RealLife: the continuum limit of *Larger than Life* cellular automata

Marcus Pivato
Trent University

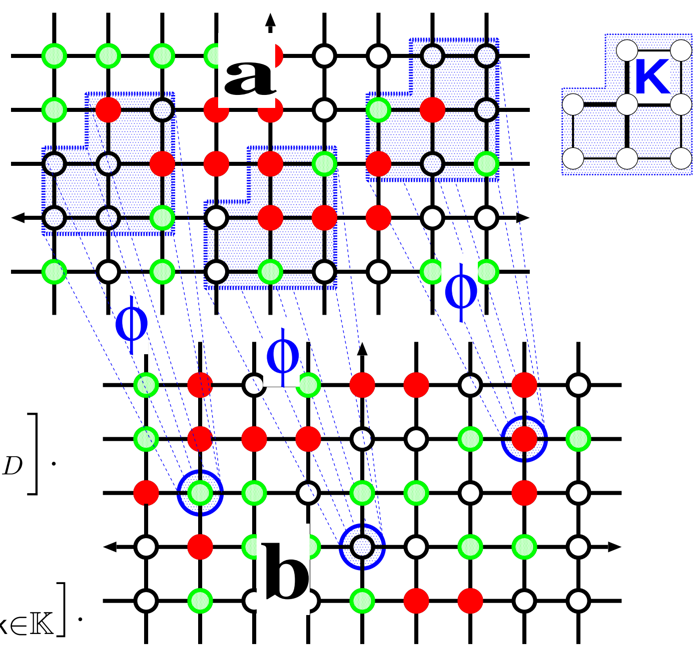
http://xaravve.trentu.ca/pivato/Research/reallife.pdf

This research supported by NSERC Canada.

Cellular Automata

Notational conventions:

- $\mathbb{Z}^D$:= D-dimensional lattice.
- $\mathcal{A} := \{0, 1\}$ (alphabet of local states).
- $\mathcal{A}^{\mathbb{Z}^D}$:= space of $\mathbb{Z}^D$-indexed *configurations* $\mathbf{a} := \left[a_{\mathsf{z}}\big|_{\mathsf{z}\in\mathbb{Z}^D}\right]$.
- Cellular automaton $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$, computed by applying a *local rule* ϕ at every point in space.



Neighbourhood:
$\mathbb{K} \subset \mathbb{Z}^D$ (finite set)

Local rule: ϕ: $\mathcal{A}^{\mathbb{K}} \longrightarrow \mathcal{A}$

Let $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$, $\quad \mathbf{a} := \left[a_{\mathsf{z}}\big|_{\mathsf{z}\in\mathbb{Z}^D}\right]$.

$\forall \mathsf{z} \in \mathbb{Z}^D$, let $b_{\mathsf{z}} := \phi\left[a_{(\mathsf{k}+\mathsf{z})}\big|_{\mathsf{k}\in\mathbb{K}}\right]$.

This defines new configuration $\mathbf{b} := \left[b_{\mathsf{z}}\big|_{\mathsf{z}\in\mathbb{Z}^D}\right]$.

The CA **induced by** ϕ is function Φ: $\mathcal{A}^{\mathbb{Z}^D} \circlearrowleft$ defined: $\Phi(\mathbf{a}) := \mathbf{b}$.

Example: John H. Conway's *Game of Life*

$D := 2, \quad \mathbb{K} := [-1..1]^2, \quad \text{and } \mathcal{A} := \{0,1\}.$

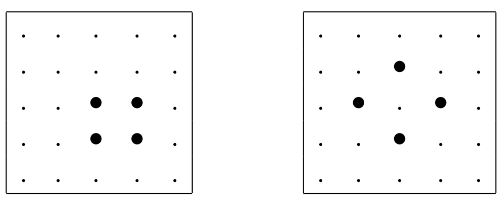
$$\Phi(\mathbf{a})_{\mathsf{z}} \;:=\; \begin{cases} 1 & \text{if } \; a_{\mathsf{z}} = 1 \text{ and } \; \mathfrak{K}(\mathbf{a})_{\mathsf{z}} \in \{3,4\} & (\text{‘survival’}); \\ 1 & \text{if } \; a_{\mathsf{z}} = 0 \text{ and } \; \mathfrak{K}(\mathbf{a})_{\mathsf{z}} = 3 & (\text{‘birth’}); \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathfrak{K}(\mathbf{a})_{\mathsf{z}} \;:=\; \sum_{\mathsf{k}\in\mathbb{K}} a_{\mathsf{z}+\mathsf{k}}$ is ‘sum of local activity’. $\mathbb{K}$:

·	·	·	·	·
·	*	*	*	·
·	*	*	*	·
·	*	*	*	·
·	·	·	·	·

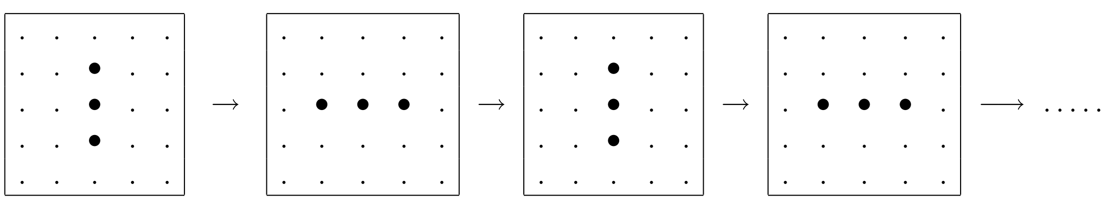
Emergence of coherent structures:

Still lifes (equilibria) e.g. ‘block’, ‘hive’:

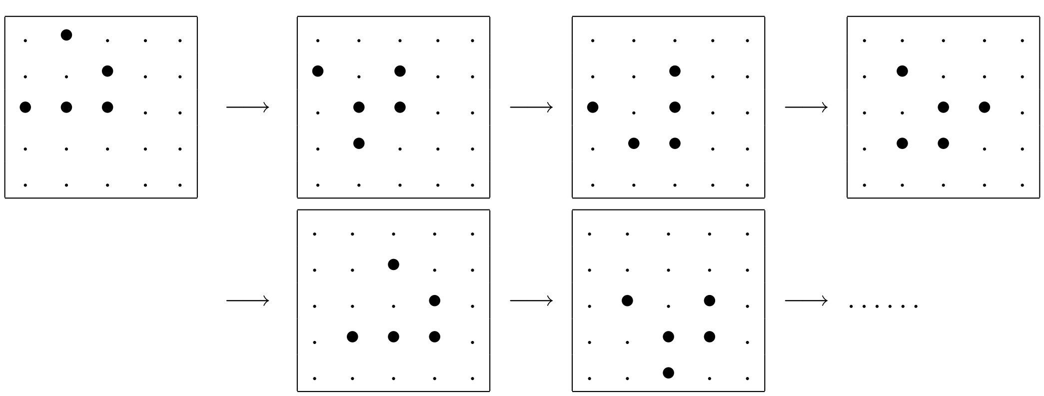


‘·’ = 0
‘•’ = 1

Oscillators (standing waves) e.g. ‘Blinker’:



Bugs (traveling waves) e.g. ‘Glider’:



- Universal computation & self-replicators (using ‘glider engineering’).

Larger than Life Cellular Automata

Larger than Life is a family of 'long-range' generalizations of Conway's ***Game of Life***, invented by Kellie Michele Evans.

$D := 2$; $\mathcal{A} := \{0, 1\}$. Now $\mathbb{K} := [-K..K] \times [-K..K]$ (for some $K > 0$).

Fix $0 \leq s_0 \leq b_0 \leq b_1 \leq s_1 \leq 1$.

$$\begin{aligned} [b_0, b_1] &= \textit{birth interval}. \\ [s_0, s_1] &= \textit{survival interval}. \end{aligned}$$

Define
$$\Phi(\mathbf{a})_{\mathsf{z}} = \begin{cases} 1 & \text{if } a_{\mathsf{z}} = 1 \text{ and } s_0 \leq \mathfrak{K}(\mathbf{a})_{\mathsf{z}} \leq s_1 \quad (\text{`survival'}); \\ 1 & \text{if } a_{\mathsf{z}} = 0 \text{ and } b_0 \leq \mathfrak{K}(\mathbf{a})_{\mathsf{z}} \leq b_1 \quad (\text{`birth'}); \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathfrak{K}(\mathbf{a})_{\mathsf{z}} := \dfrac{1}{|\mathbb{K}|} \sum_{\mathsf{k} \in \mathbb{K}} a_{\mathsf{z}+\mathsf{k}}$ is 'average local activity'.

