Positive expansiveness versus network dimension in symbolic dynamical systems

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Let $\mathcal{A}$ be a finite set (‘alphabet’), let $D \in \mathbb{N}$, and let $\mathcal{A}^{\mathbb{Z}^D}$ be the set of all $\mathbb{Z}^D$-indexed configurations of symbols from $\mathcal{A}$, with the Cantor topology.
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*If $D \geq 2$, then $\Phi$ cannot be positively expansive.*
Shereshevsky’s result

Let $\mathcal{A}$ be a finite set (‘alphabet’), let $D \in \mathbb{N}$, and let $\mathcal{A}^{\mathbb{Z}^D}$ be the set of all $\mathbb{Z}^D$-indexed *configurations* of symbols from $\mathcal{A}$, with the Cantor topology. A *cellular automaton* (CA) is a continuous function $\Phi : \mathcal{A}^{\mathbb{Z}^D} \to \mathcal{A}^{\mathbb{Z}^D}$ which commutes with all shifts. One-dimensional CA are often positively expansive. But Mark Shereshevsky (1993, 1996) showed:

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In fact, Shereshevsky’s result is much more general. Let $\mathbb{G}$ be any finitely generated group, and consider the Cantor space $\mathcal{A}^{\mathbb{G}}$. 
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In fact, Shereshevsky’s result is much more general. Let $G$ be any finitely generated group, and consider the Cantor space $A^G$. A CA is now a continuous function $\Phi : A^G \rightarrow A^G$ which commutes with all $G$-shifts.
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$G$ has \textit{dimension $D$} if a ball of radius $r$ in the Cayley digraph of $G$ has cardinality of order $O(r^D)$. 

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In fact, Shereshevsky’s result is much more general. Let \( \mathbb{G} \) be any finitely generated group, and consider the Cantor space \( \mathcal{A}^{\mathbb{G}} \). A CA is now a continuous function \( \Phi: \mathcal{A}^{\mathbb{G}} \rightarrow \mathcal{A}^{\mathbb{G}} \) which commutes with all \( \mathbb{G} \)-shifts.

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In fact, Shereshevsky’s result is much more general. Let $\mathbb{G}$ be any finitely generated group, and consider the Cantor space $\mathcal{A}^\mathbb{G}$. A CA is now a continuous function $\Phi : \mathcal{A}^\mathbb{G} \rightarrow \mathcal{A}^\mathbb{G}$ which commutes with all $\mathbb{G}$-shifts.

$\mathbb{G}$ has \textit{dimension} $D$ if a ball of radius $r$ in the Cayley digraph of $\mathbb{G}$ has cardinality of order $\mathcal{O}(r^D)$. Let $\mathcal{X} \subseteq \mathcal{A}^\mathbb{G}$ be a closed, $\Phi$-invariant, shift-invariant subset. Shereshevsky showed:

\textit{If $D \geq 2$, and $\mathcal{X}$ has nonzero topological entropy, then the topological dynamical system $(\mathcal{X}, \Phi)$ cannot be positively expansive.}
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We will generalize Shereshevsky’s result to a much broader class of symbolic dynamical systems. These are systems like a CA, but having an ‘irregular’ network topology.
Symbolic dynamical systems

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**Examples.**
Let $\mathbb{V}$ be an infinite set of ‘vertices’. Endow $A^\mathbb{V}$ with the Cantor topology. Let $\mathcal{X} \subseteq A^\mathbb{V}$ be a closed subset. Let $\Phi : \mathcal{X} \to \mathcal{X}$ be a continuous function. We will call the triple $(A^\mathbb{V}, \mathcal{X}, \Phi)$ a **symbolic dynamical system**.

**Examples.**

- **Subshift.** $\mathbb{V} = \mathbb{Z}$ or $\mathbb{N}$; $\Phi =$ shift map; $\mathcal{X} =$ shift-invariant subset.
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- **Cellular automaton.** $\mathcal{V} = \mathbb{Z}^D \times \mathbb{N}^d$ (or some group/monoid); $\mathcal{X} = \mathcal{A}^\mathcal{V}$, and $\Phi$ commutes with all shifts.
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- **$p$-ary odometer.** $\mathcal{V} = \mathbb{N}$; $\mathcal{A} = \mathbb{Z}/p$; $\mathcal{X} = \mathcal{A}^\mathbb{N}$; $\Phi =$ successor map.
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- **Automaton network.** $\mathcal{V}$ is a directed graph, in which all vertices have finite in-degree. Each vertex has a finite state automaton (FSA) with statespace $\mathcal{A}$, which takes input from all its neighbours in the digraph. The map $\Phi$ encodes the simultaneous updating of all the FSAs.
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In fact, we shall now see that any symbolic dynamical system can be seen as an automaton network.
Let $x \in A^V$ and let $B \subseteq V$. 
Let $x \in \mathcal{A}^V$ and let $\mathcal{B} \subseteq V$. We define $x_\mathcal{B} := [x_b]_{b \in \mathcal{B}}$ (an element of $\mathcal{A}^\mathcal{B}$).
Let $x \in A^V$ and let $B \subseteq V$. We define $x_B := [x_b]_{b \in B}$ (an element of $A^B$).

**Lemma.** Let $(A^V, X, \Phi)$ be a symbolic dynamical system.
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**Lemma.** Let $(A^V, \mathcal{X}, \Phi)$ be a symbolic dynamical system. For all $v \in V$, there exists a finite subset $B(v, 1) \subseteq V$ (the input neighbourhood for $v$) and a function $\phi_v : A^{B(v,1)} \to A$ (the local rule at $v$), such that for all $x \in \mathcal{X}$, we have
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Let $\mathbf{x} \in \mathcal{A}^V$ and let $\mathcal{B} \subseteq V$. We define $\mathbf{x}_\mathcal{B} := [x_b]_{b \in \mathcal{B}}$ (an element of $\mathcal{A}^\mathcal{B}$).

**Lemma.** Let $(\mathcal{A}^V, \mathcal{X}, \Phi)$ be a symbolic dynamical system. For all $v \in V$, there exists a finite subset $\mathcal{B}(v, 1) \subseteq V$ (the input neighbourhood for $v$) and a function $\phi_v : \mathcal{A}^{\mathcal{B}(v,1)} \to \mathcal{A}$ (the local rule at $v$), such that for all $\mathbf{x} \in \mathcal{X}$, we have $\Phi(\mathbf{x})_v = \phi_v(\mathbf{x}_{\mathcal{B}(v,1)})$.

**Example.** If $\Phi : \mathcal{A}^{\mathbb{Z}^D} \to \mathcal{A}^{\mathbb{Z}^D}$ is a cellular automaton, then this is just the Curtis-Hedlund-Lyndon Theorem.
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For any $v, w \in V$, write $v \rightarrow w$ if $v \in B(w, 1)$.
Let $x \in \mathcal{A}^V$ and let $B \subseteq V$. We define $x_B := \{x_b\}_{b \in B}$ (an element of $\mathcal{A}^B$).

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For any \( v, w \in V \), write \( v \rightarrow w \) if \( v \in B(w, 1) \). This defines a directed graph structure on \( V \), called the network of \( \Phi \).

**Example.** The network of a subshift on \( \mathcal{A}^\mathbb{N} \).

\[ \begin{array}{cccccccc}
0 & - & 1 & - & 2 & - & 3 & - & 4 & - & 5 & - & 6 & - & 7 & - & 8 & \ldots
\end{array} \]
Let $x \in A^V$ and let $B \subseteq V$. We define $x_B := [x_b]_{b \in B}$ (an element of $A^B$).

**Lemma.** Let $(A^V, \mathcal{X}, \Phi)$ be a symbolic dynamical system. For all $v \in V$, there exists a finite subset $B(v, 1) \subseteq V$ (the input neighbourhood for $v$) and a function $\phi_v : A^{B(v,1)} \rightarrow A$ (the local rule at $v$), such that for all $x \in \mathcal{X}$, we have $\Phi(x)_v = \phi_v (x_{B(v,1)})$.

**Example.** If $\Phi : A^Z \rightarrow A^Z$ is a cellular automaton, then this is just the Curtis-Hedlund-Lyndon Theorem.

For any $v, w \in V$, write $v \rightarrow w$ if $v \in B(w, 1)$. This defines a directed graph structure on $V$, called the *network* of $\Phi$.

**Example.** The network of a subshift on $A^Z$. 

![Network Diagram]
Let \( x \in \mathcal{X}^V \) and let \( B \subseteq V \). We define \( x_B := [x_b]_{b \in B} \) (an element of \( \mathcal{X}^B \)).

**Lemma.** Let \((\mathcal{X}^V, \mathcal{X}, \Phi)\) be a symbolic dynamical system. For all \( v \in V \), there exists a finite subset \( B(v, 1) \subseteq V \) (the input neighbourhood for \( v \)) and a function \( \phi_v : \mathcal{X}^{B(v,1)} \rightarrow \mathcal{X} \) (the local rule at \( v \)), such that for all \( x \in \mathcal{X} \), we have \( \Phi(x)_v = \phi_v(x_{B(v,1)}) \).

**Example.** If \( \Phi : \mathcal{X}^{\mathbb{Z}^D} \rightarrow \mathcal{X}^{\mathbb{Z}^D} \) is a cellular automaton, then this is just the Curtis-Hedlund-Lyndon Theorem.

For any \( v, w \in V \), write \( v \rightarrow w \) if \( v \in B(w, 1) \). This defines a directed graph structure on \( V \), called the network of \( \Phi \).

**Example.** The network of CA on \( \mathbb{Z}^2 \) (von Neumann neighbourhood)
The network of a symbolic dynamical system

Let $x \in \mathcal{A}^V$ and let $B \subseteq V$. We define $x_B := [x_b]_{b \in B}$ (an element of $\mathcal{A}^B$).

**Lemma.** Let $(\mathcal{A}^V, \mathcal{X}, \Phi)$ be a symbolic dynamical system. For all $v \in V$, there exists a finite subset $B(v, 1) \subseteq V$ (the input neighbourhood for $v$) and a function $\phi_v : \mathcal{A}^{B(v, 1)} \rightarrow \mathcal{A}$ (the local rule at $v$), such that for all $x \in \mathcal{X}$, we have $\Phi(x)_v = \phi_v (x_{B(v, 1)})$.

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**Example.** The network of cellular automaton on $\mathbb{Z}^2$ (Moore neighbourhood)
Let \( x \in \mathcal{A}^V \) and let \( B \subseteq V \). We define \( x_B := [x_b]_{b \in B} \) (an element of \( \mathcal{A}^B \)).

**Lemma.** Let \((\mathcal{A}^V, \mathcal{X}, \Phi)\) be a symbolic dynamical system. For all \( v \in V \), there exists a finite subset \( B(v, 1) \subseteq V \) (the input neighbourhood for \( v \)) and a function \( \phi_v : \mathcal{A}^{B(v, 1)} \rightarrow \mathcal{A} \) (the local rule at \( v \)), such that for all \( x \in \mathcal{X} \), we have \( \Phi(x)_v = \phi_v(x_{B(v, 1)}) \).

**Example.** If \( \Phi : \mathcal{A}^\mathbb{Z}^D \rightarrow \mathcal{A}^\mathbb{Z}^D \) is a cellular automaton, then this is just the Curtis-Hedlund-Lyndon Theorem.

For any \( v, w \in V \), write \( v \rightarrow w \) if \( v \in B(w, 1) \). This defines a directed graph structure on \( V \), called the network of \( \Phi \).

