The Ergodic Theory of Cellular Automata

Marcus Pivato
Department of Mathematics, Trent University

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Contents

Glossary

Configuration space and the shift: Let $\mathbb{M}$ be a finitely generated group or monoid (usually abelian). Typically, $\mathbb{M} = \mathbb{N} := \{0, 1, 2, \ldots\}$ or $\mathbb{M} = \mathbb{Z} := \{\ldots, -1, 0, 1, 2, \ldots\}$, or $\mathbb{M} = \mathbb{N}^E$,

$\mathbb{Z}^D$, or $\mathbb{Z}^D \times \mathbb{N}^E$ for some $D, E \in \mathbb{N}$. In some applications, $\mathbb{M}$ could be nonabelian (although usually amenable), but to avoid notational complexity we will generally assume $\mathbb{M}$ is abelian and additive, with operation '+'.

Let $\mathcal{A}$ be a finite set of symbols (called an *alphabet*). Let $\mathcal{A}^{\mathbb{M}}$ denote the set of all functions $\mathbf{a} : \mathbb{M} \longrightarrow \mathcal{A}$, which we regard as $\mathbb{M}$-indexed *configurations* of elements in $\mathcal{A}$. We write such a configuration as $\mathbf{a} = [a_{\mathsf{m}}]_{\mathsf{m} \in \mathbb{M}}$, where $a_{\mathsf{m}} \in \mathcal{A}$ for all $\mathsf{m} \in \mathbb{M}$, and refer to $\mathcal{A}^{\mathbb{M}}$ as *configuration space*.

Treat $\mathcal{A}$ as a discrete topological space; then $\mathcal{A}$ is compact (because it is finite), so $\mathcal{A}^{\mathbb{M}}$ is compact in the Tychonoff product topology. In fact, $\mathcal{A}^{\mathbb{M}}$ is a *Cantor space*: it is compact, perfect, totally disconnected, and metrizable. For example, if $\mathbb{M} = \mathbb{Z}^D$, then the standard metric on $\mathcal{A}^{\mathbb{Z}^D}$ is defined $d(\mathbf{a}, \mathbf{b}) = 2^{-\Delta(\mathbf{a},\mathbf{b})}$, where $\Delta(\mathbf{a}, \mathbf{b}) := \min\{|\mathsf{z}| \; ; \; a_{\mathsf{z}} \neq b_{\mathsf{z}}\}$.

Any $\mathsf{v} \in \mathbb{M}$, determines a continuous *shift map* $\sigma^{\mathsf{v}} : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathcal{A}^{\mathbb{M}}$ defined by $\sigma^{\mathsf{v}}(\mathbf{a})_{\mathsf{m}} = a_{\mathsf{m}+\mathsf{v}}$ for all $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$ and $\mathsf{m} \in \mathbb{M}$. The set $\{\sigma^{\mathsf{v}}\}_{\mathsf{v} \in \mathbb{M}}$ is then a continuous $\mathbb{M}$-action on $\mathcal{A}^{\mathbb{M}}$, which we denote simply by "σ".

If $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$ and $\mathbb{U} \subset \mathbb{M}$, then we define $\mathbf{a}_{\mathbb{U}} \in \mathcal{A}^{\mathbb{U}}$ by $\mathbf{a}_{\mathbb{U}} := [a_{\mathsf{u}}]_{\mathsf{u} \in \mathbb{U}}$. If $\mathsf{m} \in \mathbb{M}$, then strictly speaking, $\mathbf{a}_{\mathsf{m}+\mathbb{U}} \in \mathcal{A}^{\mathsf{m}+\mathbb{U}}$; however, it will often be convenient to 'abuse notation' and treat $\mathbf{a}_{\mathsf{m}+\mathbb{U}}$ as an element of $\mathcal{A}^{\mathbb{U}}$ in the obvious way.

Cellular automata: Let $\mathbb{H} \subset \mathbb{M}$ be some finite subset, and let $\phi : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$ be a function (called a *local rule*). The *cellular automaton* (CA) determined by ϕ is the function $\Phi : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathcal{A}^{\mathbb{M}}$ defined by $\Phi(\mathbf{a})_{\mathsf{m}} = \phi(\mathbf{a}_{\mathsf{m}+\mathbb{H}})$ for all $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$ and $\mathsf{m} \in \mathbb{M}$. Curtis, Hedlund and Lyndon showed that cellular automata are exactly the continuous transformations of $\mathcal{A}^{\mathbb{M}}$ which commute with all shifts. (Hedlund, 1969, Theorem 3.4). We refer to $\mathbb{H}$ as the *neighbourhood* of Φ. For example, if $\mathbb{M} = \mathbb{Z}$, then typically $\mathbb{H} := [-\ell...r] := \{-\ell, 1-\ell, \ldots, r-1, r\}$ for some *left radius* $\ell \geq 0$ and *right radius* $r \geq 0$. If $\ell \geq 0$, then ϕ can either define CA on $\mathcal{A}^{\mathbb{N}}$ or define a *one-sided* CA on $\mathcal{A}^{\mathbb{Z}}$. If $\mathbb{M} = \mathbb{Z}^D$, then typically $\mathbb{H} \subseteq [-R...R]^D$, for some *radius* $R \geq 0$. Normally we assume that ℓ, r, and R are chosen to be minimal. Several specific classes of CA will be important to us:

Linear CA Let $(\mathcal{A}, +)$ be a finite abelian group (e.g. $\mathcal{A} = \mathbb{Z}_{/p}$, where $p \in \mathbb{N}$; usually p is prime). Then Φ is a *linear* CA (LCA) if the local rule ϕ has the form

$$\phi(\mathbf{a}_{\mathbb{H}}) \quad := \quad \sum_{\mathsf{h} \in \mathbb{H}} \varphi_{\mathsf{h}}(a_{\mathsf{h}}), \qquad \forall \; \mathbf{a}_{\mathbb{H}} \in \mathcal{A}^{\mathbb{H}}, \tag{0.1}$$

where $\varphi_{\mathsf{h}} : \mathcal{A} \longrightarrow \mathcal{A}$ is an endomorphism of $(\mathcal{A}, +)$, for each $\mathsf{h} \in \mathbb{H}$. We say that Φ has *scalar coefficients* if, for each $\mathsf{h} \in \mathbb{H}$, there is some scalar $c_{\mathsf{h}} \in \mathbb{Z}$, so that $\varphi_{\mathsf{h}}(a_{\mathsf{h}}) := c_{\mathsf{h}} \cdot a_{\mathsf{h}}$; then $\phi(\mathbf{a}_{\mathbb{H}}) := \sum_{\mathsf{h} \in \mathbb{H}} c_{\mathsf{h}} a_{\mathsf{h}}$. For example, if $\mathcal{A} = (\mathbb{Z}_{/p}, +)$, then *all* endomorphisms are scalar multiplications, so *all* LCA have scalar coefficients.

If $c_{\mathsf{h}} = 1$ for all $\mathsf{h} \in \mathbb{H}$, then Φ has local rule $\phi(\mathbf{a}_{\mathbb{H}}) := \sum_{\mathsf{h} \in \mathbb{H}} a_{\mathsf{h}}$; in this case, Φ is called an *additive cellular automaton*; see ADDITIVE CELLULAR AUTOMATA for more information.

Affine CA If $(\mathcal{A}, +)$ is a finite abelian group, then an *affine CA* is one with a local rule $\phi(\mathbf{a}_{\mathbb{H}}) := c + \sum_{\mathsf{h} \in \mathbb{H}} \phi_{\mathsf{h}}(a_{\mathsf{h}})$, where c is some constant and where $\varphi_{\mathsf{h}} : \mathcal{A} \longrightarrow \mathcal{A}$ are endomorphisms of $(\mathcal{A}, +)$. Thus, Φ is an LCA if $c = 0$.

Permutative CA Suppose $\Phi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ has local rule $\phi : \mathcal{A}^{[-\ell...r]} \longrightarrow \mathcal{A}$. Fix $\mathbf{b} = [b_{1-\ell}, \ldots, b_{r-1}, b_r] \in \mathcal{A}^{(-\ell...r]}$. For any $a \in \mathcal{A}$, define $[a\,\mathbf{b}] := [a, b_{1-\ell}, \ldots, b_{r-1}, b_r] \in \mathcal{A}^{[-\ell...r]}$. We then define the function $\phi_{\mathbf{b}} : \mathcal{A}{\longrightarrow}\mathcal{A}$ by $\phi_{\mathbf{b}}(a) := \phi\left([a\,\mathbf{b}]\right)$. We say that Φ is *left-permutative* if $\phi_{\mathbf{b}} : \mathcal{A}{\longrightarrow}\mathcal{A}$ is a permutation (i.e. a bijection) for all $\mathbf{b} \in \mathcal{A}^{(-\ell \ldots r]}$.

Likewise, given $\mathbf{b} = [b_{-\ell}, \ldots, b_{r-1}] \in \mathcal{A}^{[-\ell \ldots r)}$ and $c \in \mathcal{A}$, define $[\mathbf{b}\,c] := [b_{-\ell}, b_{1-\ell} \ldots, b_{r-1}, c] \in \mathcal{A}^{[-\ell \ldots r]}$, and define ${}_{\mathbf{b}}\phi : \mathcal{A}{\longrightarrow}\mathcal{A}$ by ${}_{\mathbf{b}}\phi(c) := \phi\left([\mathbf{b}\,c]\right)$; then Φ is *right-permutative* if ${}_{\mathbf{b}}\phi : \mathcal{A}{\longrightarrow}\mathcal{A}$ is a permutation for all $\mathbf{b} \in \mathcal{A}^{[-\ell \ldots r)}$. We say Φ is *bipermutative* if it is both left- and right-permutative. More generally, if $\mathbb{M}$ is any monoid, $\mathbb{H} \subset \mathbb{M}$ is any neighbourhood, and $\mathsf{h} \in \mathbb{H}$ is any fixed coordinate, then we define h-*permutativity* for a CA on $\mathcal{A}^{\mathbb{M}}$ in the obvious fashion.

