; \mathcal{P}\left(\varsigma^{\ell}(\mathtt{t})\right), \text{ for all } \ell \in \mathbb{L}.$$

$\mathbf{p} := \mathcal{P}_\varsigma(\mathtt{t})$ is a **quasisturmian configuration**



We get function $\mathcal{P}_\varsigma : \widetilde{\mathbf{T}} \longrightarrow \mathcal{A}^{\mathbb{L}}$ (where $\widetilde{\mathbf{T}} \subset \mathbf{T}$ dense G_δ) such that

$$\mathcal{P}_\varsigma \circ \varsigma^{\ell} \quad = \quad \sigma^{\ell} \circ \mathcal{P}_\varsigma, \qquad \text{for all } \ell \in \mathbb{L}.$$

$\mathfrak{P} := \text{closure}\left[\mathcal{P}_\varsigma(\widetilde{\mathbf{T}})\right] \subset \mathcal{A}^{\mathbb{L}}$ is a **quasisturmian shift** on $\mathcal{A}^{\mathbb{L}}$

If $\mathcal{P}$ is a *measurable* partition, then $\mu := \mathcal{P}_\varsigma(\lambda)$ is a **quasisturmian measure** on $\mathcal{A}^{\mathbb{L}}$.

The standard topology on $\mathcal{A}^{\mathbb{L}}$ is induced by **Cantor metric:**

$$\forall \mathbf{p},\mathbf{q} \in \mathcal{A}^{\mathbb{L}}, \quad d_C(\mathbf{p},\mathbf{q}) := 2^{-D(\mathbf{p},\mathbf{q})}; \quad D(\mathbf{p},\mathbf{q}) \ := \ \min\{|\ell| \ ; \ \ell \in \mathbb{L}, \ p_\ell \neq q_\ell\}.$$

Let $\mathfrak{QS}_\varsigma \ := \ \{$Quasisturmian sequences in $\mathcal{A}^{\mathbb{L}}\}$.

Then $\mathfrak{QS}_\varsigma$ is σ-invariant, d_C-dense subset of $\mathcal{A}^{\mathbb{L}}$ (but not d_C-closed).

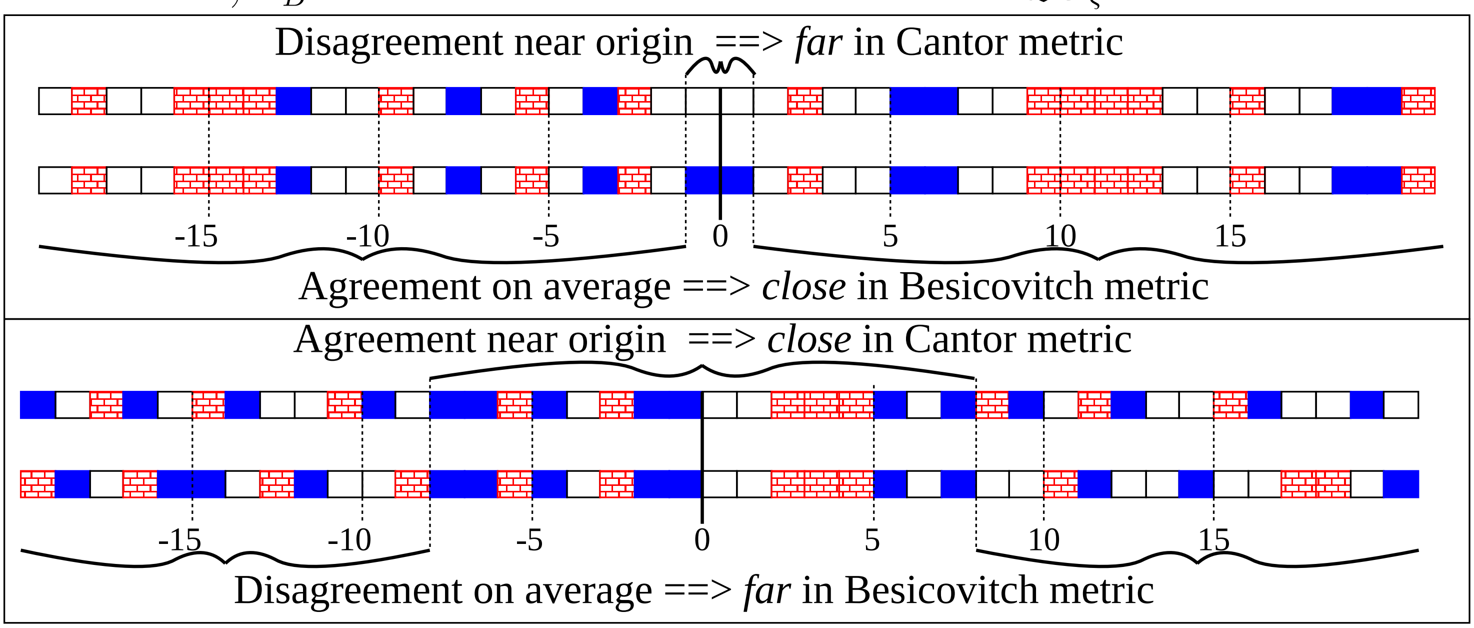
More useful is **Besicovitch (pseudo)metric:** $\forall \mathbf{p},\mathbf{q} \in \mathcal{A}^{\mathbb{L}}$,

$$d_B(\mathbf{p},\mathbf{q}) \ := \ \overline{\mathsf{density}}\,(\ell \in \mathbb{L} \ ; \ p_\ell \neq q_\ell) \ = \ \limsup_{N\to\infty} \frac{\#\{\mathsf{b} \in \mathbb{B}(N) \ ; \ p_{\mathsf{b}} \neq q_{\mathsf{b}}\},}{(2N)^D}$$

(*pseudo*metric, because $d_B(\mathbf{p},\mathbf{q}) = 0$ if $\mathbf{p} \neq \mathbf{q}$ on set of density zero.)

Lemma: *If* $\mathbf{p},\mathbf{q} \in \mathfrak{QS}_\varsigma$, *then* $\Big(\ d_B(\mathbf{p},\mathbf{q}) = 0 \ \Big) \iff \Big(\ \mathbf{p} = \mathbf{q} \ \Big)$.

