Positive expansiveness versus network dimension in symbolic dynamical systems

Marcus Pivato

Department of Mathematics, Trent University
Peterborough, Ontario, Canada
marcuspivato@trentu.ca

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We will generalize Shereshevsky's result to a much broader class of *symbolic dynamical systems*. These are systems like a CA, but having an 'irregular' network topology.

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In fact, we shall now see that *any* symbolic dynamical system can be seen as an automaton network.

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The network of a symbolic dynamical system

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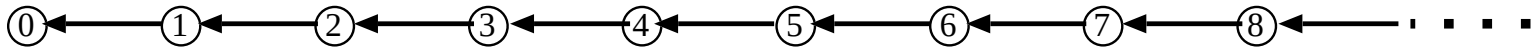

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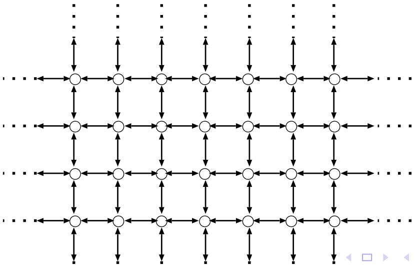

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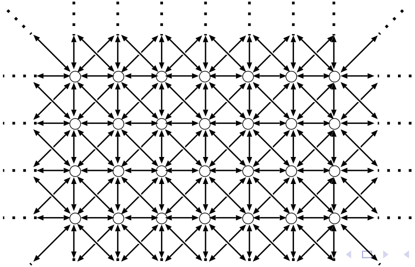
Example. The network of cellular automaton on $\mathbb{Z}^2$ (Moore neighbourhood)



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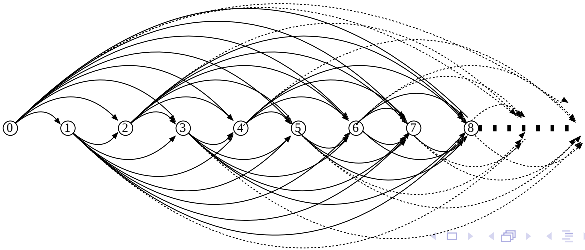
Example. The network of an odometer.



Network Dimension

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If $\underline{\dim}(\mathbb{V}, \bullet\!\!\rightarrow) = \overline{\dim}(\mathbb{V}, \bullet\!\!\rightarrow)$, then denote common value by $\dim(\mathbb{V}, \bullet\!\!\rightarrow)$.
Example. If $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ is a CA, then $\dim(\mathbb{Z}^D, \bullet\!\!\rightarrow) = D$.

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More generally, if $\mathbb{G}$ is a group, and $\Phi : \mathcal{A}^{\mathbb{G}} \longrightarrow \mathcal{A}^{\mathbb{G}}$ is a CA, then $\dim(\mathbb{G}, \bullet\!\!\!\to) =$ the dimension of the group $\mathbb{G}$.

Entropy

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Example. (a) $h(\mathcal{A}^{\mathbb{V}}) = \log_2 |\mathcal{A}|$.

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Example. (a) $h(\mathcal{A}^{\mathbb{V}}) = \log_2 |\mathcal{A}|$.
(b) Suppose $\mathbb{V} = \mathbb{Z}^D$ (with Cayley digraph), and $\mathcal{X} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ is a D-dimensional subshift. Then $h(\mathcal{X})$ is the (D-dimensional) topological entropy of $\mathcal{X}$.

Local independence and subisometries

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Subisometries and subsymmetries

Recall: a *subisometry* of $(\mathbb{V}, \multimap)$ is an injection $\tau : \mathbb{V} \longrightarrow \mathbb{V}$ such that, for all $\mathsf{v}, \mathsf{w} \in \mathbb{V}$, we have $(\mathsf{v} \multimap \mathsf{w}) \iff (\tau(\mathsf{v}) \multimap \tau(\mathsf{w}))$.