**Example.** The network of an odometer.

\[0 1 2 3 4 5 6 7 8\]
For any $U \subset V$, define $B(U, 1) := U \cup \{v \in V : \exists u \in U : v \xrightarrow{\bullet} u\}$. 
For any $U \subseteq V$, define $B(U, 1) := U \cup \{ v \in V ; \exists u \in U : v \rightarrow u \}$. Then inductively define $B(U, n + 1) := B(B(U, n), 1)$ for all $n \in \mathbb{N}$. 
For any \( U \subset V \), define \( B(U, 1) := U \cup \{ v \in V ; \exists u \in U : v \rightarrow u \} \).

Then inductively define \( B(U, n + 1) := B(B(U, n), 1) \) for all \( n \in \mathbb{N} \).

If \( v \in V \), then \( B(v, r) \) is the set of all \( w \in V \) such that there exists a directed path \( w = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_s = v \) with \( s \leq r \).
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Define $\dim_v(V, \rightarrow) := \liminf_{r \to \infty} \frac{\log |B(v, r)|}{\log(r)}$ ("lower dimension")
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If \( \dim_v(V, \rightarrow) = \dim_v(V, \rightarrow) \), then denote common value by \( \dim_v(V, \rightarrow) \).
Network Dimension

For any \( U \subset V \), define \( \mathcal{B}(U, 1) := U \cup \{ v \in V ; \exists u \in U : v \rightarrow u \} \).
Then inductively define \( \mathcal{B}(U, n + 1) := \mathcal{B}(\mathcal{B}(U, n), 1) \) for all \( n \in \mathbb{N} \).
If \( v \in V \), then \( \mathcal{B}(v, r) \) is the set of all \( w \in V \) such that there exists a directed path \( w = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_s = v \) with \( s \leq r \).

Define \( \underline{\dim}_v(V, \rightarrow) := \liminf_{r \to \infty} \frac{\log |\mathcal{B}(v, r)|}{\log(r)} \) ( "lower dimension")
and \( \overline{\dim}_v(V, \rightarrow) := \limsup_{r \to \infty} \frac{\log |\mathcal{B}(v, r)|}{\log(r)} \) ( "upper dimension").

If \( \underline{\dim}_v(V, \rightarrow) = \overline{\dim}_v(V, \rightarrow) \), then denote common value by \( \dim_v(V, \rightarrow) \).

Define \( \overline{\dim}(V, \rightarrow) := \sup \{ \overline{\dim}_v(V, \rightarrow) ; v \in V \} \);
For any \( U \subset V \), define \( B(U, 1) := U \cup \{ v \in V ; \exists u \in U : v \rightarrow u \} \).

Then inductively define \( B(U, n + 1) := B(B(U, n), 1) \) for all \( n \in \mathbb{N} \).

If \( v \in V \), then \( B(v, r) \) is the set of all \( w \in V \) such that there exists a directed path \( w = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_s = v \) with \( s \leq r \).

Define \( \underline{\dim}_v(\mathbb{V}, \rightarrow) := \lim \inf_{r \to \infty} \frac{\log |B(v, r)|}{\log(r)} \) ("lower dimension")

and \( \overline{\dim}_v(\mathbb{V}, \rightarrow) := \lim \sup_{r \to \infty} \frac{\log |B(v, r)|}{\log(r)} \) ("upper dimension").

If \( \underline{\dim}_v(\mathbb{V}, \rightarrow) = \overline{\dim}_v(\mathbb{V}, \rightarrow) \), then denote common value by \( \dim_v(\mathbb{V}, \rightarrow) \).

Define \( \underline{\dim}(\mathbb{V}, \rightarrow) := \sup \{ \overline{\dim}_v(\mathbb{V}, \rightarrow) ; v \in \mathbb{V} \} \);

and \( \overline{\dim}(\mathbb{V}, \rightarrow) := \inf \{ \underline{\dim}_v(\mathbb{V}, \rightarrow) ; v \in \mathbb{V} \} \).
For any $U \subset V$, define $B(U, 1) := U \cup \{v \in V ; \exists u \in U : v \rightarrow u\}$. Then inductively define $B(U, n + 1) := B(B(U, n), 1)$ for all $n \in \mathbb{N}$. If $v \in V$, then $B(v, r)$ is the set of all $w \in V$ such that there exists a directed path $w = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_s = v$ with $s \leq r$.

Define $\underline{\dim}_v(V, \rightarrow) := \liminf_{r \to \infty} \frac{\log |B(v, r)|}{\log(r)}$ ("lower dimension")

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If $\underline{\dim}_v(V, \rightarrow) = \overline{\dim}_v(V, \rightarrow)$, then denote common value by $\dim_v(V, \rightarrow)$.

Define $\underline{\dim}(V, \rightarrow) := \sup \{\underline{\dim}_v(V, \rightarrow) ; v \in V\}$;

and $\overline{\dim}(V, \rightarrow) := \inf \{\overline{\dim}_v(V, \rightarrow) ; v \in V\}$.

If $\underline{\dim}(V, \rightarrow) = \overline{\dim}(V, \rightarrow)$, then denote common value by $\dim(V, \rightarrow)$. 
For any $U \subset V$, define $B(U, 1) := U \cup \{v \in V ; \exists u \in U : v \xrightarrow{} u\}$. Then inductively define $B(U, n + 1) := B(B(U, n), 1)$ for all $n \in \mathbb{N}$. If $v \in V$, then $B(v, r)$ is the set of all $w \in V$ such that there exists a directed path $w = v_1 \xrightarrow{} v_2 \xrightarrow{} \cdots \xrightarrow{} v_s = v$ with $s \leq r$.

Define $\underline{\dim}_v(V, \xrightarrow{}) := \liminf_{r \to \infty} \frac{\log |B(v, r)|}{\log(r)}$ ("lower dimension") and $\overline{\dim}_v(V, \xrightarrow{}) := \limsup_{r \to \infty} \frac{\log |B(v, r)|}{\log(r)}$ ("upper dimension").

If $\underline{\dim}_v(V, \xrightarrow{}) = \overline{\dim}_v(V, \xrightarrow{})$, then denote common value by $\dim_v(V, \xrightarrow{})$.

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If $\dim(V, \xrightarrow{}) = \overline{\dim}(V, \xrightarrow{})$, then denote common value by $\dim(V, \xrightarrow{})$.

**Example.** If $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ is a CA, then $\dim(\mathbb{Z}^D, \xrightarrow{}) = D$. 
Network Dimension

For any $U \subset V$, define $B(U, 1) := U \cup \{v \in V ; \exists u \in U : v \rightarrow u\}$. Then inductively define $B(U, n + 1) := B(B(U, n), 1)$ for all $n \in \mathbb{N}$. If $v \in V$, then $B(v, r)$ is the set of all $w \in V$ such that there exists a directed path $w = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_s = v$ with $s \leq r$.

Define $\dim_v(V, \rightarrow) := \liminf_{r \to \infty} \frac{\log |B(v, r)|}{\log(r)}$ ("lower dimension")

and $\bar{\dim}_v(V, \rightarrow) := \limsup_{r \to \infty} \frac{\log |B(v, r)|}{\log(r)}$ ("upper dimension").

If $\dim_v(V, \rightarrow) = \bar{\dim}_v(V, \rightarrow)$, then denote common value by $\dim_v(V, \rightarrow)$.

Define $\bar{\dim}(V, \rightarrow) := \sup \{\bar{\dim}_v(V, \rightarrow) ; v \in V\}$;
and $\dim(V, \rightarrow) := \inf \{\dim_v(V, \rightarrow) ; v \in V\}$.

If $\dim(V, \rightarrow) = \bar{\dim}(V, \rightarrow)$, then denote common value by $\dim(V, \rightarrow)$.

Example. If $\Phi : A^{\mathbb{Z}^D} \rightarrow A^{\mathbb{Z}^D}$ is a CA, then $\dim(\mathbb{Z}^D, \rightarrow) = D$.
More generally, if $G$ is a group, and $\Phi : A^G \rightarrow A^G$ is a CA, then $\dim(G, \rightarrow) =$ the dimension of the group $G$. 
Let $(\mathbb{V}, \bullet \rightarrow)$ be a digraph, and let $\mathcal{X} \subseteq \mathcal{A}^\mathbb{V}$ be a closed subset.
Let $(V, \rightarrow)$ be a digraph, and let $\mathcal{X} \subseteq A^V$ be a closed subset. For any $B \subseteq V$, define $\mathcal{X}_B := \{x_B; x \in \mathcal{X}\}$. 
Let \((V, \rightarrow)\) be a digraph, and let \(X \subseteq A^V\) be a closed subset. For any \(B \subseteq V\), define \(X_B := \{x_B; x \in X\}\). For any \(v \in V\), we define:

\[
h_v(X) := \liminf_{r \to \infty} \frac{\log_2 |X_{B(v,r)}|}{|B(v, r)|}
\]

(“lower topological entropy”)
Let \((V, \rightarrow)\) be a digraph, and let \(\mathcal{X} \subseteq \mathcal{A}^V\) be a closed subset. For any \(B \subseteq V\), define \(\mathcal{X}_B := \{x_B; x \in \mathcal{X}\}\). For any \(v \in V\), we define:

\[
\underline{h}(\mathcal{X}) := \liminf_{r \to \infty} \frac{\log_2 |\mathcal{X}_B(v, r)|}{|B(v, r)|} \quad \text{("lower topological entropy")}
\]

and \(\overline{h}(\mathcal{X}) := \limsup_{r \to \infty} \frac{\log_2 |\mathcal{X}_B(v, r)|}{|B(v, r)|} \quad \text{("upper topological entropy")}
\]
Let $(\mathcal{V}, \rightarrow)$ be a digraph, and let $\mathcal{X} \subseteq \mathcal{A}^\mathcal{V}$ be a closed subset. For any $\mathcal{B} \subseteq \mathcal{V}$, define $\mathcal{X}_\mathcal{B} := \{\mathbf{x}_\mathcal{B}; \mathbf{x} \in \mathcal{X}\}$. For any $\mathbf{v} \in \mathcal{V}$, we define:

\[
\underline{h}_\mathbf{v}(\mathcal{X}) := \lim_{r \to \infty} \inf \frac{\log_2 |\mathcal{X}_{\mathbf{B}(\mathbf{v}, r)}|}{|\mathcal{B}(\mathbf{v}, r)|} \quad (\text{"lower topological entropy"})
\]

and

\[
\bar{h}_\mathbf{v}(\mathcal{X}) := \lim_{r \to \infty} \sup \frac{\log_2 |\mathcal{X}_{\mathbf{B}(\mathbf{v}, r)}|}{|\mathcal{B}(\mathbf{v}, r)|} \quad (\text{"upper topological entropy"}).
\]

Clearly, $0 \leq \underline{h}_\mathbf{v}(\mathcal{X}) \leq \bar{h}_\mathbf{v}(\mathcal{X}) \leq \log_2 |\mathcal{A}|$. 