For example, suppose $(\mathcal{A}, +)$ is an abelian group and Φ is an affine CA on $\mathcal{A}^{\mathbb{Z}}$ with local rule $\phi(\mathbf{a}_{\mathbb{H}}) = c + \sum_{h=-\ell}^{r} \phi_h(a_h)$. Then Φ is left-permutative iff $\phi_{-\ell}$ is an automorphism, and right-permutative iff ϕ_r is an automorphism. If $\mathcal{A} = \mathbb{Z}_{/p}$, and p is prime, then *every* nontrivial endomorphism is an automorphism (because it is multiplication by a nonzero element of $\mathbb{Z}_{/p}$, which is a field), so in this case, *every* affine CA is permutative in every coordinate of its neighbourhood (and in particular, bipermutative). If $\mathcal{A} \neq \mathbb{Z}_{/p}$, however, then not all affine CA are permutative.

Permutative CA were introduced by Hedlund (1969)[§6], and are sometimes called *permutive* CA. Right permutative CA on $\mathcal{A}^{\mathbb{N}}$ are also called *toggle automata*.

Subshifts: A *subshift* is a closed, σ-invariant subset $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$. For any $\mathbb{U} \subset \mathbb{M}$, let $\mathbf{X}_{\mathbb{U}} := \{\mathbf{x}_{\mathbb{U}} \,;\, \mathbf{x} \in \mathbf{X}\} \subset \mathcal{A}^{\mathbb{U}}$. We say $\mathbf{X}$ is a *subshift of finite type* (SFT) if there is some finite $\mathbb{U} \subset \mathbb{M}$ such that $\mathbf{X}$ is entirely described by $\mathbf{X}_{\mathbb{U}}$, in the sense that $\mathbf{X} = \left\{\mathbf{x} \in \mathcal{A}^{\mathbb{M}} \,;\, \mathbf{x}_{\mathbb{U}+\mathsf{m}} \in \mathbf{X}_{\mathbb{U}},\ \forall \mathsf{m} \in \mathbb{M}\right\}$.

In particular, if $\mathbb{M} = \mathbb{Z}$, then a (two-sided) *Markov subshift* is an SFT $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}}$ determined by a set $\mathbf{X}_{\{0,1\}} \subset \mathcal{A}^{\{0,1\}}$ of *admissible transitions*; equivalently, $\mathbf{X}$ is the set of all bi-infinite directed paths in a digraph whose vertices are the elements of $\mathcal{A}$, with an edge $a \rightsquigarrow b$ iff $(a, b) \in \mathbf{X}_{\{0,1\}}$. If $\mathbb{M} = \mathbb{N}$, then a *one-sided Markov subshift* is a subshift of $\mathcal{A}^{\mathbb{N}}$ defined in the same way.

If $D \geq 2$, then an SFT in $\mathcal{A}^{\mathbb{Z}^D}$ can be thought of as the set of admissible 'tilings' of $\mathbb{R}^D$ by Wang tiles corresponding to the elements of $\mathbf{X}_{\mathbb{U}}$. (Wang tiles are unit squares [or (hyper)cubes] with various 'notches' cut into their edges [or (hyper)faces] so that they can only be juxtaposed in certain ways.)

A subshift $\mathbf{X} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ is *strongly irreducible* (or *topologically mixing*) if there is some $R \in \mathbb{N}$ such that, for any disjoint finite subsets $\mathbb{V}, \mathbb{U} \subset \mathbb{Z}^D$ separated by a distance of at least R, and for any $\mathbf{u} \in \mathbf{X}_{\mathbb{U}}$ and $\mathbf{v} \in \mathbf{X}_{\mathbb{V}}$, there is some $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{x}_{\mathbb{U}} = \mathbf{u}$ and $\mathbf{x}_{\mathbb{V}} = \mathbf{v}$.

Please see SYMBOLIC DYNAMICS for more about subshifts.

Measures: For any finite subset $\mathbb{U} \subset \mathbb{M}$, and any $\mathbf{b} \in \mathcal{A}^{\mathbb{U}}$, let $\langle \mathbf{b} \rangle := \left\{\mathbf{a} \in \mathcal{A}^{\mathbb{M}} \,;\, \mathbf{a}_{\mathbb{U}} := \mathbf{b}\right\}$ be the *cylinder set* determined by $\mathbf{b}$. Let $\mathfrak{B}$ be the sigma-algebra on $\mathcal{A}^{\mathbb{M}}$ generated by all cylinder sets. A (probability) measure μ on $\mathcal{A}^{\mathbb{M}}$ is a countably additive function $\mu : \mathfrak{B}{\longrightarrow}[0, 1]$ such that $\mu[\mathcal{A}^{\mathbb{M}}] = 1$. A measure on $\mathcal{A}^{\mathbb{M}}$ is entirely determined by its values on cylinder sets. We will be mainly concerned with the following classes of measures:

Bernoulli measure Let β_0 be a probability measure on $\mathcal{A}$. The *Bernoulli measure* induced by β_0 is the measure β on $\mathcal{A}^{\mathbb{M}}$ such that, for any finite subset $\mathbb{U} \subset \mathbb{M}$, and any $\mathbf{a} \in \mathcal{A}^{\mathbb{U}}$, if $U := |\mathbb{U}|$, then $\beta[\langle \mathbf{a} \rangle] = \prod_{\mathsf{h} \in \mathbb{H}} \beta_0(a_{\mathsf{h}})$.

Invariant measure Let μ be a measure on $\mathcal{A}^{\mathbb{M}}$, and let $\Phi : \mathcal{A}^{\mathbb{M}}{\longrightarrow}\mathcal{A}^{\mathbb{M}}$ be a cellular automaton. The measure $\Phi\mu$ is defined by $\Phi\mu(\mathbf{B}) = \mu(\Phi^{-1}(\mathbf{B}))$, for any $\mathbf{B} \in \mathfrak{B}$. We say that μ is

Φ-*invariant* (or that Φ is μ-*preserving*) if $\Phi\mu = \mu$. For more information, see ERGODIC THEORY: BASIC EXAMPLES AND CONSTRUCTIONS.

Uniform measure Let $A := |\mathcal{A}|$. The uniform measure η on $\mathcal{A}^{\mathbb{M}}$ is the Bernoulli measure such that, for any finite subset $\mathbb{U} \subset \mathbb{M}$, and any $\mathbf{b} \in \mathcal{A}^{\mathbb{U}}$, if $U := |\mathbb{U}|$, then $\mu[\langle \mathbf{b} \rangle] = 1/A^U$.

The *support* of a measure μ is the smallest closed subset $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ such that $\mu[\mathbf{X}] = 1$; we denote this by $\mathsf{supp}(\mu)$. We say μ has *full support* if $\mathsf{supp}(\mu) = \mathcal{A}^{\mathbb{M}}$ —equivalently, $\mu[\mathbf{C}] > 0$ for *every* cylinder subset $\mathbf{C} \subset \mathcal{A}^{\mathbb{M}}$.

Notation: Let $\mathsf{CA}(\mathcal{A}^{\mathbb{M}})$ denote the set of all cellular automata on $\mathcal{A}^{\mathbb{M}}$. If $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$, then let $\mathsf{CA}(\mathbf{X})$ be the subset of all $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$ such that $\Phi(\mathbf{X}) \subseteq \mathbf{X}$. Let $\mathfrak{M}_{eas}(\mathcal{A}^{\mathbb{M}})$ be the set of all probability measures on $\mathcal{A}^{\mathbb{M}}$, and let $\mathfrak{M}_{eas}(\mathcal{A}^{\mathbb{M}}; \Phi)$ be the subset of Φ-invariant measures. If $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$, then let $\mathfrak{M}_{eas}(\mathbf{X})$ be the set of probability measures μ with $\mathsf{supp}(\mu) \subseteq \mathbf{X}$, and define $\mathfrak{M}_{eas}(\mathbf{X}; \Phi)$ in the obvious way.

Font conventions: Upper case calligraphic letters $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots)$ denote finite alphabets or groups. Upper-case bold letters $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots)$ denote subsets of $\mathcal{A}^{\mathbb{M}}$ (e.g. subshifts), lowercase bold-faced letters $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots)$ denote elements of $\mathcal{A}^{\mathbb{M}}$, and Roman letters $(a, b, c, \ldots)$ are elements of $\mathcal{A}$ or ordinary numbers. Lower-case sans-serif $(\ldots, \mathsf{m}, \mathsf{n}, \mathsf{p})$ are elements of $\mathbb{M}$, upper-case hollow font $(\mathbb{U}, \mathbb{V}, \mathbb{W}, \ldots)$ are subsets of $\mathbb{M}$. Upper-case Greek letters $(\Phi, \Psi, \ldots)$ are functions on $\mathcal{A}^{\mathbb{M}}$ (e.g. CA, block maps), and lower-case Greek letters $(\phi, \psi, \ldots)$ are other functions (e.g. local rules, measures.)

1 Introduction

The study of CA as symbolic dynamical systems began with Hedlund (1969), and the study of CA as measure-preserving systems began with Coven and Paul (1974) and Willson (1975). The ergodic theory of CA is important for several reasons:

- CA are topological dynamical systems. We can gain insight into the topological dynamics of a CA by identifying its invariant measures, and then studying the corresponding measurable dynamics (see also CHAOTIC BEHAVIOUR OF CA and TOPOLOGICAL DYNAMICS OF CA, as well as Blanchard et al. (1997) and Kůrka (2001)).

- CA are often proposed as stylized models of spatially distributed systems in statistical physics —for example, as microscale models of hydrodynamics, or of atomic lattices (see CA MODELLING OF PHYSICAL SYSTEMS and LATTICE GASES AND CA). In this context, the distinct invariant measures of a CA correspond to distinct 'phases' of the physical system (see PHASE TRANSITIONS IN CA).

- CA can also act as information-processing systems (see CA AS MODELS OF PARALLEL COMPUTATION and CA, UNIVERSALITY OF). Ergodic theory studies the 'informational' aspect of dynamical systems, so it is particularly suited to explicitly 'informational' dynamical systems like CA.