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For any $\mathsf{v} \in \mathbb{V}$, let $\mathrm{speed}(\mathsf{v}, \tau) := \lim_{n \to \infty} \dfrac{d[\mathsf{v}, \tau^n(\mathsf{v})]}{n}$. This limit is well-defined and constant on each undirected-path component of $(\mathbb{V}, \bullet\!\!\to)$. We say that τ is a *moving* subsymmetry if $\mathrm{speed}(\mathsf{v}, \tau) > 0$ for all $\mathsf{v} \in \mathbb{V}$.

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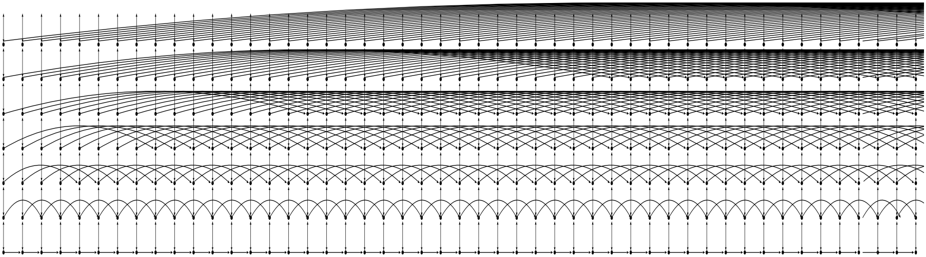

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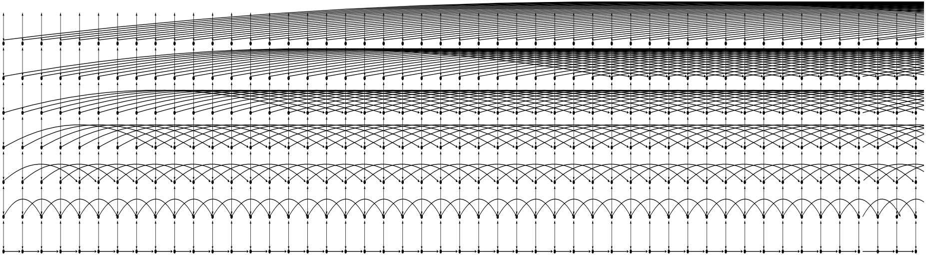

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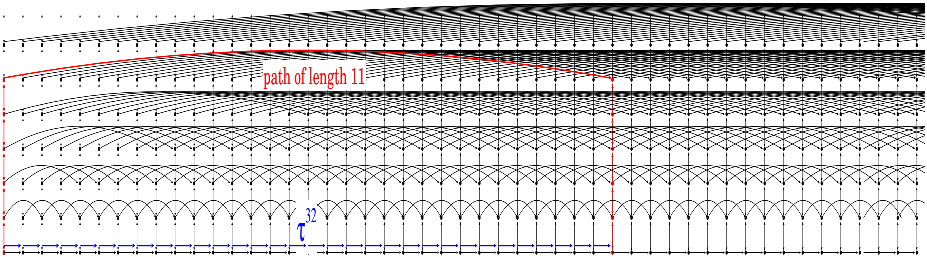

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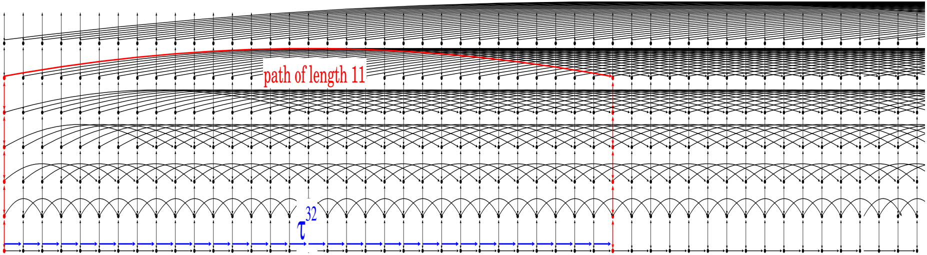