Let \((V, \rightarrow)\) be a digraph, and let \(X \subseteq A^V\) be a closed subset. For any \(B \subseteq V\), define \(X_B := \{x_B; x \in X\}\). For any \(v \in V\), we define:

\[
\underline{h}_v(X) := \lim_{r \to \infty} \inf \frac{\log_2 |X_B(v, r)|}{|B(v, r)|} \quad \text{("lower topological entropy")}
\]

and

\[
\bar{h}_v(X) := \lim_{r \to \infty} \sup \frac{\log_2 |X_B(v, r)|}{|B(v, r)|} \quad \text{("upper topological entropy")}
\]

Clearly, \(0 \leq \underline{h}_v(X) \leq \bar{h}_v(X) \leq \log_2 |A|\). We define

\[
\underline{h}(X) := \inf_{v \in V} \underline{h}_v(X)
\]
Let \((V, \cdot \mapsto)\) be a digraph, and let \(\mathcal{X} \subseteq A^V\) be a closed subset. For any \(B \subseteq V\), define \(\mathcal{X}_B := \{x_B; x \in \mathcal{X}\}\). For any \(v \in V\), we define:

\[
\underline{h}_v(\mathcal{X}) := \liminf_{r \to \infty} \frac{\log_2 |\mathcal{X}_B(v, r)|}{|B(v, r)|} \quad \text{("lower topological entropy")}
\]

and \(\overline{h}_v(\mathcal{X}) := \limsup_{r \to \infty} \frac{\log_2 |\mathcal{X}_B(v, r)|}{|B(v, r)|} \quad \text{("upper topological entropy")}
\]

Clearly, \(0 \leq \underline{h}_v(\mathcal{X}) \leq \overline{h}_v(\mathcal{X}) \leq \log_2 |A|\). We define

\[
\underline{h}(\mathcal{X}) := \inf_{v \in V} \underline{h}_v(\mathcal{X}) \quad \text{and} \quad \overline{h}(\mathcal{X}) := \sup_{v \in V} \overline{h}_v(\mathcal{X}).
\]
Let \((V, \rightarrow)\) be a digraph, and let \(X \subseteq A^V\) be a closed subset. For any \(B \subseteq V\), define \(X_B := \{x_B; x \in X\}\). For any \(v \in V\), we define:

\[
\underline{h}_v(X) := \liminf_{r \to \infty} \frac{\log_2 |X_{B(v,r)}|}{|B(v, r)|} \quad (\text{“lower topological entropy”})
\]

and \(\overline{h}_v(X) := \limsup_{r \to \infty} \frac{\log_2 |X_{B(v,r)}|}{|B(v, r)|} \quad (\text{“upper topological entropy”}).\)

Clearly, \(0 \leq \underline{h}_v(X) \leq \overline{h}_v(X) \leq \log_2 |A|\). We define

\[
\underline{h}(X) := \inf_{v \in V} \underline{h}_v(X) \quad \text{and} \quad \overline{h}(X) := \sup_{v \in V} \overline{h}_v(X).
\]

If \(\underline{h}(X) = \overline{h}(X)\), then we denote their common value by \(h(X)\).
Let $(V, \cdot \rightarrow)$ be a digraph, and let $X \subseteq A^V$ be a closed subset. For any $B \subseteq V$, define $X_B := \{x_B; x \in X\}$. For any $v \in V$, we define:

$$h_v(X) := \lim \inf_{r \to \infty} \frac{\log_2 |X_{B(v,r)}|}{|B(v, r)|} \quad \text{("lower topological entropy")},$$

and

$$\bar{h}_v(X) := \lim \sup_{r \to \infty} \frac{\log_2 |X_{B(v,r)}|}{|B(v, r)|} \quad \text{("upper topological entropy")}.$$

Clearly, $0 \leq h_v(X) \leq \bar{h}_v(X) \leq \log_2 |A|$. We define

$$\underline{h}(X) := \inf_{v \in V} h_v(X) \quad \text{and} \quad \bar{h}(X) := \sup_{v \in V} \bar{h}_v(X).$$

If $\underline{h}(X) = \bar{h}(X)$, then we denote their common value by $h(X)$.

**Example.** (a) $h(A^V) = \log_2 |A|$. 
Let \((V, \rightarrow)\) be a digraph, and let \(X \subseteq A^V\) be a closed subset. For any \(B \subseteq V\), define \(X_B := \{x_B; x \in X\}\). For any \(v \in V\), we define:

\[
\underline{h}_v(X) := \liminf_{r \to \infty} \frac{\log_2 |X_{B(v,r)}|}{|B(v, r)|}
\]

(“lower topological entropy")

and \(\overline{h}_v(X) := \limsup_{r \to \infty} \frac{\log_2 |X_{B(v,r)}|}{|B(v, r)|}\)

(“upper topological entropy”).

Clearly, \(0 \leq \underline{h}_v(X) \leq \overline{h}_v(X) \leq \log_2 |A|\). We define

\[
\underline{h}(X) := \inf_{v \in V} \underline{h}_v(X) \quad \text{and} \quad \overline{h}(X) := \sup_{v \in V} \overline{h}_v(X).
\]

If \(\underline{h}(X) = \overline{h}(X)\), then we denote their common value by \(h(X)\).

**Example.** (a) \(h(A^V) = \log_2 |A|\).

(b) Suppose \(V = \mathbb{Z}^D\) (with Cayley digraph), and \(X \subseteq A^{\mathbb{Z}^D}\) is a \(D\)-dimensional subshift. Then \(h(X)\) is the \((D\)-dimensional) topological entropy of \(X\).
Let \((\mathcal{V}, \rightarrow)\) be a digraph, and let \(\mathcal{X} \subseteq \mathcal{A}^\mathcal{V}\) be a closed subset.
Let \((\mathcal{V}, \bullet)\) be a digraph, and let \(\mathcal{X} \subseteq \mathcal{A}^\mathcal{V}\) be a closed subset. Say that \(\mathcal{X}\) is \emph{weakly independent} if there is some constant \(\epsilon > 0\) such that, for any disjoint balls \(B_1, \ldots, B_N \subset \mathcal{V}\),

\[
\log_2 |\mathcal{X}_{B_1 \sqcup \cdots \sqcup B_N}| \geq \epsilon \sum_{n=1}^{N} \log_2 |\mathcal{X}_{B_n}|.
\]
Let \((V, \cdot \rightarrow)\) be a digraph, and let \(\mathcal{X} \subseteq A^V\) be a closed subset. Say that \(\mathcal{X}\) is \textit{weakly independent} if there is some constant \(\varepsilon > 0\) such that, for any disjoint balls \(B_1, \ldots, B_N \subset V\),

\[
\log_2 |\mathcal{X}_{B_1 \sqcup \cdots \sqcup B_N}| \geq \varepsilon \sum_{n=1}^{N} \log_2 |\mathcal{X}_{B_n}|.
\]

This is a ‘topological mixing’ condition:
Let \((\mathcal{V}, \bullet \leftrightarrow)\) be a digraph, and let \(\mathcal{X} \subseteq \mathcal{A}^{\mathcal{V}}\) be a closed subset. Say that \(\mathcal{X}\) is \textit{weakly independent} if there is some constant \(\epsilon > 0\) such that, for any disjoint balls \(\mathcal{B}_1, \ldots, \mathcal{B}_N \subset \mathcal{V}\),

\[
\log_2 |\mathcal{X}_{\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_N}| \geq \epsilon \sum_{n=1}^{N} \log_2 |\mathcal{X}_{\mathcal{B}_n}|.
\]

This is a ‘topological mixing’ condition: the information contained in balls \(\mathcal{B}_1, \ldots, \mathcal{B}_{N-1}\) has limited power to predict the contents of ball \(\mathcal{B}_N\).
Let \((V, \bullet \rightarrow)\) be a digraph, and let \(\mathcal{X} \subseteq A^V\) be a closed subset. Say that \(\mathcal{X}\) is **weakly independent** if there is some constant \(\epsilon > 0\) such that, for any disjoint balls \(B_1, \ldots, B_N \subset V\),

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\log_2 |\mathcal{X}_{B_1 \sqcup \cdots \sqcup B_N}| \geq \epsilon \sum_{n=1}^{N} \log_2 |\mathcal{X}_{B_n}|.
\]

This is a ‘topological mixing’ condition: the information contained in balls \(B_1, \ldots, B_{N-1}\) has limited power to predict the contents of ball \(B_N\).

**Example.**
Local independence and subisometries

Let \((V, \bullet \to)\) be a digraph, and let \(\mathcal{X} \subseteq \mathcal{A}^V\) be a closed subset. Say that \(\mathcal{X}\) is \textit{weakly independent} if there is some constant \(\epsilon > 0\) such that, for any disjoint balls \(B_1, \ldots, B_N \subset V\),

\[
\log_2 |\mathcal{X}_{B_1 \sqcup \cdots \sqcup B_N}| \geq \epsilon \sum_{n=1}^{N} \log_2 |\mathcal{X}_{B_n}|.
\]

This is a ‘topological mixing’ condition: the information contained in balls \(B_1, \ldots, B_{N-1}\) has limited power to predict the contents of ball \(B_N\).

\textbf{Example.} For all \(v \in A\), let \(A_v \subseteq A\) with \(|A_v| \geq 2\).
Local independence and subisometries

Let \((V, \cdot \rightarrow)\) be a digraph, and let \(X \subseteq A^V\) be a closed subset. Say that \(X\) is weakly independent if there is some constant \(\epsilon > 0\) such that, for any disjoint balls \(B_1, \ldots, B_N \subset V\),

\[
\log_2 |X_{B_1 \sqcup \ldots \sqcup B_N}| \geq \epsilon \sum_{n=1}^{N} \log_2 |X_{B_n}|.
\]

This is a ‘topological mixing’ condition: the information contained in balls \(B_1, \ldots, B_{N-1}\) has limited power to predict the contents of ball \(B_N\).

**Example.** For all \(v \in A\), let \(A_v \subseteq A\) with \(|A_v| \geq 2\). Let \(X := \prod_{v \in V} A_v \subseteq A^V\); then \(h(X) \geq 1\), and \(X\) is weakly independent.
Local independence and subisometries

Let \((V, \rightarrow)\) be a digraph, and let \(\mathcal{X} \subseteq A^V\) be a closed subset. Say that \(\mathcal{X}\) is \textit{weakly independent} if there is some constant \(\epsilon > 0\) such that, for any disjoint balls \(B_1, \ldots, B_N \subseteq V\),

\[
\log_2 |\mathcal{X}_{B_1 \sqcup \cdots \sqcup B_N}| \geq \epsilon \sum_{n=1}^{N} \log_2 |\mathcal{X}_{B_n}|.
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\textbf{Example.} For all \(v \in A\), let \(A_v \subseteq A\) with \(|A_v| \geq 2\). Let \(\mathcal{X} := \prod_{v \in V} A_v \subseteq A^V\); then \(h(\mathcal{X}) \geq 1\), and \(\mathcal{X}\) is weakly independent. In particular, the space \(\mathcal{X} = A^V\) itself is weakly independent.
Let \((V, \bullet\rightarrow)\) be a digraph, and let \(X \subseteq A^V\) be a closed subset. Say that \(X\) is weakly independent if there is some constant \(\epsilon > 0\) such that, for any disjoint balls \(B_1, \ldots, B_N \subset V\),

\[
\log_2 |X_{B_1 \sqcup \cdots \sqcup B_N}| \geq \epsilon \sum_{n=1}^{N} \log_2 |X_{B_n}|.
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A subisometry of \((V, \bullet\rightarrow)\) is an injection \(\tau : V \rightarrow V\) such that, for all \(v, w \in V\), we have \((v \bullet\rightarrow w) \iff (\tau(v) \bullet\rightarrow \tau(w))\).
Local independence and subisometries

Let \((V, \bullet \to)\) be a digraph, and let \(X \subseteq A^V\) be a closed subset. Say that \(X\) is weakly independent if there is some constant \(\epsilon > 0\) such that, for any disjoint balls \(B_1, \ldots, B_N \subseteq V\),

\[
\log_2 |X_{B_1 \sqcup \cdots \sqcup B_N}| \geq \epsilon \sum_{n=1}^{N} \log_2 |X_{B_n}|.
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This is a ‘topological mixing’ condition: the information contained in balls \(B_1, \ldots, B_{N-1}\) has limited power to predict the contents of ball \(B_N\).