Example: In Conway's ***Life***, $K = 1$, $s_0 = b_0 = b_1 = \frac{1}{3}$, and $s_1 = \frac{4}{9}$.

Evans' ***Larger than Life*** CA usually have

$$0.2 \leq s_0 \leq b_0 \leq 0.27 \leq 0.3 \leq b_1 \leq 0.35 \leq s_1 \leq 0.5.$$

More generally, let $\mathbb{K}$ be any large 'neighbourhood' of origin, and let $\mathfrak{K}(\mathbf{a})_{\mathsf{z}} := \sum_{\mathsf{k} \in \mathbb{K}} c_{\mathsf{k}} a_{\mathsf{z}+\mathsf{k}}$ for any positive coefficients $\{c_k\}_{\mathsf{k} \in \mathbb{K}}$ with $\sum_{\mathsf{k} \in \mathbb{K}} c_{\mathsf{k}} = 1$.

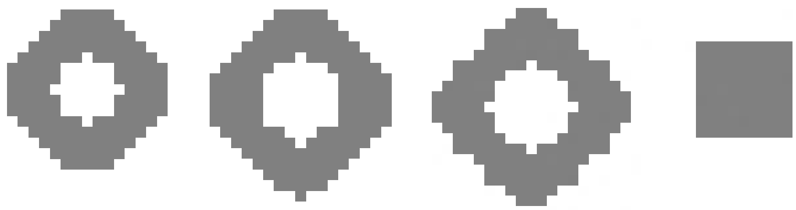
Evans' LtL CA exhibit similar phenomena to Conway's ***Life***:

- Emergence of complex, persistent structures ('life forms')
- Universal computation.

Life forms

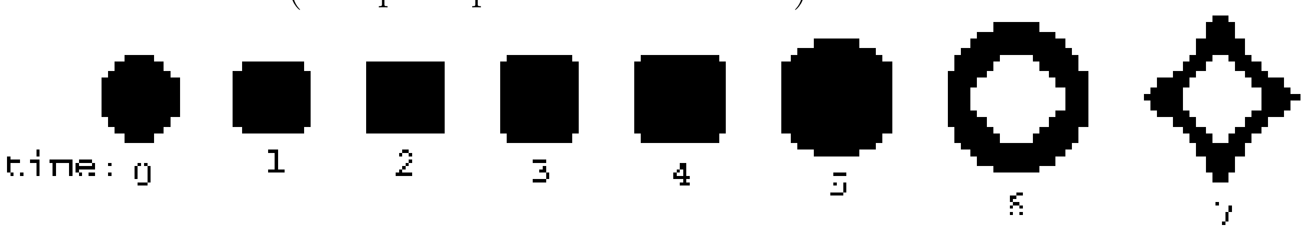
Experimentally, Evans found that LtL CA possess many *life forms:*

- *Still lifes* (compactly supported fixed points).



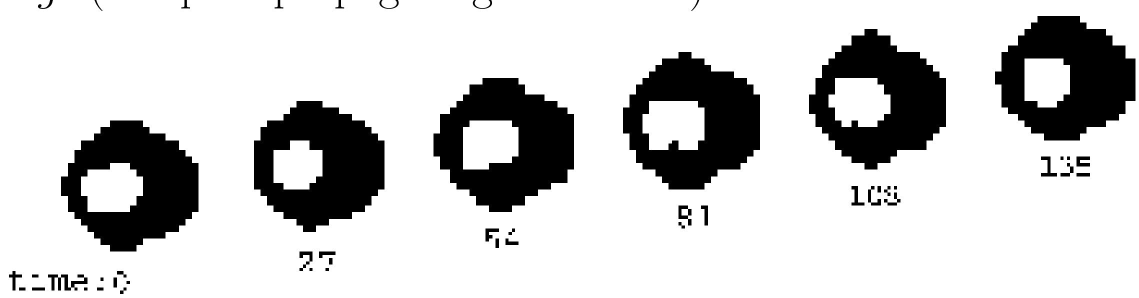
[Graphics by M.P.]

- *Oscillators* (compact periodic solutions).



[Graphics by K.M. Evans]

- *Bugs* (compact propagating structures).



[Graphics by K.M. Evans]

Evolution properties

If the parameters (s_0, b_0, b_1, s_1) are slowly varied, then the resulting life forms 'evolve' as a function of (s_0, b_0, b_1, s_1).

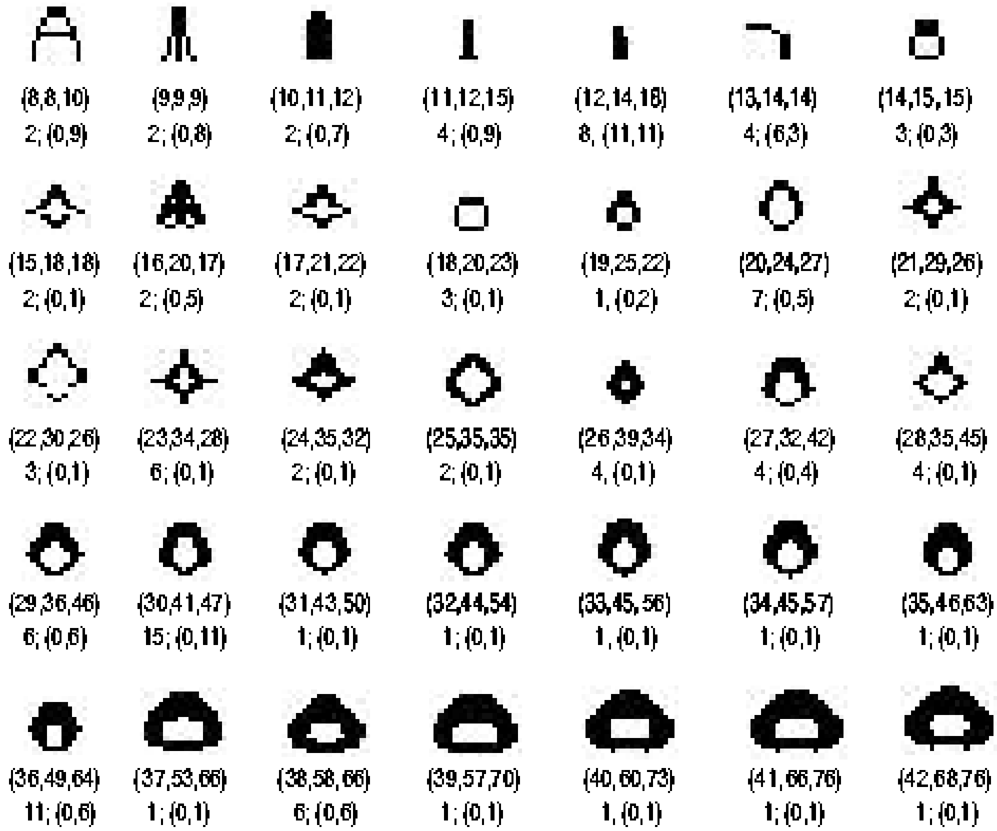


Fig. 18: Range 5 bug collection. Below each bug are the parameters $(\beta_1 = \delta_1, \beta_2, \delta_2)$ for the LtL rule that supports the bug. Also depicted are τ and $\vec{d} = (d_1, d_2)$, the bug's period and displacement vector, respectively.

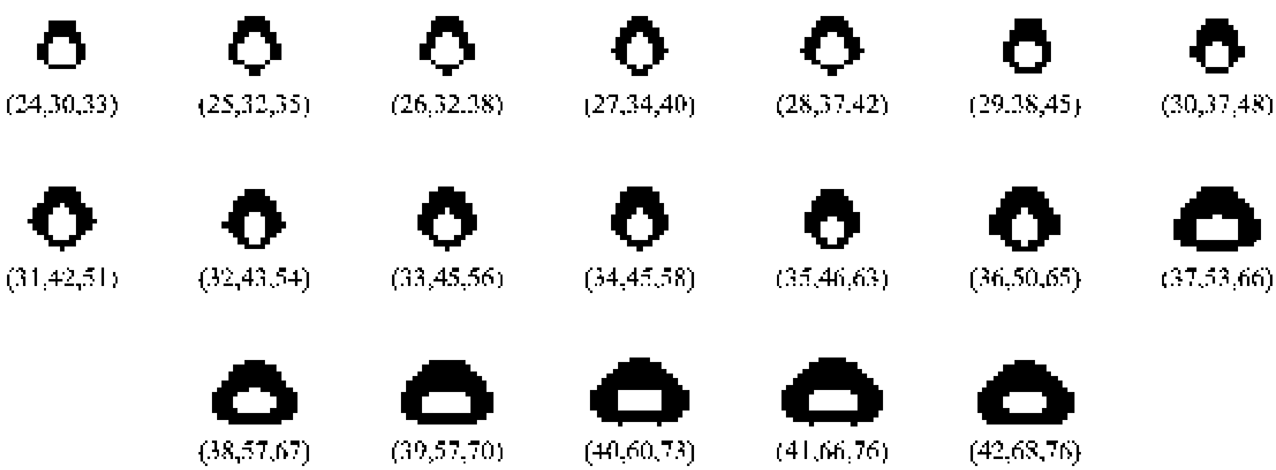


Fig. 10. Collection of range 5 translation invariant bugs. Below each is its supporting rule, $(\beta_1 = \delta_1, \beta_2, \delta_2)$.