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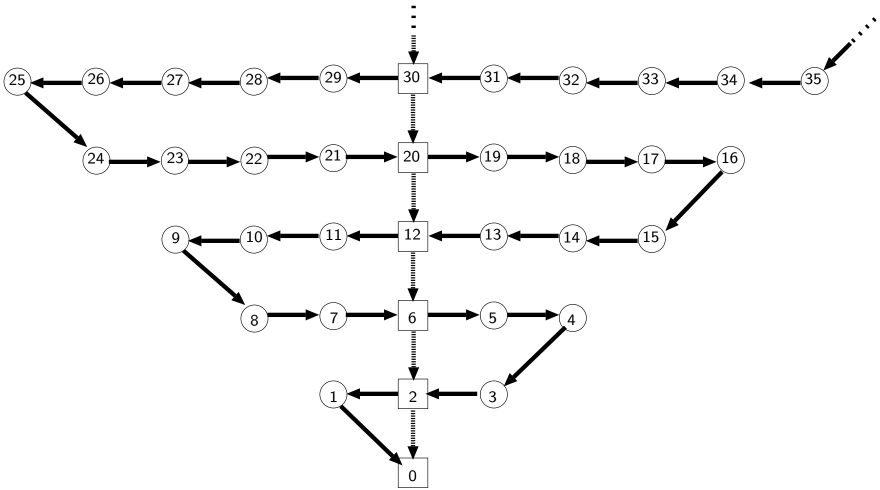
- The existence of a moving subsymmetry is also probably unnecessary. However, *some* condition is required beyond merely superlinear connectivity and nonzero entropy. This is shown by the next counterexample....

An expansive system of dimension two



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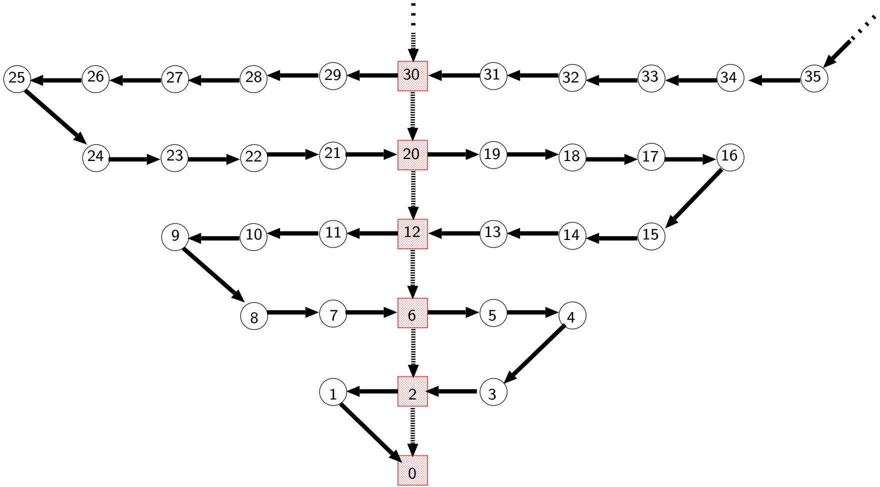
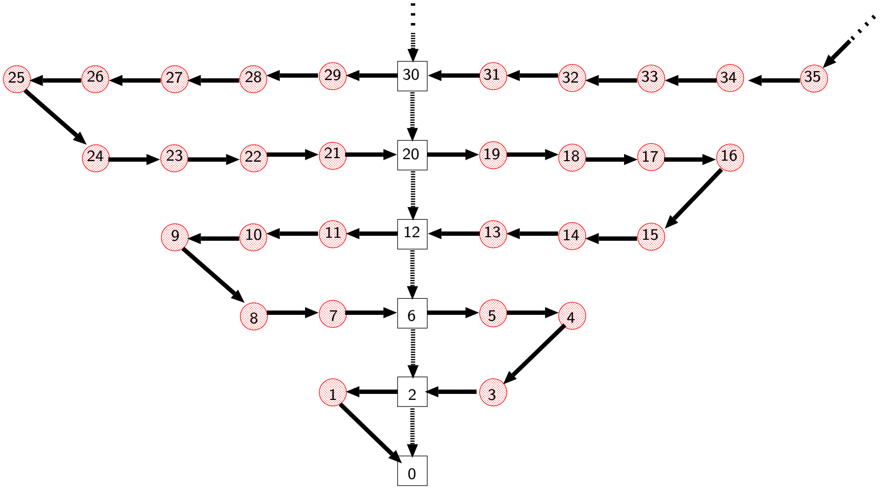
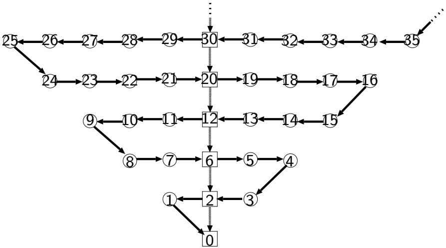