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Let \((V, \bullet \rightarrow)\) be a digraph, and let \(X \subseteq A^V\) be a closed subset. Say that \(X\) is \textit{weakly independent} if there is some constant \(\epsilon > 0\) such that, for any disjoint balls \(B_1, \ldots, B_N \subset V\),

\[
\log_2 |X_{B_1 \sqcup \cdots \sqcup B_N}| \geq \epsilon \sum_{n=1}^{N} \log_2 |X_{B_n}|.
\]

This is a ‘topological mixing’ condition: the information contained in balls \(B_1, \ldots, B_{N-1}\) has limited power to predict the contents of ball \(B_N\).

\textbf{Example.} For all \(v \in A\), let \(A_v \subseteq A\) with \(|A_v| \geq 2\). Let \(X := \prod_{v \in V} A_v \subseteq A^V\); then \(h(X) \geq 1\), and \(X\) is weakly independent. In particular, the space \(X = A^V\) itself is weakly independent.

A \textit{subisometry} of \((V, \bullet \rightarrow)\) is an injection \(\tau : V \rightarrow V\) such that, for all \(v, w \in V\), we have \((v \bullet \rightarrow w) \iff (\tau(v) \bullet \rightarrow \tau(w))\). Thus, for all \(v \in V\) and \(r > 0\), we have \(\tau[B(v, r)] \subseteq B[\tau(v), r]\) (with equality if \(\tau\) is surjective).

\textbf{Example.}
Local independence and subisometries

Let \((V, \circlearrowright)\) be a digraph, and let \(X \subseteq A^V\) be a closed subset. Say that \(X\) is **weakly independent** if there is some constant \(\epsilon > 0\) such that, for any disjoint balls \(B_1, \ldots, B_N \subset V\),

\[
\log_2 |X_{B_1 \cup \cdots \cup B_N}| \geq \epsilon \sum_{n=1}^{N} \log_2 |X_{B_n}|.
\]

This is a ‘topological mixing’ condition: the information contained in balls \(B_1, \ldots, B_{N-1}\) has limited power to predict the contents of ball \(B_N\).

**Example.** For all \(v \in A\), let \(A_v \subseteq A\) with \(|A_v| \geq 2\). Let \(X := \prod_{v \in V} A_v \subseteq A^V\); then \(h(X) \geq 1\), and \(X\) is weakly independent. In particular, the space \(X = A^V\) itself is weakly independent.

A subisometry of \((V, \circlearrowright)\) is an injection \(\tau : V \rightarrow V\) such that, for all \(v, w \in V\), we have \((v \circlearrowright w) \iff (\tau(v) \circlearrowright \tau(w))\). Thus, for all \(v \in V\) and \(r > 0\), we have \(\tau[\mathbb{B}(v, r)] \subseteq \mathbb{B}[\tau(v), r]\) (with equality if \(\tau\) is surjective).

**Example.** Let \((V, \circlearrowright) = \text{Cayley digraph of a group/monoid (e.g. } \mathbb{Z}^D \times \mathbb{N}^d)\).
Let \((V, \cdot \mapsto)\) be a digraph, and let \(X \subseteq A^V\) be a closed subset. Say that \(X\) is \textit{weakly independent} if there is some constant \(\epsilon > 0\) such that, for any disjoint balls \(B_1, \ldots, B_N \subset V,\)

\[
\log_2 |X_{\sqcup B_1} \sqcup \cdots \sqcup B_N| \geq \epsilon \sum_{n=1}^{N} \log_2 |X_{B_n}|.
\]

This is a ‘topological mixing’ condition: the information contained in balls \(B_1, \ldots, B_{N-1}\) has limited power to predict the contents of ball \(B_N\).

\textbf{Example.} For all \(v \in A\), let \(A_v \subseteq A\) with \(|A_v| \geq 2\). Let \(X := \prod_{v \in V} A_v \subseteq A^V\); then \(h(X) \geq 1\), and \(X\) is weakly independent. In particular, the space \(X = A^V\) itself is weakly independent.

A \textit{subisometry} of \((V, \cdot \mapsto)\) is an injection \(\tau : V \longrightarrow V\) such that, for all \(v, w \in V\), we have \((v \cdot \mapsto w) \iff (\tau(v) \cdot \mapsto \tau(w))\). Thus, for all \(v \in V\) and \(r > 0\), we have \(\tau[\mathbb{B}(v, r)] \subseteq \mathbb{B}[\tau(v), r]\) (with equality if \(\tau\) is surjective).

\textbf{Example.} Let \((V, \cdot \mapsto) = \text{Cayley digraph of a group/monoid (e.g. } \mathbb{Z}^D \times \mathbb{N}^d)\). Fix \(w \in V\). Define \(\tau : V \longrightarrow V\) by \(\tau(v) := v + w\). Then \(\tau\) is a subisometry.
Recall: a subisometry of \((V, \bullet \mapsto)\) is an injection \(\tau : V \rightarrow V\) such that, for all \(v, w \in V\), we have \((v \bullet \mapsto w) \iff (\tau(v) \bullet \mapsto \tau(w))\).
Subisometries and subsymmetries

Recall: a subisometry of \((V, \bullet\to)\) is an injection \(\tau : V \to V\) such that, for all \(v, w \in V\), we have \((v \bullet\to w) \iff (\tau(v) \bullet\to \tau(w))\).

For any \(a \in A^V\), define \(\tau(a) := a'\), where \(a'_v := a_{\tau(v)}\) for all \(v \in V\).
Recall: a subisometry of \((\mathcal{V}, \bullet\rightarrow)\) is an injection \(\tau: \mathcal{V} \rightarrow \mathcal{V}\) such that, for all \(v, w \in \mathcal{V}\), we have \((v \bullet\rightarrow w) \iff (\tau(v) \bullet\rightarrow \tau(w))\).

For any \(a \in \mathcal{A}^\mathcal{V}\), define \(\tau(a) := a'\), where \(a'_v := a_{\tau(v)}\) for all \(v \in \mathcal{V}\). This yields a surjection \(\tau: \mathcal{A}^\mathcal{V} \rightarrow \mathcal{A}^\mathcal{V}\).
Recall: a subisometry of $(V, \cdot \mapsto)$ is an injection $\tau : V \rightarrow V$ such that, for all $v, w \in V$, we have $(v \cdot \mapsto w) \iff (\tau(v) \cdot \mapsto \tau(w))$.

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Let $\mathcal{X} \subseteq A^V$ be a closed subset. Let $\Phi : \mathcal{X} \rightarrow \mathcal{X}'$ be a continuous map.
Recall: a subisometry of $(\mathbb{V}, \bullet \mapsto)$ is an injection $\tau: \mathbb{V} \longrightarrow \mathbb{V}$ such that, for all $v, w \in \mathbb{V}$, we have $(v \bullet \mapsto w) \iff (\tau(v) \bullet \mapsto \tau(w))$.

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Recall: a *subisometry* of \((V, \bullet \to)\) is an injection \(\tau : V \to V\) such that, for all \(v, w \in V\), we have \((v \bullet \to w) \iff (\tau(v) \bullet \to \tau(w))\).

For any \(a \in A^V\), define \(\tau(a) := a'\), where \(a'_v := a_{\tau(v)}\) for all \(v \in V\). This yields a surjection \(\tau : A^V \to A^V\).

Let \(X \subseteq A^V\) be a closed subset. Let \(\Phi : X \to X\) be a continuous map. We say \(\tau\) is a *subsymmetry* of the symbolic dynamical system \((A^V, X, \Phi)\) if \(\tau(X) = X\) and \(\tau \circ \Phi = \Phi \circ \tau\).

**Example.** Let \(\Phi : A^{\mathbb{Z}^D} \to A^{\mathbb{Z}^D}\) be a CA, and let \(X \subseteq A^{\mathbb{Z}^D}\) be a subshift with \(\Phi(X) = X\). Then any shift map is a subsymmetry of \((A^{\mathbb{Z}^D}, X, \Phi)\).
Recall: a subisometry of \((V, \cdot \to)\) is an injection \(\tau : V \to V\) such that, for all \(v, w \in V\), we have \((v \cdot \to w) \iff (\tau(v) \cdot \to \tau(w))\).

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For any \(v, w \in V\), let \(d(v, w)\) be the length of the shortest undirected path in \((V, \cdot \to)\) from \(v\) to \(w\) (or \(\infty\) if there is no such path); then \(d\) is a metric on each undirected-path component of \(V\).
Subisometries and subsymmetries

Recall: a subisometry of $(V, \bullet \mapsto)$ is an injection $\tau : V \rightarrow V$ such that, for all $v, w \in V$, we have $(v \bullet \mapsto w) \iff (\tau(v) \bullet \mapsto \tau(w))$.

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For any $v, w \in V$, let $d(v, w)$ be the length of the shortest undirected path in $(V, \bullet \mapsto)$ from $v$ to $w$ (or $\infty$ if there is no such path); then $d$ is a metric on each undirected-path component of $V$.

For any $v \in V$, let $\text{speed}(v, \tau) := \lim_{n \rightarrow \infty} \frac{d[v, \tau^n(v)]}{n}$. 
Subisometries and subsymmetries

Recall: a subisometry of \((V, \rightarrow)\) is an injection \(\tau : V \longrightarrow V\) such that, for all \(v, w \in V\), we have \((v \rightarrow w) \iff (\tau(v) \rightarrow \tau(w))\).

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Example. Let \(\Phi : A^{\mathbb{Z}^D} \longrightarrow A^{\mathbb{Z}^D}\) be a CA, and let \(X \subseteq A^{\mathbb{Z}^D}\) be a subshift with \(\Phi(X) = X\). Then any shift map is a subsymmetry of \((A^{\mathbb{Z}^D}, X, \Phi)\).

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For any \(v \in V\), let \(\text{speed}(v, \tau) := \lim_{n \to \infty} \frac{d[v, \tau^n(v)]}{n}\). This limit is well-defined and constant on each undirected-path component of \((V, \rightarrow)\).
Subisometries and subsymmetries

Recall: a subisometry of \((\nabla, \bullet\mapsto)\) is an injection \(\tau : \nabla \longrightarrow \nabla\) such that, for all \(v, w \in \nabla\), we have \((v \bullet\mapsto w) \iff (\tau(v) \bullet\mapsto \tau(w))\).

For any \(a \in \mathcal{A}^\nabla\), define \(\tau(a) := a', \) where \(a' := a_{\tau(v)}\) for all \(v \in \nabla\). This yields a surjection \(\tau : \mathcal{A}^\nabla \longrightarrow \mathcal{A}^\nabla\).

Let \(\mathcal{X} \subseteq \mathcal{A}^\nabla\) be a closed subset. Let \(\Phi : \mathcal{X} \longrightarrow \mathcal{X}\) be a continuous map. We say \(\tau\) is a subsymmetry of the symbolic dynamical system \((\mathcal{A}^\nabla, \mathcal{X}, \Phi)\) if \(\tau(\mathcal{X}) = \mathcal{X}\) and \(\tau \circ \Phi = \Phi \circ \tau\).

