Incoherent majorities: the McGarvey problem in judgement aggregation

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- Also, what can wrong with these methods?

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Finally, the rule F is *monotone* if the following is true: for all $\mathbf{v}, \mathbf{w} \in \{\pm 1\}^{\mathcal{I}}$ if $v_i \geq w_i$ for all $i \in \mathcal{I}$, then $F(\mathbf{v}) \geq F(\mathbf{w})$.

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Theorem (May, 1952) *Majority vote is the only binary voting rule which is anonymous, neutral, and monotone.*

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Electorate Profile	
Preferences	#
$A \succ B \succ C \succ D$	10
$A \succ C \succ D \succ B$	9
$A \succ D \succ B \succ C$	11
$B \succ C \succ D \succ A$	22
$C \succ D \succ B \succ A$	23
$D \succ B \succ C \succ A$	25
Total	100

Consider an election with four candidates A, B, C, and D.
(e.g. 10% of the voters prefer A to B, prefer B to C, and prefer C to D.)

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$A \succ C \succ D \succ B$	9	9			
$A \succ D \succ B \succ C$	11	11			
$B \succ C \succ D \succ A$	22		22		
$C \succ D \succ B \succ A$	23			23	
$D \succ B \succ C \succ A$	25				25
Total	100	30	22	23	25
Verdict:		A wins.			

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Clearly A wins under plurality vote, with 30% of the vote.

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A versus B, C, and D							
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$A \succ B \succ C \succ D$	10	10		10		10	
$A \succ C \succ D \succ B$	9	9		9		9	
$A \succ D \succ B \succ C$	11	11		11		11	
$B \succ C \succ D \succ A$	22		22		22		22
$C \succ D \succ B \succ A$	23		23		23		23
$D \succ B \succ C \succ A$	25		25		25		25
Total	100	30	70	30	70	30	70
Verdict:		$B \succ A$		$C \succ A$		$D \succ A$	

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Problem: 70% prefer B to A. . . .

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This is called pairwise majority vote.

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Total	100	67	33	66	34	33	67
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The majority's apparently 'cyclical' preference ordering

$$\cdots \succ A \succ B \succ C \succ A \succ B \succ C \succ A \succ B \succ C \succ A \succ B \succ C \succ \cdots$$

is called a *Condorcet cycle*.

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- However, we will now turn to a completely different social choice problem...

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Unfortunately, it is susceptible to paradoxical outcomes similar to the Condorcet paradox. For example, let p and q be two statements, and let $\mathcal{K} := \{p,\ q,\ p\&q\}$. Consider the following profile of three voters:

#	p	q	p&q
Alice	T	T	T
Bob	T	F	F
Carl	F	T	F
Majority	T	T	F

$\longleftarrow$ Contradiction

This is called the *doctrinal paradox* (Kornhauser & Sager, 1986) or the discursive dilemma (List & Pettit, 2002).

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Let $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}} \subset \{\pm 1\}^{\mathcal{K}}$ be the set of all *transitive* tournaments —i.e. strict preference relations on $\mathcal{A}$. ($\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ is called the *permutahedron*.)

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Propositionwise majority vote on $\mathcal{X}^{\mathrm{pr}}_{\mathcal{A}}$ is thus equivalent to Condorcet's pairwise majority vote: define a collective preference order by using majority rule to decide questions of the form, "Is $a \prec b$?" for all $a, b \in [1 \ldots A]$.

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To see this, let $\mathcal{A}$ be a set with $|\mathcal{A}| \geq 3$. Let $\mathcal{K} \subset \mathcal{A} \times \mathcal{A}$ be a subset containing exactly one of (a, b) or (b, a) for each $a \neq b \in \mathcal{A}$;
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Let $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}} \subset \{\pm 1\}^{\mathcal{K}}$ be the set of all *transitive* tournaments —i.e. strict preference relations on $\mathcal{A}$. ($\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ is called the *permutahedron*.)
Propositionwise majority vote on $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ is thus equivalent to Condorcet's pairwise majority vote: define a collective preference order by using majority rule to decide questions of the form, "Is $a \prec b$?" for all $a, b \in [1 \ldots A]$.
In this setting, the 'doctrinal paradox' takes the following form:

#	$a \prec b$	$b \prec c$	$a \prec c$
Alice ($a \prec b \prec c$)	T	T	T
Bob ($c \prec a \prec b$)	T	F	F
Carl ($b \prec c \prec a$)	F	T	F
Majority	T	T	F

$\longleftarrow$ $a \prec b \prec c \prec a$ (a cycle)

The McGarvey problem

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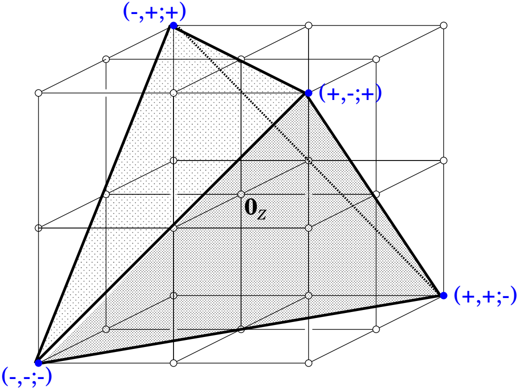
- (b5) $\mathrm{cone}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$.