Let $\mathbb{V}$ be the digraph shown above. Let ${\color{red}\mathbb{V}_\Box}$:= set of 'box' vertices, indexed by $\mathbb{M} := \{0, 2, 6, 12, 20, \ldots, m_k, \ldots\}$, where ${\color{red}m_k} := \sum_{i=0}^{k} 2j$.



Let $\mathbb{V}$ be the digraph shown above. Let $\mathbb{V}_\square$:= set of 'box' vertices, indexed by $\mathbb{M} := \{0, 2, 6, 12, 20, \ldots, m_k, \ldots\}$, where $m_k := \sum_{i=0}^{k} 2j$.
Let $\mathbb{V}_\circ$:= set of 'circle' vertices; then $\mathbb{V} = \mathbb{V}_\square \sqcup \mathbb{V}_\circ$.



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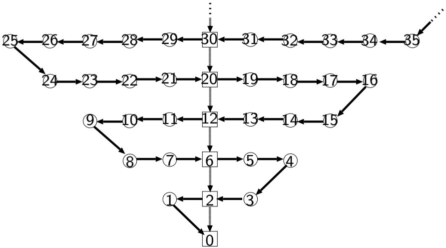

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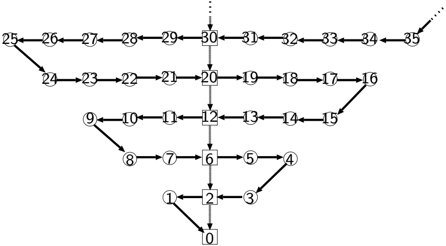

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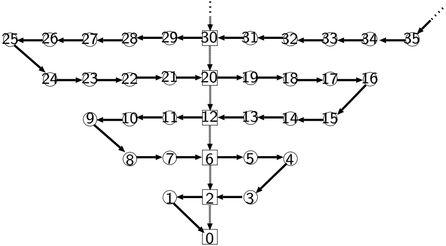

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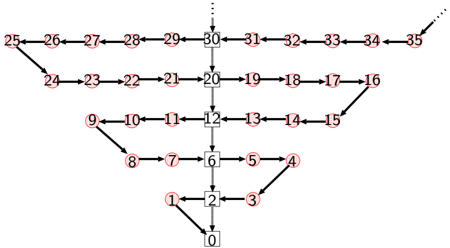
An expansive system of dimension two



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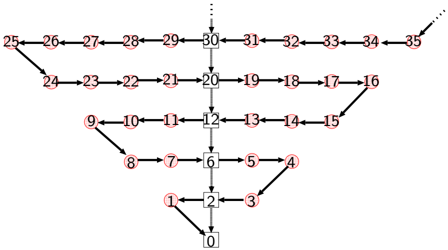
An expansive system of dimension two



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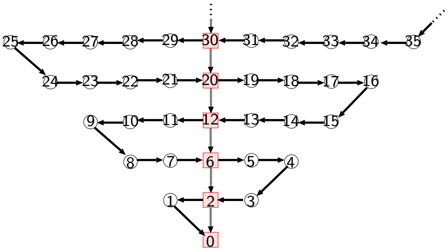

