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Pivato, Marcus (3-TREN)

Cellular automata versus quasisturmian shifts. (English. English summary)

Ergodic Theory Dynam. Systems **25** (2005), *no.* 5, 1583–1632.

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Invariant measures for bipermutative cellular automata. (English. English summary)

Discrete Contin. Dyn. Syst. **12** (2005), *no.* 4, 723–736.

The antecedent of this article is a result proved in [B. Host, A. Maass and S. A. Martínez, Discrete Contin. Dyn. Syst. **9** (2003), no. 6, 1423–1446; MR2017675 (2004i:37022)] that states that for one-dimensional affine cellular automata Φ on a finite field, a Φ-invariant measure of positive entropy, which is shift(σ)-invariant and σ-ergodic too, is by necessity the uniform Bernoulli measure. In this spirit, the article considers one-dimensional cellular automata Φ with a configuration space equipped with a topological group structure such that σ is a group automorphism. If Φ is a group endomorphism of the configuration space and is specified by a local rule that is a right-sided nearest neighbor group multiplication then a measure which is Φ-invariant, σ-invariant, totally σ-ergodic and has positive entropy is necessarily the uniform Bernoulli measure. Another result is proved in this article for one-dimensional cellular automata Φ on nonabelian groups with a nearest neighbor multiplication local rule. In this situation a measure which is Φ-invariant, Φ-ergodic and σ-invariant is necessarily supported on a right coset of a subgroup and is uniformly distributed on this coset.

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Interior symmetry and local bifurcation in coupled cell networks. (English. English summary)

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Limit measures for affine cellular automata. II. (English. English summary)

Ergodic Theory Dynam. Systems **24** (2004), *no.* 6, 1961–1980.

Introduction: "Let $\mathcal{A}$ be a finite abelian group, with the discrete topology. If $\mathbb{M}$ is any set, then $\mathcal{A}^{\mathbb{M}}$ is a compact abelian group when endowed with the Tikhonov product topology and componentwise addition. If $(\mathbb{M},+)$ is an abelian monoid (for example, a lattice: $\mathbb{Z}^D\times\mathbb{N}^E$), then the action of $\mathbb{M}$ on itself by translation induces a natural shift action of $\mathbb{M}$ on the configuration space: for all $\mathsf{e}\in\mathbb{M}$ and $\mathbf{a}\in\mathcal{A}^{\mathbb{M}}$, define $\sigma^{\mathsf{e}}[\mathbf{a}]=[b_{\mathsf{m}|\mathsf{m}\in\mathbb{M}}]$ where for all $\mathsf{m}\in\mathbb{M}$, $b_{\mathsf{m}}=a_{\mathsf{e}+\mathsf{m}}$.

"A linear cellular automaton (LCA) is a continuous endomorphism $\mathfrak{F}\colon\mathcal{A}^{\mathbb{M}}\hookleftarrow$ which commutes with all shift maps. If μ is a measure on $\mathcal{A}^{\mathbb{M}}$, it is natural to consider the sequence of measures $\{\mathfrak{F}^n\mu|_{n\in\mathbb{N}}\}$ and ask whether this sequence converges in the weak* topology on the space $\mathcal{M}[\mathcal{A}^{\mathbb{M}}]$ of Borel probability measures on $\mathcal{A}^{\mathbb{M}}$. If $\{\mathfrak{F}^n\mu|_{n\in\mathbb{N}}\}$ does not itself converge, we might hope at least for convergence in density (i.e., convergence of a subsequence $\{\mathfrak{F}^j\mu|_{j\in\mathbb{J}}\}$, where $\mathbb{J}\subset\mathbb{N}$ is a subset of Cesàro density 1) or convergence of the Cesàro average $(1/N)\sum_{n=1}^N\mathfrak{F}^n\mu$.

"Let η denote the Haar measure on $\mathcal{A}^{\mathbb{M}}$. Since η is invariant under the algebraic operations of $\mathcal{A}^{\mathbb{M}}$, it seems like a natural limit point for $\{\mathfrak{F}^n\mu|_{n\in\mathbb{N}}\}$. Indeed, D. Lind [Phys. D **10** (1984), no. 1-2, 36–44; MR0762651 (86g:68128)] has shown that, if $\mathcal{A}=Z_2$, $\mathfrak{F}$ is the automaton defined by $\mathfrak{F}(\mathbf{a})_0=a_{(-1)}+a_1$ and μ is any Bernoulli measure, then

$$\text{weak}^*-\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N\mathfrak{F}^n\mu=\eta.$$

Lind also showed that $\{\mathfrak{F}^n\mu|_{n\in\mathbb{N}}\}$ does not converge to η: convergence fails along the subsequence $\{\mathfrak{F}^{(2^n)}\mu|_{n\in\mathbb{N}}\}$.