For example, consider the 'truth functional' judgement space:

$$\mathcal{X}_{\mathrm{XOR}} \quad := \quad \left\{\mathbf{x} \in \{\pm 1\}^3 \,;\, x_3 \;=\; (x_1 \text{ XOR } x_2)\right\}.$$



Clearly, $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$; thus, $\mathcal{X}_{\mathrm{XOR}}$ is McGarvey.

Next consider the 'truth functional' judgement space:

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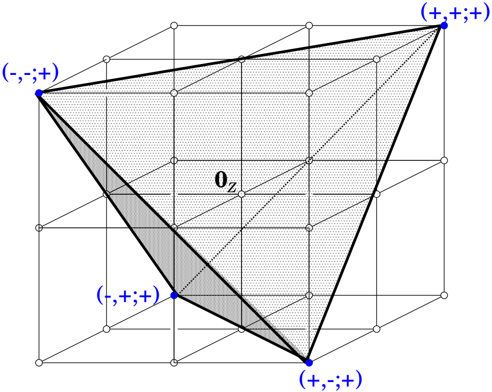

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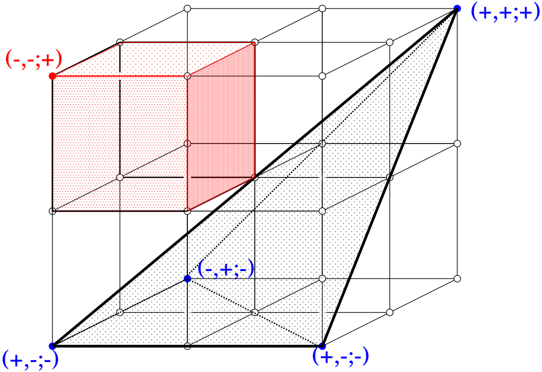

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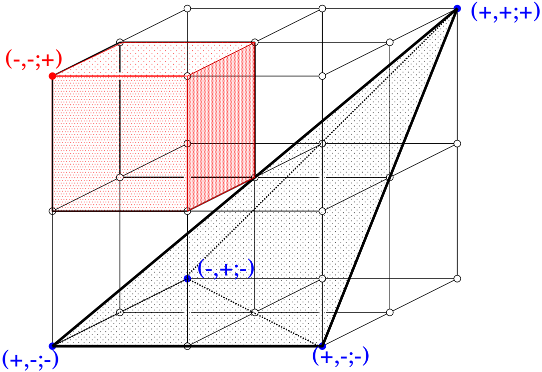

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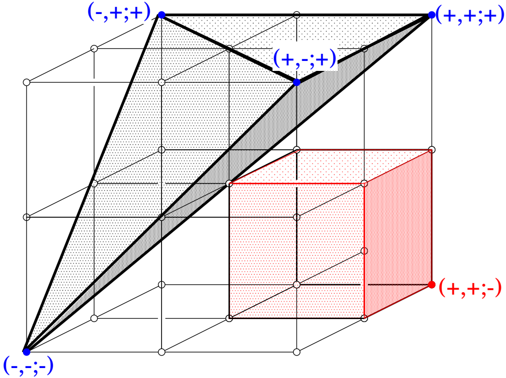

If $\mathbf{x} := (+1, +1; -1)$, then $\mathrm{conv}(\mathcal{X}_{\mathrm{OR}})$ does *not* intersect $\mathcal{O}_{\mathbf{x}}$.

Example: truth-functional aggregation in $\mathbb{R}^3$ (OR)

Finally, consider the 'truth functional' judgement space:

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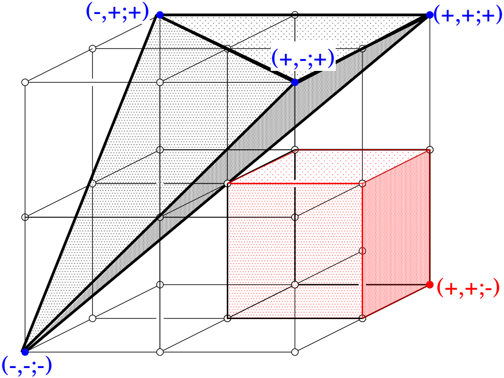

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Example: Convexities

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The set $\mathfrak{C}$ of all convex subsets of $\mathcal{K}$ is then a convexity structure on $\mathcal{K}$.

