

Interior symmetry and local bifurcation in coupled cell networks

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Abstract. A coupled cell system is a network of dynamical systems, or ‘cells’, coupled together. Such systems can be represented schematically by a directed graph whose nodes correspond to cells and whose edges represent couplings. A symmetry of a coupled cell system is a permutation of the cells and edges that preserves all internal dynamics and all couplings. It is well known that symmetry can lead to patterns of synchronized cells, rotating waves, multirhythms, and synchronized chaos. Recently, the introduction of a less stringent form of symmetry, the ‘symmetry groupoid’, has shown that global group-theoretic symmetry is not the only mechanism that can create such states in a coupled cell system. The symmetry groupoid consists of structure-preserving bijections between certain subsets of the cell network, the input sets. Here, we introduce a concept intermediate between the groupoid symmetries and the global group symmetries of a network: ‘interior symmetry’. This concept is closely related to the groupoid structure, but imposes stronger constraints of a group-theoretic nature. We develop the local bifurcation theory of coupled cell systems possessing interior symmetries, by analogy with symmetric bifurcation theory. The main results are analogues for ‘synchrony-breaking’ bifurcations of the Equivariant Branching Lemma for steady-state bifurcation, and the Equivariant Hopf Theorem for bifurcation to time-periodic states.

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1. Introduction

A ‘coupled cell system’ is a system of ordinary differential equations (ODEs) whose structure is that of a finite number of subsystems (‘cells’) that are coupled together. That is, the dynamics of a given cell depends on the states of some of the other cells, as well as its own state. Which cells are coupled can be specified by a ‘network’ or directed graph G , whose nodes are the cells and whose (directed) edges correspond to couplings. That is, cell i influences cell j if there is a directed edge from node i to node j .

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Abstractly, the topology of the network (especially when equipped with ‘labels’ specifying when two cells or two couplings are ‘the same’ (Stewart *et al.*, 2003) determines a class \mathcal{F}_G^P of ‘admissible’ vector fields. The corresponding ODEs are precisely the coupled cell systems whose ‘architecture’ is that of G . It turns out that many aspects of pattern-formation in coupled cell systems are primarily consequences of the network topology, and that the precise choice of an admissible vector field is of secondary significance.

At this level of generality, any dynamical system can be thought of as a coupled cell system, and conversely. However, there is a significant difference in viewpoint. A coupled cell system is equipped with a distinguished set of ‘observables’, the states of the cells. Specifically, let $\mathcal{C} = \{1, \dots, N\}$ and let the cells be indexed by $c \in \mathcal{C}$. Equip each cell c with a phase space P_c (a finite-dimensional smooth manifold, which for simplicity we shall take to be $\mathbf{R}^{k(c)}$ where $k(c)$ is the number of ‘degrees of freedom’ of cell c). Then the ‘total phase space’ for the system is

$$P = P_1 \times \cdots \times P_N$$

and the state of the system at time t is $(x_1(t), \dots, x_N(t))$, where $x_c(t) \in P_c$ is the state of cell c at time t . The distinguished observables are then the natural projections $\pi_c: P \rightarrow P_c$ from the total phase space to the phase spaces of the cells. These projections are an important part of the structure, so that, for example, the only permitted changes of variables are ones that preserve these projections (Golubitsky and Stewart, 2002a; 2002b).

The existence of distinguished observables makes it meaningful to compare the dynamics of distinct cells. For example, we may say that cells i, j are ‘synchronous’ if $x_i(t) = x_j(t)$ for all $t \in \mathbf{R}$, or that they are ‘phase-related’ with ‘phase difference’ θ if both x_i, x_j are time-periodic and $x_j(t) = x_i(t + \theta)$. There now exists a huge literature on synchrony, phase-relations, and more exotic dynamical phenomena in coupled cell systems, often referred to simply as ‘networks’: see, for example, Kuramoto (1984), Pecora and Carroll (1990), Boccaletti *et al.* (2001), Wang (2002), and Watts and Strogatz (1998). The potential range of applications of such systems is also huge, including communication via the Internet, the spread of epidemics, food webs in ecosystems, metabolic networks in the cell, neural circuits, networks of gene expression, commercial supply chains, electrical power grids, transport networks, and crowd flow.

One strand in the literature focuses on the effect of the overall symmetry of the network on the formation of spatial and spatio-temporal patterns (Golubitsky and Stewart, 2002a; 2002b; Dionne *et al.*, 1996a; 1996b). Applications include animal locomotion (Buono, 1998; 2001; Buono and Golubitsky, 2001; Collins and Stewart, 1992; 1993a; 1993b; Golubitsky *et al.*; 1998) and speciation (Cohen and Stewart, 2000; Elmhirst, 1998; 2001; 2004; Dias and Stewart, 2003; Stewart, 2003a; 2003b; Stewart *et al.*, 2003; Vincent and Vincent, 2000). A continuum analogue has been applied to visual perception (Bressloff *et al.*, 2001a; 2001b; 2002).

The formal setting for this theory is the symmetry group of the network: the set of permutations of the nodes that preserves the coupling structure. However, the existence of global network symmetries is a very strong constraint, and it turns out that for many purposes a weaker notion is preferable, the ‘symmetry groupoid’ of the network (Stewart *et al.*, 2003; Golubitsky *et al.*, 2004; 2005).

A groupoid is a generalization of a group, in which products of elements are not always defined (Higgins, 1971). The symmetry groupoid is a formalization of the ‘local symmetries’ of a coupled cell network that relate subsets of the network to each other. In particular, the ‘admissible’ vector fields — those specified by the network topology — are precisely those that are equivariant under the action of the symmetry groupoid (Stewart *et al.* 2003).

In this paper we study a concept of ‘symmetry’ that is intermediate between the global group symmetries of G and the local groupoid symmetries. We refer to this concept as ‘interior symmetry’. Roughly speaking, an interior symmetry is a symmetry of some subset of cells, that fixes all cells that receive inputs from cells outside that subset. Interior symmetries have a strong influence on pattern formation, and the details of this influence differ from both the group and groupoid cases.

Our main objective in this paper is to initiate the ‘synchrony-breaking’ bifurcation theory of not necessarily symmetric coupled cell systems in the presence of interior symmetries, and in particular to prove analogues of the Equivariant Branching Lemma and the Equivariant Hopf Theorem (Golubitsky and Stewart, 2002; Golubitsky *et al.*, 1988). Section 2 reviews the pertinent features of coupled cell systems and their associated networks. In particular we give a formal definition of a coupled cell network, explain its graphical representation, discuss admissible vector fields, and recall the concept of a balanced equivalence relation. Section 3 defines interior symmetries and establishes some of their properties. Section 4 sets up the required properties of linearizations. Section 5 states and proves an analogue of the Equivariant Branching Lemma, while section 6 states and proves an analogue of the Equivariant Equivariant Hopf Theorem. We conclude with a typical simulation for the time-periodic case.