Example. Let \(\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}\) be a CA, and let \(\mathcal{X} \subseteq \mathcal{A}^{\mathbb{Z}^D}\) be a subshift with \(\Phi(\mathcal{X}) = \mathcal{X}\). Then any shift map is a subsymmetry of \((\mathcal{A}^{\mathbb{Z}^D}, \mathcal{X}, \Phi)\).

For any \(v, w \in \nabla\), let \(d(v, w)\) be the length of the shortest undirected path in \((\nabla, \bullet\mapsto)\) from \(v\) to \(w\) (or \(\infty\) if there is no such path); then \(d\) is a metric on each undirected-path component of \(\nabla\).

For any \(v \in \nabla\), let \(\text{speed}(v, \tau) := \lim_{n \rightarrow \infty} \frac{d[v, \tau^n(v)]}{n}\). This limit is well-defined and constant on each undirected-path component of \((\nabla, \bullet\mapsto)\). We say that \(\tau\) is a moving subsymmetry if \(\text{speed}(v, \tau) \geq 0\) for all \(v \in \nabla\).
Recall: $\tau$ is a \textit{moving} subsymmetry if $\text{speed}(v, \tau) > 0$ for all $v \in \mathbb{V}$. 
Recall: $\tau$ is a moving subsymmetry if $\text{speed}(v, \tau) > 0$ for all $v \in \mathbb{V}$.

**Example.** Let $(\mathbb{V}, \bullet\rightarrow)$ be the Cayley digraph of a group or monoid (e.g. $\mathbb{Z}^D \times \mathbb{N}^d$).
Recall: $\tau$ is a moving subsymmetry if $\text{speed}(v, \tau) > 0$ for all $v \in V$.

**Example.** Let $(V, \rightarrow)$ be the Cayley digraph of a group or monoid (e.g. $\mathbb{Z}^D \times \mathbb{N}^d$). Fix $w \in V$. Define $\tau : V \rightarrow V$ by $\tau(v) := v + w$. Then $\tau$ is a moving subsymmetry. (In fact, $\text{speed}(v, \tau) = |w|$).
Recall: $\tau$ is a *moving* subsymmetry if $\text{speed}(v, \tau) > 0$ for all $v \in \mathbb{V}$.

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**Nonexample.** Let $\mathbb{V} = \mathbb{Z} \times \mathbb{N}$, with the digraph structure shown below.
Recall: \( \tau \) is a \textit{moving} subsymmetry if \( \text{speed}(v, \tau) > 0 \) for all \( v \in \mathbb{V} \).

\textbf{Example.} Let \((\mathbb{V}, \rightarrow)\) be the Cayley digraph of a group or monoid (e.g. \( \mathbb{Z}^D \times \mathbb{N}^d \)). Fix \( w \in \mathbb{V} \). Define \( \tau : \mathbb{V} \rightarrow \mathbb{V} \) by \( \tau(v) := v + w \). Then \( \tau \) is a moving subsymmetry. (In fact, \( \text{speed}(v, \tau) = |w| \)).

\textbf{Nonexample.} Let \( \mathbb{V} = \mathbb{Z} \times \mathbb{N} \), with the digraph structure shown below.

Define subisometry \( \tau : \mathbb{V} \rightarrow \mathbb{V} \) by \( \tau(z, n) = (z + 1, n) \).
Moving subsymmetries

Recall: \( \tau \) is a *moving* subsymmetry if \( \text{speed}(v, \tau) > 0 \) for all \( v \in \mathbb{V} \).

**Example.** Let \((\mathbb{V}, \bullet \rightarrow)\) be the Cayley digraph of a group or monoid (e.g. \( \mathbb{Z}^D \times \mathbb{N}^d \)). Fix \( w \in \mathbb{V} \). Define \( \tau : \mathbb{V} \rightarrow \mathbb{V} \) by \( \tau(v) := v + w \). Then \( \tau \) is a moving subsymmetry. (In fact, \( \text{speed}(v, \tau) = |w| \)).

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Define subisometry \( \tau : \mathbb{V} \rightarrow \mathbb{V} \) by \( \tau(z, n) = (z + 1, n) \). Then \( \text{speed}(\tau, v) = 0 \), for all \( v \in \mathbb{V} \), because for any \( k \in \mathbb{N} \), there is a path from \( v \) to \( \tau^{(2^k)}(v) \) of length at most \( 2k + 1 \).
Recall: $\tau$ is a moving subsymmetry if $\text{speed}(\nu, \tau) > 0$ for all $\nu \in \mathbb{V}$.

**Example.** Let $(\mathbb{V}, \bullet \rightarrow)$ be the Cayley digraph of a group or monoid (e.g. $\mathbb{Z}^D \times \mathbb{N}^d$). Fix $w \in \mathbb{V}$. Define $\tau: \mathbb{V} \rightarrow \mathbb{V}$ by $\tau(\nu) := \nu + w$. Then $\tau$ is a moving subsymmetry. (In fact, $\text{speed}(\nu, \tau) = |w|$).

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Define subisometry $\tau: \mathbb{V} \rightarrow \mathbb{V}$ by $\tau(\nu) = (\nu + 1, \eta)$. Then $\text{speed}(\tau, \nu) = 0$, for all $\nu \in \mathbb{V}$, because for any $k \in \mathbb{N}$, there is a path from $\nu$ to $\tau^{(2k)}(\nu)$ of length at most $2k + 1$. Thus, $\tau$ is not a moving subsymmetry.
A symbolic dynamical system \((A^\forall, \mathcal{X}, \Phi)\) is *positively expansive* if it is topologically conjugate to a one-sided shift.

We now come to our generalizations of Shereshevsky’s result.
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**Theorem 1.**
A symbolic dynamical system \((\mathcal{A}^\mathbb{V}, \mathcal{X}, \Phi)\) is *positively expansive* if it is topologically conjugate to a one-sided shift.

We now come to our generalizations of Shereshevsky’s result.

**Theorem 1.** Let \(\Phi : \mathcal{A}^\mathbb{V} \longrightarrow \mathcal{A}^\mathbb{V}\) be a continuous self-map with a *moving subsymmetry*. 

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**Theorem 1.** Let \(\Phi : \mathcal{A}^V \rightarrow \mathcal{A}^V\) be a continuous self-map with a moving subsymmetry. If \(\dim(\mathcal{V}, \rightarrow) > 1\), then the system \((\mathcal{A}^V, \Phi)\) is *not* positively expansive.
A symbolic dynamical system \((A^\mathcal{V}, \mathcal{X}, \Phi)\) is \textit{positively expansive} if it is topologically conjugate to a one-sided shift.

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**Theorem 1.** Let \(\Phi : A^\mathcal{V} \rightarrow A^\mathcal{V}\) be a continuous self-map with a moving subsymmetry. If \(\dim(\mathcal{V}, \bullet) > 1\), then the system \((A^\mathcal{V}, \Phi)\) is not positively expansive.

**Theorem 2.**
A symbolic dynamical system \((\mathcal{A}^\mathcal{V}, \mathcal{X}, \Phi)\) is \textit{positively expansive} if it is topologically conjugate to a one-sided shift.

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**Theorem 2.** Let \((\mathcal{A}^\mathcal{V}, \mathcal{X}, \Phi)\) be a symbolic dynamical system with a moving subsymmetry, and suppose \(\mathcal{X}\) is \textit{weakly independent}.
Main results

A symbolic dynamical system \((A^V, X, \Phi)\) is *positively expansive* if it is topologically conjugate to a one-sided shift.

We now come to our generalizations of Shereshevsky’s result.

**Theorem 1.** Let \(\Phi : A^V \to A^V\) be a continuous self-map with a moving subsymmetry. If \(\dim(V, \implies) > 1\), then the system \((A^V, \Phi)\) is not positively expansive.

**Theorem 2.** Let \((A^V, X, \Phi)\) be a symbolic dynamical system with a moving subsymmetry, and suppose \(X\) is weakly independent.

- If \(\overline{h}(X) > 0\) and \(\dim(V, \implies) > 1\) then \((X, \Phi)\) is not positively expansive.
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**Theorem 1.** Let \(\Phi : \mathcal{A}^\mathcal{V} \rightarrow \mathcal{A}^\mathcal{V}\) be a continuous self-map with a moving subsymmetry. If \(\dim(\mathcal{V}, \rightarrow) > 1\), then the system \((\mathcal{A}^\mathcal{V}, \Phi)\) is not positively expansive.

**Theorem 2.** Let \((\mathcal{A}^\mathcal{V}, \mathcal{X}, \Phi)\) be a symbolic dynamical system with a moving subsymmetry, and suppose \(\mathcal{X}\) is weakly independent.

\begin{itemize}
  \item If \(\overline{h(\mathcal{X})} > 0\) and \(\overline{\dim(\mathcal{V}, \rightarrow)} > 1\) then \((\mathcal{X}, \Phi)\) is not positively expansive.
  
  \item If \(\dim_v(\mathcal{V}, \rightarrow) = \overline{\dim_v(\mathcal{V}, \rightarrow)}\) for all \(v \in \mathcal{V}\), and \(\overline{h(\mathcal{X})} > 0\), and \(\overline{\dim(\mathcal{V}, \rightarrow)} > 1\), then \((\mathcal{X}, \Phi)\) is not positively expansive.
\end{itemize}
In fact, Theorems 1 and 2 are both special cases of a more general result.
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\liminf_{r \to \infty} \frac{|B(v, r)|}{r} = \infty.
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\(B(v, r)\) := upstream ball of radius \(r\) around \(v\).
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**Example.** If \(\dim_v(\mathbb{V}, \bullet \rightarrow) > 1\), then \(v\) has superlinear connectivity.
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**Example.** If \(\dim_v(V, \rightarrow) > 1\), then \(v\) has superlinear connectivity. In particular, if \(V\) is a Cayley digraph of a group with dimension \(> 1\), then every vertex has superlinear connectivity.
More main results

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**Theorem 0.** Let \((A^V, \mathcal{X}, \Phi)\) be a symbolic dynamical system with a moving subsymmetry. If \(\mathcal{X}\) is weakly independent, and there exists some \(v \in V\) with superlinear connectivity such that \(\bar{h}_v(\mathcal{X}) > 0\), then the system \((\mathcal{X}, \Phi)\) is not positively expansive.
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**Theorem 0.** Let $(\mathcal{A}^\mathbb{V}, \mathcal{X}, \Phi)$ be a symbolic dynamical system with a moving subsymmetry. If $\mathcal{X}$ is weakly independent, and there exists some $v \in \mathbb{V}$ with superlinear connectivity such that $\overline{h}_v(\mathcal{X}) > 0$, then the system $(\mathcal{X}, \Phi)$ is not positively expansive.

**Proof sketch.** (by contradiction)
More main results

In fact, Theorems 1 and 2 are both special cases of a more general result. In a digraph \((\mathcal{V}, \rightarrow)\), a vertex \(v \in \mathcal{V}\) has *superlinear connectivity* if

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**Example.** If \(\dim_v(\mathcal{V}, \rightarrow) > 1\), then \(v\) has superlinear connectivity.

In particular, if \(\mathcal{V}\) is a Cayley digraph of a group with dimension \(\geq 1\), then every vertex has superlinear connectivity.

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**Proof sketch.** (by contradiction) If \((\mathcal{X}, \Phi)\) is positively expansive, then there is some finite ‘window’ \(\mathcal{W} \subset \mathcal{V}\) such that for any \(x \in \mathcal{X}\), the data \([x_\mathcal{W}, \Phi(x)_\mathcal{W}, \Phi^2(x)_\mathcal{W}, \Phi^3(x)_\mathcal{W}, \ldots \Phi^t(x)_\mathcal{W}, \ldots]\) completely encodes \(x\).
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In fact, Theorems 1 and 2 are both special cases of a more general result. In a digraph $\langle V, \rightarrow \rangle$, a vertex $v \in V$ has superlinear connectivity if

$$\liminf_{r \to \infty} \frac{|B(v, r)|}{r} = \infty.$$  ($B(v, r) :=$ upstream ball of radius $r$ around $v$.)