In §2, we characterize the invariant measures for various classes of CA. Then, in §3, we investigate which measures are 'generic' in the sense that they arise as the attractors for some large class of initial conditions. In §4 we study the mixing and spectral properties of CA as measure-preserving dynamical systems. Finally, in §5, we look at entropy.

These sections are logically independent, and can be read in any order.

2 Invariant measures for CA

2A The uniform measure vs. surjective cellular automata

The uniform measure η plays a central role in the ergodic theory of cellular automata, because of the following result.

Theorem 2A.1 *Let* $\mathbb{M} = \mathbb{Z}^D \times \mathbb{N}^E$, *let* $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$ *and let* η *be the uniform measure on* $\mathcal{A}^{\mathbb{M}}$. *Then* $\Big(\Phi$ *preserves* $\eta\Big) \iff \Big(\Phi$ *is surjective*$\Big)$.

Proof sketch: " $\Longrightarrow$ " If Φ preserves η, then Φ must map $\mathsf{supp}(\eta)$ onto itself. But $\mathsf{supp}(\eta) = \mathcal{A}^{\mathbb{M}}$; hence Φ is surjective.

"$\Longleftarrow$" The case $D = 1$ follows from a result of W.A. Blankenship and Oscar S. Rothaus, which first appeared in (Hedlund, 1969, Theorem 5.4). The Blankenship-Rothaus Theorem states that, if $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ is surjective and has neighbourhood $[-\ell...r]$, then for any $k \in \mathbb{N}$ and any $\mathbf{a} \in \mathcal{A}^k$, the Φ-preimage of the cylinder set $\langle \mathbf{a} \rangle$ is a disjoint union of exactly $A^{r+\ell}$ cylinder sets of length $k+r+\ell$; it follows that $\mu[\Phi^{-1}\langle \mathbf{a} \rangle] = A^{r+\ell}/A^{k+r+\ell} = A^{-k} = \mu\langle \mathbf{a} \rangle$. This result was later reproved by Kleveland (1997)[Theorem 5.1]. The special case $\mathcal{A} = \{0, 1\}$ was also proved by Shirvani and Rogers (1991)[Theorem 2.4].

The case $D \geq 2$ follows from the multidimensional version of the Blankenship-Rothaus Theorem, which was proved by Maruoka and Kimura (1976)[Theorem 2] (their proof assumes that $D = 2$ and that Φ has a 'quiescent' state, but neither hypothesis is essential). Alternately, "$\Longleftarrow$" follows from recent, more general results of Meester, Burton, and Steif; see Example 2B.4 below. □

Example 2A.2: Let $\mathbb{M} = \mathbb{Z}$ or $\mathbb{N}$ and consider CA on $\mathcal{A}^{\mathbb{M}}$.

(a) Say that Φ is *bounded-to-one* if there is some $B \in \mathbb{N}$ such that every $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$ has at most B preimages. Then $\Big(\Phi$ is bounded-to-one$\Big) \iff \Big(\Phi$ is surjective$\Big)$.

(b) Any posexpansive CA on $\mathcal{A}^{\mathbb{M}}$ is surjective (see §2D below).

(c) Any left- or right-permutative CA on $\mathcal{A}^{\mathbb{Z}}$ (or right-permutative CA on $\mathcal{A}^{\mathbb{N}}$) is surjective. This includes, for example, most linear CA.

Hence, in any of these cases, Φ preserves the uniform measure. ◊

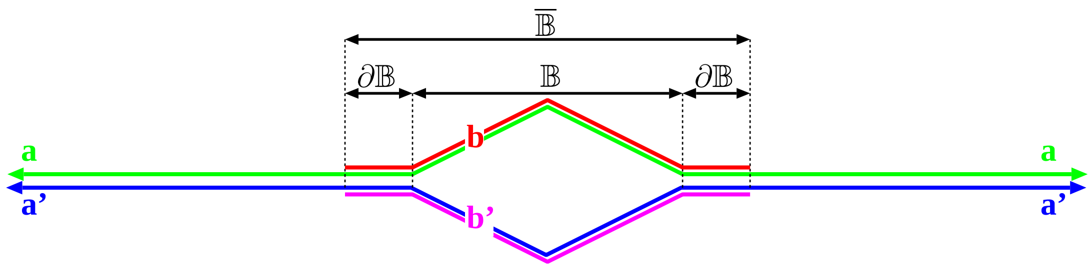

Figure 1: A 'diamond' in $\mathcal{A}^{\mathbb{Z}}$.

Proof: For **(a)**, see (Hedlund, 1969, Theorem 5.9) or (Lind and Marcus, 1995, Corollary 8.1.20, p.271). For **(b)**, see (Blanchard and Maass, 1997, Proposition 2.2) in the case $\mathcal{A}^{\mathbb{N}}$; their argument also works for $\mathcal{A}^{\mathbb{Z}}$.

Part **(c)** follows from (b) because any permutative CA is posexpansive (Proposition 2D.1 below). There is also a simple direct proof for a right-permutative CA on $\mathcal{A}^{\mathbb{N}}$: using right-permutativity, you can systematically construct a preimage of any desired image sequence, one entry at a time. See (Hedlund, 1969, Theorem 6.6) for the proof in $\mathcal{A}^{\mathbb{Z}}$. □

The surjectivity of a one-dimensional CA can be determined in finite time using certain combinatorial tests (see REVERSIBLE CA). However, for $D \geq 2$, it is formally undecidable whether an arbitrary CA on $\mathcal{A}^{\mathbb{Z}^D}$ is surjective (see THE TILING PROBLEM AND UNDECIDABILITY IN CA). This problem is sometimes refered to as the *Garden of Eden problem*, because an element of $\mathcal{A}^{\mathbb{Z}^D}$ with no Φ-preimage is called a *Garden of Eden* (GOE) configuration for Φ (because it could only ever occur at the 'beginning of time'). However, it is known that a CA is surjective if it is 'almost injective' in a certain sense, which we now specify.

Let $(\mathbb{M}, +)$ be any monoid, and let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$ have neighbourhood $\mathbb{H} \subset \mathbb{M}$. If $\mathbb{B} \subset \mathbb{M}$ is any subset, then we define

$$\overline{\mathbb{B}} \;:=\; \mathbb{B} + \mathbb{H} \;=\; \{\mathsf{b} + \mathsf{h}\,;\ \mathsf{b} \in \mathbb{B},\ \mathsf{h} \in \mathbb{H}\}; \qquad \text{and} \qquad \partial\mathbb{B} \;:=\; \overline{\mathbb{B}} \cap \overline{\mathbb{B}^{\complement}}.$$

If $\mathbb{B}$ is finite, then so is $\overline{\mathbb{B}}$ (because $\mathbb{H}$ is finite). If Φ has local rule $\phi : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$, then ϕ induces a function $\Phi_{\mathbb{B}} : \mathcal{A}^{\overline{\mathbb{B}}} \longrightarrow \mathcal{A}^{\mathbb{B}}$ in the obvious fashion. A $\mathbb{B}$-*bubble* (or $\mathbb{B}$-*diamond*) is a pair $\mathbf{b}, \mathbf{b}' \in \mathcal{A}^{\overline{\mathbb{B}}}$ such that:

$$\mathbf{b} \neq \mathbf{b}'; \qquad \mathbf{b}_{\partial\mathbb{B}} = \mathbf{b}'_{\partial\mathbb{B}}; \qquad \text{and} \qquad \Phi_{\mathbb{B}}(\mathbf{b}) = \Phi_{\mathbb{B}}(\mathbf{b}').$$

Suppose $\mathbf{a}, \mathbf{a}' \in \mathcal{A}^{\mathbb{M}}$ are two configurations such that

$$\mathbf{a}_{\overline{\mathbb{B}}} \;=\; \mathbf{b}, \qquad \mathbf{a}'_{\overline{\mathbb{B}}} \;=\; \mathbf{b}', \qquad \text{and} \qquad \mathbf{a}_{\mathbb{B}^{\complement}} \;=\; \mathbf{a}'_{\mathbb{B}^{\complement}}.$$

Then it is easy to verify that $\Phi(\mathbf{a}) = \Phi(\mathbf{a}')$. We say that $\mathbf{a}$ and $\mathbf{a}'$ form a *mutually erasable pair* (because Φ 'erases' the difference between $\mathbf{a}$ and $\mathbf{a}'$.) Figure 1 is a schematic representation of this structure in the case $D = 1$ (hence the term 'diamond'). If $D = 2$, then $\mathbf{a}$ and $\mathbf{a}'$ are like two membranes which are glued together everywhere except for a $\mathbb{B}$-shaped 'bubble'. We say that Φ is ***pre-injective*** if any (and thus, all) of the following three conditions hold:

- Φ admits no bubbles.
- Φ admits no mutually erasable pairs.
- For any $\mathbf{c} \in \mathcal{A}^{\mathbb{M}}$, if $\mathbf{a}, \mathbf{a}' \in \Phi^{-1}\{\mathbf{c}\}$ are distinct, then $\mathbf{a}$ and $\mathbf{a}'$ must differ in infinitely many locations.

For example, any injective CA is preinjective (because a mutually erasable pair for Φ gives two distinct Φ-preimages for some point). More to the point, however, if $\mathbb{B}$ is finite, and Φ admits a $\mathbb{B}$-bubble $(\mathbf{b}, \mathbf{b}')$, then we can embed N disjoint copies of $\mathbb{B}$ into $\mathbb{M}$, and thus, by making various choices between $\mathbf{b}$ and $\mathbf{b}'$ on different translates, we obtain a configuration with 2^N distinct Φ-preimages (where N is arbitrarily large). But if some configurations in $\mathcal{A}^{\mathbb{M}}$ have such a large number of preimages, then other configurations in $\mathcal{A}^{\mathbb{M}}$ must have very few preimages, or even none. By extrapolating this combinatorial argument along a Følner sequence of subsets of $\mathbb{M}$ (each packed with disjoint copies of $\mathbb{B}$), one can prove:

2A.3. Garden of Eden Theorem. *Let $\mathbb{M}$ be a finitely generated amenable group* (e.g. $\mathbb{M} = \mathbb{Z}^D$). *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$.*

(a) *Φ is surjective if and only if Φ is pre-injective.*

(b) *Let $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ be a strongly irreducible SFT such that $\Phi(\mathbf{X}) \subseteq \mathbf{X}$. Then $\Phi(\mathbf{X}) = \mathbf{X}$ if and only if $\Phi_{|\mathbf{X}}$ is pre-injective.*

Proof: **(a)** The case $\mathbb{M} = \mathbb{Z}^2$ was originally proved by Moore (1963) and Myhill (1963), while the case $\mathbb{M} = \mathbb{Z}$ was implicit in Hedlund (1969)[Lemma 5.11 and Theorems 5.9 and 5.12]. The case when $\mathbb{M}$ is a finite-dimensional group was proved by Machì and Mignosi (1993). Finally, the general case was proved by Ceccherini-Silberstein et al. (1999)[Theorem 3].