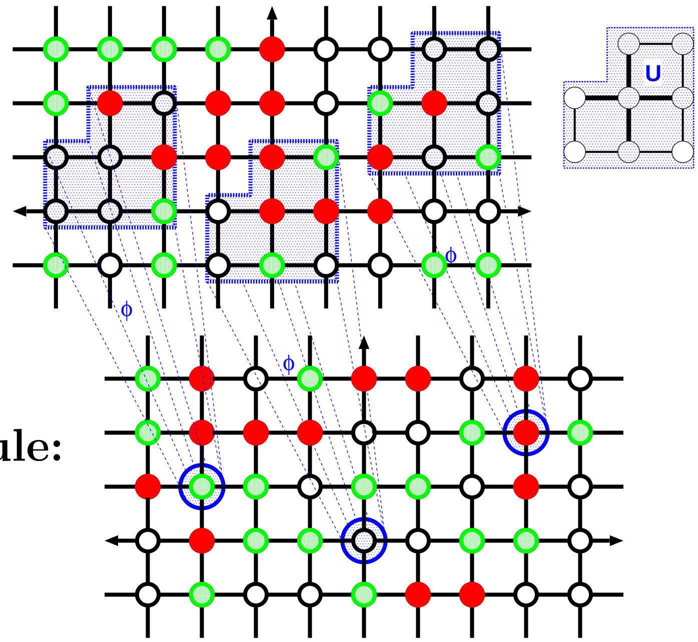
Hence, d_B *is a true metric when restricted to* $\mathfrak{QS}_\varsigma$. □



Cellular automata are Besicovitch-continuous; topological dynamics studied by F. Blanchard, E. Formenti, and P. Kurka [1997].

Neighbourhood:

$\mathbb{U} \subset \mathbb{L}$ (finite set)



Local transformation rule:

ϕ: $\mathcal{A}^{\mathbb{U}} \longrightarrow \mathcal{A}$

The CA **induced by** ϕ is function Φ: $\mathcal{A}^{\mathbb{L}} \circlearrowleft$, so that for $\forall\ \mathbf{a} \in \mathcal{A}^{\mathbb{L}}$,

$$\Phi(\mathbf{a}) \ := \ [b_\ell|_{\ell \in \mathbb{L}}], \qquad \text{where, } \forall \ell \in \mathbb{L}, \quad b_\ell \ = \ \phi\big[a_{(u+\ell)}|_{u \in \mathbb{U}}\big].$$

Equivalently, a CA is a continuous transformation Φ: $\mathcal{A}^{\mathbb{L}} \circlearrowleft$ that commutes with all shifts, ie. $\forall \ell \in \mathbb{L}, \quad \Phi \circ \sigma^\ell \ = \ \sigma^\ell \circ \Phi$.

Example: $\mathbb{L} := \mathbb{Z}$; $\qquad \mathbb{U} := \{0, 1\}$, $\qquad \mathcal{A} := \mathbb{Z}_{/2} = \{0, 1\}$.

Define $\phi : \mathbb{Z}_{/2}{}^{\{0,1\}} \longrightarrow \mathbb{Z}_{/2}$ by $\phi(\mathbf{a}) := a_0 + a_1 \pmod 2$.

This yields **linear cellular automaton** $\Phi \ = \ \mathrm{Id} + \sigma$.

…																1		…
…															1	1		…
…														1		1		…
…													1	1	1	1		…
…												1				1		…
…											1	1			1	1		…
…										1		1		1		1		…
…									1	1	1	1	1	1	1	1		…
…								1								1		…
…							1	1							1	1		…
…						1		1						1		1		…
…					1	1	1	1					1	1	1	1		…
…				1				1				1				1		…
…			1	1			1	1			1	1			1	1		…
…		1		1		1		1		1		1		1		1		…
…	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1		…
…1																1		…

Theorem 1: [Hof & Knill, 1995; M.P. 2004] *Let* $\Phi\colon \mathcal{A}^{\mathbb{L}} \circlearrowleft$ *be a CA.*

(a) *If* $\mathbf{p} \in \mathcal{A}^{\mathbb{L}}$ *is a QS sequence, then* $\Phi(\mathbf{p})$ *is also a QS sequence. Thus,* $\Phi(\mathfrak{QS}_\varsigma) \subseteq \mathfrak{QS}_\varsigma$.

(b) *If* $\mathfrak{P} \subset \mathcal{A}^{\mathbb{L}}$ *is a QS shift, then* $\Phi(\mathfrak{P})$ *is also a QS shift.*

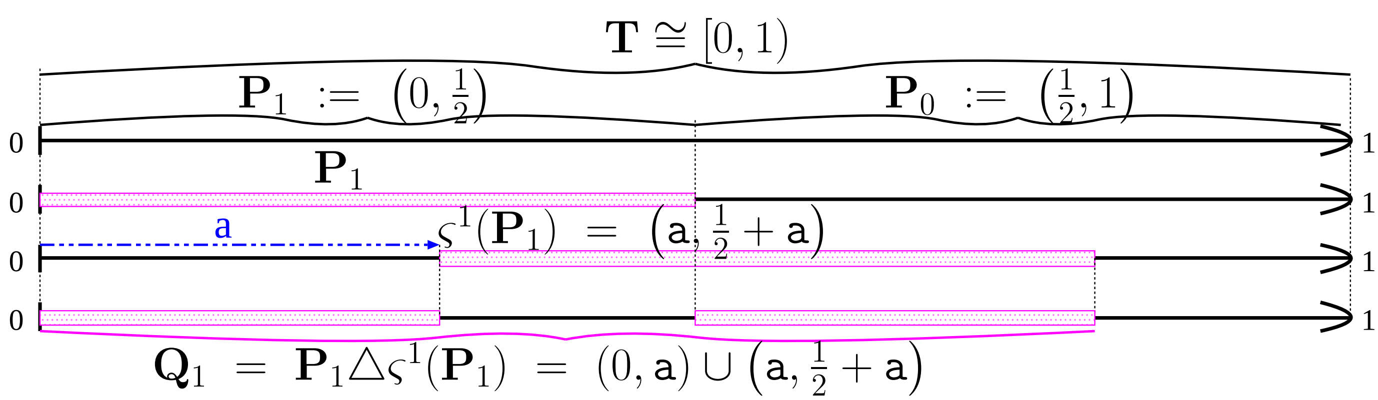
(c) *If* μ *is a QS measure, then* $\Phi(\mu)$ *is also a QS measure.* $\square$

Proof idea: Let $\mathcal{P} = \{\mathbf{P}_a\}_{a\in\mathcal{A}}$. Define partition $\mathcal{Q} := \{\mathbf{Q}_a\}_{a\in\mathcal{A}}$ by

$$\forall\, a \in \mathcal{A}, \qquad \mathbf{Q}_a \quad := \quad \bigcup_{\substack{\mathbf{c}\in\mathcal{A}^{\mathbb{B}} \\ \phi(\mathbf{c})=a}} \ \bigcap_{\mathsf{b}\in\mathbb{B}} \varsigma^{-\mathsf{b}}(\mathbf{P}_{c_\mathsf{b}}) \quad \subset \quad \mathbf{T}.$$

Example: $\mathbb{L} = \mathbb{Z}$; $\mathcal{A} = \mathbb{Z}_{/2}$; $\Phi := \mathrm{Id} + \sigma$.