2. Coupled cell systems and network topology

We recall the coupled cell network formalism. There are two versions: the original version of [1], and the ‘multiarrow’ generalization of (Golubitsky *et al.*, 2005), which has important technical advantages (at the expense of slight complications in the definition of a network). We therefore use the multiarrow version.

2.1. Formal definition of a coupled cell network

Definition 1. In the ‘multiarrow formalism’, a ‘coupled cell network’ G consists of:

- (a) A finite set $\mathcal{C} = \{1, \dots, N\}$ of ‘nodes’ or ‘cells’.
- (b) An equivalence relation $\sim_{\mathcal{C}}$ on cells in \mathcal{C} .
The ‘type’ or ‘cell label’ of cell c is the $\sim_{\mathcal{C}}$ -equivalence class $[c]_{\mathcal{C}}$ of c .
- (c) A finite set \mathcal{E} of ‘edges’ or ‘arrows’.
- (d) An equivalence relation $\sim_{\mathcal{E}}$ on edges in \mathcal{E} . The ‘type’ or ‘coupling label’ of edge e is the $\sim_{\mathcal{E}}$ -equivalence class $[e]_{\mathcal{E}}$ of e .
- (e) Two maps $\mathcal{H}: \mathcal{E} \rightarrow \mathcal{C}$ and $\mathcal{T}: \mathcal{E} \rightarrow \mathcal{C}$.
For $e \in \mathcal{E}$ we call $\mathcal{H}(e)$ the ‘head’ of e and $\mathcal{T}(e)$ the ‘tail’ of e .

Moreover, we require the following consistency condition:

- (f) Equivalent arrows have equivalent tails and heads. That is, if $e_1, e_2 \in \mathcal{E}$ and $e_1 \sim_{\mathcal{E}} e_2$, then

$$\mathcal{H}(e_1) \sim_{\mathcal{C}} \mathcal{H}(e_2) \quad \mathcal{T}(e_1) \sim_{\mathcal{C}} \mathcal{T}(e_2).$$

Definition 2. Let $c \in \mathcal{C}$. Then the ‘input set’ of c is

$$I(c) = \{e \in \mathcal{E} : \mathcal{H}(e) = c\} \tag{1}$$

An element of $I(c)$ is called an ‘input edge’ or ‘input arrow’ of c .

Definition 3. The relation \sim_I of ‘input equivalence’ on \mathcal{C} is defined by $c \sim_I d$ if and only if there exists an arrow type preserving bijection

$$\beta : I(c) \rightarrow I(d). \tag{2}$$

That is, for every input arrow $i \in I(c)$

$$i \sim_E \beta(i). \tag{3}$$

Such a bijection β is an ‘input isomorphism’ from c to d . Let $B(c, d)$ denote the set of all input isomorphisms from c to d . Then the (disjoint) union

$$\mathcal{B}_G = \bigcup_{c, d \in \mathcal{C}} B(c, d)$$

is a groupoid (Higgins, 1971; Brown, 1987). We call \mathcal{B}_G the ‘groupoid of the network’.

2.2. *Graphical representation*

We can represent abstract classes of networks with the same general ‘architecture’ by labelled directed graphs. Figure 1 shows a simple example.

Here the node labels, drawn as the two circles and the square, indicate cells; the symbols show that cells 1 and 2 have the same cell type, whereas cell 3 is different. There are three types of edge label, drawn as different styles of arrow. These occur in three edge-types: two identical arrows from cell 3 to cells 1 and 2, respectively; two identical arrows between cells 1 and 2, one in each direction; an arrow from cell 1 to cell 3.

We now explain how to interpret such a diagram as being representative of a class of vector fields.

2.3. *Admissible vector fields*

We now define a class \mathcal{F}_G^P of ‘admissible’ vector fields corresponding to a given coupled cell network G on a total phase space P . For each cell in \mathcal{C} select a ‘cell phase space’ P_c . This is a smooth manifold of dimension ≥ 1 , assumed for simplicity to be a non-zero finite-dimensional vector space over \mathbf{R} . We require

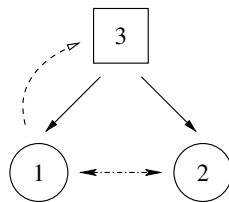


Figure 1. A 3-cell network without symmetry.

$$c \sim_C d \implies P_c = P_d$$

and we employ the same coordinate systems on P_c and P_d . The corresponding ‘total phase space’ is

$$P = \prod_{c \in \mathcal{C}} P_c$$

with coordinate system

$$x = (x_c)_{c \in \mathcal{C}}.$$

More generally, let $\mathcal{D} = \{d_1, \dots, d_s\}$ be any finite s -tuple of cells in \mathcal{C} (so that in particular the same cell can appear more than once in \mathcal{D}). Define

$$P_{\mathcal{D}} = P_{d_1} \times \dots \times P_{d_s}.$$

Further, write

$$x_{\mathcal{D}} = (x_{d_1}, \dots, x_{d_s})$$

where $x_{d_j} \in P_{d_j}$. In particular, for a given cell c the ‘internal phase space’ is P_c and the ‘coupling phase space’ is

$$P_{\mathcal{T}(I(c))} = P_{\mathcal{T}(i_1)} \times \dots \times P_{\mathcal{T}(i_s)}$$

where $\mathcal{T}(I(c))$ denotes the ordered set of cells $(\mathcal{T}(i_1), \dots, \mathcal{T}(i_s))$ as the arrows i_k run through $I(c)$.

Suppose $c, d \in \mathcal{C}$ and $c \sim_I d$. For any $\beta \in B(c, d)$, define the ‘pullback map’

$$\beta^* : P_{\mathcal{T}(I(d))} \rightarrow P_{\mathcal{T}(I(c))}$$

by

$$(\beta^*(z))_{\mathcal{T}(i)} = z_{\mathcal{T}(\beta(i))} \tag{4}$$

for all $i \in I(c)$ and $z \in P_{\mathcal{T}(I(d))}$. We use pullback maps to relate different components of a vector field associated with a given coupled cell network. Specifically, the class of vector fields that is encoded by a coupled cell network is given in Definition 4.

Definition 4. A vector field $f : P \rightarrow P$ is ‘ \mathcal{B}_G -equivariant’ or ‘G-admissible’ if:

- (a) For all $c \in \mathcal{C}$ the component $f_c(x)$ depends only on the internal phase space variables x_c and the coupling phase space variables $x_{\mathcal{T}(I(c))}$; that is, there exists $\hat{f}_c : P_c \times P_{\mathcal{T}(I(c))} \rightarrow P_c$ such that

$$f_c(x) = \hat{f}_c(x_c, x_{\mathcal{T}(I(c))}) \tag{5}$$

- (b) For all $c, d \in \mathcal{C}$ and $\beta \in B(c, d)$

$$\hat{f}_d(x_d, x_{\mathcal{T}(I(d))}) = \hat{f}_c(x_d, \beta^*(x_{\mathcal{T}(I(d))})) \tag{6}$$

for all $x \in P$.