**Example.** If $\dim_v(V, \rightarrow) > 1$, then $v$ has superlinear connectivity. In particular, if $V$ is a Cayley digraph of a group with dimension $> 1$, then every vertex has superlinear connectivity.

**Theorem 0.** Let $(A^V, \mathcal{X}, \Phi)$ be a symbolic dynamical system with a moving subsymmetry. If $\mathcal{X}$ is weakly independent, and there exists some $v \in V$ with superlinear connectivity such that $h_v(\mathcal{X}) > 0$, then the system $(\mathcal{X}, \Phi)$ is not positively expansive.

**Proof sketch.** (by contradiction) If $(\mathcal{X}, \Phi)$ is positively expansive, then there is some finite ‘window’ $\mathcal{W} \subset V$ such that for any $x \in \mathcal{X}$, the data $[x_{\mathcal{W}}, \Phi(x)_{\mathcal{W}}, \Phi^2(x)_{\mathcal{W}}, \Phi^3(x)_{\mathcal{W}}, \ldots \Phi^t(x)_{\mathcal{W}}, \ldots]$ completely encodes $x$. But if $v \in V$ has superlinear connectivity and $h_v(\mathcal{X}) > 0$, then the information content of $x_{B(v, r)}$ grows superlinearly as $r \to \infty$. 

In fact, Theorems 1 and 2 are both special cases of a more general result.

In a digraph $(\mathcal{V}, \rightarrow)$, a vertex $v \in \mathcal{V}$ has superlinear connectivity if

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**Theorem 0.** Let $(A^\mathcal{V}, \mathcal{X}, \Phi)$ be a symbolic dynamical system with a moving subsymmetry. If $\mathcal{X}$ is weakly independent, and there exists some $v \in \mathcal{V}$ with superlinear connectivity such that $\overline{h}_v(\mathcal{X}) > 0$, then the system $(\mathcal{X}, \Phi)$ is not positively expansive.

**Proof sketch.** (by contradiction) If $(\mathcal{X}, \Phi)$ is positively expansive, then there is some finite ‘window’ $\mathcal{W} \subset \mathcal{V}$ such that for any $x \in \mathcal{X}$, the data $[x_\mathcal{W}, \Phi(x)_\mathcal{W}, \Phi^2(x)_\mathcal{W}, \Phi^3(x)_\mathcal{W}, \ldots, \Phi^t(x)_\mathcal{W}, \ldots]$ completely encodes $x$.

But if $v \in \mathcal{V}$ has superlinear connectivity and $\overline{h}_v(\mathcal{X}) > 0$, then the information content of $x_{B(v, r)}$ grows superlinearly as $r \to \infty$. It is impossible to transmit all this information through $\mathcal{W}$ quickly enough as $t \to \infty$. 
Theorem 0. Let \((\mathcal{A}^V, \mathcal{X}, \Phi)\) be a symbolic dynamical system with a moving subsymmetry. If \(\mathcal{X}\) is weakly independent, and there exists some \(v \in V\) with superlinear connectivity such that \(h_v(\mathcal{X}) > 0\), then the system \((\mathcal{X}, \Phi)\) is not positively expansive.

Remarks.
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Remarks.

- Theorems 0 - 2 apply even if the moving subsymmetry \(\tau\) and its iterates are the only symmetries of the system \((A^V, \mathcal{X}, \Phi)\).
Theorem 0. Let \((AV, X, \Phi)\) be a symbolic dynamical system with a moving subsymmetry. If \(X\) is weakly independent, and there exists some \(v \in V\) with superlinear connectivity such that \(h_v(X) > 0\), then the system \((X, \Phi)\) is not positively expansive.

Remarks.

- Theorems 0 - 2 apply even if the moving subsymmetry \(\tau\) and its iterates are the only symmetries of the system \((AV, X, \Phi)\).
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**Theorem 0.** Let \((\mathcal{A}^V, \mathcal{X}, \Phi)\) be a symbolic dynamical system with a moving subsymmetry. If \(\mathcal{X}\) is weakly independent, and there exists some \(v \in V\) with superlinear connectivity such that \(\bar{h}_v(\mathcal{X}) > 0\), then the system \((\mathcal{X}, \Phi)\) is not positively expansive.

**Remarks.**

> Theorems 0 - 2 apply even if the moving subsymmetry \(\tau\) and its iterates are the only symmetries of the system \((\mathcal{A}^V, \mathcal{X}, \Phi)\). In particular, we do not require the symmetry group of \((\mathcal{A}^V, \mathcal{X}, \Phi)\) to itself have growth dimension greater than 1.

> The ‘weak independence’ condition in Theorems 0 and 2 is probably unnecessary. (It is absent from Shereshevsky’s original result).
Main results: remarks

**Theorem 0.** Let \((A^V, \mathcal{X}, \Phi)\) be a symbolic dynamical system with a moving subsymmetry. If \(\mathcal{X}\) is weakly independent, and there exists some \(v \in V\) with superlinear connectivity such that \(h_v(\mathcal{X}) > 0\), then the system \((\mathcal{X}, \Phi)\) is not positively expansive.

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- Theorems 0 - 2 apply even if the moving subsymmetry \(\tau\) and its iterates are the only symmetries of the system \((A^V, \mathcal{X}, \Phi)\). In particular, we do not require the symmetry group of \((A^V, \mathcal{X}, \Phi)\) to itself have growth dimension greater than 1.

- The ‘weak independence’ condition in Theorems 0 and 2 is probably unnecessary. (It is absent from Shereshevsky’s original result). However, it is not clear how to dispense with it.
Theorem 0. Let $(A^V, \mathcal{X}, \Phi)$ be a symbolic dynamical system with a moving subsymmetry. If $\mathcal{X}$ is weakly independent, and there exists some $v \in V$ with superlinear connectivity such that $\overline{h}_v(\mathcal{X}) > 0$, then the system $(\mathcal{X}, \Phi)$ is not positively expansive.

Remarks.

- Theorems 0 - 2 apply even if the moving subsymmetry $\tau$ and its iterates are the only symmetries of the system $(A^V, \mathcal{X}, \Phi)$. In particular, we do not require the symmetry group of $(A^V, \mathcal{X}, \Phi)$ to itself have growth dimension greater than 1.

- The ‘weak independence’ condition in Theorems 0 and 2 is probably unnecessary. (It is absent from Shereshevsky’s original result). However, it is not clear how to dispense with it.

- The existence of a moving subsymmetry is also probably unnecessary. However, some condition is required beyond merely superlinear connectivity and nonzero entropy. This is shown by the next counterexample....
Let $\mathcal{V}$ be the digraph shown above.
Let $\mathbb{V}$ be the digraph shown above. Let $\mathbb{V}_{\Box} := \text{set of 'box' vertices, indexed by } M := \{0, 2, 6, 12, 20, \ldots, m_k, \ldots\}$, where $m_k := \sum_{i=0}^{k} 2^j$. 
Let $\mathbb{V}$ be the digraph shown above. Let $\mathbb{V}_{\Box} := \text{set of ‘box’ vertices, indexed by } \mathbb{M} := \{0, 2, 6, 12, 20, \ldots, m_k, \ldots\}$, where $m_k := \sum_{i=0}^{k} 2j$. Let $\mathbb{V}_{\circ} := \text{set of ‘circle’ vertices};$ then $\mathbb{V} = \mathbb{V}_{\Box} \cup \mathbb{V}_{\circ}$. 
Let $V$ be the digraph shown above. Let $V_{\square} := \text{set of 'box' vertices, indexed by } M := \{0, 2, 6, 12, 20, \ldots, m_k, \ldots\}$, where $m_k := \sum_{i=0}^{k} 2j$. Let $V_{\circ} := \text{set of 'circle' vertices; then } V = V_{\square} \cup V_{\circ}$. Let $A := \mathbb{Z}/2 \times \mathbb{Z}/2$. 
Let $\mathbb{V}$ be the digraph shown above. Let $\mathbb{V}^{\Box} := \text{set of 'box' vertices}$, indexed by $\mathbb{M} := \{0, 2, 6, 12, 20, \ldots, m_k, \ldots\}$, where $m_k := \sum_{i=0}^{k} 2j$.

Let $\mathbb{V}_o := \text{set of 'circle' vertices}$; then $\mathbb{V} = \mathbb{V}^{\Box} \cup \mathbb{V}_o$.

Let $\mathcal{A} := \mathbb{Z}/2 \times \mathbb{Z}/2$. Thus $\forall \ n \in \mathbb{N}$, the state of vertex $n$ is an ordered pair $(a_n, b_n)$, where $a_n, b_n \in \mathbb{Z}/2$. 
Let $\mathcal{V}$ be the digraph shown above. Let $\mathcal{V}_{\square} := \text{set of ‘box’ vertices, indexed by } M := \{0, 2, 6, 12, 20, \ldots, m_k, \ldots\}$, where $m_k := \sum_{i=0}^{k} 2j$. Let $\mathcal{V}_c := \text{set of ‘circle’ vertices;}$ then $\mathcal{V} = \mathcal{V}_{\square} \cup \mathcal{V}_c$.

Let $\mathcal{A} := \mathbb{Z}/2 \times \mathbb{Z}/2$. Thus $\forall \ n \in \mathbb{N}$, the state of vertex $n$ is an ordered pair $\left(\frac{a_n}{b_n}\right)$, where $a_n, b_n \in \mathbb{Z}/2$. Let $\mathcal{X} := \{a \in \mathcal{A}^\mathcal{V}; b_n = 0, \forall \ n \in \mathcal{V}_c\}$. 

An expansive system of dimension two
Let $V$ be the digraph shown above. Let $V_\square := \text{set of 'box' vertices, indexed by } \mathbb{M} := \{0, 2, 6, 12, 20, \ldots, m_k, \ldots\}$, where $m_k := \sum_{i=0}^{k} 2j$.

Let $V_\circ := \text{set of 'circle' vertices};$ then $V = V_\square \sqcup V_\circ$.

Let $A := \mathbb{Z}/2 \times \mathbb{Z}/2$. Thus $\forall n \in \mathbb{N}$, the state of vertex $n$ is an ordered pair $\left(\frac{a_n}{b_n}\right)$, where $a_n, b_n \in \mathbb{Z}/2$. Let $\mathcal{X} := \{a \in A^V; b_n = 0, \forall n \in V_\circ\}$.

Thus, if $\mathcal{X}_n$ is the projection of $\mathcal{X}$ onto vertex $n$, then $\mathcal{X}_n = \mathbb{Z}/2 \times \mathbb{Z}/2$ if $n \in V_\square$, and $\mathcal{X}_n = \mathbb{Z}/2 \times \{0\}$ if $n \in V_\circ$. 
Let $\mathbb{V}$ be the digraph shown above. Let $\mathbb{V}_\square := \text{set of 'box' vertices, indexed by } M := \{0, 2, 6, 12, 20, \ldots, m_k, \ldots\}$, where $m_k := \sum_{i=0}^{k} 2j$. Let $\mathbb{V}_\circ := \text{set of 'circle' vertices};$ then $\mathbb{V} = \mathbb{V}_\square \sqcup \mathbb{V}_\circ$.