[Graphics by K.M. Evans]

Scaling properties

Observation: The life forms in longer-range LtL CA appear to be rescaled, 'high resolution' versions of those in shorter range LtL CA.

K=25 K=50 K=75 K=100

[Graphics by M.P.]

$s_0 = b_0 = \frac{706}{2601} < b_1 = \frac{958}{2601} < s_1 = \frac{1216}{2601}$ and $\mathbb{K} = [-K..K]^2$, where $K = 25$, 50, 75, or 100.

Conjecture: (Evans) *The life forms for LtL CA converge to continuum limits, which are life forms for a "Euclidean automaton"; a translationally-equivariant transformation of* $\mathcal{A}^{\mathbb{R}^2}$.

Questions:

- What is the 'right' definition of 'Euclidean automaton'?
- Which Euclidean automata (EA) are the continuum limits of LtL CA?
- Do life forms of LtL CA 'evolve' toward life forms of these limit EA?
- How do the dynamics of the limit EA vary as the parameters (s_0, b_0, b_1, s_1) are varied? As the neighbourhood $\mathbb{K}$ is varied?

  How do life forms evolve as parameters/nhood change?

Euclidean Automata and *RealLife*

Let $D \geq 1$. Let λ be the D-dimensional Lebesgue measure on $\mathbb{R}^D$, and let $\mathbf{L}^\infty := \mathbf{L}^\infty(\mathbb{R}^D, \lambda)$.

Let $\mathcal{A} := \{0, 1\}$. Let $\mathcal{A}^{\mathbb{R}^D} \subset \mathbf{L}^\infty$ be the set of all **configurations**: Borel-measurable functions $\mathbf{a} : \mathbb{R}^D \longrightarrow \mathcal{A}$.

(Any $\mathbf{a} \in \mathcal{A}^{\mathbb{R}^D}$ is equivalent to a measurable subset of $\mathbb{R}^D$).

If $\vec{v} \in \mathbb{R}^D$, then define the shift map $\sigma^{\vec{v}} : \mathcal{A}^{\mathbb{R}^D} \longrightarrow \mathcal{A}^{\mathbb{R}^D}$ by $\sigma^{\vec{v}}(\mathbf{a}) = \mathbf{a}'$, where $\mathbf{a}'(x) = \mathbf{a}(x + v)$ for all $x \in \mathbb{R}^D$.

A **Euclidean automaton** (EA) is a function $\Phi : \mathcal{A}^{\mathbb{R}^D} \longrightarrow \mathcal{A}^{\mathbb{R}^D}$ which commutes with all shifts, and which is *determined by local information*, meaning that there is some compact neighbourhood $\mathbb{K} \subset \mathbb{R}^D$ around zero so that, if $\mathbf{a}, \mathbf{a}' \in \mathcal{A}^{\mathbb{R}^D}$, and $\mathbf{a}|_{\mathbb{K}} = \mathbf{a}'|_{\mathbb{K}}$, then $\Phi(\mathbf{a})(0) = \Phi(\mathbf{a}')(0)$.

Let $\mathcal{K} := \left\{\mathfrak{k} \in \mathbf{L}^\infty(\mathbb{R}^D; \mathbb{R}^+) \; ; \; \text{compact support}, \; \int_{\mathbb{R}^D} \mathfrak{k} = 1\right\}$ ('*kernels*').

Fix $\mathfrak{k} \in \mathcal{K}$. If $\mathbf{a} \in \mathcal{A}^{\mathbb{R}^D}$, then $\underbrace{\mathfrak{k} * \mathbf{a}(x) := \int_{\mathbb{R}^D} \mathfrak{k}(y) \cdot \mathbf{a}(x - y) \, d\lambda[y]}_{\text{average local activity}}$.

Example: Let $\mathbb{K} :=$ compact neighbourhood of zero (eg. a ball), and $\mathfrak{k} := \lambda[\mathbb{K}]^{-1} \mathbb{1}_{\mathbb{K}}$, then $\mathfrak{k} * \mathbf{a}(x) \;=\; \lambda[\mathbb{K}]^{-1} \int_{\mathbb{K}} \mathbf{a}(x - k) \, d\lambda[k]$.

RealLife: Fix $0 \leq s_0 \leq b_0 < b_1 \leq s_1 \leq 1$. Define $\Phi : \mathcal{A}^{\mathbb{R}^D} \hookleftarrow$ by

$$\forall \, \mathbf{a} \in \mathcal{A}^{\mathbb{R}^D}, \; \Phi(\mathbf{a})(x) \;=\; \begin{cases} 1 & \text{if } \; \mathbf{a}(x) = 1 \text{ and } \quad s_0 \leq \mathfrak{k} * \mathbf{a}(x) \leq s_1 \;\; \text{('survival')} \\ 1 & \text{if } \; \mathbf{a}(x) = 0 \text{ and } \quad b_0 \leq \mathfrak{k} * \mathbf{a}(x) \leq b_1 \;\; \text{('birth')} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Theta := \{(s_0, b_0, b_1, s_1) \; ; \; 0 \leq s_0 \leq b_0 < b_1 \leq s_1 \leq 1\}$ ('*thresholds*').

Then Φ depends on a choice of $\mathfrak{k} \in \mathcal{K}$ and $(s_0, b_0, b_1, s_1) \in \Theta$.

Φ is called *RealLife* because it is the continuum limit of a sequences of LtL CA with 'birth interval' $[b_0, b_1]$ and 'survival interval' $[s_0, s_1]$.

Continuity Properties

Let $\mathbf{L}^1 := \mathbf{L}^1(\mathbb{R}^D, \lambda)$. Let ${}^1\!\mathcal{A}^{\mathbb{R}^D} := \mathcal{A}^{\mathbb{R}^D} \cap \mathbf{L}^1 = \left\{ \begin{array}{r} \text{configurations whose supports} \\ \text{have finite measure} \end{array} \right\}$.

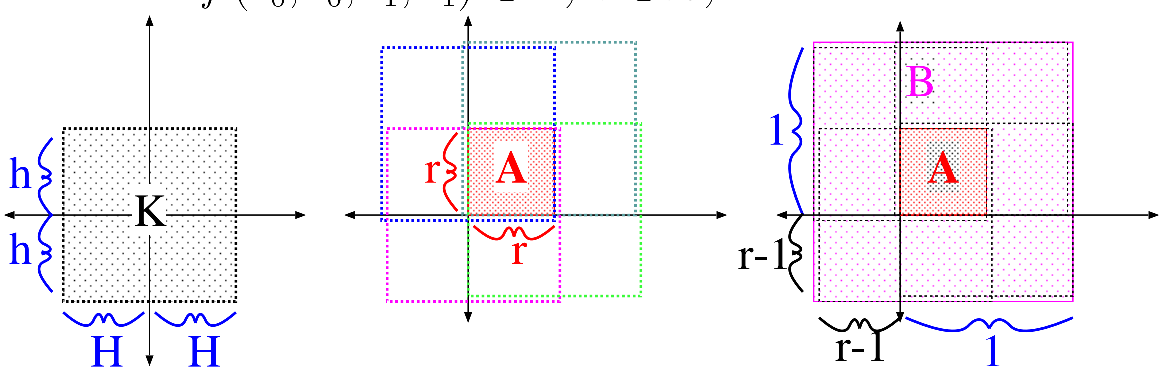
Then $\Phi({}^1\!\mathcal{A}^{\mathbb{R}^D}) \subseteq {}^1\!\mathcal{A}^{\mathbb{R}^D}$. Extend Φ to function $\Phi : \mathbf{L}^1 \circlearrowleft$ as follows: Let $\mathfrak{b} := \mathbb{1}_{[b_0,b_1]}$ and $\mathfrak{s} := \mathbb{1}_{[s_0,s_1]}$. For any $\mathbf{a} \in \mathbf{L}^1$, define

$$\Phi(\mathbf{a})(x) \quad := \quad \mathbf{a}(x) \cdot \mathfrak{s}\left(\mathfrak{k} * \mathbf{a}(x)\right) + \left(1 - \mathbf{a}(x)\right) \cdot \mathfrak{b}\left(\mathfrak{k} * \mathbf{a}(x)\right), \quad \forall\, x \in \mathbb{R}^D.$$