"Later, Ferrari, Maass, Martínez and Ney showed similar Cesàro convergence results in a variety of special cases [P. A. Ferrari et al., Ergodic Theory Dynam. Systems **20** (2000), no. 6, 1657–1670; MR1804951 (2001k:37017); A. Maass and S. A. Martínez, in *Cellular automata and complex systems (Santiago, 1996)*, 37–54, Kluwer Acad. Publ., Dordrecht, 1999;MR1672858]. Recently, in Part I [Ergodic Theory Dynam. Systems **22** (2002), no. 4, 1269–1287; MR1926287 (2004b:37017)], we developed broad sufficient conditions for convergence. The concepts of harmonic mixing for measures and diffusion for LCA were introduced: if μ is a harmonically mixing probability measure and $\mathfrak{F}$ a diffusive LCA, then $\{\mathfrak{F}^n\mu|_{n\in\mathbb{N}}\}$ weak* converges to η in density and, thus, also in the Cesàro mean.

"This paper is a continuation of Part I. First we will extend the results on the diffusion of LCA to a broader class of abelian groups: in §3, to the case when $\mathcal{A} = Z_n$, for any $n \in \mathbb{N}$, and then in §4, to the case when $\mathcal{A} = (\mathbb{Z}/p^r)^J$ (p prime, $J, r \in \mathbb{N}$). Next, in §5, we demonstrate harmonic mixing for any Markov random field on $\mathcal{A}^{(\mathbb{Z}^D)}$ with full support."

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Dynam. Sys. **22**(4) (2002), 1269–1287. MR1926287 (2004b:37017)

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Symmetry groupoids and patterns of synchrony in coupled cell networks. (English. English summary)

SIAM J. Appl. Dyn. Syst. **2** (2003), *no.* 4, 609–646 (*electronic*).

In this paper the authors introduce the formalism of symmetry groupoids for networks, which generalizes the more common concept of symmetry based on group actions. This formalism is applied to networks of coupled dynamical systems, and it allows one to explain the formation of patterns of synchrony in cases where the network does not share the symmetry of such patterns.

The aim of the paper is attained by establishing two results about the consequences that the existence of groupoid symmetries in a coupled cell network G has for the dynamics generated by G-admissible vector fields.

Prerequisites.

A coupled cell network G is a directed graph with cells (vertices) in the set $\mathcal{C}$ and arrows in $\mathcal{E}\subset\mathcal{C}\times\mathcal{C}$, supplied with equivalence relations $\sim_{\mathcal{C}}$ and $\sim_{\mathcal{E}}$ on cells and arrows respectively. In addition it is assumed that $\Delta_{\mathcal{C}}:=\{(c,c)\colon\ c\in\mathcal{C}\}$ is contained in $\mathcal{E}$, that the equivalence relations $\sim_{\mathcal{C}}$ and $\sim_{\mathcal{E}}$ are compatible, and that arrows in $\Delta_{\mathcal{C}}$ are never equivalent to arrows in $\mathcal{E}\smallsetminus\Delta_{\mathcal{C}}$.

Two cells $c,d\in\mathcal{C}$ are input equivalent, which is denoted $c\sim_I d$, if there exists a bijection

$$\beta\colon I(c):=\{i\in\mathcal{C}\colon\ (i,c)\in\mathcal{E}\}\to I(d):=\{i\in\mathcal{C}\colon\ (i,d)\in\mathcal{E}\}$$

such that $(i,c)\sim_{\mathcal{E}}(\beta(i),d)$ for each $i\in I(c)$. Such a bijection is called an input isomorphism. The collection of all input isomorphisms from c to its input equivalent cell d is denoted $B(c,d)$, and the disjoint union $\mathcal{B}_G=\dot{\bigcup}_{c,d\in\mathcal{C}}B(c,d)$ is the symmetry groupoid of the coupled cell network G.