For brevity, we write (6) as

$$f_{\beta(c)}(x) = f_c(\beta^*(x)) \tag{7}$$

for all $x \in P$. However, when using (7) it is necessary to remember that $f_d(x)$ depends only on the internal phase space variables x_d and the coupling phase space variables in $x_{\mathcal{T}(I(d))}$. Otherwise, $\beta^*(x)$ is not defined.

Observe that self-coupling is allowed (that is, P_c can be one of the factors in $P_{\mathcal{T}(I(c))}$) and multiple arrows between two cells are allowed (since the tail of two arrows terminating in $I(c)$ can be the same cell).

As in Lemma 4.5 (Stewart *et al.*, 2003), any ‘ \mathcal{B}_G -equivariant’ vector field F is determined by one mapping f for each input equivalence class of cells. Each component f_c of F is invariant under the vertex group $B(c, c)$. Indeed, if we choose one such invariant function for one c in each \sim_I -equivalence class, and transfer it to all other components of F using pullbacks, then we have specified a unique admissible vector field.

Example 1. Consider the network of figure 1. Cell phase spaces P_1 and P_2 are identical and equal to \mathbf{R}^k , whereas $P_3 = \mathbf{R}^\ell$ is not necessarily the same. Indeed, we do not identify P_3 with P_2 even if $\ell = k$. Coordinates on P take the form (x_1, x_2, x_3) . Admissible vector fields are those of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2, x_3) \\ \dot{x}_2 &= f(x_2, x_1, x_3) \\ \dot{x}_3 &= g(x_3, x_1) \end{aligned} \tag{8}$$

where $x_1, x_2 \in \mathbf{R}^k$ and $x_3 \in \mathbf{R}^\ell$.

If the arrows between cells 1 and 2 are deleted, this network is the ‘master/slave’ system of equation of Pecora and Carroll (1990) with $(x_1, x_2, x_3) = (w, w', v)$. They study synchronization of w, w' ; that is, of cells 1 and 2. We now look at why such synchronization is reasonable.

2.4. *Balanced equivalence relations*

We formalize the concept of synchrony. We adopt a strong definition: cells c, d are ‘synchronous’ if $x_c(t) = x_d(t)$ for all $t \in \mathbf{R}$. This condition depends on the dynamic trajectory $x(t)$; that is, it depends on initial conditions. The point of view is that the first step in understanding patterns of synchrony is to consider existence issues. Questions such as the dynamic stability of the synchronous state, or various weaker forms of synchrony, can then be placed in context.

Abstractly, the relation ‘ c and d are synchronous’ is an equivalence relation on the set of cells, so we formalize patterns of synchrony in terms of an equivalence relation \bowtie on \mathcal{C} . We assume that \bowtie is a refinement of $\sim_{\mathcal{C}}$; that is, if $c \bowtie d$, then c and d have the same cell labels. (Otherwise ‘synchronous’ makes no sense.) The corresponding ‘polydiagonal’ is

$$\Delta_{\bowtie} = \{x \in P : x_c = x_d \text{ whenever } c \bowtie d \quad \forall c, d \in \mathcal{C}\}.$$

This space is well defined since x_c and x_d lie in the same space $P_c = P_d$. The polydiagonal Δ_{\bowtie} is a linear subspace of P . It is precisely the subspace on which all cells in the same \bowtie -equivalence class are synchronous.

Patterns of ‘robust’ synchrony are classified by a special type of equivalence relation:

Definition 5. An equivalence relation \bowtie on \mathcal{C} is ‘balanced’ if for every $c, d \in \mathcal{C}$ with $c \bowtie d$, there exists an input isomorphism $\beta \in B(c, d)$ such that $T(i) \bowtie T(\beta(i))$ for all $i \in I(c)$.

In particular, $B(c, d) \neq \emptyset$ implies $c \sim_I d$. Hence, balanced equivalence relations refine input equivalence \sim_I .

Definition 6. Let \bowtie be an equivalence relation on \mathcal{C} that refines $\sim_{\mathcal{C}}$. Then \bowtie is ‘robustly polysynchronous’ if Δ_{\bowtie} is invariant under every vector field $f \in \mathcal{F}_G^P$. That is

$$f(\Delta_{\bowtie}) \subseteq \Delta_{\bowtie}$$

for all $f \in \mathcal{F}_G^P$. Equivalently, if $x(t)$ is a trajectory of any $f \in \mathcal{F}_G^P$, with initial condition $x(0) \in \Delta_{\bowtie}$, then $x(t) \in \Delta_{\bowtie}$ for all $t \in \mathbf{R}$.

The main theorem on balanced equivalence relations is Theorem 4.3 of (Golubitsky *et al.*, 2005):

Theorem 1. *Let \bowtie be an equivalence relation on a coupled cell network. Then \bowtie is robustly polysynchronous if and only if \bowtie is balanced.*

Example 2. We continue with our running example, the 3-cell network of figure 1. There is an equivalence relation \bowtie for which $1 \bowtie 2$; its equivalence classes are $\{1, 2\}$ and $\{3\}$. The corresponding polydiagonal is $\Delta_{\bowtie} = \{x : x_1 = x_2\} = \{(x, x, y)\}$. On this subspace the differential equations become

$$\begin{aligned} \dot{x} &= f(x, x, y) \\ \dot{x} &= f(x, x, y) \\ \dot{y} &= g(y, x). \end{aligned} \tag{9}$$

Since the first two equations are identical, Δ_{\bowtie} is invariant under all admissible vector fields. Thus the synchrony is ‘robust’.

The relation \bowtie is also balanced. The only condition to verify is that cells 1 and 2, which are \bowtie -equivalent but distinct, have input sets that are isomorphic by an isomorphism that preserves \bowtie -equivalence classes. The input sets are

$$I(1) = \{1, 2, 3\} \quad I(2) = \{2, 1, 3\}$$

where as a convention we list the ‘base point’ c of $I(c)$ first. The permutation $\beta : I(1) \rightarrow I(2)$ with $\beta(1) = 2, \beta(2) = 1, \beta(3) = 3$ is an input isomorphism that preserves \bowtie -equivalence classes. That is, \bowtie is balanced as claimed.

3. Interior symmetries

The main concept involved in the bifurcation theorems that we discuss in later sections is the notion of an ‘interior symmetry’. Roughly, this is a permutation of cells that preserves a certain amount of input structure. Before defining ‘interior

symmetry’ we recall (Golubitsky and Stewart, 2002; *et al.*, 1988) that a ‘symmetry’ γ of a differential equation

$$\dot{x} = f(x)$$

with phase space $X = \mathbf{R}^k$ is a linear map $x \mapsto \gamma x$ on X that commutes with f ; that is, γ satisfies

$$f(\gamma x) = \gamma f(x) \tag{10}$$

for all $x \in X$. Next we discuss symmetry in the context of coupled systems.