$\mathbf{Q}_1 = \mathbf{P}_1 \triangle \varsigma(\mathbf{P}_1)$ and $\mathbf{Q}_0 = \mathbf{Q}_1^{\complement} = \Big(\mathbf{P}_0 \cap \varsigma(\mathbf{P}_0)\Big) \sqcup \Big(\mathbf{P}_1 \cap \varsigma(\mathbf{P}_1)\Big)$.



We write: '$\mathcal{Q} = \Phi_\varsigma(\mathcal{P})$'. Thus, Φ induces a map $\Phi_\varsigma : \mathcal{A}^{\mathbf{T}} \longrightarrow \mathcal{A}^{\mathbf{T}}$, where

$$\mathcal{A}^{\mathbf{T}} \quad := \quad \{\mathcal{A}\text{-labelled partitions of } \mathbf{T}\}.$$

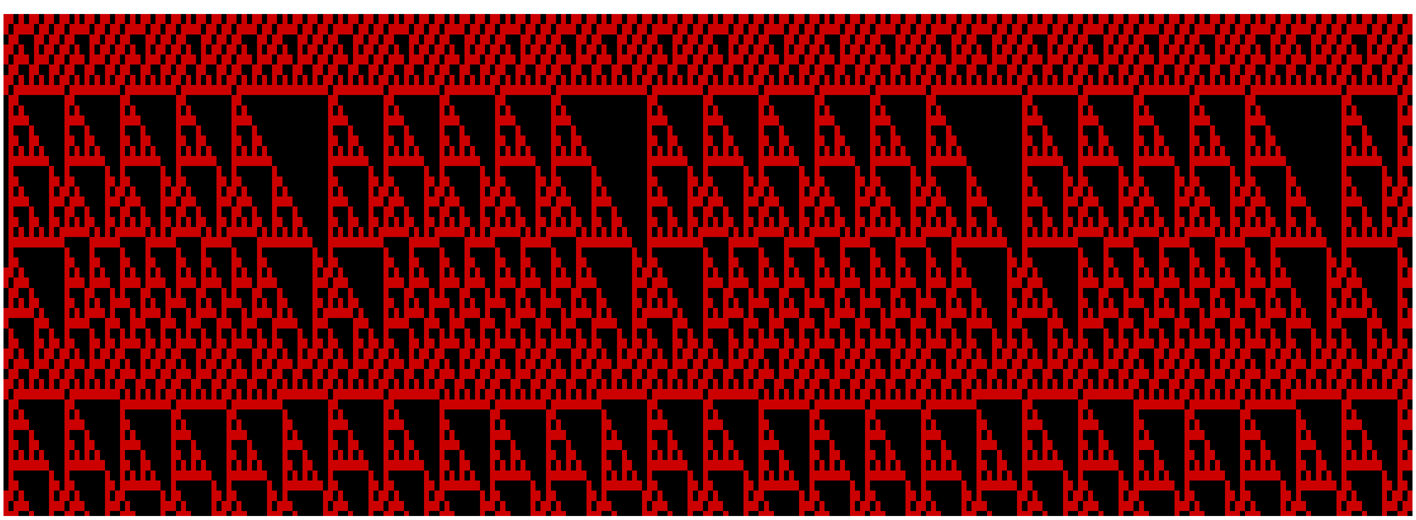
Lemma 2: *Let* $\mathcal{P} \in \mathcal{A}^{\mathbf{T}}$ *and let* $\mathcal{Q} = \Phi_\varsigma(\mathcal{P})$. *Then*

(a) *If* $\mathcal{P}$ *is open then* $\mathcal{Q}$ *is open. If* $\mathtt{t} \in \widetilde{\mathbf{T}}$, *then* $\Phi\left(\mathcal{P}_\varsigma(\mathtt{t})\right) = \mathcal{Q}_\varsigma(\mathtt{t})$.

(b) *If* $\mathfrak{P}$ = *QS shift of* $\mathcal{P}$, *then* $\Phi(\mathfrak{P})$ = *QS shift of* $\mathcal{Q}$.

(c) *If* μ = *QS measure of* $\mathcal{P}$, *then* $\Phi(\mu)$ = *QS measure of* $\mathcal{Q}$. $\square$

Fifty iterations of Φ_ς on a partition of $\mathbf{T}$.



Fifty iterations of Φ on a corresponding quasisturmian sequence.

Let $\mathcal{A}^{\mathbf{T}} := \{\mathcal{A}$-labelled measurable partitions of $\mathbf{T}\}$.

Define **symmetric difference** metric on $\mathcal{A}^{\mathbf{T}}$:

$$d_{\triangle}(\mathcal{P},\mathcal{Q}) \quad = \quad \sum_{a\in\mathcal{A}} \lambda\left(\mathbf{P}_a \triangle \mathbf{Q}_a\right), \qquad \text{for any } \mathcal{P},\mathcal{Q}\in\mathcal{A}^{\mathbf{T}}.$$

$(\mathcal{A}^{\mathbf{T}}, d_{\triangle})$ is a complete and bounded metric space (but not compact).

${}^{o}\mathcal{A}^{\mathbf{T}} := \{\mathcal{A}$-labelled open partitions of $\mathbf{T}\}$ is $d_{\triangle}$-dense subset of $\mathcal{A}^{\mathbf{T}}$.

Theorem *Let $\Phi : \mathcal{A}^{\mathbb{L}}\circlearrowleft$ be any cellular automaton. Then*

(a) $\Phi_{\varsigma}: \mathcal{A}^{\mathbf{T}}\to\mathcal{A}^{\mathbf{T}}$ *is $d_{\triangle}$-Lipschitz.*

(b) $(\mathcal{A}^{\mathbf{T}}, d_{\triangle}, \Phi_{\varsigma})$ *is a topological dynamical system, and*

(c) $\Phi_{\varsigma}({}^{o}\mathcal{A}^{\mathbf{T}}) \subset {}^{o}\mathcal{A}^{\mathbf{T}}$, *so* $({}^{o}\mathcal{A}^{\mathbf{T}}, d_{\triangle}, \Phi_{\varsigma})$ *is a subsystem.* □

Let ${}^{o}\mathcal{A}_0^{\mathbf{T}} := \{\mathcal{A}$-labelled open partitions of $\mathbf{T}$ with $0\in\widetilde{\mathbf{T}}\}$.