(b) The case $\mathbb{M} = \mathbb{Z}$ is (Lind and Marcus, 1995, Corollary 8.1.20) (actually this holds for any sofic subshift); see also Fiorenzi (2000). The general case is (Fiorenzi, 2003, Corollary 4.8). □

2A.4. Incompressibility Corollary. *Suppose $\mathbb{M}$ is a finitely generated amenable group and $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$. If Φ is injective, then Φ is surjective.* □

Remarks 2A.5: (a) A *cellular network* is a CA-like system defined on an infinite, locally finite digraph, with different local rules at different nodes. By assuming a kind of 'amenability' for this digraph, and then imposing some weak global statistical symmetry conditions on the local rules, Gromov (1999)[Theorem 8.F'] has generalized the GOE Theorem 2A.3 to a large class of such cellular networks (which he calls 'endomorphisms of symbolic algebraic varieties'). See also Ceccherini-Silberstein et al. (2004).

(b) In the terminology suggested by Gottschalk (1973), Incompressibility Corollary 2A.4 says that the group $\mathbb{M}$ is *surjunctive*; Gottschalk claims that 'surjunctivity' was first proved for all residually finite groups by Lawton (unpublished). For a recent direct proof (not using the GOE theorem), see (Weiss, 2000, Theorem 1.6). Weiss also defines *sofic* groups (a class containing both residually finite groups and amenable groups) and shows that Corollary 2A.4 holds whenever $\mathbb{M}$ is a sofic group (Weiss, 2000, Theorem 3.2).

(c) If $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ is an SFT such that $\Phi(\mathbf{X}) \subseteq \mathbf{X}$, then Corollary 2A.4 holds as long as $\mathbf{X}$ is 'semi-strongly irreducible'; see Fiorenzi (2004)[Corollary 4.10]. $\diamondsuit$

2B Invariance of maxentropy measures

If $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^D}$ is any subshift with topological entropy $h_{\rm top}(\mathbf{X}, \sigma)$, and $\mu \in \mathfrak{M}_{\rm eas}(\mathbf{X}, \sigma)$ has measurable entropy $h(\mu, \sigma)$, then in general, $h(\mu, \sigma) \leq h_{\rm top}(\mathbf{X}, \sigma)$; we say μ is a *measure of maximal entropy* (or *maxentropy measure*) if $h(\mu, \sigma) = h_{\rm top}(\mathbf{X}, \sigma)$. [See Example 5C.1(a) for definitions.]

Every subshift admits one or more maxentropy measures. If $D = 1$ and $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}}$ is an irreducible subshift of finite type (SFT), then Parry (1964)[Theorem 10] showed that $\mathbf{X}$ admits a *unique* maxentropy measure $\eta_{\mathbf{X}}$ (now called the *Parry measure*); see (Walters, 1982, Theorem 8.10, p.194) or (Lind and Marcus, 1995, §13.3, pp.443-444). Theorem 2A.1 is then a special case of the following result:

Theorem 2B.1 (Coven, Paul, Meester and Steif)

Let $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^D}$ *be an SFT having a unique maxentropy measure* $\eta_{\mathbf{X}}$*, and let* $\Phi \in \mathsf{CA}(\mathbf{X})$*. Then* Φ *preserves* $\eta_{\mathbf{X}}$ *if and only if* $\Phi(\mathbf{X}) = \mathbf{X}$*.*

Proof: The case $D = 1$ is (Coven and Paul, 1974, Corollary 2.3). The case $D \geq 2$ follows from (Meester and Steif, 2001, Theorem 2.5(iii)), which states: if $\mathbf{X}$ and $\mathbf{Y}$ are SFTs, and $\Phi : \mathbf{X} \longrightarrow \mathbf{Y}$ is a factor mapping, and μ is a maxentropy measure on $\mathbf{X}$, then $\Phi(\mu)$ is a maxentropy measure on $\mathbf{Y}$. $\square$

For example, if $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}}$ is an irreducible SFT and $\eta_{\mathbf{X}}$ is its Parry measure, and $\Phi(\mathbf{X}) = \mathbf{X}$, then Theorem 2B.1 says $\Phi(\eta_{\mathbf{X}}) = \eta_{\mathbf{X}}$, as observed by Coven and Paul (1974)[Theorem 5.1]. Unfortunately, higher-dimensional SFTs do *not*, in general, have unique maxentropy measures. Burton and Steif (1994) provided a plethora of examples of such nonuniqueness, but they also gave a sufficient condition for uniqueness of the maxentropy measure, which we now explain.

Let $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^D}$ be an SFT and let $\mathbb{U} \subset \mathbb{Z}^D$. For any $\mathbf{x} \in \mathbf{X}$, let $\mathbf{x}_{\mathbb{U}} := [x_{\mathsf{u}}]_{\mathsf{u} \in \mathbb{U}}$ be its 'projection' to $\mathcal{A}^{\mathbb{U}}$, and let $\mathbf{X}_{\mathbb{U}} := \{\mathbf{x}_{\mathbb{U}} \ ; \ \mathbf{x} \in \mathbf{X}\} \subseteq \mathcal{A}^{\mathbb{U}}$. Let $\mathbb{V} := \mathbb{U}^{\complement} \subset \mathbb{Z}^D$. For any $\mathbf{u} \in \mathcal{A}^{\mathbb{U}}$ and $\mathbf{v} \in \mathcal{A}^{\mathbb{V}}$, let $[\mathbf{uv}]$ denote the element of $\mathcal{A}^{\mathbb{Z}^D}$ such that $[\mathbf{uv}]_{\mathbb{U}} = \mathbf{u}$ and $[\mathbf{uv}]_{\mathbb{V}} = \mathbf{v}$. Let

$$\mathbf{X}^{(\mathbf{u})} \quad := \quad \left\{\mathbf{v} \in \mathcal{A}^{\mathbb{V}} \ ; \ [\mathbf{uv}] \in \mathbf{X}\right\}$$

be the set of all "$\mathbf{X}$-admissible completions" of $\mathbf{u}$ (thus, $\mathbf{X}^{(\mathbf{u})} \neq \emptyset \ \Leftrightarrow \ \mathbf{u} \in \mathbf{X}_{\mathbb{U}}$). If $\mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{Z}^D})$, and $\mathbf{u} \in \mathcal{A}^{\mathbb{U}}$, then let $\mu^{(\mathbf{u})}$ denote the *conditional measure* on $\mathcal{A}^{\mathbb{V}}$ induced by $\mathbf{u}$. If $\mathbb{U}$ is finite, then $\mu^{(\mathbf{u})}$ is just the restriction of μ to the cylinder set $\langle \mathbf{u} \rangle$. If $\mathbb{U}$ is infinite, then the precise definition of $\mu^{(\mathbf{u})}$ involves a 'disintegration' of μ into 'fibre measures' (we will suppress the details).

Let $\mu_{\mathbb{U}}$ be the projection of μ onto $\mathcal{A}^{\mathbb{U}}$. If $\mathsf{supp}\,(\mu) \subseteq \mathbf{X}$, then $\mathsf{supp}\,(\mu_{\mathbb{U}}) \subseteq \mathbf{X}_{\mathbb{U}}$, and for any $\mathbf{u} \in \mathcal{A}^{\mathbb{U}}$, $\mathsf{supp}\left(\mu^{(\mathbf{u})}\right) \subseteq \mathbf{X}^{(\mathbf{u})}$. We say that μ is a *Burton-Steif measure* on $\mathbf{X}$ if:

(1) $\mathsf{supp}\,(\mu) = \mathbf{X}$; and

(2) For any $\mathbb{U} \subset \mathbb{Z}^D$ whose complement $\mathbb{U}^{\complement}$ is finite, and for $\mu_{\mathbb{U}}$-almost any $\mathbf{u} \in \mathbf{X}_{\mathbb{U}}$, the measure $\mu^{(\mathbf{u})}$ is uniformly distributed on the (finite) set $\mathbf{X}^{(\mathbf{u})}$.

For example, if $\mathbf{X} = \mathcal{A}^{\mathbb{Z}^D}$, then the only Burton-Steif measure is the uniform Bernoulli measure. If $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}}$ is an irreducible SFT, then the only Burton-Steif measure is the Parry measure. If $r > 0$ and $\mathbb{B} := [-r...r]^D \subset \mathbb{Z}^D$, and $\mathbf{X}$ is an SFT determined by a set of admissible words $\mathbf{X}_{\mathbb{B}} \subset \mathcal{A}^{\mathbb{B}}$, then it is easy to check that any Burton-Steif measure μ on $\mathbf{X}$ must be a Markov random field with interaction range r.