Each cell $c\in\mathcal{C}$ is associated to a cell phase space P_c, which is assumed to be a finite-dimensional real vector space. It is required that $P_c=P_d$ whenever $c\sim_{\mathcal{C}}d$. This defines a total phase space, which is the Cartesian product $P=\prod_{c\in\mathcal{C}}P_c$. Each input isomorphism $\beta\in B(c,d)$ induces a pullback map $\beta^*\colon\prod_{i\in I(d)}P_i\to\prod_{j\in I(c)}P_j$ such that $\beta^*(x_i)=x_{\beta(j)}$. A vector field $f\colon P\to P$ is G-admissible if for each $c\in\mathcal{C}$, $f_c(x)$ depends only on $\{x_i\colon\ i\in I(c)\}$. It is also required that for all

$c, d \in \mathcal{C}$ and $\beta \in B(c, d)$, $f_{\beta(c)}(x) = f(\beta^*(\{x_i\colon\ i \in I(c)\}))$.

A cell equivalence relation $\bowtie$ defines the polydiagonal subspace $\Delta_{\bowtie} := \{x \in P\colon\ x_c = x_d \text{ whenever } c \bowtie d\, \forall c, d \in C\} \subset P$. The trajectory $x(t)$ of the vector field $f\colon P \to P$ is $\bowtie$-polysynchronous if $c \bowtie d$ implies $x_c(t) = x_d(t)$ for all $t \in \mathbb{R}$. A polysynchronous state $x \in \Delta_{\bowtie}$ is what one calls a pattern of synchrony.

The relation $\bowtie$ is robustly polysynchronous if $f(\Delta_{\bowtie}) \subset \Delta_{\bowtie}$ for each G-admissible vector field. The relation $\bowtie$ is balanced if for all $c, d \in \mathcal{C}$ such that $c \bowtie d$ and $c \neq d$, there exists $\gamma \in B(c, d)$ such that $i \bowtie \gamma(i)$ for all $i \in I(c)$.

A map $\varphi\colon \mathcal{C}_1 \to \mathcal{C}_2$ from the cells of the networks G_1 to the cells of the network G_2 is a factor map if: it is onto, (d, d') is an arrow of G_2 if and only if $(d, d') = (\varphi(c), \varphi(c'))$ for some arrow (c, c') of G_1, and β_2 is an input isomorphism of G_2 if and only if $\beta_2 \circ \varphi = \varphi \circ \beta_1$ for some input isomorphisms β_1 of G_2.

A choice of a phase space $P = \prod_{c \in \mathcal{C}_1} P_c$ for the network G_1 determines a corresponding phase space $\overline{P} := \prod_{d \in \mathcal{C}_2} P_c$, $c \in \varphi^{-1}(\{d\})$, for the network G_2. The vector field $f\colon P \to P$ induces a vector field for $\overline{f}\colon \overline{P} \to \overline{P}$ as follows. Let $\alpha\colon \overline{P} \to P$ be such that $\alpha(y)_c = y_d$ for each $c \in \varphi^{-1}(\{d\})$; then $\overline{f}(y) = \alpha^{-1}(f(\alpha(y)))$ for each $y \in \overline{P}$.

Results.

The first main result gives a complete characterization of the robustly polysynchronous equivalence relations on $\mathcal{C}$. It establishes that an equivalence relation $\bowtie$ is robustly polysynchronous if and only if it is balanced.

The second main result establishes that, given a factor map $\varphi\colon G_1 \to G_2$, any G_1-admissible vector field in the chosen phase space $f\colon P \to P$ induces a G_2-admissible vector field in the corresponding phase space $\overline{f}\colon \overline{P} \to \overline{P}$.

Comments.

It is from the second result mentioned above that one can explain the existence of patterns of synchrony in a network which does not share the symmetry of such patterns. This comes from the fact that a network may admit quotient networks which have symmetries even when the original network has none.

The article proceeds in a very pedagogical way, illustrating each new notion with clear examples. It starts with an example-based informal discussion about symmetries and invariant subspaces, and the relation between them. Though it is a mathematical paper whose aim is to establish rigorous results, it may be easily followed by a nonspecialist, and could be very useful for physicists and engineers working in applications. *Edgardo Ugalde* (San Luis Potosí)

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Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.

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MR2016936 (2004h:60020) 60E07 62F10

Pivato, Marcus (3-TREN); **Seco, Luis** (3-TRNT)

Estimating the spectral measure of a multivariate stable distribution via spherical harmonic analysis. (English. English summary)

J. Multivariate Anal. **87** (2003), *no.* 2, 219–240.

Summary: "A new method is developed for estimating the spectral measure of a multivariate stable probability measure, by representing the measure as a sum of spherical harmonics."