Then ${}^{o}\mathcal{A}_0^{\mathbf{T}}$ is a ς-invariant, Φ_{ς}-invariant, comeager subset of ${}^{o}\mathcal{A}^{\mathbf{T}}$.

Theorem: *Define* $\xi_{\varsigma} : {}^{o}\mathcal{A}_0^{\mathbf{T}} \longrightarrow \mathfrak{Q}\mathfrak{S}_{\varsigma}$ *by* $\xi_{\varsigma}(\mathcal{P}) := \mathcal{P}_{\varsigma}(0)$. *Then:*

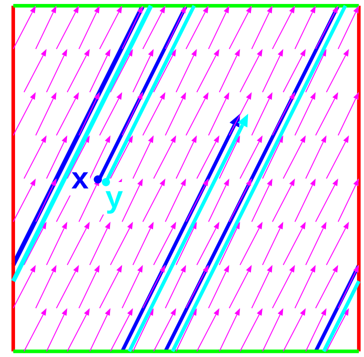
(a) $\xi_{\varsigma}\left({}^{o}\mathcal{A}_0^{\mathbf{T}}\right) = \mathfrak{Q}\mathfrak{S}_{\varsigma}$.

(b) *If* $\mathcal{P},\mathcal{Q}\in {}^{o}\mathcal{A}_0^{\mathbf{T}}$, *then* $d_{\triangle}(\mathcal{P},\mathcal{Q}) \;=\; 2\cdot d_B\Big(\xi_{\varsigma}(\mathcal{P}), \xi_{\varsigma}(\mathcal{Q})\Big)$.

(c) $\xi_{\varsigma}\circ\Phi_{\varsigma} \;=\; \Phi\circ\xi_{\varsigma}$. *Also, for any* $\ell\in\mathbb{L}$, $\quad \xi_{\varsigma}\circ\varsigma^{\ell} \;=\; \sigma^{\ell}\circ\xi_{\varsigma}$.

(e) ξ_{ς} *is top.dyn.sys.isomorphism* $({}^{o}\mathcal{A}_0^{\mathbf{T}}, d_{\triangle}, \Phi_{\varsigma}, \varsigma) \cong (\mathfrak{Q}\mathfrak{S}_{\varsigma}, d_B, \Phi, \sigma)$.
□

Idea: Study action of Φ on $\mathfrak{Q}\mathfrak{S}_{\varsigma}$ via action of Φ_{ς} on ${}^{o}\mathcal{A}_0^{\mathbf{T}}$, ${}^{o}\mathcal{A}^{\mathbf{T}}$ & $\mathcal{A}^{\mathbf{T}}$.



Top.dyn.sys. $(\mathbf{X}, d, \varphi)$ is **equicontinuous** if, for $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ such that $\forall\ x, y \in \mathbf{X}$,

$$\Big(\ d(x,y) < \delta\ \Big) \Longrightarrow \Big(\ d\left(\varphi^n(x), \varphi^n(y)\right) < \epsilon \text{ for all } n \in \mathbb{N}\ \Big).$$

Theorem: *If $(\mathcal{A}^{\mathbb{L}}, d_C, \Phi)$ is equicont. then $({}^{o}\!\mathcal{A}_0^{\mathbf{T}}, d_{\triangle}, \Phi_{\varsigma})$ is equicont.* $\Box$

Let $\xi > 0$. The top.dyn.sys. $(\mathfrak{Q}\mathfrak{S}_{\varsigma}, d_B, \Phi)$ is (positively) **ξ-expansive** if, for $\forall\ \mathbf{p}, \mathbf{q} \in \mathfrak{Q}\mathfrak{S}_{\varsigma}$ with $\mathbf{q} \neq \mathbf{p}$, $\exists\ n \in \mathbb{N}$ so that $d_B\left(\Phi^n(\mathbf{p}), \Phi^n(\mathbf{q})\right) > \xi$.

Proposition [Blanchard, Formenti & Kurka, 1997]

If $\Phi : \mathcal{A}^{\mathbb{Z}}\circlearrowleft$ is any CA, then $(\mathcal{A}^{\mathbb{Z}}, \Phi, d_B)$ is <u>not</u> expansive.

Proof idea: Construct $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$ which is 'nonexpansive' for all CA.. $\Box$

But $\mathbf{a}$ is *not* QS, so this proof does *not* apply to $(\mathfrak{Q}\mathfrak{S}_{\varsigma}, d_B, \Phi)$.

If $\mathtt{t} \in \mathbb{T}^1$, let

$$\mathfrak{Q}\mathfrak{S}_{\mathtt{t}} \quad := \quad \{\text{QS sequences induced by rotation-by-}\mathtt{t}\} \quad \subset \quad \mathcal{A}^{\mathbb{Z}}.$$

Theorem: *Let $\mathcal{A} = \mathbb{Z}_{/2}$ and let $\Phi := \mathrm{Id} + \sigma$.*

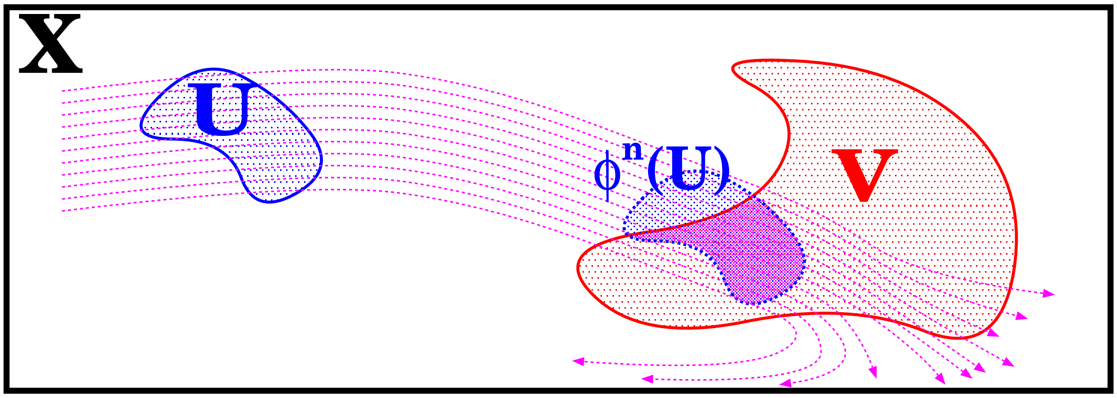
For $\forall_{\lambda}\ \mathtt{t} \in \mathbf{T}$, the top.dyn.sys. $(\mathfrak{Q}\mathfrak{S}_{\mathtt{t}}, d_B, \Phi)$ <u>is</u> expansive. $\Box$

Proof Idea: Sufficient to show expansive at [...0000...]. (Φ is linear).