Theorem 2B.2 (Burton and Steif) *Let $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift of finite type.*

(a) *Any maxentropy measure on $\mathbf{X}$ is a Burton-Steif measure.*

(b) *If $\mathbf{X}$ is strongly irreducible, then any Burton-Steif measure on $\mathbf{X}$ is a maxentropy measure for $\mathbf{X}$.*

Proof: **(a)** and **(b)** are Propositions 1.20 and 1.21 of Burton and Steif (1995), respectively. For a proof in the case when $\mathbf{X}$ is a symmetric nearest-neighbour subshift of finite type, see Propositions 1.19 and 4.1 of Burton and Steif (1994), respectively. □

Any subshift admits at least one maxentropy measure, so any SFT admits at least one Burton-Steif measure. Theorems 2B.1 and 2B.2 together imply:

Corollary 2B.3 *If $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^D}$ is an SFT which admits a unique Burton-Steif measure $\eta_{\mathbf{X}}$, then $\eta_{\mathbf{X}}$ is the unique maxentropy measure for $\mathbf{X}$. Thus, if $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$ and $\Phi(\mathbf{X}) = \mathbf{X}$, then $\Phi(\eta_{\mathbf{X}}) = \eta_{\mathbf{X}}$.* □

Example 2B.4: If $\mathbf{X} = \mathcal{A}^{\mathbb{Z}^D}$, then we get Theorem 2A.1, because the the unique Burton-Steif measure on $\mathcal{A}^{\mathbb{Z}^D}$ is the uniform Bernoulli measure. ◊

Remark 2B.5: If $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ is a subshift admitting a unique maxentropy measure μ, and $\mathsf{supp}(\mu) = \mathbf{X}$, then Weiss (2000)[Theorem 4.2] has observed that $\mathbf{X}$ automatically satisfies Incompressibility Corollary 2A.4. In particular, this applies to any SFT having a unique Burton-Steif measure. ♢

2C Periodic invariant measures

If $P \in \mathbb{N}$, then a sequence $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$ is *P-periodic* if $\sigma^P(\mathbf{a}) = \mathbf{a}$. If $A := |\mathcal{A}|$, then there are exactly A^P such sequences, and a measure μ on $\mathcal{A}^{\mathbb{Z}}$ is called *P-periodic* if μ is supported entirely on these P-periodic sequences. More generally, if $\mathbb{M}$ is any monoid and $\mathbb{P} \subset \mathbb{M}$ is any submonoid, then a configuration $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$ is *$\mathbb{P}$-periodic* if $\sigma^{\mathsf{p}}(\mathbf{a}) = \mathbf{a}$ for all $\mathsf{p} \in \mathbb{P}$. (For example, if $\mathbb{M} = \mathbb{Z}$ and $\mathbb{P} := P\mathbb{Z}$, then the $\mathbb{P}$-periodic configurations are the P-periodic sequences). Let $\mathcal{A}^{\mathbb{M}/\mathbb{P}}$ denote the set of $\mathbb{P}$-periodic configurations. If $P := |\mathbb{M}/\mathbb{P}|$, then $|\mathcal{A}^{\mathbb{M}/\mathbb{P}}| = A^P$. A measure μ is called *$\mathbb{P}$-periodic* if $\mathsf{supp}(\mu) \subseteq \mathcal{A}^{\mathbb{M}/\mathbb{P}}$.

Proposition 2C.1 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$. If $\mathbb{P} \subset \mathbb{M}$ is any submonoid and $|\mathbb{M}/\mathbb{P}|$ is finite, then there exists a $\mathbb{P}$-periodic, Φ-invariant measure.*

Proof sketch: If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$, then $\Phi(\mathcal{A}^{\mathbb{M}/\mathbb{P}}) \subseteq \mathcal{A}^{\mathbb{M}/\mathbb{P}}$. Thus, if μ is $\mathbb{P}$-periodic, then $\Phi^t(\mu)$ is $\mathbb{P}$-periodic for all $t \in \mathbb{N}$. Thus, the Cesàro limit of the sequence $\{\Phi^t(\mu)\}_{t=1}^{\infty}$ is $\mathbb{P}$-periodic and Φ-invariant. This Cesàro limit exists because $\mathcal{A}^{\mathbb{M}/\mathbb{P}}$ is finite. □

These periodic measures have finite (hence discrete) support, but by convex-combining them, it is easy to obtain (nonergodic) Φ-invariant measures with countable, dense support. When studying the invariant measures of CA, we usually regard these periodic measures (and their convex combinations) as somewhat trivial, and concentrate instead on invariant measures supported on aperiodic configurations.

2D Posexpansive and permutative CA

Let $\mathbb{B} \subset \mathbb{M}$ be a finite subset, and let $\mathcal{B} := \mathcal{A}^{\mathbb{B}}$. If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$, then we define a continuous function $\Phi_{\mathbb{B}}^{\mathbb{N}} : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathcal{B}^{\mathbb{N}}$ by

$$\Phi_{\mathbb{B}}^{\mathbb{N}}(\mathbf{a}) \quad := \quad [\mathbf{a}_{\mathbb{B}};\ \Phi(\mathbf{a})_{\mathbb{B}};\ \Phi^2(\mathbf{a})_{\mathbb{B}};\ \Phi^3(\mathbf{a})_{\mathbb{B}};\ \ldots] \in \mathcal{B}^{\mathbb{N}}. \tag{2.1}$$

Clearly, $\Phi_{\mathbb{B}}^{\mathbb{N}} \circ \Phi = \sigma \circ \Phi_{\mathbb{B}}^{\mathbb{N}}$. We say that Φ is *$\mathbb{B}$-posexpansive* if $\Phi_{\mathbb{B}}^{\mathbb{N}}$ is injective. Equivalently, for any $\mathbf{a}, \mathbf{a}' \in \mathcal{A}^{\mathbb{M}}$, if $\mathbf{a} \neq \mathbf{a}'$, then there is some $t \in \mathbb{N}$ such that $\Phi^t(\mathbf{a})_{\mathbb{B}} \neq \Phi^t(\mathbf{a}')_{\mathbb{B}}$. We say Φ is *positively expansive* (or *posexpansive*) if Φ is $\mathbb{B}$-posexpansive for some finite $\mathbb{B}$ (it is easy to see that this is equivalent to the usual definition of positive expansiveness a topological dynamical system).

Thus, if $\mathbf{X} := \Phi_{\mathbb{B}}^{\mathbb{N}}(\mathcal{A}^{\mathbb{M}}) \subset \mathcal{B}^{\mathbb{N}}$, then $\mathbf{X}$ is a compact, shift-invariant subset of $\mathcal{B}^{\mathbb{N}}$, and $\Phi_{\mathbb{B}}^{\mathbb{N}} : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathbf{X}$ is an isomorphism from the system $(\mathcal{A}^{\mathbb{M}}, \Phi)$ to the one-sided subshift $(\mathbf{X}, \sigma)$, which is sometimes called the *canonical factor* or *column shift* of Φ. The easiest examples of posexpansive CA are one-dimensional, permutative automata.

Proposition 2D.1 **(a)** *Suppose $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{N}})$ has neighbourhood $[r...R]$, where $0 \leq r < R$. Let $\mathbb{B} := [0....R)$ and let $\mathcal{B} := \mathcal{A}^{\mathbb{B}}$. Then*

$$\Big(\Phi \text{ is right permutative}\Big) \iff \Big(\Phi \text{ is } \mathbb{B}\text{-posexpansive, and } \Phi_{\mathbb{B}}^{\mathbb{N}}(\mathcal{A}^{\mathbb{N}}) = \mathcal{B}^{\mathbb{N}}\Big).$$

(b) *Suppose $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ has neighbourhood $[-L...R]$, where $-L < 0 < R$. Let $\mathbb{B} := [-L....R)$, and let $\mathcal{B} := \mathcal{A}^{\mathbb{B}}$. Then*

$$\Big(\Phi \text{ is bipermutative}\Big) \iff \Big(\Phi \text{ is } \mathbb{B}\text{-posexpansive, and } \Phi_{\mathbb{B}}^{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}) = \mathcal{B}^{\mathbb{N}}\Big).$$

Thus, one-sided, right-permutative CA and two-sided, bipermutative CA are both topologically conjugate the one-sided full shift $(\mathcal{B}^{\mathbb{N}}, \sigma)$, where $\mathcal{B}$ is an alphabet with $|\mathcal{A}|^{R+L}$ symbols (setting $L = 0$ in the one-sided case).

Proof: Suppose $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$ (where $\mathbb{M} = \mathbb{N}$ or $\mathbb{Z}$). Draw a picture of the spacetime diagram for Φ. For any $t \in \mathbb{N}$, and any $\mathbf{b} \in \mathcal{B}^{[0...t)}$, observe how (bi)permutativity allows you to reconstruct a unique $\mathbf{a}_{[-tL...tR)} \in \mathcal{A}^{[-tL...tR)}$ such that $\mathbf{b} = (\mathbf{a}_{\mathbb{B}}, \Phi(\mathbf{a})_{\mathbb{B}}, , \Phi^2(\mathbf{a})_{\mathbb{B}}, \ldots, \Phi^{t-1}(\mathbf{a})_{\mathbb{B}})$. By letting $t \to \infty$, we see that the function $\Phi_{\mathbb{B}}^{\mathbb{N}}$ is a bijection between $\mathcal{A}^{\mathbb{M}}$ and $\mathcal{B}^{\mathbb{N}}$. □

Remark 2D.2: (a) The idea of Proposition 2D.1 is implicit in (Hedlund, 1969, Theorem 6.7), but it was apparently first stated explicitly by Shereshevsky and Afraĭmovich (1992/93)[Theorem 1]. It was later rediscovered by Kleveland (1997)[Corollary 7.3] and Fagnani and Margara (1998)[Theorem 3.2].

(b) Proposition 2D.1(b) has been generalized to higher dimensions by Allouche and Skordev (2003)[Proposition 1], which states that a permutative CA on $\mathcal{A}^{\mathbb{Z}^D}$ (with $D \geq 2$) is conjugate to a full shift $(\mathcal{K}^{\mathbb{N}}, \sigma)$, where $\mathcal{K}$ is an uncountable, compact space. ◇

Proposition 2D.1 is quite indicative of the general case. Posexpansiveness occurs only in one-dimensional CA, in which it takes a very specific form. To explain this, suppose $(\mathbb{M}, \cdot)$ is a group with finite generating set $\mathbb{G} \subset \mathbb{M}$. For any $r > 0$, let $\mathbb{B}(r) := \{\mathsf{g}_1 \cdot \mathsf{g}_2 \cdots \mathsf{g}_r \ ; \ \mathsf{g}_1, \ldots, \mathsf{g}_r \in \mathbb{G}\}$. The ***dimension*** (or ***growth degree***) of $(\mathbb{M}, \cdot)$ is defined $\dim(\mathbb{M}, \cdot) := \limsup_{r\to\infty} \log |\mathbb{B}(r)| \,/\, \log(r)$. It can be shown that this number is independent of the choice of generating set $\mathbb{G}$, and is always an integer. For example, $\dim(\mathbb{Z}^D, +) = D$. If $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ is a subshift, then we define its topological entropy $h_{\text{top}}(\mathbf{X})$ with respect to $\dim(\mathbb{M})$ in the obvious fashion [see Example 5C.1(a)].