For any $\mathcal{J} \subseteq \mathcal{K}$, define $\boldsymbol{\chi}^{\mathcal{J}} \in \{\pm 1\}^{\mathcal{K}}$ by $\chi_j^{\mathcal{J}} := 1$ for all $j \in \mathcal{J}$ and $\chi_k^{\mathcal{J}} := -1$ for all $k \in \mathcal{K} \setminus \mathcal{J}$.

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(For example, the metric graph convexity is McGarvey.)

Symmetric spaces

For any $\mathcal{X} \subset \mathbb{R}^{\mathcal{K}}$, the *symmetry group* of $\mathcal{X}$ is the set $\Gamma_{\mathcal{X}}$ of all invertible linear transformations $\gamma : \mathbb{R}^{\mathcal{K}} \longrightarrow \mathbb{R}^{\mathcal{K}}$ such that $\gamma(\mathcal{X}) = \mathcal{X}$.

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Clearly, $\mathcal{X}$ cannot be McGarvey unless $\mathrm{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$. Advantage of Proposition 3 over Theorem 1(b2): it is easier to verify that $\mathrm{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$ than to verify that $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$. The next result is often sufficient.

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Lemma 4. *Let $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$. Suppose, for every $j \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ such that $x_j \neq y_j$, but $x_k = y_k$ for all $k \in \mathcal{K} \setminus \{j\}$. Then $\operatorname{span}(\mathcal{X}) = \mathbb{R}^{\mathcal{K}}$.*

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- For any $(a, b) \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}_{\mathcal{N}}^{\mathrm{pr}}$ such that $x_{a,b} \neq y_{a,b}$, but $\mathbf{x}$ and $\mathbf{y}$ agree in every other coordinate.
(Example:
  - let $\mathbf{x}$ represent an ordering of the form $a \prec b \prec c_3 \prec c_4 \prec \cdots \prec c_N$,
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- Thus, Proposition 3(b) yields McGarvey's original result: $\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}$ is McGarvey.

Example: Additively separable preferences (1)

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A binary relation "$\succ$" on $\mathcal{A}$ is *separable* if, for each $\mathcal{M} \subseteq [1 \ldots L]$, there is a binary relation "$\underset{\mathcal{M}}{\succ}$" on $\mathcal{A}_{\mathcal{M}}$ such that, for all $\mathbf{a}_{\mathcal{M}}, \mathbf{b}_{\mathcal{M}} \in \mathcal{A}_{\mathcal{M}}$ and all $\mathbf{c}_{\mathcal{M}^\complement} \in \mathcal{A}_{\mathcal{M}^\complement}$, we have

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Every additively separable preference relation is separable (but not conversely).

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- Can be shown: $\mathrm{Maj}(\mathcal{X}_{\mathcal{A}}^{\mathrm{sep}}) = \mathcal{X}_{\mathcal{A}}^{\mathrm{sep}}$; thus, $\mathcal{X}_{\mathcal{A}}^{\mathrm{sepr}}$ and $\mathcal{X}_{\mathcal{A}}^{\mathrm{add}}$ are *not* McGarvey.

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- **Explanation:** $\mathcal{X}_{\mathcal{A}}^{\mathrm{sep}}$ has many 'redundant' coordinates, which always agree in value because of the separability constraints.
- **Idea:** Eliminate redundant coordinates, to obtain a subset $\widetilde{\mathcal{K}} \subset \mathcal{K}$ such that no two coordinates of $\widetilde{\mathcal{K}}$ are related by separability constraints.

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- Recall: $\mathcal{X}_{\mathcal{A}}^{\mathrm{sep}} \subset \{\pm 1\}^{\mathcal{K}}$ is set of all separable tournaments over $\mathcal{A}$.
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We can derive Shelah's result as an easy consequence of Proposition 3(a).

(Most of the work: showing that RHS implies that $\mathrm{span}(\mathcal{X}_{\mathcal{T}}) = \mathbb{R}^{\mathcal{K}}$.)

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Claim B. $\mathrm{span}(\mathcal{X}^{\rm eq}_{\mathcal{A}}) = \mathbb{R}^{\mathcal{K}}$.
Proof. For any $\{n, m\} \in \mathcal{K}$, there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\rm eq}_{\mathcal{A}}$ such that $x_{n,m} \neq y_{n,m}$, but $\mathbf{x}$ and $\mathbf{y}$ agree in every other coordinate.