Small 'perturbations' of [...0000...] are magnified along certain powers of 2 (use *Fermat Property:* for $\forall\ n \in \mathbb{N}$, $\Phi^{2^n} = \mathrm{Id} + \sigma^{2^n}$.) $\Box$

A top.dyn.sys. $(\mathbf{X}, d, \varphi)$ is **transitive** if, for any open sets $\mathbf{U}, \mathbf{V} \subset \mathbf{X}$, $\exists\ n \in \mathbb{N}$ so that $\mathbf{U} \cap \varphi^{-n}(\mathbf{V}) \neq \emptyset$.

$(\mathbf{X}, d, \varphi)$ is **topologically mixing** if, for any open sets $\mathbf{U}, \mathbf{V} \subset \mathbf{X}$, $\exists\ N \in \mathbb{N}$ so that $\mathbf{U} \cap \varphi^{-n}(\mathbf{V}) \neq \emptyset$ for all $n > N$.



Theorem:

- *If* $({}^{o}\mathcal{A}^{\mathbf{T}}, d_{\triangle}, \Phi_{\varsigma})$ *is transitive, then* $(\mathcal{A}^{\mathbb{L}}, d_C, \Phi)$ *is transitive.*
- *If* $({}^{o}\mathcal{A}^{\mathbf{T}}, d_{\triangle}, \Phi_{\varsigma})$ *is top.mixing, then* $(\mathcal{A}^{\mathbb{L}}, d_C, \Phi)$ *is top.mixing.* $\Box$

Question: $\exists$ CA Φ such that $({}^{o}\mathcal{A}^{\mathbf{T}}, d_{\triangle}, \Phi_{\varsigma})$ is transitive or top.mixing?

Conjecture: *If* $\Phi\colon \mathcal{A}^{\mathbb{L}} \circlearrowleft$ *is surjective, then* $\Phi_\varsigma\colon \mathcal{A}^{\mathbf{T}} \circlearrowleft$ *is surjective.*

Counterexample: $\mathcal{A} = \{0,1\} = \mathbb{Z}_{/2}$. For any $\mathcal{P} \in \mathcal{A}^{\mathbf{T}}$, if $\mathcal{P} = \{\mathbf{P}_0, \mathbf{P}_1\}$, then let $\overline{\mathcal{P}} := \{\overline{\mathbf{P}}_0, \overline{\mathbf{P}}_1\}$, where $\overline{\mathbf{P}}_0 := \mathbf{P}_1$ and $\overline{\mathbf{P}}_1 := \mathbf{P}_0$.

Lemma: *Let* $\Phi := \mathrm{Id} + \sigma$. *If* $\mathcal{P} \in \Phi_\varsigma(\mathcal{A}^{\mathbf{T}})$, *then* $\overline{\mathcal{P}} \notin \Phi_\varsigma(\mathcal{A}^{\mathbf{T}})$. □

Thus $\Phi_\varsigma(\mathcal{A}^{\mathbf{T}})$ only fills 'half' of $\mathcal{A}^{\mathbf{T}}$. But $\Phi_\varsigma(\mathcal{A}^{\mathbf{T}})$ is still dense in $\mathcal{A}^{\mathbf{T}}$...

Theorem *Let* $\Phi : \mathcal{A}^{\mathbb{Z}} \circlearrowleft$ *be a cellular automaton. TFAE:*

(a) Φ *is surjective onto* $\mathcal{A}^{\mathbb{Z}}$.

(b) $\Phi_\varsigma({}^o\!\mathcal{A}^{\mathbf{T}})$ *is* $d_\triangle$*-dense in* ${}^o\!\mathcal{A}^{\mathbf{T}}$, *and* $\Phi_\varsigma(\mathcal{A}^{\mathbf{T}})$ *is* $d_\triangle$*-dense in* $\mathcal{A}^{\mathbf{T}}$.

(c) $\Phi(\mathfrak{QS}_\varsigma)$ *is* d_B*-dense in* $\mathfrak{QS}_\varsigma$.

(d) $\Phi(\mathfrak{QS}_\varsigma)$ *is* d_C*-dense in* $\mathfrak{QS}_\varsigma$.

Proof idea: "**(a)⟹(b)**" Irrational rotations are *rank one*, so we can 'tile' any QS sequence $\mathbf{p} = \mathcal{P}_\varsigma(\mathtt{t})$ with some word $\mathbf{w}$.

Let $\mathbf{v}$ be a Φ-preimage of $\mathbf{w}$. Build $\mathcal{Q} \in {}^o\!\mathcal{A}^{\mathbf{T}}$ such that $\mathbf{q} := \mathcal{Q}_\varsigma(\mathtt{t})$ is 'tiled' with $\mathbf{v}$ (and 'tilings' of $\mathbf{p}$ and $\mathbf{q}$ are 'aligned'). Then $\Phi(\mathbf{q}) \underset{d_B}{\frown} \mathbf{p}$. Thus, $\Phi_\varsigma(\mathcal{Q}) \underset{d_\triangle}{\frown} \mathcal{P}$.

"**(b)⟹(c)**" Top.dyn.sys. isomorphism $({}^o\!\mathcal{A}_0^{\mathbf{T}}, d_\triangle, \Phi_\varsigma) \cong (\mathfrak{QS}_\varsigma, d_B, \Phi)$.

"**(c)⟹(d)**" Use 'tiling' argument to show:

If $\mathfrak{X} \subset \mathfrak{QS}_\varsigma$ *is* σ*-invariant and* d_B*-dense in* $\mathfrak{QS}_\varsigma$, *then* $\mathfrak{X}$ *is also* d_C*-dense in* $\mathfrak{QS}_\varsigma$.