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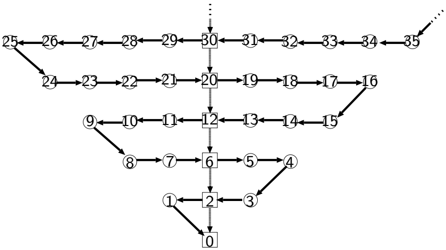

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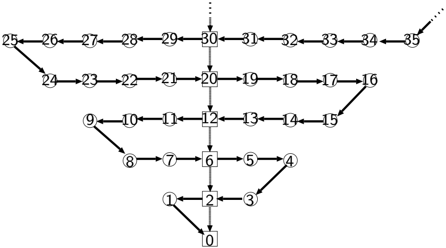

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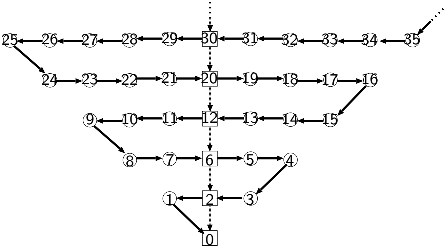

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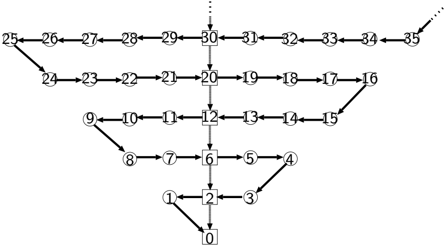

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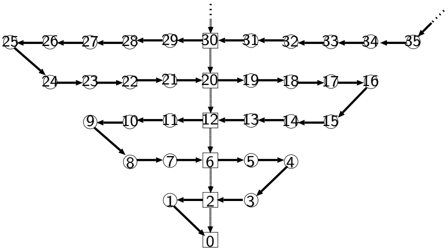

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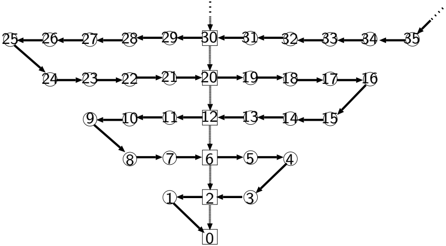
Proof sketch.

An expansive system of dimension two



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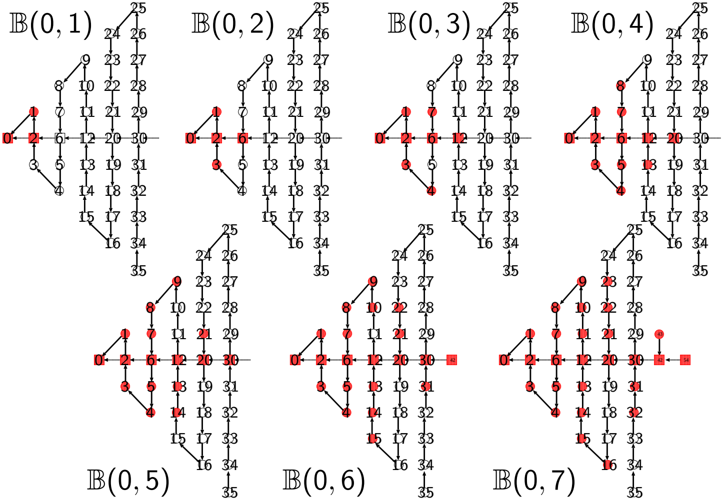
Proof sketch. (a) is obvious from the definition of $\mathcal{X}$.



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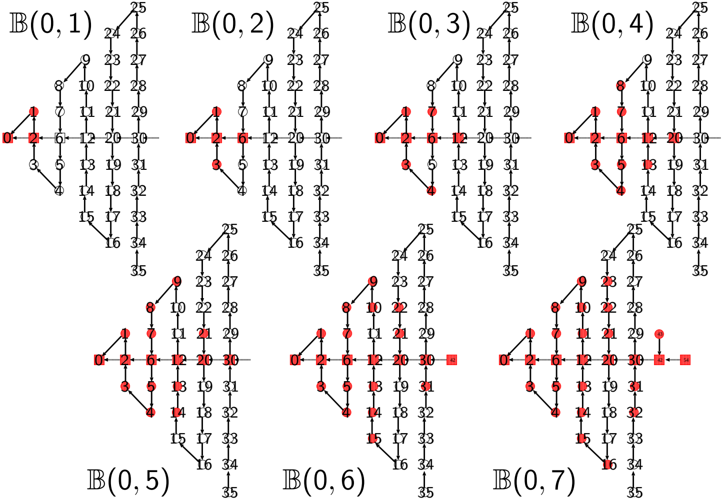