In the multiarrow formulation, a ‘network symmetry’ consists of two bijections $\gamma : \mathcal{C} \rightarrow \mathcal{C}$ and $\gamma_E : \mathcal{E} \rightarrow \mathcal{E}$ where γ preserves input equivalence and γ_E preserves edge equivalence; that is, $\gamma(c) \sim_I c$ and $\gamma_E(i) \sim_E i$. In addition, the two bijections must satisfy the consistency conditions

$$\gamma(\mathcal{H}(i)) = \mathcal{H}(\gamma_E(i)) \quad \text{and} \quad \gamma(\mathcal{T}(i)) = \mathcal{T}(\gamma_E(i)). \tag{11}$$

Next we relate network symmetries to (ODE) symmetries. Suppose that the input equivalence preserving cell permutation γ is a network symmetry in the sense that there is an edge equivalence preserving arrow permutation γ_E that satisfies (11). Then γ commutes with every G -admissible vector field $f : P \rightarrow P$; that is, γ and f satisfy (10). Note that equivariance holds independently of the choice of γ_E because of the invariance of the component f_c under the vertex group $B(c, c)$.

Finally, we introduce interior symmetries: an important concept for the local bifurcation theory of coupled cell networks.

Definition 7. Let $\mathcal{S} \subseteq \mathcal{C}$ be a subset of cells, and let $I(\mathcal{S}) = \{i \in \mathcal{E} : \mathcal{H}(i) \in \mathcal{S}\}$. The pair (σ, σ_E) is an ‘interior symmetry’ on \mathcal{S} if $\sigma : \mathcal{C} \rightarrow \mathcal{C}$ is an input equivalence preserving permutation that is the identity on the complement of \mathcal{S} , $\sigma_E : \mathcal{E} \rightarrow \mathcal{E}$ is an edge equivalence preserving permutation that is the identity on the complement of $I(\mathcal{S})$, and the pair satisfies (11) for all $i \in I(\mathcal{S})$.

Given a vertex permutation σ , there is often a unique edge permutation σ_E satisfying the consistency condition (11). In this case, we can refer to σ ‘itself’ as an interior symmetry (because σ_E is implicitly defined by σ). We will adopt this convention throughout the remainder of the paper.

The ‘interior symmetry group’ $\Sigma_{\mathcal{S}}$ is the set of all interior symmetries on \mathcal{S} . It is obviously a group. The interior symmetry group $\Sigma_{\mathcal{C}}$ on the whole network \mathcal{C} is the usual symmetry group Γ of the entire coupled cell system.

Example 3. We now relate our running example, the 3-cell network of figure 1, to interior symmetries. Because of the (dotted) arrow from cell 1 to cell 3, the permutation $(1\ 2) \in \mathbf{S}_3$ is not a group symmetry of the network, but it is an interior symmetry on the subset $\mathcal{S} = \{1, 2\}$.

A state (x_1, x_2, x_3) is fixed by this interior symmetry if and only if $x_1 = x_2$, that is, cells 1 and 2 are synchronous. That is, they lie in the polydiagonal $\Delta_{\bowtie} = \{x : x_1 = x_2\} = \{(x, x, y)\}$ corresponding to the balanced equivalence relation \bowtie whose equivalence classes are $\{1, 2\}, \{3\}$.

In the context of interior symmetry we can characterize the polydiagonal group-theoretically, as a fixed-point subspace. Specifically, groupoid-equivariance of the

cell system implies that

$$f_S(\sigma x_S, x_{C \setminus S}) = \sigma f_S(x_S, x_{C \setminus S}) \tag{12}$$

where f_S is the cell system vector field on the cells in S and $\sigma \in \Sigma_S$.

Suppose that $T \subset \Sigma_S$ is a subgroup. Then

$$\text{Fix}_P(T) = \{(x_S, x_{C \setminus S}) \in P : \delta x_S = x_S \quad \forall \delta \in T\} \tag{13}$$

Proposition 1. *Let T be a subgroup of Σ_S , and let f be a G -admissible vector field. Then the subspace (T) is flow-invariant for f .*

Proof. The equivalence relation on cells defined by $i \bowtie j$ if $i, j \in S$ lie on the same T orbit is a balanced equivalence relation. Apply Theorem 6.5 of (Stewart *et al.*, 2003) to show that the subspace

$$\Delta_{\bowtie} = \{x \in P : x_c = x_d \quad \text{whenever } c \bowtie d \quad \forall c, d \in C\}$$

is flow-invariant. Finally note that $\text{Fix}(T) = \Delta_{\bowtie}$. □

For related observations see Dias and Stewart (2004).

4. Linear theory and local bifurcation theory

With these preliminaries out of the way, we now turn to the main aim of this paper, which is to analyse some special types of local bifurcation in coupled cell systems: ‘synchrony-breaking’ bifurcations. These occur when a synchronous state loses stability and bifurcates to a state with less synchrony. Such bifurcations can be considered to be a generalization of symmetry-breaking bifurcations in symmetric coupled cell systems.

In the context of symmetry-breaking, there are two main local bifurcation theorems (Golubitsky and Stewart, 2002; Golubitsky *et al.*, 1988). The Equivariant Branching Lemma (see chapter XIII, section 3 of (Golubitsky *et al.*, 1988)) proves the existence of certain branches of symmetry-breaking steady states; the Equivariant Hopf Theorem (see chapter XVI, section 4 of Golubitsky *et al.*, 1988) proves the existence of certain branches of spatio-temporal symmetry-breaking time-periodic states. We will prove analogues of both of these theorems for coupled cell systems in a groupoid-equivariant setting. The analogue of the Equivariant Branching Lemma is a natural generalization of the symmetric case, but the analogue of the Equivariant Hopf Theorem has novel and rather restrictive features. In particular, instead of proving the existence of states with certain spatio-temporal symmetries, we prove the existence of states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having spatio-temporal symmetries.

Interior symmetries induce extra structure on linearized eigenvalues, and this structure controls the synchrony-breaking local bifurcations.

Example 4. The network of figure 1 has the admissible vector fields in (8). Let $x_0 = (x_1, x_1, x_3)$ be a synchronous equilibrium and let J be the Jacobian matrix at

that equilibrium. Then

$$J = \begin{bmatrix} a & b & c \\ b & a & c \\ d & 0 & e \end{bmatrix}$$

where a, b are $k \times k$ matrices, c is a $k \times \ell$ matrix, d is an $\ell \times k$, and e is an $\ell \times \ell$ matrix.

In this example $\mathcal{S} = \{1, 2\}$, $\Sigma_{\mathcal{S}} = \mathbf{Z}_2$, and $\text{Fix}_P(\Sigma_{\mathcal{S}}) = \{(x, x, y); x \in \mathbf{R}^k, y \in \mathbf{R}^{\ell}\}$. Let $W = \{(x, -x, 0); x \in \mathbf{R}^k\}$; then W is a complementary subspace to $\text{Fix}_P(\Sigma_{\mathcal{S}})$ on which $\Sigma_{\mathcal{S}}$ acts non-trivially (as $-I$). Since $\text{Fix}_P(\Sigma_{\mathcal{S}})$ is flow-invariant, the matrix J written in the decomposition $W \oplus \text{Fix}_P(\Sigma_{\mathcal{S}})$ as \hat{J} is block lower triangular. Indeed

$$\hat{J} = \left[\begin{array}{c|cc} a-b & 0 & 0 \\ \hline 0 & a+b & c \\ d & d & e \end{array} \right].$$

Thus synchrony-breaking bifurcations occur when the eigenvalues of $J|_W$ are critical (that is, the $k \times k$ matrix $a - b$ has zero real part eigenvalues). In (17) and Lemma 1 (a) we show that this kind of decomposition occurs generally for linearizations about a synchronous equilibrium supported by an interior symmetry group.