If $\mathbf{a} \in {}^1\!\mathcal{A}^{\mathbb{R}^D}$, let $\alpha := \mathfrak{k} * \mathbf{a}$ and $M(\mathbf{a}) := \lambda\left[\alpha^{-1}\{s_0, b_0, s_1, b_1\}\right]$. Define

$${}^0\!\mathcal{A}^{\mathbb{R}^D} \quad := \quad \left\{\mathbf{a} \in {}^1\!\mathcal{A}^{\mathbb{R}^D} \,;\, M(\mathbf{a}) = 0\right\}.$$

Theorem 1: *If $(s_0, b_0, b_1, s_1) \in \Theta$, $\mathfrak{k} \in \mathcal{K}$, then Φ is $\mathbf{L}^1$-continuous on ${}^0\!\mathcal{A}^{\mathbb{R}^D}$.*



${}^0\!\mathcal{A}^{\mathbb{R}^D} \subsetneq {}^1\!\mathcal{A}^{\mathbb{R}^D}$: Let $D = 2$; let $\mathbb{K} := \left[\frac{-1}{2}, \frac{1}{2}\right]^2 \subset \mathbb{R}^2$ and $\mathfrak{k} := \mathbb{1}_{\mathbb{K}}$. Let $r := \sqrt{s_0}$ and $\mathbf{A} := [0, r]^2$, so $\lambda[\mathbf{A}] = s_0$. If $\mathbf{a} := \mathbb{1}_{\mathbf{A}}$, and $\alpha := \mathfrak{k} * \mathbf{a}$, then $\alpha(x) = s_0$ for $\forall\, x \in [r-1, 1]^2$. Thus, $M(\mathbf{a}) = \lambda\left([r-1, 1]^2\right) = (2-r)^2 > 0$, so $\mathbf{a} \notin {}^0\!\mathcal{A}^{\mathbb{R}^D}$.

Φ is not $\mathbf{L}^1$-continuous on ${}^1\!\mathcal{A}^{\mathbb{R}^D}$: If $b_0 > s_0$, then $\Phi(\mathbf{a}) = \mathbf{a}$. Let $\epsilon > 0$; let $r' := r - \epsilon/2$; let $\mathbf{A}' := [0, r']^2$; let $\mathbf{a}' := \mathbb{1}_{\mathbf{A}'}$. Then $\|\mathbf{a} - \mathbf{a}'\|_1 < \epsilon$. But $\lambda[\mathbf{A}'] < s_0$, so $\Phi(\mathbf{a}') = \mathbf{o}$ (zero config.), so $\|\Phi(\mathbf{a}) - \Phi(\mathbf{a}')\|_1 = \lambda[\mathbf{A}] = s_0$.

Theorem 2: *Fix $(s_0, b_0, b_1, s_1) \in \Theta$. If $\mathfrak{k} \in \mathcal{K}$ is almost continuous*$^{(*)}$ *then ${}^0\!\mathcal{A}^{\mathbb{R}^D}$ is a σ-invariant, dense $G\delta$ subset of ${}^1\!\mathcal{A}^{\mathbb{R}^D}$.*

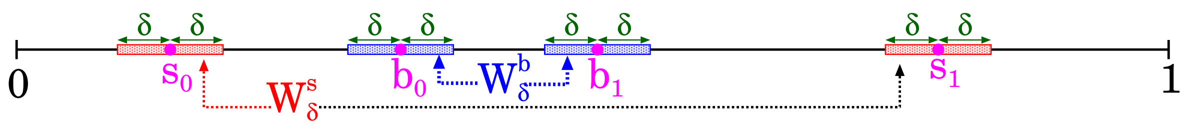
$(*)$ i.e. $\mathfrak{k}$ is continuous on open $\mathbf{Y} \subseteq \mathbb{R}^D$ and $\mathbf{Y}^{\complement}$ is 'thin': $\lim_{\epsilon \to 0} \lambda\left[\mathbb{B}(\mathbf{Y}^{\complement}, \epsilon)\right] = 0$.

To prove Theorem 1, we use

Lemma 1: *Let* $\mathbf{a}, \mathbf{a}' \in \mathbf{L}^1$. *Let* $\alpha := \mathfrak{k} * \mathbf{a}$ *and* $\alpha' := \mathfrak{k} * \mathbf{a}'$. *Then:*

(a) $\|\Phi(\mathbf{a}) - \Phi(\mathbf{a}')\|_1 \ \leq \ 2\|\mathbf{a} - \mathbf{a}'\|_1 + \|\mathfrak{s} \circ \alpha - \mathfrak{s} \circ \alpha'\|_1 + \|\mathfrak{b} \circ \alpha - \mathfrak{b} \circ \alpha'\|_1$.

(b) *For any* $\delta \geq 0$, *define*



$$\mathbf{W}^s_\delta \ := \ (s_0 - \delta, s_0 + \delta) \cup (s_1 - \delta, s_1 + \delta) \quad \text{and} \quad M^s_{\mathbf{a}}(\delta) \ := \ \lambda\left[\alpha^{-1}(\mathbf{W}^S_\delta)\right];$$
$$\mathbf{W}^b_\delta \ := \ (b_0 - \delta, b_0 + \delta) \cup (b_1 - \delta, b_1 + \delta) \quad \text{and} \quad M^b_{\mathbf{a}}(\delta) \ := \ \lambda\left[\alpha^{-1}(\mathbf{W}^b_\delta)\right].$$

Then $\|\mathfrak{s} \circ \alpha' - \mathfrak{s} \circ \alpha\|_1 \ \leq \ M^s_{\mathbf{a}}(\|\alpha' - \alpha\|_\infty)$

and $\|\mathfrak{b} \circ \alpha' - \mathfrak{b} \circ \alpha\|_1 \ \leq \ M^b_{\mathbf{a}}(\|\alpha' - \alpha\|_\infty)$.

(c) *Let* $K := \|\mathfrak{k}\|_\infty$. *Then* $\|\alpha' - \alpha\|_\infty \leq K \cdot \|\mathbf{a} - \mathbf{a}'\|_1$. *(Young's Inequality)*

(d) *If* $M(\mathbf{a}) = 0$, *then* $\lim_{\delta \to 0} M^s_{\mathbf{a}}(\delta) \ = \ 0 \ = \ \lim_{\delta \to 0} M^b_{\mathbf{a}}(\delta)$. *(Radon property)*

Proof of Theorem 1: Let $\mathbf{a} \in {}^0\!\mathcal{A}^{\mathbb{R}^D}$ and $\mathbf{a}' \in \mathbf{L}^1$, with $\|\mathbf{a} - \mathbf{a}'\|_1 < \delta$. If $\alpha := \mathfrak{k} * \mathbf{a}$ and $\alpha' := \mathfrak{k} * \mathbf{a}'$, then

$$\|\Phi(\mathbf{a}) - \Phi(\mathbf{a}')\|_1 \ \underset{\text{(a)}}{\leq} \ 2\|\mathbf{a} - \mathbf{a}'\|_1 \ + \ \|\mathfrak{s} \circ \alpha - \mathfrak{s} \circ \alpha'\|_1 \ + \ \|\mathfrak{b} \circ \alpha - \mathfrak{b} \circ \alpha'\|_1$$
$$\underset{\text{(bc)}}{\leq} \ 2\delta \ + \ M^s_{\mathbf{a}}(K\delta) \ + \ M^b_{\mathbf{a}}(K\delta).$$

(a) is by Lemma 1(a). (bc) is by Lemma 1(b,c). But $M(\mathbf{a}) = 0$, so $M^s_{\mathbf{a}}(K\delta) \ + \ M^b_{\mathbf{a}}(K\delta) \xrightarrow[\delta \to 0]{} 0$, by Lemma 1(d). □

Proof sketch of Theorem 2: If $\mathbf{a} \not\in {}^0\!\mathcal{A}^{\mathbb{R}^D}$, then $\lambda[\alpha^{-1}\{r\}] > 0$ for some $r \in \{s_0, b_0, b_1, s_1\}$. This is a 'fragile' property, which can be disrupted by small perturbation. Example: approximate $\mathbf{a}$ with disjoint union of boxes; then slightly shrink box sizes. □

Life forms and Evolution: Kernel Convergence

Life forms: $\mathbf{a} \in \mathcal{A}^{\mathbb{R}^D}$ is a **still life** for Φ if $\Phi(\mathbf{a}) = \mathbf{a}$.