"**(d)⟹(a)**" $\mathfrak{QS}_\varsigma$ is d_C-dense in $\mathcal{A}^{\mathbb{L}}$. Thus, $\Phi(\mathcal{A}^{\mathbb{L}})$ is dense in $\mathcal{A}^{\mathbb{L}}$; thus $\Phi(\mathcal{A}^{\mathbb{L}}) = \mathcal{A}^{\mathbb{L}}$ (compactness). □

Theorem *Let $\Phi\colon \mathcal{A}^{\mathbb{L}} \circlearrowleft$ be a CA. $\exists$ dense G_δ subset ${}^*\mathcal{A}^{\mathbf{T}} \subset \mathcal{A}^{\mathbf{T}}$, with $\Phi_\varsigma({}^*\mathcal{A}^{\mathbf{T}}) \subseteq {}^*\mathcal{A}^{\mathbf{T}}$, so that, for $\forall\ \mathcal{P} \in {}^*\mathcal{A}^{\mathbf{T}}$, the following dichotomies hold:*

(a) *If μ is QS measure induced by $\mathcal{P}$, then either Φ is constant (μ-æ), or Φ is injective (μ-æ).*

(b) *If $\mathcal{P} \in {}^o\mathcal{A}^{\mathbf{T}}$ and $\mathfrak{P}$ is the QS shift induced by $\mathcal{P}$, then either $\Phi|_{\mathfrak{P}}$ is constant, or $\Phi|_{\mathfrak{P}}$ is injective.*

Proof idea: Say that $\mathcal{P}$ is *simple* if $\mathcal{P}$ has no translational symmetries.

Lemma: *Suppose $\mathcal{P}$ is simple.*

(a) *If $\mathcal{P} \in \mathcal{A}^{\mathbf{T}}$ then the map $\mathcal{P}_\varsigma : \widetilde{\mathbf{T}} \longrightarrow \mathcal{A}^{\mathbb{L}}$ is injective (λ-æ).*

(b) *If $\mathcal{P} \in {}^o\mathcal{A}^{\mathbf{T}}$ then $\mathcal{P}_\varsigma : \widetilde{\mathbf{T}} \longrightarrow \mathcal{P}_\varsigma(\widetilde{\mathbf{T}}) \subset \mathcal{A}^{\mathbb{L}}$ is a homeomorphism with respect to both to the d_C and d_B metrics on $\widetilde{\mathfrak{P}}$.* $\square$

Corollary: *Suppose $\mathcal{P}$ and $\mathcal{Q} = \Phi_\varsigma(\mathcal{P})$ are both simple.*

(a) *If μ is QS measure of $\mathcal{P}$, then Φ is injective (μ-æ).*

(b) *If $\mathcal{P} \in {}^o\mathcal{A}^{\mathbf{T}}$ and $\mathfrak{P}$ is QS shift of $\mathcal{P}$, then $\Phi|_{\mathfrak{P}}$ is injective.* $\square$

Strategy: Find a dense G_δ set ${}^*\mathcal{A}^{\mathbf{T}} \subset \mathcal{A}^{\mathbf{T}}$ of simple partitions whose Φ_ς-images are also simple.

Theorem: *Let $\Phi\colon \mathcal{A}^{\mathbb{L}} \circlearrowleft$ be a CA. Let μ be QS measure generated by $\mathcal{P} \in \mathcal{A}^{\mathbf{T}}$. Then:*

$$\Big(\mu \text{ is } \Phi\text{-invariant} \Big) \iff \Big(\Phi_\varsigma(\mathcal{P}) \;=\; \boldsymbol{\rho}^{\mathtt{t}}(\mathcal{P}) \text{ for some } \mathtt{t} \in \mathbf{T} \Big).$$

($\boldsymbol{\rho}^{\mathtt{t}}$ is **rotation map** *ie. $\boldsymbol{\rho}^{\mathtt{t}}(\mathtt{s}) \;:=\; \mathtt{s} + \mathtt{t}$, for all $\mathtt{s} \in \mathbf{T}$).*

Proof Sketch: If $\mathcal{Q} = \Phi_\varsigma(\mathcal{P})$, then $\mathcal{Q}$ also generates μ. Must show:

Claim: *If μ is generated by $\mathcal{P} \in \mathcal{A}^{\mathbf{T}}$, then:*

$$\{\textit{Partitions of } \mathbf{T} \textit{ generating } \mu\} \quad = \quad \{\boldsymbol{\rho}^{\mathtt{t}}(\mathcal{P})\ ;\ \mathtt{t} \in \mathbf{T}\}.$$

Claim proof sketch: If $\mathcal{P}$ simple, then $\mathcal{P}_\varsigma\colon \widetilde{\mathbf{T}} \longrightarrow \mathcal{A}^{\mathbb{L}}$ injective (λ-æ).

If $\mathcal{Q}$ also generates μ, and $\mathcal{Q}$ simple, then $\mathcal{Q}_\varsigma\colon \widetilde{\mathbf{T}} \longrightarrow \mathcal{A}^{\mathbb{L}}$ is injective (λ-æ). (If $\mathcal{P}$, $\mathcal{Q}$ not simple, then replace with 'quotient' partitions)

Thus, get commuting diagram:

$$\begin{tikzcd} (\mathbf{T},\lambda,\varsigma) \arrow[rr, dotted, "\varphi"] \arrow[dr, "\mathcal{P}_\varsigma"'] & & (\mathbf{T},\lambda,\varsigma) \arrow[dl, "\mathcal{Q}_\varsigma"] \\ & (\mathbf{T},\lambda,\varsigma) & \end{tikzcd}$$

$\varphi := \mathcal{Q}_\varsigma^{-1} \circ \mathcal{P}_\varsigma$ is measure-preserving endomorphism of torus rotation system $(\mathbf{T},\lambda,\varsigma)$. Any endomorphism must be a rotation. Claim follows.□

Corollary: *If $\mathcal{A} = \mathbb{Z}_{/2}$, then $\Phi := \mathrm{Id} + \sigma$ has no QS invariant measures.*

Proof idea: If $\mathcal{P} \in \mathcal{A}^{\mathbf{T}}$, then $\Phi_\varsigma^{2^{n_k}}(\mathcal{P}) \xrightarrow[k\to\infty]{} \mathcal{O}$ (trivial partition) for some sequence $\{n_k\}_{k=1}^{\infty}$. ("Niltropy")

But this is impossible if Φ_ς is acting as rotation. □

Question: If μ is a QS measure, then $\exists$ $\text{wk}^*\text{-}\lim_{N\to\infty} \dfrac{1}{N} \sum_{n=1}^{N} \Phi^n \mu$?

If $\mathcal{A}$ is an abelian group (eg. $\mathcal{A} = \mathbb{Z}_{/n}$) then so is $\mathcal{A}^{\mathbb{L}}$. A **linear CA** is a CA $\Phi : \mathcal{A}^{\mathbb{L}} \circlearrowleft$ that is also a group homomorphism.

If μ is prob.measure on $\mathcal{A}^{\mathbb{L}}$, then Φ **asymptotically randomizes** μ if $\text{wk}^*\text{–}\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \Phi^n \mu \;=\; \eta$, ($\eta :=$ uniform ('Haar') measure on $\mathcal{A}^{\mathbb{L}}$.)