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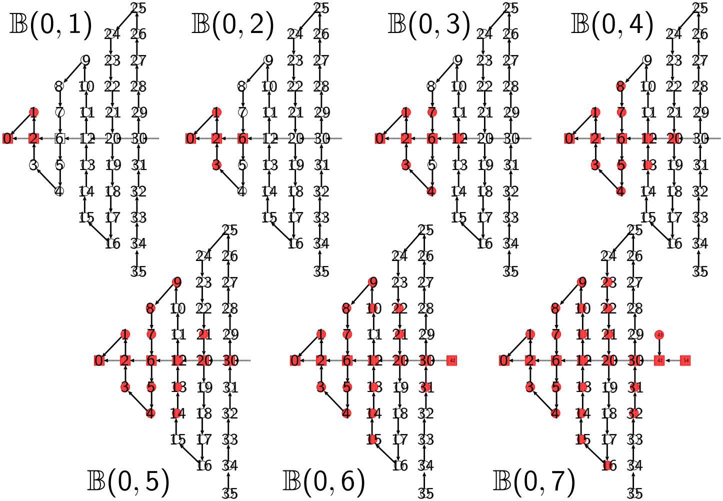

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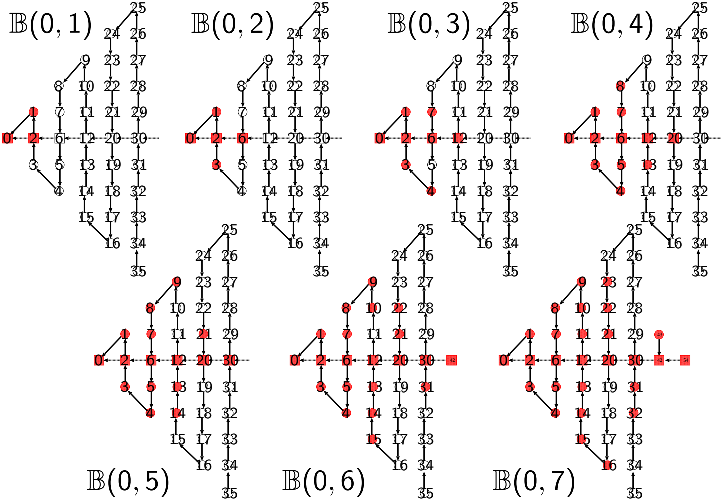

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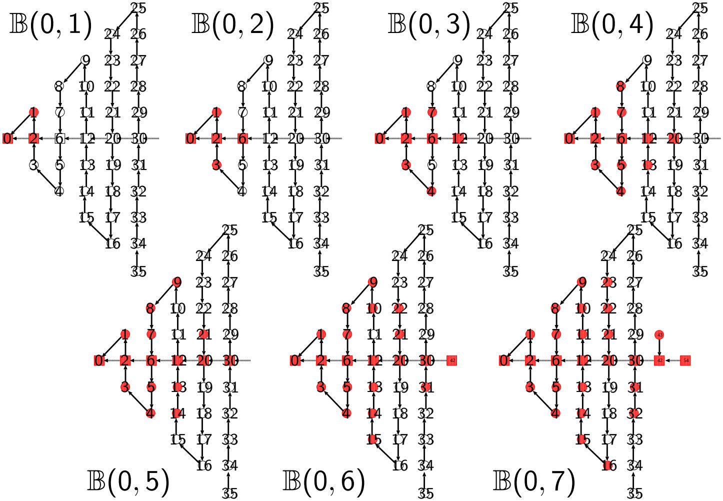

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Proof sketch. (c) Straightforward computation shows that the data x_0, $\Phi(\mathbf{x})_0$, $\Phi^2(\mathbf{x})_0$, $\Phi^3(\mathbf{x})_0$, $\Phi^4(\mathbf{x})_0$, $\ldots$, $\Phi^t(\mathbf{x})_0$, $\ldots$ is sufficient to reconstruct $\mathbf{x}$, for any $\mathbf{x} \in \mathcal{X}$.