If $P \in \mathbb{N}$, then $\mathbf{a}$ is a P**-oscillator** if $\Phi^P(\mathbf{a}) = \mathbf{a}$.

If $P \in \mathbb{N}$ and $\vec{v} \in \mathbb{R}^D$, then $\mathbf{a}$ is a P**-periodic bug** with **velocity** $\vec{v}$ if $\Phi^P(\mathbf{a}) \;=\; \sigma^{P\vec{v}}(\mathbf{a})$.

Evolution: If threshold parameters (s_0, b_0, b_1, s_1) change continuously, or kernel $\mathfrak{k}$ changes continuously, then the corresponding *RealLife* EA Φ should also change continuously, and its 'life forms' should continuously 'evolve' as a function of (s_0, b_0, b_1, s_1) and $\mathfrak{k}$.

Formally: let $\{\Phi_n : \mathbf{L}^1 \hookleftarrow\}_{n=1}^{\infty}$ be sequence of RealLife EA. Let $\mathfrak{A} \subset \mathbf{L}^1$ be σ-invariant subset. $\{\Phi_n\}_{n=1}^{\infty}$ evolves to Φ on $\mathfrak{A}$ if, for any $\{\mathbf{a}_n\}_{n=1}^{\infty} \subset \mathbf{L}^1$ with $\mathbf{L}^1\text{–}\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a} \in \mathfrak{A}$, the following holds:

(a) If $\Phi_n(\mathbf{a}_n) = \mathbf{a}_n$ for all $n \in \mathbb{N}$, then $\Phi(\mathbf{a}) \;=\; \mathbf{a}$. (still lifes $\xrightarrow[n\to\infty]{}$ still lifes)

(b) Let $P \in \mathbb{N}$, and suppose $\Phi^p(\mathbf{a}) \in \mathfrak{A}$ for all $p \in [0...P)$.

- **[i]** If $\Phi_n^P(\mathbf{a}_n) = \mathbf{a}_n$ for all $n \in \mathbb{N}$, then $\Phi^P(\mathbf{a}) \;=\; \mathbf{a}$. (oscillators$\rightarrow$oscillators)
- **[ii]** If $\vec{v} \in \mathbb{R}^D$ and $\Phi_n^P(\mathbf{a}_n) = \sigma^{P\vec{v}}(\mathbf{a}_n)$ for all $n \in \mathbb{N}$, then $\Phi^P(\mathbf{a}) = \sigma^{P\vec{v}}(\mathbf{a})$. (bugs $\xrightarrow[n\to\infty]{}$ bugs)

Theorem 3: *Fix $(s_0, b_0, b_1, s_1) \in \Theta$. Let $\{\mathfrak{k}_n\}_{n=1}^{\infty} \subset \mathcal{K}$. For all $n \in \mathbb{N}$, let $\Phi_n : \mathbf{L}^1 \longrightarrow \mathbf{L}^1$ be the RealLife EA defined by (s_0, b_0, b_1, s_1) and $\mathfrak{k}_n$. Suppose $\mathbf{L}^1\text{–}\lim_{n\to\infty} \mathfrak{k}_n = \mathfrak{k}$. Let Φ be the RealLife EA defined by (s_0, b_0, b_1, s_1) and $\mathfrak{k}$. Then*

(a) $\mathbf{L}^1\text{–}\lim_{n\to\infty} \Phi_n(\mathbf{a}) \;=\; \Phi(\mathbf{a})$, *for all* $\mathbf{a} \in {}^0\!\mathcal{A}^{\mathbb{R}^D}$.

(b) *If* $\sup_{n\in\mathbb{N}} \|\mathfrak{k}_n\|_\infty < \infty$, *then* $\{\Phi_n\}_{n=1}^{\infty}$ *evolves to* Φ *on* ${}^0\!\mathcal{A}^{\mathbb{R}^D}$.

Proof of Thm 3(a): Fix $\mathbf{a} \in {}^0\!\mathcal{A}^{\mathbb{R}^D}$.

$$\begin{array}{rrcl} \text{Then} & \Phi(\mathbf{a}) & = & \mathbf{a}\cdot(\mathfrak{s}\circ\alpha) \;+\; (1-\mathbf{a})\cdot(\mathfrak{b}\circ\alpha), \qquad \text{where } \alpha := \mathfrak{k} * \mathbf{a}. \\ \text{and} & \Phi_n(\mathbf{a}) & = & \mathbf{a}\cdot(\mathfrak{s}\circ\alpha_n) \;+\; (1-\mathbf{a})\cdot(\mathfrak{b}\circ\alpha_n), \quad \text{where } \alpha_n := \mathfrak{k}_n * \mathbf{a}. \\ \text{Thus,} & \Phi_n(\mathbf{a}) - \Phi(\mathbf{a}) & & \\ & & = & \mathbf{a}\cdot(\mathfrak{s}\circ\alpha_n - \mathfrak{s}\circ\alpha) \;+\; (1-\mathbf{a})\cdot(\mathfrak{b}\circ\alpha_n - \mathfrak{b}\circ\alpha). \\ \text{Thus,} & \|\Phi_n(\mathbf{a}) - \Phi(\mathbf{a})\|_1 & & \\ & & \leq & \|\mathbf{a}\|_\infty\cdot\|\mathfrak{s}\circ\alpha_n - \mathfrak{s}\circ\alpha\|_1 \;+\; \|1-\mathbf{a}\|_\infty\cdot\|\mathfrak{b}\circ\alpha_n - \mathfrak{b}\circ\alpha\|_1. \\ & & \leq & \|\mathfrak{s}\circ\alpha_n - \mathfrak{s}\circ\alpha\|_1 \;+\; \|\mathfrak{b}\circ\alpha_n - \mathfrak{b}\circ\alpha\|_1 \\ & & \underset{(\flat)}{\leq} & M^s_{\mathbf{a}}(\|\alpha_n - \alpha\|_\infty) \;+\; M^b_{\mathbf{a}}(\|\alpha_n - \alpha\|_\infty), \end{array}$$

where $(\flat)$ is by Lemma 1(b). But

$$\|\alpha_n - \alpha\|_\infty \;=\; \|(\mathfrak{k}_n - \mathfrak{k}) * \mathbf{a}\|_\infty \;\underset{(*)}{\leq}\; \|\mathfrak{k}_n - \mathfrak{k}\|_1 \cdot \|\mathbf{a}\|_\infty \;=\; \|\mathfrak{k}_n - \mathfrak{k}\|_1 \xrightarrow[n\to\infty]{\text{hypoth}} 0.$$

$(*)$ is Young's inequality. Thus, $\|\Phi_n(\mathbf{a}) - \Phi(\mathbf{a})\|_1 \xrightarrow[n\to\infty]{} 0$, by Lemma 1(d).
□

To establish 'evolution', we use equicontinuity. If $\{\Phi_n : \mathbf{L}^1 \hookleftarrow\!\!\!\supset\}_{n=1}^\infty$ are EA, and $\mathbf{a} \in \mathbf{L}^1$, then $\{\Phi_n\}_{n=1}^\infty$ is **equicontinuous** at $\mathbf{a}$ if, for any $\gamma > 0$, there is some $\delta > 0$ so that, for any $\mathbf{a}' \in \mathbf{L}^1$,

$$\Big(\|\mathbf{a}' - \mathbf{a}\|_1 < \delta\Big) \Longrightarrow \Big(\text{for all } n \in \mathbb{N},\; \|\Phi_n(\mathbf{a}') - \Phi_n(\mathbf{a})\|_1 < \gamma\Big).$$

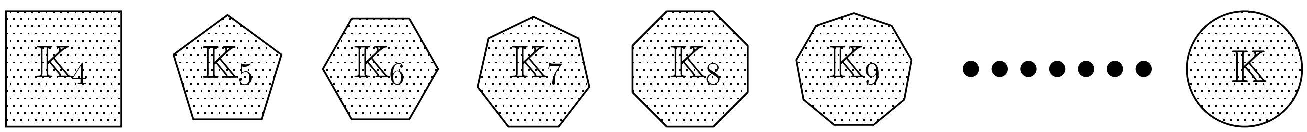
Proposition A: *Let $\mathfrak{A} \subset \mathbf{L}^1$ be a σ-invariant subset. If $\{\Phi_n\}_{n=1}^\infty$ is equicontinuous and converges to Φ at all points in $\mathfrak{A}$, then $\{\Phi_n\}_{n=1}^\infty$ evolves to Φ on $\mathfrak{A}$.*