Theorem 1 [D.Lind, 1984]

$$\left(\begin{array}{l} \bullet\ \mathcal{A} = \mathbb{Z}_{/2} \text{ and } \mathbb{L} = \mathbb{Z}. \\ \bullet\ \Phi = \sigma^{-1} + \sigma^{1}.\ (\text{“N.N. XOR”}) \\ \bullet\ \mu \text{ is a } \textbf{Bernoulli measure}. \end{array}\right) \Longrightarrow \left(\begin{array}{l} \Phi \text{ asymptotically} \\ \text{randomizes } \mu \end{array}\right)$$

Theorem 2 [P.Ferrari, P.Ney, A.Maass & S.Martínez, 1998]

$$\left(\begin{array}{l} \bullet\ p \text{ prime};\quad \mathcal{A} = \mathbb{Z}_{/(p^n)};\quad \mathbb{L} = \mathbb{N}. \\ \bullet\ \Phi = \varphi_0\cdot\sigma^0 + \varphi_1\cdot\sigma^1.\ (\text{“Ledrappier”}) \\ \quad \varphi_0 \text{ and } \varphi_1 \text{ are relativ. prime to } p. \\ \bullet\ \mu \text{ is a } \textbf{Markov measure}. \\ \quad \text{All transition probabilities nonzero.} \end{array}\right) \Longrightarrow \left(\begin{array}{l} \Phi \text{ asympt.} \\ \text{randomizes } \mu \end{array}\right)$$

Theorem 3 [R.Yassawi & M.P., 2000]

$$\left(\begin{array}{l} \bullet\ \text{Let } \mathcal{A} = \mathbb{Z}_{/p}\ (p \text{ prime}). \\ \bullet\ \text{Let } \mathbb{L} = \mathbb{Z}^D \times \mathbb{N}^d \text{ be any lattice.} \\ \bullet\ \text{Let } \Phi\colon \mathcal{A}^{\mathbb{L}} \circlearrowleft \text{ be any nontrivial linear} \\ \quad \text{cellular automaton (ie. not a shift).} \\ \bullet\ \mu \text{ a } \textbf{harmonically mixing} \text{ measure.} \end{array}\right) \Longrightarrow \left(\begin{array}{l} \Phi \text{ asympt.} \\ \text{randomizes } \mu \end{array}\right)$$

Harmonic Mixing: eg. Bernoulli measures, Markov chains, or Markov random fields with 'full support'.
Y&P [2002, 2003] extends Thm.3 to $\mathcal{A} = \mathbb{Z}_{/n}$ ($\forall\, n \in \mathbb{N}$) and other abelian groups, and to μ supported on subshifts of finite type and sofic shifts.

Theorem: *$\exists$ $\mathbf{S} \subseteq \mathbb{T}^1$ (dense G_δ) so that, for any $\mathtt{s} \in \mathbf{S}$ and $\mathcal{P} \in {}^o\!\mathcal{A}^{\mathbf{T}}$, if μ is the QS measure generated by $\mathcal{P}$ under $\mathtt{s}$-rotation, then μ is <u>not</u> asymptotically randomized by $\Phi = \mathrm{Id} + \sigma$.*

Proof Heuristic: *Fermat Property:* For $\forall\, n \in \mathbb{N}$, $\Phi^{2^n} = \mathrm{Id} + \sigma^{2^n}$.

Thus $\Phi_\varsigma^{2^n} = \mathrm{Id} + \varsigma^{2^n}$.

Now suppose $\mathtt{s} = 0.1\underbrace{0}_{1}1\underbrace{000}_{3}1\underbrace{0000000}_{7}1\underbrace{00\ldots0}_{15}1\ldots$

Dyadic Recurrence: $\exists\ \{n_k\}_{k=1}^\infty$ s.t. $d(2^{n_k} \cdot \mathtt{s},\ 0) < 2^{-n_k}$.

Thus, $\Phi_\varsigma^{2^n}(\mathcal{P}) = \mathcal{P} + \varsigma^{2^n}(\mathcal{P}) \sim \mathcal{P} + \mathcal{P} = \mathcal{O}$. ($\mathcal{O}$ = trivial partition).

Thus, $\Phi_\varsigma^{2^n+k}(\mathcal{P}) \sim \Phi^k(\mathcal{O}) \sim \mathcal{O}$ for $k \in \left[0...2^{n-4}\right]$

Thus, $\Phi^{2^n+k}(\mu)[0000] \geq \frac{1}{8} > \frac{1}{16} = \eta[0000]$ for $k \in \left[0...2^{n-4}\right]$.

Thus, $\mathrm{wk}^*\!-\!\lim_{N\to\infty} \dfrac{1}{N}\sum_{n=1}^{N} \Phi^n\mu \neq \eta$: it gives too much mass to $[0000]$. $\Box$

Corollary: $\exists$ weak* cluster point μ_∞ of $\left\{\dfrac{1}{N}\sum_{n=1}^{N}\Phi^n\mu\right\}_{N=1}^{\infty}$; $\mu_\infty \neq \eta$.

Now, $\Phi(\mu_\infty) = \mu_\infty$, so $h(\mu_\infty, \sigma) = 0$ [Host, Maass, Martinez, 2004]. But also μ_∞ is not QS.

Question: What is μ_∞?

Remark: $\lambda[\mathbf{S}] = 0$. Can we find a larger set of 'nonrandomizing' irrational rotations? **Conjecture:** Yes.

Question: Can *any* CA asymptotically randomize *any* QS measure?

Conjecture: No.

Hof & Knill [1995] saw empirically that Φ_ς 'chops' partition $\mathcal{P}$ into tiny bits. Empirically, # of bits in $\Phi_\varsigma^n(\mathcal{P})$ grew polynomially with n.

Idea: Measure 'chopping' via growth in size of $\partial\mathcal{P} := \bigcup_{a\in\mathcal{A}} \partial\mathbf{P}_a$.

Let $\lceil\partial\mathcal{P}\rceil$ be some measure of size of $\partial\mathcal{P}$. For example:

- If $\mathbf{T} = \mathbb{T}^1$, then $\lceil\partial\mathcal{P}\rceil$ is *cardinality* of $\partial\mathcal{P}$.
- If $\mathbf{T} = \mathbb{T}^2$, then $\lceil\partial\mathcal{P}\rceil$ is *length* of $\partial\mathcal{P}$.
- If $\mathbf{T} = \mathbb{T}^3$, then $\lceil\partial\mathcal{P}\rceil$ is *area* of $\partial\mathcal{P}$.
- If $\partial\mathcal{P}$ is α-dim. fractal, then $\lceil\partial\mathcal{P}\rceil$ is α-dim. Hausdorff measure.