Célestin C. Kokonendji (F-PAU-AM)

MR1966329 (2005b:37020) 37B15 37A05

Pivato, Marcus (3-TRNT)

Multiplicative cellular automata on nilpotent groups: structure, entropy, and asymptotics. (English. English summary)

J. Statist. Phys. **110** (2003), *no.* 1-2, 247–267.

Let $(\mathcal{B},\cdot)$ be a finite group, $\mathbb{M}$ a lattice, and $\mathbb{V}\subset\mathbb{M}$ a finite set of coordinates. Fix an ordering $v\colon\{0,1,\dots,I\}\to\mathbb{V}$. A multiplicative cellular automaton (MCA) $\mathfrak{G}\colon\mathcal{B}^{\mathbb{M}}\to\mathcal{B}^{\mathbb{M}}$ has local map $\mathfrak{g}\colon\mathcal{B}^{\mathbb{V}}\to\mathcal{B}$ of the form

$$\mathfrak{g}(\mathbf{b}|_{\mathbb{V}})=g\cdot\prod_{i=0}^{I}\mathfrak{g}_i(b_{v[i]}),$$

where $g\in\mathcal{B}$, and for each $i=0,\dots,I$, $\mathfrak{g}_i$ is an endomorphism of the group $(\mathcal{B},\cdot)$.

In this paper the author first shows how a pseudoproduct decomposition $\mathcal{B}=\mathcal{A}\star\mathcal{C}$ leads to a decomposition $\mathfrak{G}=\mathfrak{F}\star\mathfrak{H}$ of the MCA as a skew product of a MCA $\mathfrak{H}\colon\mathcal{C}^{\mathbb{M}}\to\mathcal{C}^{\mathbb{M}}$, and a so-called multiplicative relative cellular automaton $\mathfrak{F}\colon\mathcal{A}^{\mathbb{M}}\times\mathcal{C}^{\mathbb{M}}\to\mathcal{A}^{\mathbb{M}}$.

Among other interesting applications, this decomposition allows the author to extend his previous result with R. Yassawi [Ergodic

Theory Dynam. Systems **22** (2002), no. 4, 1269–1287; MR1926287 (2004b:37017)] concerning the weak convergence of the iterates of a harmonic measure under the action of an affine cellular automaton. In the present paper the author proves that for $(\mathcal{B},\cdot)$ nilpotent and $\mu\in\mathcal{M}[\mathcal{B}^{\mathbb{M}}]$ harmonic, if $\mathfrak{G}\colon\mathbb{B}^{\mathbb{M}}\to\mathbb{B}^{\mathbb{M}}$ is multiplicative, then

$$\text{wk}^*\lim_{\mathbb{J}\ni j\to\infty}\mathfrak{G}^j\mu=\eta_{\mathcal{B}},$$

where $\mathbb{J}\subset\mathbb{N}$ is a set of density 1, and $\eta_{\mathcal{B}}$ is the uniform measure in $\mathcal{B}^{\mathbb{M}}$. This implies in particular the weak convergence of the Cesàro averages $(1/N)\sum_{j=0}^{N-1}\mathfrak{G}^j\mu$ towards the uniform measure $\eta_{\mathcal{B}}$.

Another application of the decomposition $\mathfrak{G}=\mathfrak{F}\star\mathfrak{H}$ concerns the Abramov formula for the entropy of a skew product.

Edgardo Ugalde (San Luis Potosí)

MR1938472 (2003j:37020) 37B15 68Q80

Pivato, Marcus (3-TREN)

Conservation laws in cellular automata. (English. English summary)

Nonlinearity **15** (2002), *no.* 6, 1781–1793.

Given a discrete abelian group X and a finite discrete space A, A^X denotes the space of maps from X to A. A mapping $F\colon A^X\to A^X$ is a cellular automaton if F commutes with all shifts σ_x, where for each $x\in X$ the shift $\sigma_x\colon A^X\to A^X$ is defined by $(\sigma_x f)(y)=f(x+y)$ for each $f\in A^X$ and $y\in X$. Given another abelian group $(G,+)$ and a map $\varphi\colon A\to G$, one can consider the induced operator $S\varphi\colon D\subset A^X\to G$ defined by $S\varphi(f)=\sum_{x\in X}\varphi(f(x))$, where $D\subset A^X$ is the subset consisting of all f such that only finitely many terms in the sum are nonzero in G. The paper studies the question of when such a map $S\varphi$ is invariant under a cellular automaton F. Several characterizations of such conservation laws are given. The question of determining when there exist cellular automata for which a given $S\varphi$ is invariant is also raised, and detailed results are given in the case that both groups G and X are the integers. *Michael Hurley* (1-CWR)

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Pivato, Marcus (1-HST); **Yassawi, Reem** (3-TREN)

Limit measures for affine cellular automata. (English. English summary)

Ergodic Theory Dynam. Systems **22** (2002), *no.* 4, 1269–1287.