Let $\mathcal{A} := \mathbb{Z}/2 \times \mathbb{Z}/2$. Thus $\forall \ n \in \mathbb{N}$, the state of vertex $n$ is an ordered pair $\left(\frac{a_n}{b_n}\right)$, where $a_n, b_n \in \mathbb{Z}/2$. Let $\mathcal{X} := \{a \in \mathcal{A}^\mathbb{V}; \ b_n = 0, \ \forall \ n \in \mathbb{V}_\circ\}$. Thus, if $\mathcal{X}_n$ is the projection of $\mathcal{X}$ onto vertex $n$, then $\mathcal{X}_n = \mathbb{Z}/2 \times \mathbb{Z}/2$ if $n \in \mathbb{V}_\square$, and $\mathcal{X}_n = \mathbb{Z}/2 \times \{0\}$ if $n \in \mathbb{V}_\circ$.

For any $n \in \mathbb{V}_\circ$, define $\phi_n : \mathcal{X}_{n+1} \longrightarrow \mathcal{X}_n$ by $\phi_n \left(\frac{a_{n+1}}{b_{n+1}}\right) = \left(\frac{a_{n+1}}{0}\right)$ (i.e. $\phi_n$ just copies the first coordinate of vertex $n + 1$ into vertex $n$).
For any $n \in \mathbb{V}_\circ$, define $\phi_n : \mathcal{X}_{n+1} \rightarrow \mathcal{X}_n$ by $\phi_n \left( \begin{array}{c} a_{n+1} \\ b_{n+1} \end{array} \right) = \left( \begin{array}{c} a_{n+1} \\ 0 \end{array} \right)$. 
An expansive system of dimension two

For any $n \in \mathbb{V}_\circ$, define $\phi_n : X_{n+1} \rightarrow X_n$ by $\phi_n \left( \begin{array}{c} a_{n+1} \\ b_{n+1} \end{array} \right) = \left( \begin{array}{c} a_{n+1} \\ 0 \end{array} \right)$.

For any $m_k \in \mathbb{V}_\square$, define $\phi_{m_k} : X_{(m_k)+1} \times X_{m(k+1)} \rightarrow X_{m_k}$ as follows:

$$
\phi_{m_k} \left( \begin{array}{c} a(m_k)+1 \\ 0 \\ a_{m(k+1)} \\ b_{m(k+1)} \end{array} \right) := \left( \begin{array}{c} a(m_k)+1 \\ a_{m(k+1)} + b_{m(k+1)} \end{array} \right).
$$
For any \( n \in \mathbb{V}_\circ \), define \( \phi_n : \mathcal{X}_{n+1} \to \mathcal{X}_n \) by \( \phi_n \left( \begin{array}{c} a_{n+1} \\ b_{n+1} \end{array} \right) = \left( \begin{array}{c} a_{n+1} \\ 0 \end{array} \right) \).

For any \( m_k \in \mathbb{V}_\square \), define \( \phi_{m_k} : \mathcal{X}_{(m_k)+1} \times \mathcal{X}_{m(k+1)} \to \mathcal{X}_{m_k} \) as follows:

\[
\phi_{m_k} \left( \begin{array}{c} a(m_k)+1 \\ 0 \\
\end{array}, \begin{array}{c} a_{m(k+1)} \\ b_{m(k+1)} \end{array} \right) := \left( \begin{array}{c} a(m_k)+1 \\ a_{m(k+1)} + b_{m(k+1)} \end{array} \right)
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Theorem.
An expansive system of dimension two

For any $n \in \mathbb{V}_\circ$, define $\phi_n : \mathcal{X}_{n+1} \rightarrow \mathcal{X}_n$ by $\phi_n \left( \begin{pmatrix} a_n+1 \\ b_{n+1} \end{pmatrix} \right) = \left( \begin{pmatrix} a_n+1 \\ 0 \end{pmatrix} \right)$.

For any $m_k \in \mathbb{V}_\square$, define $\phi_{m_k} : \mathcal{X}_{(m_k)+1} \times \mathcal{X}_{m(k+1)} \rightarrow \mathcal{X}_{m_k}$ as follows:

$$\phi_{m_k} \left( \begin{pmatrix} a(m_k)+1 \\ 0 \end{pmatrix} , \begin{pmatrix} a_{m(k+1)}+1 \\ b_{m(k+1)} \end{pmatrix} \right) := \begin{pmatrix} a(m_k)+1 \\ a_{m(k+1)} + b_{m(k+1)} \end{pmatrix}.$$ 

**Theorem.** (a) $h(\mathcal{X}) \geq 1$, and $\mathcal{X}$ is weakly independent; and
For any \( n \in \mathbb{V}_\circ \), define \( \phi_n : \mathcal{X}_{n+1} \rightarrow \mathcal{X}_n \) by 
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\phi_{m_k} \left( \begin{pmatrix} a_{(m_k)+1} \\ 0 \\ b_{m(k+1)} \end{pmatrix}, \begin{pmatrix} a_{m(k+1)} \\ b_{m(k+1)} \end{pmatrix} \right) := \begin{pmatrix} a_{(m_k)+1} \\ a_{m(k+1)} + b_{m(k+1)} \end{pmatrix}.
\]

**Theorem.** (a) \( h(\mathcal{X}) \geq 1 \), and \( \mathcal{X} \) is weakly independent; and (b) \( \dim_v(\mathbb{V}, \rightarrow) = 2 \) for all \( v \in \mathbb{V} \); but
An expansive system of dimension two

For any \( n \in V_\circ \), define \( \phi_n : X_{n+1} \rightarrow X_n \) by
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\]
For any \( m_k \in V_\square \), define \( \phi_{m_k} : X_{m(k+1)+1} \times X_{m(k+1)} \rightarrow X_{m_k} \) as follows:
\[
\phi_{m_k} \left( \begin{pmatrix} a_{m(k)+1} \\ b_{m(k)+1} \\ 0 \end{pmatrix} , \begin{pmatrix} a_{m(k+1)} \\ b_{m(k+1)} \end{pmatrix} \right) := \begin{pmatrix} a_{m(k)+1} \\ a_{m(k+1)} + b_{m(k+1)} \end{pmatrix}.
\]

**Theorem.** (a) \( h(X) \geq 1 \), and \( X \) is weakly independent; and
(b) \( \dim_v(V, \rightarrow) = 2 \) for all \( v \in V \); but
(c) The system \( (X, \Phi) \) is positively expansive.
An expansive system of dimension two

For any \( n \in \mathbb{V}_\circ \), define \( \phi_n : X_{n+1} \rightarrow X_n \) by 
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**Proof sketch.**
For any \( n \in \mathbb{N} \) define \( \phi_n : \mathcal{X}_{n+1} \rightarrow \mathcal{X}_n \) by
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\phi_n \left( \begin{array}{c} a_{n+1} \\ b_{n+1} \end{array} \right) = \left( \begin{array}{c} a_n \\ 0 \end{array} \right).
\]
For any \( m_k \in \mathbb{N} \) define \( \phi_{m_k} : \mathcal{X}(m_k)_{n+1} \times \mathcal{X}(k)_{m_{k+1}} \rightarrow \mathcal{X}_{m_k} \) as follows:
\[
\phi_{m_k} \left( \left( \begin{array}{c} a(m_k) \\ 0 \end{array} \right), \left( \begin{array}{c} a_{m(k+1)} \\ b_{m(k+1)} \end{array} \right) \right) := \left( \begin{array}{c} a(m_k)+1 \\ a_{m(k+1)} + b_{m(k+1)} \end{array} \right).
\]

**Theorem.** (a) \( h(\mathcal{X}) \geq 1 \), and \( \mathcal{X} \) is weakly independent; and
(b) \( \dim_{\mathbb{V}}(\mathcal{V}, \mathcal{X}) = 2 \) for all \( v \in \mathcal{V} \); but
(c) The system \( (\mathcal{X}, \Phi) \) is positively expansive.

**Proof sketch.** (a) is obvious from the definition of \( \mathcal{X} \).
Theorem. (a) $h(\mathcal{X}) \geq 1$, and $\mathcal{X}$ is weakly independent; and
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Proof sketch. (b) (Case $v = 0$) The figure shows $\mathcal{B}(0, r)$ for $r = 1, 2, \ldots, 7$. 
Theorem. (a) \( h(\mathcal{X}) \geq 1 \), and \( \mathcal{X} \) is weakly independent; and (b) \( \dim_v(\mathcal{V}, \to) = 2 \) for all \( v \in \mathcal{V} \); but (c) The system \( (\mathcal{X}, \Phi) \) is positively expansive.

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Theorem. (a) \( h(\mathcal{X}) \geq 1 \), and \( \mathcal{X} \) is weakly independent; and
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Theorem. (a) $h(\mathcal{X}) \geq 1$, and $\mathcal{X}$ is weakly independent; and
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Proof sketch. (b) (Case $v = 0$) The figure shows $B(0, r)$ for $r = 1, 2, \ldots, 7$. Clearly, $|B(0, r)|$ grows quadratically as $r \to \infty$. Thus, $\dim_0(\mathcal{V}, \rightarrow) = 2$. The proof for other $v \in \mathcal{V}$ is similar.
Theorem. (a) $h(\mathcal{X}) \geq 1$, and $\mathcal{X}$ is weakly independent; and
(b) $\dim_v(\mathcal{V}, \cdot \rightarrow) = 2$ for all $v \in \mathcal{V}$; but
(c) The system $(\mathcal{X}, \Phi)$ is positively expansive.

Proof sketch. (c) Straightforward computation shows that the data $x_0, \Phi(x)_0, \Phi^2(x)_0, \Phi^3(x)_0, \Phi^4(x)_0, \ldots, \Phi^t(x)_0, \ldots$ is sufficient to reconstruct $x$, for any $x \in \mathcal{X}$. 
Question. Is the network dimension of a symbolic dynamical system invariant under topological conjugacy?
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But network dimension is invariant under a stronger kind of conjugacy. Given a symbolic dynamical system $(\mathcal{A}^\mathcal{V}, \mathcal{X}, \Phi)$, one can define a metric $d$ on $\mathcal{X}$ such that $\Phi$ is Lipschitz relative to $d$. 
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But network dimension *is* invariant under a stronger kind of conjugacy. Given a symbolic dynamical system $(\mathcal{A}^\mathcal{X}, \mathcal{X}, \Phi)$, one can define a metric $d$ on $\mathcal{X}$ such that $\Phi$ is *Lipschitz* relative to $d$. That is: there is some constant $\lambda > 0$ such that, for all $x, y \in \mathcal{X}$, we have $d(\Phi(x), \Phi(y)) \leq \lambda \cdot d(x, y)$. 
**Question.** Is the network dimension of a symbolic dynamical system invariant under topological conjugacy?

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But network dimension is invariant under a stronger kind of conjugacy. Given a symbolic dynamical system \((A^V, \mathcal{X}, \Phi)\), one can define a metric \(d\) on \(\mathcal{X}\) such that \(\Phi\) is Lipschitz relative to \(d\). That is: there is some constant \(\lambda > 0\) such that, for all \(x, y \in \mathcal{X}\), we have \(d(\Phi(x), \Phi(y)) \leq \lambda \cdot d(x, y)\).