We discuss steady-state and Hopf bifurcations from ‘polysynchronous’ equilibria (that is, equilibria in $\text{Fix}_P(\Sigma_{\mathcal{S}})$) that break interior symmetry.

The action of the group $\Sigma_{\mathcal{S}}$ decomposes \mathcal{S} as

$$\mathcal{S} = \mathcal{S}_1 \cup \dots \cup \mathcal{S}_k$$

where each \mathcal{S}_j is an orbit of the action. Let

$$W = \left\{ x \in P : x_j = 0 \quad \forall j \in \mathcal{C} \setminus \mathcal{S} \quad \text{and} \quad \sum_{i \in \mathcal{S}_\ell} x_i = 0 \text{ for } 1 \leq \ell \leq k \right\}. \tag{14}$$

Note that W is a $\Sigma_{\mathcal{S}}$ -invariant subspace.

We can write the state space P as

$$P = W \oplus \text{Fix}_P(\Sigma_{\mathcal{S}}). \tag{15}$$

In particular, (14) implies that vectors in W , when written in coupled cell coordinates, have zero components on all cells not in \mathcal{S} .

Bifurcation theory concerns changes in solutions of an ODE as parameters are varied, so we introduce an explicit bifurcation parameter $\lambda \in \mathbf{R}$. We assume that f (hence also its components f_c) depend on λ , and that the ODE

$$\dot{x} = f(x, \lambda) \tag{16}$$

has a ‘trivial’ equilibrium $x_0 \in \text{Fix}_P(\Sigma_{\mathcal{S}})$. In the present context, we assume that

$$f(x_0, \lambda) \equiv 0$$

and that the bifurcation occurs at $\lambda = 0$. Let $L = (df)_{x_0}$. Proposition 1 implies that $\text{Fix}_P(\Sigma_{\mathcal{S}})$ is invariant under L , so that L has the block form

$$L = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \tag{17}$$

with respect to the decomposition (15); that is, $A : W \rightarrow W$ and $B : \text{Fix}_P(\Sigma_S) \rightarrow \text{Fix}_P(\Sigma_S)$. Thus the eigenvalues of L are the eigenvalues of A , together with those of B .

Local bifurcation (steady-state or Hopf) occurs when some eigenvalue of L has zero real part. That eigenvalue is either associated with A or with B , and it is the former case that concerns us here. We say that f undergoes a bifurcation at x_0 that ‘breaks interior symmetry’ if A has an eigenvalue with zero real part. In this case, steady-state bifurcation occurs when A has a zero eigenvalue, and Hopf bifurcation occurs when A has a conjugate pair of purely imaginary eigenvalues. In the Hopf case, we may assume (after rescaling time if necessary) that the purely imaginary eigenvalues of A are $\pm i$. We assume that the centre subspace $E(L) = E(A)$ and that the centre subspace $E(A)$ is equal to $\ker A$ in steady-state bifurcation, and to the real eigenspace

$$E(A) = \{x \in P : (A^2 + 1)x = 0\}$$

in Hopf bifurcation. The structure of L in (17) has several important implications:

Lemma 1

- (a) A commutes with the action of Σ_S on W .
- (b) A vector $u \in \text{Fix}_P(\Sigma_S)$ is an eigenvector of B with eigenvalue μ if and only if u is an eigenvector of L with eigenvalue μ .
- (c) If $w \in W$ is an eigenvector of A with eigenvalue μ , then there exists an eigenvector v of L with eigenvalue μ of the form

$$v = w + u$$

where $u \in \text{Fix}_P(\Sigma_S)$.

- (d) The centre subspace $E(A)$ is Σ_S -invariant.

Proof. Part (a) follows from Proposition 1 and the chain rule. Parts (b) and (c) are consequences of the block form of L . Part (d) follows because part (a) implies that all eigenspaces of A are Σ_S -invariant. □

In our bifurcation studies we shall assume:

- (1) Critical eigenvalues μ (0 in steady-state bifurcation and $\pm i$ in Hopf bifurcation) extend uniquely and smoothly to eigenvalues $\mu(\lambda)$ for λ near 0.
- (2) The ‘eigenvalue crossing condition’

$$\frac{d}{d\lambda} \text{Re}(\mu)(0) \neq 0 \tag{18}$$

is valid.

5. Interior symmetry branching lemma

Recall (Golubitsky and Stewart, 2002; Dionne *et al.*, 1996a; 1996b) that an ‘axial’ subgroup of a group action on a vector space is an isotropy subgroup whose fixed-point subspace is one-dimensional. Recall that Lemma 1(d) implies that Σ_S acts on $\ker A$. We can now prove a generalization of the Equivariant Branching Lemma to coupled cell systems.

Theorem 2. *Assume that $\ker L$ and $\ker A$ have the same dimension. Let $T \subset \Sigma_S$ be an axial subgroup of the action of Σ_S on $\ker A$, and assume the eigenvalue crossing*

condition (18). Then there exists a unique branch of equilibria, bifurcating from $(x_0, 0)$, with symmetry group T .

Proof. By Proposition 1, $\text{Fix}_P(T)$ is flow-invariant. Now, T is an axial subgroup for the action of Σ_S on $\ker A$; hence $\dim \text{Fix}_{\ker A}(T) = 1$. But $\text{Fix}_{\ker A}(T) = \text{Fix}_P(T) \cap \ker A$; hence $\dim(\text{Fix}_P(T) \cap \ker A) = 1$, so the bifurcation of $f|_{\text{Fix}_P(T) \times \mathbf{R}}$ is a simple eigenvalue bifurcation. Moreover, since $f|_{\text{Fix}_P(\Sigma_S) \times \mathbf{R}}$ is non-singular at $\lambda = 0$, there is a unique branch of trivial solutions to $f = 0$ in $\text{Fix}_P(\Sigma_S) \times \mathbf{R}$. By the usual implicit function theorem argument (Golubitsky *et al.*, 1988), the eigenvalue crossing condition implies the existence of a unique branch of non-trivial solutions to $f = 0$ in $\text{Fix}_P(T) \times \mathbf{R}$. \square

Remark 1. Liapunov–Schmidt reduction (see chapter I, section 3 of (Golubitsky and Schaeffer, 1985)) applied to the equation $f = 0$ yields a reduced equation $g = 0$, where

$$g : \ker L \times \mathbf{R} \rightarrow \ker L$$

whose solutions are in one-to-one correspondence with the solutions of $f = 0$ near the origin. Even though Theorem 2 proves the simultaneous existence of branches of solutions to $g = 0$ corresponding to each axial subgroup T , the reduced mapping g satisfies no obvious symmetry constraints. In particular:

- (1) The group Σ_S does not act (naturally) on $\ker L$, even though it does act on $\ker A$.
- (2) The bifurcating branches can be transcritical, even when the symmetry condition $N_{\Sigma_S}(T)/T \cong \mathbf{Z}_2$ (where $N_{\Sigma_S}(T)$ is the normalizer of T in Σ_S , see (Golubitsky *et al.* 1988)) would imply that generic equivariant bifurcations are pitchfork bifurcations.