Proof sketch of Theorem 3(b): Must show that $\{\Phi_n\}_{n=1}^\infty$ is equicontinuous at every $\mathbf{a} \in {}^0\!\mathcal{A}^{\mathbb{R}^D}$. As in **Theorem 1** (continuity), we use **Lemma 1** to find a suitable δ for each γ. □

Life forms and Evolution: Kernel Convergence

Corollary: *Fix* $(s_0, b_0, b_1, s_1) \in \Theta$. *Let* $\mathbb{K} \subset \mathbb{R}^D$ *and* $\mathfrak{k} := \lambda[\mathbb{K}]^{-1}\mathbb{1}_{\mathbb{K}}$. *Let* $\{\mathbb{K}_n \subset \mathbb{R}^D\}_{n=1}^{\infty}$, *and for any* $n \in \mathbb{N}$, *let* $\Phi_n : \mathbf{L}^1 \longrightarrow \mathbf{L}^1$ *be the* RealLife *EA defined by* (s_0, b_0, b_1, s_1) *and* $\mathfrak{k}_n := \lambda[\mathbb{K}_n]^{-1}\mathbb{1}_{\mathbb{K}_n}$.

Suppose $\lim_{n\to\infty} \lambda[\mathbb{K}_n \triangle \mathbb{K}] = 0$. *Then* $\mathbf{L}^1\text{–}\lim_{n\to\infty} \Phi_n(\mathbf{a}) \;=\; \Phi(\mathbf{a})$, *for all* $\mathbf{a} \in {}^0\mathcal{A}^{\mathbb{R}^D}$, *and* $\{\Phi_n\}_{n=1}^{\infty}$ *evolves to* Φ *on* ${}^0\mathcal{A}^{\mathbb{R}^D}$.



Life forms and Evolution: Threshold Convergence

Theorem 4: *Fix* $\mathfrak{k} \in \mathcal{K}$. *Let* $\{(s_0^n, b_0^n, b_1^n, s_1^n)\}_{n=1}^{\infty} \subset \Theta$. *For each* $n \in \mathbb{N}$, *let* $\Phi_n : \mathbf{L}^1 \longrightarrow \mathbf{L}^1$ *be the* RealLife *EA defined by* $(s_0^n, b_0^n, b_1^n, s_1^n)$ *and* $\mathfrak{k}$. *Suppose* $\lim_{n\to\infty} (s_0^n, b_0^n, b_1^n, s_1^n) = (s_0, b_0, b_1, s_1)$. *Let* Φ *be the* RealLife *EA defined by* (s_0, b_0, b_1, s_1) *and* $\mathfrak{k}$. *Then*

(a) $\mathbf{L}^1\text{–}\lim_{n\to\infty} \Phi_n(\mathbf{a}) \;=\; \Phi(\mathbf{a})$, *for all* $\mathbf{a} \in {}^0\mathcal{A}^{\mathbb{R}^D}$.

(b) $\{\Phi_n\}_{n=1}^{\infty}$ *evolves to* Φ *on* ${}^0\mathcal{A}^{\mathbb{R}^D}$.

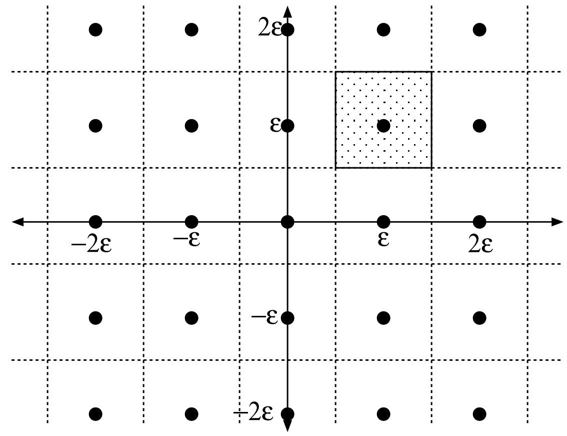
Proof: Similar strategy to proof of Theorem 3. □

Interpretation: The dynamics of Φ change *continuously* as functions of the kernel $\mathfrak{k}$ and threshold parameters (s_0, b_0, b_1, s_1).

Question: What sort of bifurcation phenomena occur as we vary (s_0, b_0, b_1, s_1) or $\mathfrak{k}$?

From *Larger than Life* to *RealLife*

For any $\epsilon > 0$, let $\mathcal{B}_\epsilon$ be the sigma-algebra generated by ϵ-length cubes with centres in lattice $\epsilon\mathbb{Z}^D \subset \mathbb{R}^D$. ('pixels')



Let $\mathbf{L}^1_\epsilon := \mathbf{L}^1(\mathbb{R}^D, \mathcal{B}_\epsilon, \lambda)$. Then $\mathbf{L}^1_\epsilon \cong \ell^1(\mathbb{Z}^D)$.

Let ${}^{\epsilon}\!\mathcal{A}^{\mathbb{R}^D} := {}^{1}\!\mathcal{A}^{\mathbb{R}^D} \cap \mathbf{L}^1_\epsilon$. Then ${}^{\epsilon}\!\mathcal{A}^{\mathbb{R}^D} \cong {}^{1}\!\mathcal{A}^{\mathbb{Z}^D}$.

We can define a 'discretized' ***RealLife*** EA $\Phi_\epsilon : \mathbf{L}^1_\epsilon \circlearrowleft$ that is dynamically isomorphic to a ***Larger than Life*** CA with radius $\sim 1/\epsilon$.

To show that ***RealLife*** is 'continuum limit' of LtL CA, we extend Φ_ϵ to a function $\Phi_\epsilon : \mathbf{L}^1 \longrightarrow \mathbf{L}^1_\epsilon$, and show that Φ_ϵ converges to Φ as $\epsilon \longrightarrow 0$.

Theorem 5: *Fix* $(s_0, b_0, s_1, b_1) \in \Theta$ *and* $\mathfrak{k} \in \mathcal{K}$. *Let* Φ *be the resulting* ***RealLife*** *EA.*

(a) *If* $\mathbf{a} \in {}^{0}\!\mathcal{A}^{\mathbb{R}^D}$, *then* $\mathbf{L}^1\text{-}\lim_{\epsilon\to 0} \Phi_\epsilon(\mathbf{a}) = \Phi(\mathbf{a})$.

(b) *If* $\lim_{n\to\infty} \epsilon_n = 0$, *then* $\{\Phi_{\epsilon_n}\}_{n=1}^{\infty}$ *evolves to* Φ *on* ${}^{0}\!\mathcal{A}^{\mathbb{R}^D}$.

Remarks: (a) We cannot simulate ***RealLife*** on computer; we can only simulate large-radius LtL CA. Theorem 5(a) says this will yield 'good approximation' of ***RealLife***.

(b) Evans empirically found that LtL CA of increasingly large radii have life forms that are virtually identical after rescaling. Theorem 5(b) suggests (but doesn't prove) that ***RealLife*** EA have life forms which are morphologically similar to those seen by Evans in LtL CA.

Discretization of *RealLife*

Let $\mathcal{M}(\epsilon\mathbb{Z}^D) = \{\text{measures on } \epsilon\mathbb{Z}^D\}$. If $\mathfrak{k} \in \mathcal{K}$, define $\bar{\mathfrak{k}}_\epsilon \in \mathcal{M}(\epsilon\mathbb{Z}^D)$ by

$$\bar{\mathfrak{k}}_\epsilon \;:=\; \sum_{\mathsf{z}\in\mathbb{Z}^D} k_{\mathsf{z}}\,\delta_{\epsilon\mathsf{z}}, \quad \text{where, for all } \mathsf{z}\in\mathbb{Z}^D, \quad \delta_{\epsilon\mathsf{z}} \;:=\; \text{point mass at } \epsilon\mathsf{z},$$

and $k_{\mathsf{z}} \;:=\; \displaystyle\int_{\mathbf{C}(\mathsf{z},\epsilon)} \mathfrak{k}(c)\;d\lambda[c]$, where $\mathbf{C}(\mathsf{z},\epsilon) := \left(\begin{matrix}\text{unique } \epsilon\text{-cube in } \mathcal{B}_\epsilon \\ \text{which contains } \epsilon\mathsf{z}\end{matrix}\right)$.