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But network dimension *is* invariant under a stronger kind of conjugacy. Given a symbolic dynamical system $(\mathcal{A}^{\mathbb{V}}, \mathcal{X}, \Phi)$, one can define a metric d on $\mathcal{X}$ such that Φ is *Lipschitz* relative to d. That is: there is some constant $\lambda > 0$ such that, for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we have $d\left(\Phi(\mathbf{x}), \Phi(\mathbf{y})\right) \ \leq \ \lambda \cdot d(\mathbf{x}, \mathbf{y})$.
(**Rough idea:** Fix $\mathsf{v} \in \mathbb{V}$. Define d so that $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, are "$d$-close" if $\mathbf{x}_{\mathbb{B}(\mathsf{v},r)} = \mathbf{y}_{\mathbb{B}(\mathsf{v},r)}$ for some large $r > 0$.)
One can then assign a dimension $\dim(\mathcal{X}, d)$ to the metric space $(\mathcal{X}, d)$. Under suitable conditions, this metric is *dimensionally compatible* with the network topology of $(\mathbb{V}, \bullet\!\!\rightarrow)$, meaning that $\dim(\mathcal{X}, d) = \dim(\mathbb{V}, \bullet\!\!\rightarrow)$.

Let $(\mathcal{X}', d')$ be another metric space, and let $\Gamma : \mathcal{X} \longrightarrow \mathcal{X}'$ be a continuous function.

Conjugacy invarance

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Let $(\mathcal{X}', d')$ be another metric space, and let $\Gamma : \mathcal{X} \longrightarrow \mathcal{X}'$ be a continuous function. Say Γ is Hölder if there are constants $\eta, \lambda \in (0, \infty)$ such that, for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$, we have $d'\left(\Gamma(\mathbf{x}_1),\ \Gamma(\mathbf{x}_2)\right) \ \leq \ \lambda \cdot d(\mathbf{x}_1, \mathbf{x}_2)^{\eta}$.

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Recall: a continuous function $\Gamma : (\mathcal{X}, d) \longrightarrow (\mathcal{X}', d')$ is Hölder if there are constants $\eta, \lambda \in (0, \infty)$ such that, for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$,

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Corollary. *Let $(\mathcal{A}^{\mathbb{V}}, \mathcal{X}_1, \Phi_1)$ and $(\mathcal{B}^{\mathbb{W}}, \mathcal{X}_2, \Phi_2)$ be two symbolic dynamical systems, and let d_1 and d_2 be dimensionally compatible Lipschitz metrics on $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively.*

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Conclusion

We have shown that the positive expansiveness of $(\mathcal{A}^{\mathbb{V}}, \mathcal{X}, \Phi)$ is related to the network dimension of the digraph $(\mathbb{V}, \bullet\!\!\rightarrow)$.

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One could also go the other way. Start with an infinite digraph $(\mathbb{V}, \bullet\!\!\rightarrow)$, and randomly generate a continuous self-map $\Phi : \mathcal{A}^{\mathbb{V}} \longrightarrow \mathcal{A}^{\mathbb{V}}$, such that $(\bullet\!\!\rightarrow)$ is the network of Φ.

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Thank you.

These presentation slides are available at

<http://euclid.trentu.ca/pivato/Research/network.pdf>

For more information, see

Positive expansiveness versus network dimension in symbolic dynamical systems, to appear in *Theoretical Computer Science* (2011).

<http://arxiv.org/abs/0907.2935>

...but it is a Hölder conjugacy invariant

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