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(Example: let $\mathbf{x}$ represent an equivalence relation where n and m are both in singleton equivalence classes, and let $\mathbf{y}$ represent the relation obtained from $\mathbf{x}$ by joining n and m together into one doubleton equivalence class).

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(Example: let $\mathbf{x}$ represent an equivalence relation where n and m are both in singleton equivalence classes, and let $\mathbf{y}$ represent the relation obtained from $\mathbf{x}$ by joining n and m together into one doubleton equivalence class).
Now apply Lemma 4. □

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Claims A, B and C and Corollary 6 imply that $\mathcal{X}_{\mathcal{A}}^{\mathrm{eq}}$ is McGarvey.

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For any $N \in \mathbb{N}$, let

$$\Delta_N^*(\mathcal{X}) \quad := \quad \left\{\mu \in \Delta(\mathcal{X})\ ;\ \forall\ \mathbf{x} \in \mathcal{X},\ \mu(\mathbf{x}) = \frac{n}{N} \text{ for some } n \in [0 \ldots N]\right\}.$$

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Example. Let $A := |\mathcal{A}|$. Stearns (1959) showed that $0.55 \cdot A/\log(A) \leq S(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}) \leq A + 2$, while Erdös and Moser (1964) showed that $S(\mathcal{X}_{\mathcal{A}}^{\mathrm{pr}}) = \Theta(A/\log(A))$.

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- We can improve the estimate in Theorem 7, by making further assumptions about the structure of $\mathcal{X}$.

For any $\mathbf{x}_1, \ldots, \mathbf{x}_K \in \{\pm 1\}^{\mathcal{K}}$, let

$$\delta(\mathbf{x}_1, \ldots, \mathbf{x}_K) \quad := \quad \min\{\|\mathbf{c}\|_\infty \,;\, \mathbf{c} \in \mathrm{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_K)\}.$$

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Proposition 8. *Let* $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$.

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Proposition 8. *Let $\mathcal{X} \subset \{\pm 1\}^{\mathcal{K}}$.*

(a) *If $\mathcal{X}$ is McGarvey, then $\sigma(\mathcal{X}) \leq \lceil 1/\delta(\mathcal{X}) \rceil$.*

Estimates of the Stearns Number

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For any $\mathbf{x}_1, \ldots, \mathbf{x}_K \in \{\pm 1\}^{\mathcal{K}}$, let

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This seems obvious. But random polytopes in $\{\pm 1\}^{\mathcal{K}}$ have stochastically 'thick' boundaries. There is a big difference between "generically, $\mathbf{0} \in \mathrm{conv}(\mathcal{X})$" and "generically, $\mathbf{0} \in \mathrm{int}\,[\mathrm{conv}\,(\mathcal{X})]$".

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For any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathcal{X}$, define $\mathrm{med}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) := \mathrm{Maj}(\mu)$, where $\mu \in \Delta(\mathcal{X})$ is defined by $\mu(\mathbf{x}_j) = \frac{1}{3}$ for $j = 1, 2, 3$.

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For all $n \in \mathbb{N}$, define $\mathrm{med}^{n+1}(\mathcal{X}) := \{\mathrm{med}(\mathbf{x}, \mathbf{y}, \mathbf{z});\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathrm{med}^n(\mathcal{X})\}$.

For any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathcal{X}$, define $\mathrm{med}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) := \mathrm{Maj}(\mu)$, where $\mu \in \Delta(\mathcal{X})$ is defined by $\mu(\mathbf{x}_j) = \frac{1}{3}$ for $j = 1, 2, 3$.
This defines a ternary operator on $\{\pm 1\}^{\mathcal{K}}$, called the *median operator*.

Let $\mathrm{med}^1(\mathcal{X}) := \{\mathrm{med}(\mathbf{x}, \mathbf{y}, \mathbf{z})\ ;\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}\}$. (Thus, $\mathcal{X} \subseteq \mathrm{med}^1(\mathcal{X})$.)

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Let $\mathrm{med}^\infty(\mathcal{X}) := \bigcup_{n=1}^{\infty} \mathrm{med}^n(\mathcal{X})$ (= smallest median space containing $\mathcal{X}$).

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Question. When does $\mathrm{Maj}(\mathcal{X}) = \mathrm{med}^\infty(\mathcal{X})$?

Merci.

These presentation slides are available at

<http://euclid.trentu.ca/pivato/Research/mcgarvey.pdf>.

The paper is available at

<http://mpra.ub.uni-muenchen.de/14823>.

It will appear in *Discrete Applied Mathematics* in 2011.

DOI: <10.1016/j.dam.2011.03.014>.

The Füredi problem

The generalized McGarvey Problem