An example illustrating the second point is the 3-cell network of figure 1, where we assume that each cell has one-dimensional internal dynamics. In this example, $S = \{1, 2\}$ and $\Sigma_S = \mathbf{Z}_2(1\ 2)$. Generically, the bifurcating branch is transcritical. To ensure this, it is enough to assume that the coupling from cell 1 to cell 3 is non-zero at linear level.

6. Interior symmetry Hopf Theorem

We now generalize the Equivariant Hopf Theorem (see chapter XVI of (Golubitsky *et al.*, 1988)) to the context of interior symmetries of coupled cell systems. We begin this process by recalling the Equivariant Hopf Theorem.

Let $x(t)$ be a 2π -periodic solution of a system of ODEs with symmetry group Γ . A ‘spatio-temporal’ symmetry of $x(t)$ is a pair $(\gamma, \theta) \in \Gamma \times \mathbf{S}^1$ such that

$$\gamma x(t) = x(t + \theta).$$

Here \mathbf{S}^1 is the circle group of phase shifts modulo the period. The group of (spatio-temporal) symmetries of a periodic solution is a subgroup $\Delta \subseteq \Gamma \times \mathbf{S}^1$. The subgroup $K = \Delta \cap \Gamma$ consists of the purely spatial symmetries of $x(t)$. It is known that K must be an isotropy subgroup of the Γ -action (Golubitsky and Stewart, 2002; Bueno and Golubitsky, 2001; Golubitsky *et al.*, 1988).

Consider now the context of Γ -equivariant vector fields $F : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n$ where Γ is a (compact Lie) group, so that

$$F(\gamma x, \lambda) = \gamma F(x, \lambda) \quad \forall \gamma \in \Gamma.$$

Assume that F has a Γ -invariant equilibrium x_0 for all λ , that is

$$F(x_0, \lambda) \equiv 0.$$

Suppose that x_0 undergoes a Hopf bifurcation at $\lambda = 0$. After rescaling time, we may suppose that

$$L = (d_x F)_{x_0}$$

has eigenvalues that include $\pm i$ when $\lambda = 0$. Note that L commutes with Γ since x_0 is a Γ -invariant equilibrium.

There is a natural action of \mathbf{S}^1 on $E(L)$ given by the exponential $\exp(sL)$, and this action commutes with Γ since L commutes with Γ . Recall (Golubitsky and Stewart, 2002; Dionne *et al.*, 1996a; 1996b) that an isotropy subgroup $\Delta \subset \Gamma \times \mathbf{S}^1$ is ‘C-axial’ if

$$\dim \text{Fix}_{E(L)}(\Delta) = 2.$$

Suppose that the eigenvalue μ that extends i at $\lambda = 0$ crosses the imaginary axis with non-zero speed as λ varies. Then the Equivariant Hopf Theorem states that for each C-axial subgroup Δ , there is a branch of periodic solutions emanating from $(x_0, 0)$ with spatio-temporal symmetry group Δ . The proof uses Liapunov–Schmidt reduction in the context of loop spaces: see chapter XIV, section 4 of (Golubitsky *et al.* 1988).

We now prove a generalization: for each C-axial subgroup Δ of the action of a group of interior symmetries on \mathcal{C} , there is also a branch of periodic solutions emanating from this bifurcation point, whose structure is related to Δ , but in a less straightforward way.

6.1. Hopf Theorem with interior symmetry

We pursue the same proof strategy to derive an analogue of the Equivariant Hopf Theorem for interior symmetries. Let G be a coupled cell network, let $\mathcal{S} \subseteq \mathcal{C}$, and suppose that there is an interior symmetry group $\Sigma_{\mathcal{S}}$. Let f be a G -admissible vector field. In general, f is not $\Sigma_{\mathcal{S}}$ -equivariant, and L does not commute with $\Sigma_{\mathcal{S}}$. However, the block matrix A defined in (17) does commute with $\Sigma_{\mathcal{S}}$, so there is a natural $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ -action on $E(A)$, where \mathbf{S}^1 acts by $\exp(sA)$. Let $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ be a subgroup. The ‘spatial subgroup’ of Δ is $K = \Delta \cap \Sigma_{\mathcal{S}}$.

Definition 8. The subgroup $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$ is ‘spatially C-axial’ if

$$\dim \text{Fix}_{E_i(A)}(\Delta) = \dim \text{Fix}_{E_i(A)}(K) = 2 \tag{19}$$

where K is the spatial subgroup of Δ .

We require the following concept:

Definition 9. The ODE (16) undergoes a ‘synchrony-breaking Hopf bifurcation’ at x_0 when $\lambda = 0$ if:

- (a) After rescaling time, the linearization $L = (df)_{x_0}$ at $\lambda = 0$ has eigenvalues $\pm i$ coming from A , as defined in Lemma 1(c).
- (b) The eigenvalue crossing condition (18) holds.

We use the term ‘synchrony-breaking’ because the equilibrium state is assumed to be in $\text{Fix}_p(\Sigma_{\mathcal{S}})$ (and hence synchronous on each $\Sigma_{\mathcal{S}}$ -orbit in \mathcal{S}), whereas the Hopf

bifurcation critical eigenvectors (corresponding to a critical eigenvalue of A in (17)) are assumed to be transverse to $\text{Fix}_P(\Sigma_S)$ (which leads to periodic solutions not in $\text{Fix}_P(\Sigma_S)$ and hence with less synchrony).

The generalization of the Equivariant Hopf Theorem is:

Theorem 3. *Consider the coupled cell system (16). Let \mathcal{S} be a subset of cells with interior symmetry group Σ_S , and let $x_0 \in \text{Fix}_P(\Sigma_S)$ be an equilibrium of f . Assume that a synchrony-breaking Hopf bifurcation occurs at x_0 when $\lambda = 0$. Let $\Delta \subset \Sigma_S \times \mathbf{S}^1$ be a spatially \mathbf{C} -axial subgroup, with spatial subgroup K . Then generically there exists a family of periodic solutions of (16), bifurcating from $(x_0, 0)$ and having period near 2π , that is synchronous on any two cells in \mathcal{S} lying in the same K -orbit. Moreover, to lowest order in the bifurcation parameter λ , the solution $x(t)$ is of the form*

$$x(t) \approx u(t) + w(t) \tag{20}$$

where $u(t) = e^{iL}u_0$ is synchronous on Σ_S group orbits of cells in \mathcal{S} and $w(t) = e^{iL}v_0$ has exact Δ spatio-temporal symmetries on cells in \mathcal{S} .