Let A be a finite abelian group and $F\colon A^K \to A^K$ be a linear cellular automaton, where K is a countable monoid like $\mathbb{N}, \mathbb{Z}$, or $\mathbb{Z}^d$. Let μ be a probability measure on A^K. In this paper a formalism is proposed so that the sequence $(F^n\mu)_{n\in\mathbb{N}}$ weakly converges in density to the Haar measure of the group, in particular, the Cesàro average.

Two notions are defined. (1) Harmonically mixing measures: A probability measure μ on A^K is called harmonically mixing if for all $\varepsilon > 0$ there is some $R > 0$ such that for any character $\psi \in \widehat{A^K}$,

$$\operatorname{rank}(\psi) > R \Rightarrow |\widehat{\mu}(\psi)| < \varepsilon.$$

In particular, it is proved that non-trivial Bernoulli measures and Markov measures whose transition matrices do not have zero entries are harmonically mixing. (2) Diffusive in density linear cellular automata: A linear cellular automaton $F\colon A^K \to A^K$ is said to be diffusive in density if for any non-trivial character $\psi \in \widehat{A^K}$ there is a subset $J \subseteq \mathbb{N}$ of Cesàro density one such that

$$\lim_{\substack{j\to\infty \\ j\in J}} \operatorname{rank}(\psi \circ F^j) = \infty.$$

The authors prove that if p is a prime number and $A = \mathbb{Z}_p$, then any non-trivial cellular automaton on $A^{\mathbb{Z}^d}$ is diffusive in density.

Using these notions the main result of the paper is stated: Theorem. If $F\colon A^K \to A^K$ is a diffusive in density linear cellular automaton and μ is a harmonically mixing probability measure on A^K, then for some $J \subseteq \mathbb{N}$ of Cesàro density one,

$$\lim_{\substack{j\to\infty \\ j\in J}} F^j\mu = \text{Haar}.$$

Alejandro Eduardo Maass (RCH-UCS-EM)

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Boland, J. (3-MMAS-MS); **Hurd, T. R.** (3-MMAS-MS);
Pivato, M. (3-TRNT); **Seco, L.** (3-TRNT)

Measures of dependence for multivariate Lévy distributions. (English. English summary)

Disordered and complex systems (*London*, 2000), 289–295, *AIP Conf. Proc.*, 553, *Amer. Inst. Phys.*, *Melville*, *NY*, 2001.

MR1820914 (2002b:37009) 37A50 37B10 60G10

Pivato, Marcus (3-TRNT)

Building a stationary stochastic process from a finite-dimensional marginal. (English. English summary)

Canad. J. Math. **53** (2001), *no.* 2, 382–413.

Let A be a finite alphabet, let $D \geq 1$ be an integer, and let $\mathfrak{U} \subseteq \mathbb{Z}^D$ be a subset. For a measure $\mu_{\mathfrak{U}}$ on $A^{\mathfrak{U}}$ we ask if $\mu_{\mathfrak{U}}$ is the projection of a shift-invariant (stationary) measure μ on $A^{\mathbb{Z}^D}$. For this to be the case $\mu_{\mathfrak{U}}$ must satisfy invariance within the set $\mathfrak{U}$—the measure $\mu_{\mathfrak{U}}$ must be locally stationary.

For $D = 1$ and $\mathfrak{U}$ being an interval the answer to the above question is given by the so-called Markov extension. Here any locally stationary measure $\mu_{\mathfrak{U}}$ is the projection of a shift-invariant measure μ.

For $D > 1$ the situation is more complex. The author provides another two necessary conditions: the entropy condition and the tiling condition. The later relates the existence of the shift-invariant measure μ to the tiling problem. If $\mu_{\mathfrak{U}}$ is the projection of μ, every element $x \in \operatorname{supp}\mu \subseteq A^{\mathbb{Z}^D}$ can be viewed as a tiling using the tiles in $\operatorname{supp}\mu_{\mathfrak{U}}$. Finally the existence of the invariant measure μ is, in general, formally undecidable. *Manfred Einsiedler* (1-PRIN)

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