Theorem 2D.3 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$.*

(a) *If $\mathbb{M} = \mathbb{Z}^D \times \mathbb{N}^E$ with $D + E \geq 2$, then Φ cannot be posexpansive.*

(b) *If $\mathbb{M}$ is any group with $\dim(\mathbb{M}) \geq 2$, and $\mathbf{X} \subseteq \mathcal{A}^{\mathbb{M}}$ is any subshift with $h_{\text{top}}(\mathbf{X}) > 0$, and $\Phi(\mathbf{X}) \subseteq \mathbf{X}$, then the system $(\mathbf{X}, \Phi)$ cannot be posexpansive.*

(c) *Suppose* $\mathbb{M} = \mathbb{Z}$ *or* $\mathbb{N}$, *and* Φ *has neighbourhood* $[-L....R] \subset \mathbb{M}$. *Let* $\overline{L} := \max\{0, L\}$, $\overline{R} := \max\{0, R\}$ *and* $\mathbb{B} := \left[-\overline{L}....\overline{R}\right)$. *If* Φ *is posexpansive, then* Φ *is* $\mathbb{B}$*-posexpansive.*

Proof: **(a)** is (Shereshevsky, 1993, Corollary 2); see also (Finelli et al., 1998, Theorem 4.4). Part **(b)** follows by applying (Shereshevsky, 1996, Theorem 1.1) to the natural extension of $(\mathbf{X}, \Phi)$.

(c) The case $\mathbb{M} = \mathbb{Z}$ is (Kůrka, 1997, Proposition 7). The case $\mathbb{M} = \mathbb{N}$ is (Blanchard and Maass, 1997, Proposition 2.3). □

Proposition 2D.1 says bipermutative CA on $\mathcal{A}^{\mathbb{Z}}$ are conjugate to full shifts. Using his formidable theory of *textile systems*, Nasu extended this to *all* posexpansive CA on $\mathcal{A}^{\mathbb{Z}}$.

2D.4. Nasu's Theorem. *Let* $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ *and let* $\mathbb{B} \subset \mathbb{Z}$. *If* Φ *is* $\mathbb{B}$*-posexpansive, then* $\Phi_{\mathbb{B}}^{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}}) \subseteq \mathcal{B}^{\mathbb{N}}$ *is a one-sided SFT which is conjugate to a one-sided full shift* $\mathcal{C}^{\mathbb{N}}$ *for some alphabet* $\mathcal{C}$ *with* $|\mathcal{C}| \geq 3$.

Proof sketch: The fact that $\mathbf{X} := \Phi_{\mathbb{B}}^{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}})$ is an SFT follows from (Kůrka, 1997, Theorem 10) or (Kůrka, 2001, Theorem 10.1). Next, Theorem 3.12(1) on p.49 of Nasu (1995) asserts that, if Φ is *any* surjective endomorphism of an irreducible, aperiodic, SFT $\mathbf{Y} \subseteq \mathcal{A}^{\mathbb{Z}}$, and $(\mathbf{Y}, \Phi)$ is itself conjugate to an SFT, then $(\mathbf{Y}, \Phi)$ is actually conjugate to a full shift $(\mathcal{C}^{\mathbb{N}}, \sigma)$ for some alphabet $\mathcal{C}$ with $|\mathcal{C}| \geq 3$. Let $\mathbf{Y} := \mathcal{A}^{\mathbb{Z}}$ and invoke Kůrka's result.

For a direct proof not involving textile systems, see (Maass, 1996, Theorem 4.9). □

Remarks 2D.5: (a) See Theorem 4A.4(d) for an 'ergodic' version of Theorem 2D.4.

(b) In contrast to Proposition 2D.1, Nasu's Theorem 2D.4 does *not* say that $\Phi_{\mathbb{B}}^{\mathbb{N}}(\mathcal{A}^{\mathbb{Z}})$ itself is a full shift —only that it is conjugate to one. ♢

If $(\mathbf{X}, \mu; \Psi)$ is a measure-preserving dynamical system (MPDS) with sigma-algebra $\mathfrak{B}$, then a ***one-sided generator*** is a finite partition $\mathfrak{P} \subset \mathfrak{B}$ such that $\bigvee_{t=0}^{\infty} \Psi^{-t}\mathfrak{P} \underset{\mu}{=} \mathfrak{B}$. If $\mathfrak{P}$ has C elements, and $\mathcal{C}$ is a finite set with $|\mathcal{C}| = C$, then $\mathfrak{P}$ induces an essentially injective function $\mathfrak{p} : \mathbf{X} \longrightarrow \mathcal{C}^{\mathbb{N}}$ such that $\mathfrak{p} \circ \Psi = \sigma \circ \mathfrak{p}$. Thus, if $\lambda := \mathfrak{p}(\mu)$, then $(\mathbf{X}, \mu; \Psi)$ is measurably isomorphic to the (one-sided) stationary stochastic process $(\mathcal{C}^{\mathbb{N}}, \lambda; \sigma)$. If Ψ is invertible, then a (two-sided) ***generator*** is a finite partition $\mathfrak{P} \subset \mathfrak{B}$ such that $\bigvee_{t=-\infty}^{\infty} \Psi^{t}\mathfrak{P} \underset{\mu}{=} \mathfrak{B}$. The Krieger Generator Theorem says every finite-entropy, invertible MPDS has a generator; indeed, if $h(\Psi, \mu) \leq \log_2(C)$, then $(\mathbf{X}, \mu; \Psi)$ has a generator with C or less elements. (See Ergodic Theory: Basic Examples and Constructions for more information.) If $|\mathcal{C}| = C$, then once again, $\mathfrak{P}$ induces a measurable isomorphism from $(\mathbf{X}, \mu; \Psi)$ to a two-sided stationary stochastic process $(\mathcal{C}^{\mathbb{Z}}, \lambda; \sigma)$, for some stationary measure λ on $\mathcal{A}^{\mathbb{Z}}$.

2D.6. Universal Representation Corollary. *Let $\mathbb{M} = \mathbb{N}$ or $\mathbb{Z}$, and let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$ have neighbourhood $\mathbb{H} \subset \mathbb{M}$. Suppose that*

either $\mathbb{M} = \mathbb{N}$, *Φ is right-permutative, and $\mathbb{H} = [r...R]$ for some $0 \leq r < R$, and then let $C := R \log_2 |\mathcal{A}|$;*

or $\mathbb{M} = \mathbb{Z}$, *Φ is bipermutative, and $\mathbb{H} = [-L...R]$, and then let $C := (\overline{L} + \overline{R}) \log_2 |\mathcal{A}|$ where $\overline{L} := \max\{0, L\}$ and $\overline{R} := \max\{0, R\}$;*

or $\mathbb{M} = \mathbb{Z}$ *and Φ is positively expansive, and $h_{\mathrm{top}}(\mathcal{A}^{\mathbb{M}}, \Phi) = \log_2(C)$ for some $C \in \mathbb{N}$.*

(a) *Let $(\mathbf{X}, \mu; \Psi)$ be any MPDS with a one-sided generator having at most C elements. Then there exists $\nu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{M}}; \Phi)$ such that the system $(\mathcal{A}^{\mathbb{M}}, \nu; \Phi)$ is measurably isomorphic to $(\mathbf{X}, \mu; \Psi)$.*

(b) *Let $(\mathbf{X}, \mu; \Psi)$ be an invertible MPDS, with measurable entropy $h(\mu, \phi) \leq \log_2(C)$. Then there exists $\nu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{M}}; \Phi)$ such that the natural extension of the system $(\mathcal{A}^{\mathbb{M}}, \nu; \Phi)$ is measurably isomorphic to $(\mathbf{X}, \mu, \Psi)$.*

Proof: Under each of the three hypotheses, Proposition 2D.1 or Theorem 2D.4 yields a topological conjugacy $\Gamma : (\mathcal{C}^{\mathbb{N}}, \sigma) \longrightarrow (\mathcal{A}^{\mathbb{M}}, \Phi)$, where $\mathcal{C}$ is a set of cardinality C.

(a) As discussed above, there is a measure λ on $\mathcal{C}^{\mathbb{N}}$ such that $(\mathcal{C}^{\mathbb{N}}, \lambda; \sigma)$ is measurably isomorphic to $(\mathbf{X}, \mu, \Psi)$. Thus, $\nu := \Gamma[\lambda]$ is a Φ-invariant measure on $\mathcal{A}^{\mathbb{M}}$, and $(\mathcal{A}^{\mathbb{M}}, \nu; \Phi)$ is isomorphic to $(\mathcal{C}^{\mathbb{N}}, \mu, \Psi)$ via Γ.