Define $\overline{\Phi}_\epsilon : \mathbf{L}^1_\epsilon \longrightarrow \mathbf{L}^1_\epsilon$ by

$$\overline{\Phi}_\epsilon(\mathbf{a}) \quad := \quad \mathbf{a}\cdot\mathfrak{s}\circ(\bar{\mathfrak{k}}_\epsilon * \mathbf{a}) \;+\; (1-\mathbf{a})\cdot\mathfrak{b}\circ(\bar{\mathfrak{k}}_\epsilon * \mathbf{a}), \qquad \forall \mathbf{a}\in\mathbf{L}^1_\epsilon.$$

Claim: $\overline{\Phi}_\epsilon(\mathbf{L}^1_\epsilon) \subseteq \mathbf{L}^1_\epsilon$, and $\overline{\Phi}_\epsilon({}^\epsilon\mathcal{A}^{\mathbb{R}^D}) \subseteq {}^\epsilon\mathcal{A}^{\mathbb{R}^D}$.

Proof: If $\mathbf{a} \in \mathbf{L}^1_\epsilon$ (ie. $\mathbf{a}$ is $\mathcal{B}_\epsilon$-measurable), then $\alpha := \bar{\mathfrak{k}}_\epsilon * \mathbf{a}$ is also $\mathcal{B}_\epsilon$-measurable. Thus, $\mathfrak{b}\circ\alpha$ and $\mathfrak{s}\circ\alpha$ are also $\mathcal{B}_\epsilon$-measurable. Thus, $\overline{\Phi}_\epsilon(\mathbf{a})$ is $\mathcal{B}_\epsilon$-measurable. $\Box$

For all $\mathbf{a}\in\mathbf{L}^1$, let $\overline{\mathbf{a}}_\epsilon \in \mathbf{L}^1_\epsilon$ be *conditional expectation* of $\mathbf{a}$ given $\mathcal{B}_\epsilon$:

$$\text{For any } x\in\mathbb{R}^D, \qquad \overline{\mathbf{a}}_\epsilon(x) \quad := \quad \frac{1}{\epsilon^D}\int_{\mathbf{C}(x,\epsilon)} \mathbf{a}(c)\;d\lambda[c],$$

where $\mathbf{C}(x,\epsilon) \;:=\;$ the unique ϵ-cube in $\mathcal{B}_\epsilon$ which contains x.

To extend $\overline{\Phi}_\epsilon$ to $\Phi_\epsilon : \mathbf{L}^1 \longrightarrow \mathbf{L}^1_\epsilon$, define $\Phi_\epsilon(\mathbf{a}) := \overline{\Phi}_\epsilon(\overline{\mathbf{a}}_\epsilon)$, $\forall\; \mathbf{a}\in\mathbf{L}^1$.

Note: $\Phi_\epsilon(\mathbf{a}) = \overline{\Phi}_\epsilon(\mathbf{a})$ for all $\mathbf{a}\in\mathbf{L}^1_\epsilon$ (because $\overline{\mathbf{a}}_\epsilon = \mathbf{a}$ for any $\mathbf{a}\in\mathbf{L}^1_\epsilon$.)

Suppress distinction between Φ_ϵ and $\overline{\Phi}_\epsilon$: write both as "Φ_ϵ".

Proof of Theorem 5

If $\mathbf{a} = \mathbb{1}_{\mathbf{A}}$ for some $\mathbf{A} \subset \mathbb{R}^D$, then define $L(\mathbf{a}) := \lambda[\partial\mathbf{A}]$. Define

$$^{\partial}\!\mathcal{A}^{\mathbb{R}^D} \quad := \quad \left\{\mathbf{a} \in {}^{0}\!\mathcal{A}^{\mathbb{R}^D} \; ; \; L(\mathbf{a}) = 0\right\}. \qquad \text{('configs with thin boundary')}$$

Claim 1: *$^{\partial}\!\mathcal{A}^{\mathbb{R}^D}$ is a $\mathbf{L}^1$-dense subset of $^{0}\!\mathcal{A}^{\mathbb{R}^D}$.*

Proof sketch: Any measurable set can be approximated by a finite disjoint union $\mathbf{U}$ of cubes. Then $\partial\mathbf{U}$ is finite disjoint union of faces, so it has zero volume. ♢

Claim 2: (a) *If $\mathbf{a} \in {}^{1}\!\mathcal{A}^{\mathbb{R}^D}$, then* $\lim_{\epsilon\to 0} \|\overline{\mathbf{a}}_\epsilon - \mathbf{a}\|_1 \;=\; 0.$

(b) *If $\mathbf{a} \in {}^{\partial}\!\mathcal{A}^{\mathbb{R}^D}$, then* $\lim_{\epsilon\to 0} \left\|\overline{\mathfrak{k}}_\epsilon * \overline{\mathbf{a}}_\epsilon - \mathfrak{k} * \mathbf{a}\right\|_\infty \;=\; 0.$

Proof sketch: (a): Martingale convergence theorem.

(b): Let $\mathbf{a} := \mathbb{1}_{\mathbf{A}}$ for $\mathbf{A} \subset \mathbb{R}^D$. Then $\overline{\mathbf{a}}_\epsilon(x) = \mathbf{a}(x) \in \{0,1\}$ for all x not ϵ-close to $\partial\mathbf{A}$. Thus, $\overline{\mathfrak{k}}_\epsilon * \overline{\mathbf{a}}_\epsilon = \mathfrak{k} * \mathbf{a}$ outside of ϵ-radius of $\partial\mathbf{A}$, while $\overline{\mathfrak{k}}_\epsilon * \overline{\mathbf{a}}_\epsilon \;\sim\; \mathfrak{k} * \mathbf{a}$ *inside* of ϵ-radius of $\partial\mathbf{A}$ (by convolutional 'smoothing'). ♢

Claim 3: *If $\mathbf{a} \in {}^{\partial}\!\mathcal{A}^{\mathbb{R}^D}$, then* $\mathbf{L}^1\text{-}\lim_{\epsilon\to 0} \Phi_\epsilon(\mathbf{a}) \;=\; \Phi(\mathbf{a}).$

Proof sketch: Similar to Theorem 3(a), but using Claim 2. ♢

Claim 4: *$\{\Phi_\epsilon\}_{\epsilon>0}$ is $\mathbf{L}^1$-equicontinuous at every $\mathbf{a} \in {}^{0}\!\mathcal{A}^{\mathbb{R}^D}$.*

Proof sketch: Similar to Theorems 3(b) and 4(b): use Lemma 1 to control continuity of Φ_ϵ. ♢

Theorem 5(a): Follows from Claims 1, 3, and 4.

Theorem 5(b): Follows from **(a)**, Claim 4, and Proposition A. □

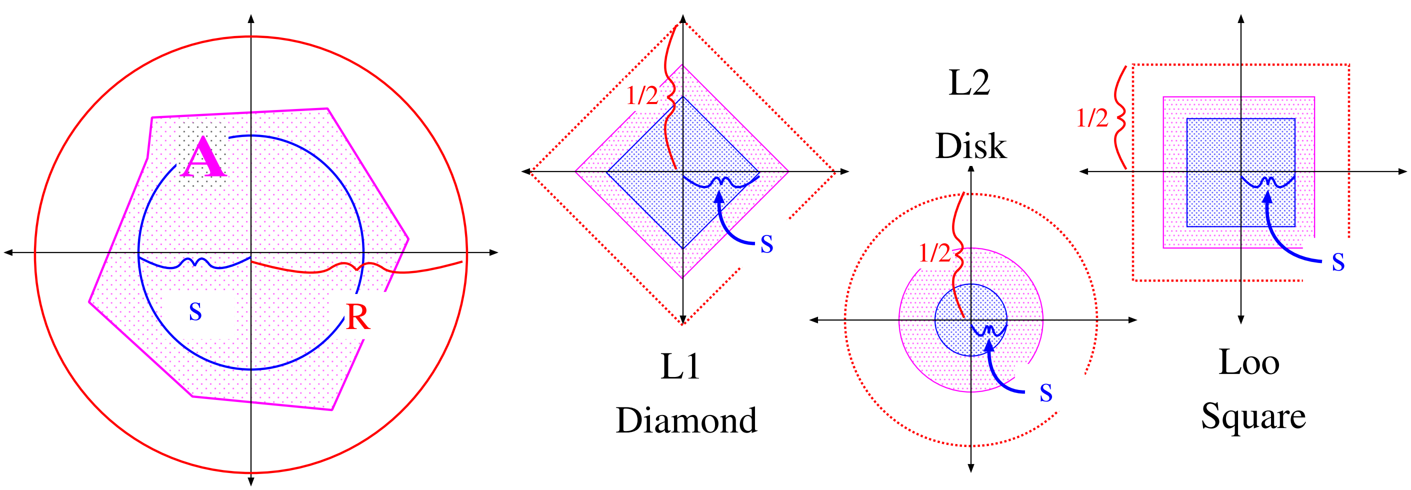
Still lifes in *RealLife*: Disks, Diamonds, and Squares

Evans empirically found many still lifes in various ***Larger than Life*** CA. Many of these still lifes are roughly 'ball'-shaped or 'annulus'-shaped (but distorted by kernel geometry and lattice anisotropy).