(Rough idea: Fix \(v \in V\). Define \(d\) so that \(x, y \in \mathcal{X}\), are “\(d\)-close” if \(x_{B(v,r)} = y_{B(v,r)}\) for some large \(r > 0\).)
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But network dimension is invariant under a stronger kind of conjugacy. Given a symbolic dynamical system \((\mathcal{A}^V, \mathcal{X}, \Phi)\), one can define a metric \(d\) on \(\mathcal{X}\) such that \(\Phi\) is Lipschitz relative to \(d\). That is: there is some constant \(\lambda > 0\) such that, for all \(x, y \in \mathcal{X}\), we have \(d(\Phi(x), \Phi(y)) \leq \lambda \cdot d(x, y)\). 

(\textbf{Rough idea:} Fix \(v \in V\). Define \(d\) so that \(x, y \in \mathcal{X}\), are “\(d\)-close” if \(x_B(v, r) = y_B(v, r)\) for some large \(r > 0\).)

One can then assign a dimension \(\text{dim}(\mathcal{X}, d)\) to the metric space \((\mathcal{X}, d)\).
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But network dimension is invariant under a stronger kind of conjugacy. Given a symbolic dynamical system \((A^V, \mathcal{X}, \Phi)\), one can define a metric \(d\) on \(\mathcal{X}\) such that \(\Phi\) is Lipschitz relative to \(d\). That is: there is some constant \(\lambda > 0\) such that, for all \(x, y \in \mathcal{X}\), we have \(d(\Phi(x), \Phi(y)) \leq \lambda \cdot d(x, y)\).

(Rough idea: Fix \(v \in V\). Define \(d\) so that \(x, y \in \mathcal{X}\), are “\(d\)-close” if \(x_{B(v, r)} = y_{B(v, r)}\) for some large \(r > 0\).)

One can then assign a dimension \(\dim(\mathcal{X}, d)\) to the metric space \((\mathcal{X}, d)\). Under suitable conditions, this metric is *dimensionally compatible* with the network topology of \((V, \rightarrow)\), meaning that \(\dim(\mathcal{X}, d) = \dim(V, \rightarrow)\).
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But network dimension *is* invariant under a stronger kind of conjugacy. Given a symbolic dynamical system \((A^\mathcal{V}, \mathcal{X}, \Phi)\), one can define a metric \(d\) on \(\mathcal{X}\) such that \(\Phi\) is *Lipschitz* relative to \(d\). That is: there is some constant \(\lambda > 0\) such that, for all \(x, y \in \mathcal{X}\), we have \(d(\Phi(x), \Phi(y)) \leq \lambda \cdot d(x, y)\).

(Rough idea: Fix \(v \in A^\mathcal{V}\). Define \(d\) so that \(x, y \in \mathcal{X}\), are “\(d\)-close” if \(x_{B(v, r)} = y_{B(v, r)}\) for some large \(r > 0\).)

One can then assign a dimension \(\dim(\mathcal{X}, d)\) to the metric space \((\mathcal{X}, d)\). Under suitable conditions, this metric is *dimensionally compatible* with the network topology of \((A^\mathcal{V}, \bullet \rightarrow)\), meaning that \(\dim(\mathcal{X}, d) = \dim(\mathcal{V}, \bullet \rightarrow)\).

Let \((\mathcal{X}', d')\) be another metric space, and let \(\Gamma : \mathcal{X} \rightarrow \mathcal{X}'\) be a continuous function.
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But network dimension *is* invariant under a stronger kind of conjugacy. Given a symbolic dynamical system \((\mathcal{A}^V, \mathcal{X}, \Phi)\), one can define a metric \(d\) on \(\mathcal{X}\) such that \(\Phi\) is *Lipschitz* relative to \(d\). That is: there is some constant \(\lambda > 0\) such that, for all \(x, y \in \mathcal{X}\), we have \(d(\Phi(x), \Phi(y)) \leq \lambda \cdot d(x, y)\).

**(Rough idea:** Fix \(v \in V\). Define \(d\) so that \(x, y \in \mathcal{X}\), are “\(d\)-close” if \(x_B(v, r) = y_B(v, r)\) for some large \(r > 0\).)**

One can then assign a dimension \(\dim(\mathcal{X}, d)\) to the metric space \((\mathcal{X}, d)\). Under suitable conditions, this metric is *dimensionally compatible* with the network topology of \((V, \bullet \rightarrow)\), meaning that \(\dim(\mathcal{X}, d) = \dim(V, \bullet \rightarrow)\).
Hölder Conjugacy invarance

Recall: a continuous function $\Gamma : (\mathcal{X}, d) \rightarrow (\mathcal{X}', d')$ is Hölder if there are constants $\eta, \lambda \in (0, \infty)$ such that, for any $x_1, x_2 \in \mathcal{X}$,

$$d'(\Gamma(x_1), \Gamma(x_2)) \leq \lambda \cdot d(x_1, x_2)^\eta.$$
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**Proposition.** Let $(\mathcal{X}, d)$ and $(\mathcal{X}', d')$ be metric spaces.
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**Proposition.** Let $(\mathcal{X}, d)$ and $(\mathcal{X}', d')$ be metric spaces. Let $\Gamma : \mathcal{X} \to \mathcal{X}'$ be a $(d, d')$-Hölder surjection. Then $\dim(\mathcal{X}, d) \geq \dim(\mathcal{X}', d')$. 


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Recall: a continuous function \( \Gamma : (\mathcal{X}, d) \rightarrow (\mathcal{X}', d') \) is Hölder if there are constants \( \eta, \lambda \in (0, \infty) \) such that, for any \( x_1, x_2 \in \mathcal{X} \),

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  d' \left( \Gamma(x_1), \Gamma(x_2) \right) \leq \lambda \cdot d(x_1, x_2)^\eta.
\]

**Proposition.** Let \((\mathcal{X}, d)\) and \((\mathcal{X}', d')\) be metric spaces.

- Let \( \Gamma : \mathcal{X} \rightarrow \mathcal{X}' \) be a \((d, d')\)-Hölder surjection. Then
  \[\dim(\mathcal{X}, d) \geq \dim(\mathcal{X}', d').\]

- If \( \Gamma \) is a \((d, d')\)-biHölder homeomorphism, then
  \[\dim(\mathcal{X}, d) = \dim(\mathcal{X}', d').\]

**Corollary.** Let \((A^\mathbb{V}, \mathcal{X}_1, \Phi_1)\) and \((B^\mathbb{W}, \mathcal{X}_2, \Phi_2)\) be two symbolic dynamical systems, and let \(d_1\) and \(d_2\) be dimensionally compatible Lipschitz metrics on \(\mathcal{X}_1\) and \(\mathcal{X}_2\) respectively.
Recall: a continuous function \( \Gamma : (X, d) \rightarrow (X', d') \) is Hölder if there are constants \( \eta, \lambda \in (0, \infty) \) such that, for any \( x_1, x_2 \in X \),

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**Proposition.** Let \( (X, d) \) and \( (X', d') \) be metric spaces.

- Let \( \Gamma : X \rightarrow X' \) be a \((d, d')\)-Hölder surjection. Then \( \dim(X, d) \geq \dim(X', d') \).
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**Corollary.** Let \( (A^V, X_1, \Phi_1) \) and \( (B^W, X_2, \Phi_2) \) be two symbolic dynamical systems, and let \( d_1 \) and \( d_2 \) be dimensionally compatible Lipschitz metrics on \( X_1 \) and \( X_2 \) respectively.

- If there is a factor mapping \( (X_1, \Phi_1) \rightarrow (X_2, \Phi_2) \) which is \((d_1, d_2)\)-Hölder, then \( \dim(V, \bullet \rightarrow_1) \geq \dim(W, \bullet \rightarrow_2) \).
Hölder Conjugacy invarance

Recall: a continuous function $\Gamma : (\mathcal{X}, d) \rightarrow (\mathcal{X}', d')$ is Hölder if there are constants $\eta, \lambda \in (0, \infty)$ such that, for any $x_1, x_2 \in \mathcal{X}$,

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Proposition. Let $(\mathcal{X}, d)$ and $(\mathcal{X}', d')$ be metric spaces.

- Let $\Gamma : \mathcal{X} \rightarrow \mathcal{X}'$ be a $(d, d')$-Hölder surjection. Then $\dim(\mathcal{X}, d) \geq \dim(\mathcal{X}', d').$

- If $\Gamma$ is a $(d, d')$-biHölder homeomorphism, then $\dim(\mathcal{X}, d) = \dim(\mathcal{X}', d').$

Corollary. Let $(\mathcal{A}^{\mathcal{V}}, \mathcal{X}_1, \Phi_1)$ and $(\mathcal{B}^{\mathcal{W}}, \mathcal{X}_2, \Phi_2)$ be two symbolic dynamical systems, and let $d_1$ and $d_2$ be dimensionally compatible Lipschitz metrics on $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively.

- If there is a factor mapping $(\mathcal{X}_1, \Phi_1) \rightarrow (\mathcal{X}_2, \Phi_2)$ which is $(d_1, d_2)$-Hölder, then $\dim(\mathcal{V}, \rightarrow_1) \geq \dim(\mathcal{W}, \rightarrow_2).$

- If $(\mathcal{X}_1, \Phi_1)$ and $(\mathcal{X}_2, \Phi_2)$ are conjugate via a bi-Hölder homeomorphism, then $\dim(\mathcal{V}, \rightarrow_1) = \dim(\mathcal{W}, \rightarrow_2).$
We have shown that the positive expansiveness of \((A^V, \mathcal{X}, \Phi)\) is related to the network dimension of the digraph \((V, \bullet \rightarrow)\).
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We have shown that the positive expansiveness of \((A^V, X, \Phi)\) is related to the network dimension of the digraph \((V, \bullet \rightarrow)\).

**Question.** What other dynamical properties of \((A^V, X, \Phi)\) are influenced by the geometry of the digraph \((V, \bullet \rightarrow)\)?

One could also go the other way. Start with an infinite digraph \((V, \bullet \rightarrow)\), and randomly generate a continuous self-map \(\Phi : A^V \rightarrow A^V\), such that \((\bullet \rightarrow)\) is the network of \(\Phi\).
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**Question.** What are the ‘generic’ (i.e. almost-certain) properties of \((\mathcal{A}^V, \Phi)\), and how do they depend on the geometry of \((\mathcal{V}, \bullet \rightarrow)\)?

**Conjecture** If \(\dim(\mathcal{V}, \bullet \rightarrow) \leq 1\), then almost surely, \((\mathcal{A}^V, \Phi)\) is equicontinuous. If \(\dim(\mathcal{V}, \bullet \rightarrow) > 1\), then almost surely, \((\mathcal{A}^V, \Phi)\) is sensitive. (The intuition here comes from percolation theory).
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**Question.** Suppose we take a symbolic dynamical system \((A^V, \Phi)\) and ‘mutate’ it, by changing the local rule at a small number of vertices.
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**Question.** Suppose we take a symbolic dynamical system \((A^V, \Phi)\) and ‘mutate’ it, by changing the local rule at a small number of vertices. What topological-dynamical properties are ‘robust’ under such mutations, and how does this depend on the geometry of \((V, \to)\)?
Thank you.

These presentation slides are available at


For more information, see


<http://arxiv.org/abs/0907.2935>
Introduction: Shereshevsky’s result

Symbolic dynamical systems
   Definition & examples
   The network of a symbolic dynamical system
   Network Dimension
   Entropy
   Local independence and subisometries
   Subisometries and subsymmetries
   Moving subsymmetries

Main results
   Theorems 1 and 2
   Theorem 0
   Remarks

An expansive system of dimension two
   Construction
   Theorem statement
   Proof sketch

Conjugacy invariance
   Dimension is not a conjugacy invariant...
...but it is a Hölder conjugacy invariant

Conclusion