Φ_ς **chops $\mathcal{P}$ on average** if $\displaystyle\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \left\lceil\partial\left(\Phi_\varsigma^n[\mathcal{P}]\right)\right\rceil = \infty.$

Equivalently, $\exists\, \mathbb{J} \subseteq \mathbb{N}$ such that

$$\mathsf{density}\,(\mathbb{J}) = 1 \quad \text{and} \quad \lim_{\mathbb{J}\ni j\to\infty} \left\lceil\partial\left(\Phi_\varsigma^n[\mathcal{P}]\right)\right\rceil = \infty.$$

Φ_ς **chops $\mathcal{P}$ intermittently** if $\displaystyle\limsup_{n\to\infty} \left\lceil\partial\left(\Phi_\varsigma^n[\mathcal{P}]\right)\right\rceil = \infty.$

Equivalent: $\exists\, \mathbb{J} \subseteq \mathbb{N}$ (zero density) so that $\displaystyle\lim_{\mathbb{J}\ni j\to\infty} \left\lceil\partial\left(\Phi_\varsigma^n[\mathcal{P}]\right)\right\rceil = \infty.$

Proposition: *If $\mathbb{L} := \mathbb{Z}^D$ and $\Phi\colon \mathcal{A}^{\mathbb{L}}\circlearrowleft$ is CA, then $\exists\, C > 0$ so that, if $\mathcal{P} \in {}^o\!\mathcal{A}^{\mathbf{T}}$ and $n \in \mathbb{N}$, then $\left\lceil\partial\left(\Phi_\varsigma^n[\mathcal{P}]\right)\right\rceil \leq C\cdot n^D \cdot \lceil\partial\mathcal{P}\rceil$.* □

Proposition: *If $(\mathfrak{Q}\mathfrak{S}_\varsigma, d_B, \Phi)$ is expansive, then Φ_ς intermittently chops all $\mathcal{P} \in {}^o\!\mathcal{A}^{\mathbf{T}}$.*

Proof idea: Let $\mathcal{P}^n := \Phi_\varsigma^n(\mathcal{P})$. Then $\mathcal{P}_\varsigma^n : \mathbf{T}\longrightarrow\mathfrak{Q}\mathfrak{S}_\varsigma$ is Lipschitz with constant proportional to $\lceil\partial\mathcal{P}^n\rceil$. If $\mathtt{s}\sim\mathtt{t}$ then $\mathcal{P}_\varsigma(\mathtt{s})\sim\mathcal{P}_\varsigma(\mathtt{t})$. But if $\mathcal{P}_\varsigma^n(\mathtt{s})\not\sim\mathcal{P}_\varsigma^n(\mathtt{t})$, then $\lceil\partial\mathcal{P}^n\rceil$ must have gotten large. □

$\mathcal{A} := \mathbb{Z}_{/2}$. A **boolean linear CA (BLCA)** is linear CA $\Phi\colon \mathcal{A}^{\mathbb{L}}\circlearrowleft$.

Theorem: *Let $\Phi\colon \mathcal{A}^{\mathbb{L}}\circlearrowleft$ be a nontrivial BLCA. For a 'generic' set of $\mathcal{P} \in {}^o\!\mathcal{A}^{\mathbf{T}}$, Φ chops $\mathcal{P}$ on average.*

If $\mathbf{T} = \mathbb{T}^1$, then Φ chops <u>all</u> $\mathcal{P} \in {}^o\!\mathcal{A}^{\mathbf{T}}$ on average.

Proof idea: Φ_ς multiplies boundary points the same way Φ multiplies a 'point mass' [.......000000010000000......] into many point masses. $\Box$

$$
\begin{array}{ccccccccccccccc|l}
\ldots & & & & & & & 1 & & & & & & & \ldots & \\
\ldots & & & & & & 1 & & 1 & & & & & & \ldots & \\
\ldots & & & & & 1 & & & & 1 & & & & & \ldots & \\
\ldots & & & & 1 & & 1 & & 1 & & 1 & & & & \ldots & \texttt{Time} \\
\ldots & & & 1 & & & & & & & & 1 & & & \ldots & \\
\ldots & & 1 & & 1 & & & & & & 1 & & 1 & & \ldots & \\
\ldots & 1 & & & & 1 & & & & 1 & & & & 1 & \ldots & \downarrow
\end{array}
$$

We can be more specific....

Theorem: *Let $\Phi = \mathrm{Id} + \sigma$. Let ς be a $\mathbb{Z}$-action on $\mathbf{T}$, and let $\mathcal{P} \in {}^o\!\mathcal{A}^{\mathbf{T}}$ be 'generic'. Then, as $n\to\infty$...*

(a) *...the <u>maximum</u> of $\left\lceil \partial\left(\Phi_\varsigma^n[\mathcal{P}]\right)\right\rceil$ grows <u>linearly</u>. $\exists K > 0$ so that*

$$K \quad \leq \quad \limsup_{n\to\infty} \frac{1}{n} \left\lceil \partial\left(\Phi_\varsigma^n[\mathcal{P}]\right)\right\rceil \quad \leq \quad \lceil \partial\mathcal{P} \rceil .$$

(b) *...the <u>minimum</u> of $\left\lceil \partial\left(\Phi_\varsigma^n[\mathcal{P}]\right)\right\rceil$ remains <u>constant</u>:*

$$\liminf_{n\to\infty} \left\lceil \partial\left(\Phi_\varsigma^n[\mathcal{P}]\right)\right\rceil \quad \leq \quad 2\lceil \partial\mathcal{P} \rceil .$$

(c) *...the <u>average</u> of $\left\lceil \partial\left(\Phi_\varsigma^n[\mathcal{P}]\right)\right\rceil$ grows like n^α, where $\alpha := \log_2\left(\frac{3}{2}\right)$.*

$$\text{If } A(N) \;:=\; \frac{1}{N}\sum_{n=0}^{N-1} \left\lceil \partial\left(\Phi_\varsigma^n[\mathcal{P}]\right)\right\rceil, \text{ then } \lim_{N\to\infty} \frac{\log\left(A(N)\right)}{\log(N)} \;=\; \alpha.$$

(d) *Both* **(a)** *and* **(b)** *are equalities for a dense set of $\mathcal{P} \in {}^o\!\mathcal{A}^{\mathbf{T}}$.* $\Box$