Proposition B: *Let $D = 2$. Let $\|\bullet\|_*$ be norm on $\mathbb{R}^2$. $\forall\ r > 0$, let $\odot(r) := \left\{x \in \mathbb{R}^D \ ;\ \|x\|_* \leq r\right\}$. Let $\mathbb{K} := \odot(1)$, and $\mathfrak{k} := \lambda[\mathbb{K}]^{-1} \cdot \mathbb{1}_{\mathbb{K}}$.*

Suppose $s_0 \leq \frac{1}{4}$. Let $R < \min\{\sqrt{b_0},\ \frac{1}{2}\}$. If $\mathbf{A} \subseteq \odot(R)$, and $s_0 \cdot \lambda[\mathbb{K}] \leq \lambda[\mathbf{A}]$ then $\mathbf{a} := \mathbb{1}_{\mathbf{A}}$ is a still life.

In particular, if $\odot(\sqrt{s_0}) \subseteq \mathbf{A} \subseteq \odot(R)$, then $\mathbf{a}$ is a still life. (There is a similar result for higher dimensions)



Examples: Let $s_0 \leq \frac{1}{4} < b_0$, and $R < \frac{1}{2}$.

ℓ^1 **norm:** For any $r > 0$, let $\mathbb{D}(r) := \left\{x = (x_1, x_2) \in \mathbb{R}^D \ ;\ |x_1| + |x_2| \leq r\right\}$ (diamond). Let $\mathfrak{k} := \frac{1}{2}\mathbb{1}_{\mathbb{D}(1)}$. Then $\mathbb{1}_{\mathbf{D}(r)}$ is a still life, $\forall\ r \in \left[\sqrt{s_0}, \frac{1}{2}\right)$.

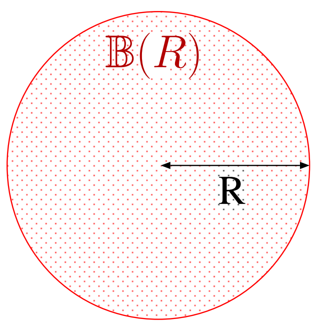
ℓ^2 **norm:** For any $r > 0$, let $\mathbb{B}(r) := \left\{x \in \mathbb{R}^2 \ ;\ |x| \leq r\right\}$ (disk). Let $\mathfrak{k} := \frac{1}{\pi}\mathbb{1}_{\mathbb{B}(1)}$. Then $\mathbb{1}_{\mathbb{B}(r)}$ is a still life for any $r \in \left[\sqrt{s_0}, \frac{1}{2}\right)$,

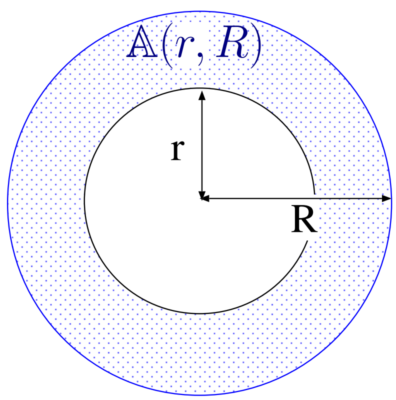
ℓ^∞ **norm:** For any $r > 0$, let $\mathbb{C}(r) := [-r, r]^2$ (square). Let $\mathfrak{k} := \frac{1}{4}\mathbb{1}_{\mathbb{C}(1)}$. Then $\mathbb{1}_{\mathbf{C}(r)}$ is a still life for any $r \in \left[\sqrt{s_0}, \frac{1}{2}\right)$.

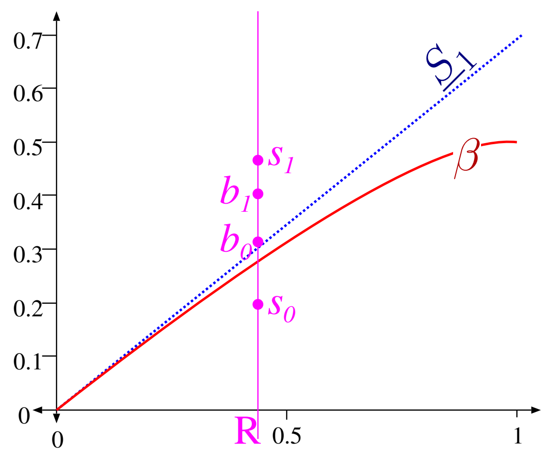
Still lifes in *RealLife*: Balls and Bubbles:

$\mathfrak{k}$ is **rotationally symmetric** if there exists function $\kappa : [0,\infty)\longrightarrow[0,\infty)$ such that $\mathfrak{k}(x) = \kappa|x|$ for all $x \in \mathbb{R}^D$.

If $R > 0$, let $\mathbb{B}(R) := \left\{x \in \mathbb{R}^D \; ; \; |x| \leq R\right\}$ and $\mathbf{b}_R := \mathbb{1}_{\mathbb{B}(R)} \in {}^1\!\mathcal{A}^{\mathbb{R}^D}$. If $r \in [0, R]$, let $\mathbb{A}(r, R) := \left\{x \in \mathbb{R}^D \; ; \; r \leq |x| \leq R\right\}$ be the **bubble** with inner radius r and outer radius R (e.g. if $D = 2$, then $\mathbb{A}(r, R)$ is an annulus). Let $\mathbf{a}_{r,R} := \mathbb{1}_{\mathbb{A}(r,R)} \in {}^1\!\mathcal{A}^{\mathbb{R}^D}$.







Proposition C: *Suppose Φ has rotationally symmetric kernel $\mathfrak{k}$.*

(a) *There are differentiable increasing functions $\underline{S}_1 : [0,\infty)\longrightarrow[0,\infty)$ and $\beta : [0,\infty)\longrightarrow[0,1]$ so that, for any $R > 0$, if $s_0 \leq \beta(R) < b_0$ and $\underline{S}_1(R) \leq s_1$, then $\mathbf{b}_R$ is a still life.*

(b) *Let $\Delta := \{(r, R) \; ; \; 0 < r < R\}$. There are differentiable functions $\beta, \underline{B}_0, \overline{B}_1, \underline{S}_1 : \Delta\longrightarrow[0,1]$ so that, for any $(r, R) \in \Delta$, if $s_0 \leq \beta(r, R) < b_0$, $\underline{S}_1(r, R) \leq s_1$, and either $\underline{B}_0(r, R) < b_0$ or $b_1 < \overline{B}_1(r, R)$ then $\mathbf{a}_{r,R}$ is a still life.*

Other results: Similar existence theorem for still lifes shaped like thin, infinitely extended 'slabs', or like gently undulating 'curtains'.

Also, still life property is 'stable' under small perturbations in a Hausdorff-style metric on ${}^1\!\mathcal{A}^{\mathbb{R}^D}$.

Open Questions

Oscillators & Bugs in *RealLife*: We have proved the existence of some *still lifes* for *RealLife* EA, but not oscillators or bugs.

Empirically, *Larger than Life* CA have oscillators and bugs. Theorem 5(b) says these should 'evolve' to oscillators and bugs for *RealLife*. But this is not a proof.

Converse of Theorem 5(b): Does existence of life forms for *RealLife* imply existence of life forms for large-radius LtL CA?

(As yet there is only empirical evidence for most life forms in LtL.)

PDEs for boundary dynamics: Simulations of *RealLife* show objects with 'smooth' boundaries, which 'smoothly' evolve over time.

Can this motion be described by a suitable system of partial differential equations?

Could these PDEs be used to prove existence of oscillators/bugs?

Computation: J.H. Conway built a universal computer in *Life*.

K.M. Evans built a universal computer in one LtL CA (*Bosco's Rule*).

Can *RealLife* simulate a universal computer?

Replication: Conway also built a 'universal constructor' in *Life*. This yields 'life forms' capable of *self-replication* (and potentially, mutation and natural selection).

Are the self-replicating structures in *RealLife*?