(b) As discussed above, there is a measure λ on $\mathcal{C}^{\mathbb{Z}}$ such that $(\mathcal{C}^{\mathbb{Z}}, \lambda; \sigma)$ is measurably isomorphic to $(\mathbf{X}, \mu, \Psi)$. Let $\lambda_{\mathbb{N}}$ be the projection of λ to $\mathcal{C}^{\mathbb{N}}$; then $(\mathcal{C}^{\mathbb{N}}, \lambda_{\mathbb{N}}; \sigma)$ is a one-sided stationary process. Thus, $\nu := \Gamma[\lambda_{\mathbb{N}}]$ is a Φ-invariant measure on $\mathcal{A}^{\mathbb{M}}$, and $(\mathcal{A}^{\mathbb{M}}, \nu; \Phi)$ is isomorphic to $(\mathcal{C}^{\mathbb{N}}, \lambda_{\mathbb{N}}; \sigma)$ via Γ. Thus, the natural extension of $(\mathcal{A}^{\mathbb{M}}, \nu; \Phi)$ is isomorphic to the natural extension of $(\mathcal{C}^{\mathbb{N}}, \lambda_{\mathbb{N}}; \sigma)$, which is $(\mathcal{C}^{\mathbb{Z}}, \lambda; \sigma)$, which is in turn isomorphic to $(\mathbf{X}, \mu; \Psi)$. □

Remark 2D.7: The Universal Representation Corollary implies that studying the measurable dynamics of the CA Φ with respect to some arbitrary Φ-invariant measure ν will generally tell us nothing whatsoever about Φ. For these measurable dynamics to be meaningful, we must pick a measure on $\mathcal{A}^{\mathbb{M}}$ which is somehow 'natural' for Φ. First, this measure should shift-invariant (because one of the defining properties of CA is that they commute with the shift). Second, we should seek a measure which has maximal Φ-entropy or is distinguished in some other way. (In general, the measures ν given by the Universal Representation Corollary will neither be σ-invariant, nor have maximal entropy for Φ.) ◊

If $\Phi_{\mathbb{N}} \in \mathsf{CA}(\mathcal{A}^{\mathbb{N}})$, and $\Phi_{\mathbb{Z}} \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ is the CA obtained by applying the same local rule to all coordinates in $\mathbb{Z}$, then $\Phi_{\mathbb{Z}}$ can *never* be posexpansive: if $\mathbb{B} = [-B...B]$, and $\mathbf{a}, \mathbf{a}' \in \mathcal{A}^{\mathbb{Z}}$ are any two sequences such that $\mathbf{a}_{(-\infty...-B)} \neq \mathbf{a}'_{(-\infty...-B)}$, then $\Phi^t(\mathbf{a})_{\mathbb{B}} = \Phi^t(\mathbf{a}')_{\mathbb{B}}$ for all $t \in \mathbb{N}$, because the local rule of Φ only propagates information to the left. Thus, in particular, the posexpansive CA on $\mathcal{A}^{\mathbb{Z}}$ are completely unrelated to the posexpansive CA on $\mathcal{A}^{\mathbb{N}}$. Nevertheless, posexpansive CA on $\mathcal{A}^{\mathbb{N}}$ behave quite similarly to those on $\mathcal{A}^{\mathbb{Z}}$.

Theorem 2D.8 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{N}})$ have neighbourhood $[r...R]$, where $0 \leq r < R$, and let $\mathbb{B} := [0...R)$. Suppose Φ is posexpansive. Then:*

(a) $\mathbf{X} := \Phi_{\mathbb{B}}^{\mathbb{N}}(\mathcal{A}^{\mathbb{N}}) \subseteq \mathcal{B}^{\mathbb{N}}$ *is a topologically mixing SFT.*

(b) *The topological entropy of Φ is $\log_2(k)$ for some $k \in \mathbb{N}$.*

(c) *If η is the uniform measure on $\mathcal{A}^{\mathbb{N}}$, then $\Phi_{\mathbb{B}}^{\mathbb{N}}(\eta)$ is the Parry measure on $\mathbf{X}$. Thus, η is the maxentropy measure for Φ.*

Proof: See (Blanchard and Maass, 1997, Corollary 3.7 and Theorems 3.8 and 3.9) or (Maass, 1996, Theorem 4.8(1,2,4)). □

Remarks: (a) See Theorem 4A.2 for an 'ergodic' version of Theorem 2D.8.

(b) The analog of Nasu's Theorem 2D.4 (i.e. conjugacy to a full shift) is *not* true for posexpansive CA on $\mathcal{A}^{\mathbb{N}}$. See Boyle et al. (1997) for a counterexample.

(c) If $\Phi : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{A}^{\mathbb{N}}$ is *invertible*, then we define the function $\Phi_{\mathbb{B}}^{\mathbb{Z}} : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{B}^{\mathbb{Z}}$ by extending the definition of $\Phi_{\mathbb{B}}^{\mathbb{N}}$ to negative times. We say that Φ is *expansive* if $\Phi_{\mathbb{B}}^{\mathbb{Z}}$ is bijective for some finite $\mathbb{B} \subset \mathbb{N}$. Expansiveness is a much weaker condition than *positive* expansiveness. Nevertheless, the analog of Theorem 2D.8(a) is true: if $\Phi : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{A}^{\mathbb{N}}$ is invertible and expansive, then $\mathcal{B}^{\mathbb{Z}}$ is conjugate to a (two-sided) subshift of finite type; see (Nasu, 2002, Theorem 1.3).

2E Measure rigidity in algebraic CA

Theorem 2A.1 makes the uniform measure η a 'natural' invariant measure for a surjective CA Φ. However, Proposition 2C.1 and Corollary 2D.6 indicate that there are many other (unnatural) Φ-invariant measures as well. Thus, it is natural to seek conditions under which the uniform measure η is the unique (or almost unique) measure which is Φ-invariant, shift-invariant, and perhaps 'nondegenerate' in some other sense —a phenomenon which is sometimes called *measure rigidity*. Measure rigidity has been best understood when Φ is compatible with an underlying algebraic structure on $\mathcal{A}^{\mathbb{M}}$.

Let $\star : \mathcal{A}^{\mathbb{M}} \times \mathcal{A}^{\mathbb{M}} \longrightarrow \mathcal{A}^{\mathbb{M}}$ be a binary operation ('multiplication') and let $\bullet^{-1} : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathcal{A}^{\mathbb{M}}$ be an unary operation ('inversion') such that $(\mathcal{A}^{\mathbb{M}}, \star)$ is a group, and suppose both operations are continuous and commute with all $\mathbb{M}$-shifts; then $(\mathcal{A}^{\mathbb{M}}, \star)$ is called a *group shift*. For example, if $(\mathcal{A}, \cdot)$ is itself a finite group, and $\mathcal{A}^{\mathbb{M}}$ is treated as a Cartesian product and endowed with componentwise multiplication, then $(\mathcal{A}^{\mathbb{M}}, \cdot)$ is a group shift. However, not all group shifts arise in this manner; see Kitchens (1987, 2000); Kitchens and Schmidt (1989, 1992) and Schmidt (1995). If $(\mathcal{A}^{\mathbb{M}}, \star)$ is a group shift, then a *subgroup shift* is a closed, shift-invariant subgroup $\mathbf{G} \subset \mathcal{A}^{\mathbb{M}}$ (i.e. $\mathbf{G}$ is both a subshift and a subgroup).

If $(\mathbf{G}, \star)$ is a subgroup shift, then the *Haar measure* on $\mathbf{G}$ is the unique probability measure $\eta_{\mathbf{G}}$ on $\mathbf{G}$ which is invariant under translation by all elements of $\mathbf{G}$. That is, if $\mathbf{g} \in \mathbf{G}$, and $\mathbf{U} \subset \mathbf{G}$ is any measurable subset, and $\mathbf{U} \star \mathbf{g} := \{\mathbf{u} \star \mathbf{g} ; \mathbf{u} \in \mathbf{U}\}$, then $\eta_{\mathbf{G}}[\mathbf{U} \star \mathbf{g}] = \eta_{\mathbf{G}}[\mathbf{U}]$. In particular, if $\mathbf{G} = \mathcal{A}^{\mathbb{M}}$, then $\eta_{\mathbf{G}}$ is just the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{M}}$.

If $(\mathcal{A}^{\mathbb{M}}, \star)$ is a group shift, and $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{M}}$ is a subgroup shift, and $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$, then Φ is called an *endomorphic* (or *algebraic*) CA on $\mathbf{G}$ if $\Phi(\mathbf{G}) \subseteq \mathbf{G}$ and $\Phi : \mathbf{G} \longrightarrow \mathbf{G}$ is an endomorphism of $(\mathbf{G}, \star)$ as a topological group. Let $\mathsf{ECA}(\mathbf{G}, \star)$ denote the set of endomorphic CA on $\mathbf{G}$. For example, suppose $(\mathcal{A}, +)$ is abelian, and let $(\mathbf{G}, \star) := (\mathcal{A}^{\mathbb{M}}, +)$ with the product group structure; then the endomorphic CA on $\mathcal{A}^{\mathbb{M}}$ are exactly the linear CA. However, if $(\mathcal{A}, \cdot)$ is a *non*abelian group, then endomorphic CA on $(\mathcal{A}^{\mathbb{M}}, \cdot)$ are *not* the same as multiplicative CA.

Even in this context, CA admit many nontrivial invariant measures. For example, it is easy to check the following:

Proposition 2E.1 *Let $\mathcal{A}^{\mathbb{M}}$ be a group shift and let $\Phi \in \mathsf{ECA}\left(\mathcal{A}^{\mathbb{M}}, \star\right)$. Let $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{M}}$ be any Φ-invariant subgroup shift; then the Haar measure on $\mathbf{G}$ is Φ-invariant.* □

For example, if $(\mathcal{A}, +)$ is any nonsimple abelian group, and $(\mathcal{A}^{\mathbb{M}}, +)$ has the product group structure, then $\mathcal{A}^{\mathbb{M}}$ admits many nontrivial subgroup shifts; see Kitchens (1987). If Φ is any linear CA on $\mathcal{A}^{\mathbb{M}}$ with scalar coefficients, then *every* subgroup shift of $\mathcal{A}^{\mathbb{M}}$ is Φ-invariant, so Proposition 2E.1 yields many nontrivial Φ-invariant measures. To isolate η as a unique measure, we must impose further restrictions. The first nontrivial results in this direction were by Host et al. (2003). Let $h(\Phi, \mu)$ be the entropy of Φ relative to the measure μ (see §5 for definition).