Proof. We adapt the proof of the Equivariant Hopf Theorem (Golubitsky *et al.*, 1988), which uses Liapunov–Schmidt reduction, to the context of a synchrony-breaking Hopf bifurcation that breaks interior symmetry. Let $\mathcal{C}_{2\pi}(P)$ be the loop space consisting of all continuous 2π -periodic functions from \mathbf{S}^1 into P , with the C^0 norm. The Hopf Theorem concerns periodic solutions to differential equations near a point where the Jacobian matrix has purely imaginary eigenvalues. We have rescaled time so that those purely imaginary eigenvalues are $\pm i$, so we look for periodic solutions with period near 2π . By introducing a perturbed period parameter τ we can rescale time again, from t to $s = (1 + \tau)t$, and consider 2π -periodic solutions to the equation

$$\mathcal{F}(v(s), \lambda, \tau) \equiv (1 + \tau) \frac{dv}{ds} - f(v(s), \lambda) = 0. \tag{21}$$

Near the trivial equilibrium $(0, 0, 0)$ these solutions are zeros of the mapping

$$\mathcal{F}: \mathcal{C}_{2\pi}^1(P) \times \mathbf{R} \times \mathbf{R} \rightarrow \mathcal{C}_{2\pi}(P)$$

defined in (21), where $\mathcal{C}_{2\pi}^1(P)$ consists of the continuously differentiable functions in $\mathcal{C}_{2\pi}(P)$ with the C^1 norm.

The linearization of \mathcal{F} about the origin is

$$\mathcal{L}(v(s)) = \frac{dv}{ds} - Lv(s)$$

and $\ker(\mathcal{L})$ consists of all functions $v(s) = \text{Re}(e^{is}v)$ where v is an eigenvector of L associated to the eigenvalue i .

As is well known (Golubitsky *et al.*, 1988), the operator \mathcal{F} is \mathbf{S}^1 -equivariant with respect to the phase shift action of \mathbf{S}^1 on loop space. In the standard Hopf Theorem, the kernel $\ker(\mathcal{L})$ is two-dimensional, and Liapunov–Schmidt reduction in the presence of symmetry leads to a reduced equation that can be solved for a unique branch of 2π -periodic solutions as long as the eigenvalue crossing condition is valid.

In the equivariant context, however, $\ker(\mathcal{L})$ may be high-dimensional. The proof of the Equivariant Hopf Theorem proceeds by restricting the Liapunov–Schmidt reduced equation to $\text{Fix}_P(\Delta)$. Since Δ is \mathbf{C} -axial, $\text{Fix}_P(\Delta)$ is two-dimensional, so

we can complete the proof as in the standard Hopf Theorem. That approach does not work in the context of interior symmetries (since the fixed-point subspace of Δ in loop space need not be \mathcal{F} -invariant). However, the fixed-point subspace of the spatial subgroup K is \mathcal{F} -invariant (by Proposition 1). Since Δ is spatially \mathbf{C} -axial, $\text{Fix}_P(K)$ is two-dimensional, and the proof now proceeds as in the equivariant case.

The form of the eigenvectors in Lemma 1(c) implies that at linear level the solution of (16), with period near 2π , is of the form

$$v(t) = w(t) + u(t)$$

where $w(t) \in \text{Fix}_W(\Delta)$ and $u(t) \in \text{Fix}_P(\Sigma_{\mathcal{S}})$. In particular, $w(t)$ has spatio-temporal symmetry Δ on cells in \mathcal{S} , and $u(t)$ is synchronous on $\Sigma_{\mathcal{S}}$ group orbits of cells in \mathcal{S} . □

Remark 2.

- (1) The theorem asserts no restrictions on $v_j(t)$ when $j \notin \mathcal{S}$.
- (2) The structure in (20) does impose genuine restrictions. Suppose, for the sake of illustration, that Δ defines a rotating wave. That is

$$\Delta = \langle (\zeta, \theta) \rangle \subseteq \Sigma_{\mathcal{S}} \times \mathbf{S}^1$$

where ζ is a k -cycle on \mathcal{S} , with $|\mathcal{S}| = k$, and $\theta = 2\pi/k$ for the period 2π case. Then on $\mathcal{S} = \{0, \dots, k-1\}$ we have

$$w_0(t) = w_1(t + 2\pi/k) = \dots = w_{k-1}(t + 2(k-1)\pi/k)$$

and

$$u_0(t) = u_1(t) = \dots = u_{k-1}(t)$$

for all t . To first order in λ , near bifurcation, the wave $v(t)$ satisfies

$$v_j(t) \approx w_j(t) + u_j(t).$$

If we define

$$y_j(t) = v_{j+1}(t) - v_j(t) \approx w_{j+1}(t) - w_j(t)$$

(subscripts modulo k) then

$$y_j(t) \approx y_0(t + 2\pi j/k)$$

is an approximate rotating wave. For a general periodic state, no such relation on differences holds.

Example 5. Consider the four-cell network whose diagram is figure 2. Set $\mathcal{S} = \{1, 2, 3\}$; observe that on the right is a \mathbf{Z}_3 -symmetric network and on the left is a network with one missing connection and \mathbf{Z}_3 interior symmetry. We compare Hopf bifurcations in the two networks.

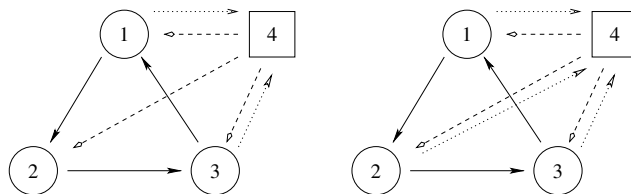


Figure 2. (Left) \mathbf{Z}_3 interior symmetry in a 4-cell example. (Right) \mathbf{Z}_3 -symmetric network.

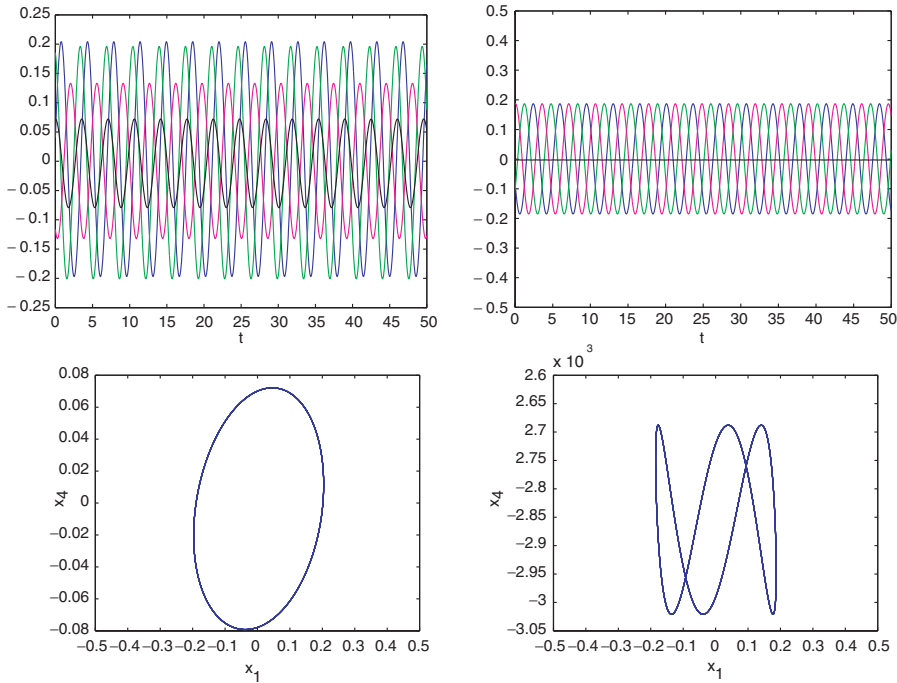


Figure 3. (Left) Z_3 interior symmetry in a 4-cell example. (Right) Z_3 -symmetric network. Upper panels show time series from all cells; lower panels show x_1 versus x_4 .

Symmetry-breaking Hopf bifurcation that breaks Z_3 symmetry leads to periodic solutions that are exact rotating waves in cells 1–3, and triple frequency motion in cell 4. Interior symmetry-breaking Hopf bifurcation leads to periodic solutions that, to first order, are the sum of a periodic rotating wave and a periodic state that is synchronous on \mathcal{S} ; in addition, cell 4 has the same frequency as cells 1–3. Therefore, the time series $x_1 - x_2, x_2 - x_3, x_3 - x_1$ form an approximate rotating wave.

In the interior symmetry case we test these conclusions using one-dimensional cells with equations

$$\begin{aligned}
 \dot{x}_1 &= -x_1 + 0.5x_1^2 - x_1^3 + \lambda x_3 + 2x_4 \\
 \dot{x}_2 &= -x_2 + 0.5x_2^2 - x_2^3 + \lambda x_1 + 2x_4 \\
 \dot{x}_3 &= -x_3 + 0.5x_3^2 - x_3^3 + \lambda x_2 + 2x_4 \\
 \dot{x}_4 &= -x_4 + 0.5x_4^2 - x_4^3 - (x_1 + x_3)
 \end{aligned}
 \tag{22}$$

where $\lambda = -2.05$. In the symmetry case we adapt (22) by changing the coupling term in the equation for \dot{x}_4 to $-(x_1 + x_2 + x_3)$.

A simulation is given in figure 3. The upper panels show superimposed time series from all cells; cell 4 in black is the smallest in amplitude. The lower panels show the triple frequency exhibited by cell 4 in the symmetric case. In figure 4 we see the rotating wave in the symmetric case on the right

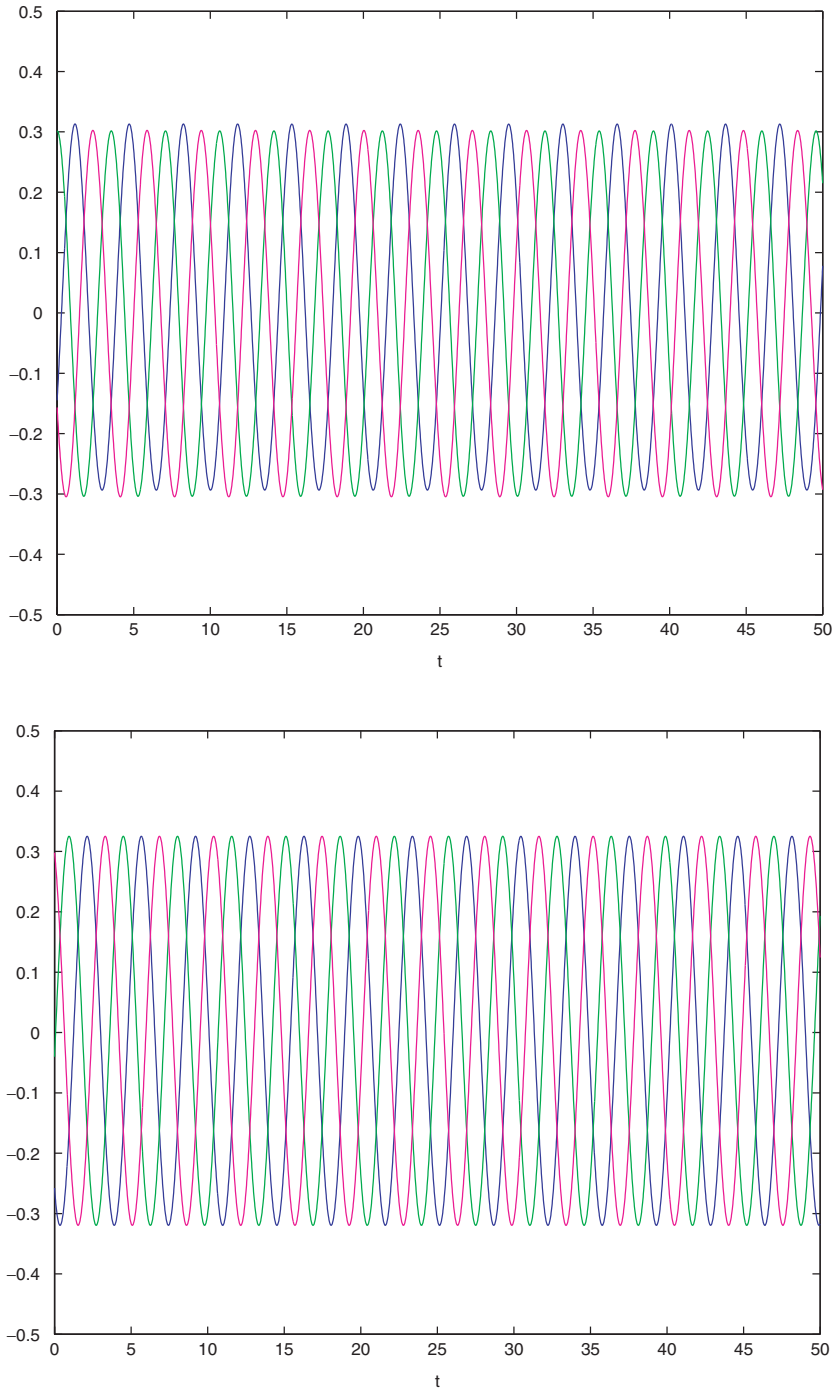


Figure 4. Simulations in cells 1–3 viewed in difference coordinates $x_1 - x_2$, $x_2 - x_3$, $x_3 - x_1$. (Left) Approximate rotating wave in \mathbf{Z}_3 interior symmetry. (Right) Exact rotating wave in \mathbf{Z}_3 -symmetric network.

and the approximate rotating wave in coordinates $x_1 - x_2$, $x_2 - x_3$, $x_3 - x_1$ on the left.

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