Proposition 2E.2 *Let $\mathcal{A} := \mathbb{Z}_{/p}$, where p is prime. Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ be a linear CA with neighbourhood $\{0, 1\}$, and let $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}}; \Phi, \sigma)$. If μ is σ-ergodic, and $h(\Phi, \mu) > 0$, then μ is the Haar measure η on $\mathcal{A}^{\mathbb{Z}}$.*

Proof: See (Host et al., 2003, Theorem 12). □

A similar idea is behind the next result, only with the roles of Φ and σ reversed. If μ is a measure on $\mathcal{A}^{\mathbb{N}}$, and $\mathbf{b} \in \mathcal{A}^{[1...\infty)}$, then we define the conditional measure $\mu^{(\mathbf{b})}$ on $\mathcal{A}$ by $\mu^{(\mathbf{b})}(a) := \mu[x_0 = a | \mathbf{x}_{[1...\infty)} = \mathbf{b}]$, where $\mathbf{x}$ is a μ-random sequence. For example, if μ is a Bernoulli measure, then $\mu^{(\mathbf{b})}(a) = \mu[x_0 = a]$, independent of $\mathbf{b}$; if μ is a Markov measure, then $\mu^{(\mathbf{b})}(a) = \mu[x_0 = a | x_1 = b_1]$.

Proposition 2E.3 *Let $(\mathcal{A}, \cdot)$ be any finite (possibly nonabelian) group, and let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{N}})$ have multiplicative local rule $\phi : \mathcal{A}^{\{0,1\}} \longrightarrow \mathcal{A}$ defined by $\phi(a_0, a_1) := a_0 \cdot a_1$. Let $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}}; \Phi, \sigma)$. If μ is Φ-ergodic, then there is some subgroup $\mathcal{C} \subset \mathcal{A}$ such that, for every $\mathbf{b} \in \mathcal{A}^{[1...\infty]}$, $\mathsf{supp}\left(\mu^{(\mathbf{b})}\right)$ is a right coset of $\mathcal{C}$, and $\mu^{(\mathbf{b})}$ is uniformly distributed on this coset.*

Proof: See (Pivato, 2005b, Theorem 3.1). □

Example 2E.4: Let Φ and μ be as in Proposition 2E.3. Let η be the Haar measure on $\mathcal{A}^{\mathbb{N}}$

(a) μ has *complete connections* if $\mathsf{supp}\left(\mu^{(\mathbf{b})}\right) = \mathcal{A}$ for μ-almost all $\mathbf{b} \in \mathcal{A}^{[1...\infty)}$. Thus, if μ has complete connections in Proposition 2E.3, then $\mu = \eta$.

(b1) Suppose $h(\mu, \sigma) \; > \; h_0 := \max\{\log_2 |\mathcal{C}| \; ; \; \mathcal{C}$ a proper subgroup of $\mathcal{A}\}$. Then $\mu = \eta$.

(b2) In particular, suppose $\mathcal{A} = (\mathbb{Z}_{/p}, +)$, where p is prime; then $h_0 = 0$. Thus, if Φ has local rule $\phi(a_0, a_1) := a_0 + a_1$, and μ is any σ-invariant, Φ-ergodic measure with $h(\mu, \sigma) > 0$, then $\mu = \eta$. This is closely analogous to Proposition 2E.2, but 'dual' to it, because the roles of Φ and σ are reversed in the ergodicity and entropy hypotheses.

(c) If $\mathcal{C} \subset \mathcal{A}$ is a subgroup, and μ is the Haar measure on the subgroup shift $\mathcal{C}^{\mathbb{N}} \subset \mathcal{A}^{\mathbb{N}}$, then μ satisfies the conditions of Proposition 2E.3. Other, less trivial possibilities also exist (Pivato, 2005b, Examples 3.2(b,c)). ♢

If μ is a measure on $\mathcal{A}^{\mathbb{Z}}$, and $\mathbf{X}, \mathbf{Y} \subset \mathcal{A}^{\mathbb{Z}}$, then we say $\mathbf{X}$ *essentially equals* $\mathbf{Y}$ and write $\mathbf{X} \underset{\mu}{=} \mathbf{Y}$ if $\mu[\mathbf{X} \triangle \mathbf{Y}] = 0$. If $n \in \mathbb{N}$, then let

$$\mathfrak{I}_n(\mu) \quad := \quad \left\{\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}} \; ; \; \sigma^n(\mathbf{X}) \underset{\mu}{=} \mathbf{X}\right\}$$

be the sigma-algebra of subsets of $\mathcal{A}^{\mathbb{Z}}$ which are 'essentially' σ^n-invariant. Thus, μ is σ-ergodic if and only if $\mathfrak{I}_1(\mu)$ is trivial (i.e. contains only sets of measure zero or one). We say μ is *totally σ-ergodic* if $\mathfrak{I}_n(\mu)$ is trivial for all $n \in \mathbb{N}$ (see Ergodicity and Mixing Properties).

Let $(\mathcal{A}^{\mathbb{Z}}, *)$ be any group shift. The identity element $\mathbf{e}$ of $(\mathcal{A}^{\mathbb{Z}}, *)$ is a constant sequence. Thus, if $\Phi \in \mathsf{ECA}\left(\mathcal{A}^{\mathbb{Z}}, *\right)$ is surjective, then $\ker(\Phi) := \left\{\mathbf{a} \in \mathcal{A}^{\mathbb{Z}} \; ; \; \Phi(\mathbf{a}) = \mathbf{e}\right\}$ is a finite, shift-invariant subgroup of $\mathcal{A}^{\mathbb{Z}}$ (i.e. a finite collection of σ-periodic sequences).

Proposition 2E.5 *Let $(\mathcal{A}^{\mathbb{Z}}, *)$ be a (possibly nonabelian) group shift, and let $\Phi \in \mathsf{ECA}\left(\mathcal{A}^{\mathbb{Z}}, *\right)$ be bipermutative, with neighbourhood $\{0, 1\}$. Let $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}}; \Phi, \sigma)$. Suppose that:*

(IE) *μ is totally ergodic for σ;* **(H)** *$h(\Phi, \mu) > 0$; and*

(K) $\ker(\Phi)$ *contains no nontrivial σ-invariant subgroups.*

Then μ is the Haar measure on $\mathcal{A}^{\mathbb{Z}}$.

Proof: See (Pivato, 2005b, Theorem 5.2). □

Example 2E.6: If $\mathcal{A} = \mathbb{Z}_{/p}$ and $(\mathcal{A}^{\mathbb{Z}}, +)$ is the product group, then Φ is a linear CA and condition (c) is automatically satisfied, so Proposition 2E.5 becomes a special case of Proposition 2E.2. ♢

If $\Phi \in \mathsf{ECA}\left(\mathcal{A}^{\mathbb{Z}}, *\right)$, then we have an increasing sequence of finite, shift-invariant subgroups $\ker(\Phi) \subseteq \ker(\Phi^2) \subseteq \ker(\Phi^3) \subseteq \cdots$. If $\mathbf{K}(\Phi) := \bigcup_{n=1}^{\infty} \ker(\Phi^n)$, then $\mathbf{K}(\Phi)$ is a countable, shift-invariant subgroup of $(\mathcal{A}^{\mathbb{Z}}, *)$.

Theorem 2E.7 *Let $(\mathcal{A}^{\mathbb{Z}}, +)$ be an abelian group shift, and let $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{Z}}$ be a subgroup shift. Let $\Phi \in \mathsf{ECA}(\mathbf{G}, +)$ be bipermutative, and let $\mu \in \mathfrak{M}_{\text{eas}}(\mathbf{G}; \Phi, \sigma)$. Suppose:*

(I) $\mathfrak{I}_{kP}(\mu) = \mathfrak{I}_1(\mu)$, *where P is the lowest common multiple of the σ-periods of all elements in $\ker(\Phi)$, and $k \in \mathbb{N}$ is any common multiple of all prime factors of $|\mathcal{A}|$.*

(H) $h(\Phi, \mu) > 0$.

Furthermore, suppose that either:

(E1) *μ is ergodic for the $\mathbb{N} \times \mathbb{Z}$ action (Φ, σ);*

(K1) *Every infinite, σ-invariant subgroup of $\mathbf{K}(\Phi) \cap \mathbf{G}$ is dense in $\mathbf{G}$;*

or:

(E2) *μ is σ-ergodic;*

(K2) *Every infinite, (Φ, σ)-invariant subgroup of $\mathbf{K}(\Phi) \cap \mathbf{G}$ is dense in $\mathbf{G}$.*

Then μ is the Haar measure on $\mathbf{G}$.

Proof: See Theorems 3.3 and 3.4 of Sablik (2007b), or Théorèmes V.4 and V.5 on p.115 of Sablik (2006). In the special case when $\mathbf{G}$ has topological entropy $\log_2(p)$ (where p is prime), Sobottka has given a different and simpler proof, by using his theory of 'quasigroup shifts' to establish an isomorphism between Φ and a linear CA on $\mathbb{Z}_{/p}$, and then invoking Theorem 2E.2. See Theorems 7.1 and 7.2 of Sobottka (2007a), or Teoremas IV.3.1 and IV.3.2 on pp.100-101 of Sobottka (2005). □

Example 2E.8: (a) Let $\mathcal{A} := \mathbb{Z}_{/p}$, where p is prime. Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ be linear, with neighbourhood $\{0, 1\}$, and let $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}}; \Phi, \sigma)$. Suppose that μ is (Φ, σ)-ergodic, $h(\Phi, \mu) > 0$, and $\mathfrak{I}_{p(p-1)}(\mu) = \mathfrak{I}_1(\mu)$. Setting $k = p$ and $P = p - 1$ in Theorem 2E.7, we conclude that μ is the Haar measure on $\mathcal{A}^{\mathbb{Z}}$. This result first appeared as (Host et al., 2003, Theorem 13).

(b) If $(\mathcal{A}^{\mathbb{Z}}, *)$ is abelian, then Proposition 2E.5 is a special case of Theorem 2E.7 [hypothesis **(IE)** of the former implies hypotheses **(I)** and **(E2)** of the latter, while **(K)** implies **(K2)**]. Note, however, that Proposition 2E.5 also applies to nonabelian groups. ◇

An *algebraic $\mathbb{Z}^D$-action* is an action of $\mathbb{Z}^D$ by automorphisms on a compact abelian group $\mathbf{G}$. For example, if $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ is an abelian subgroup shift, then σ is an algebraic $\mathbb{Z}^D$-action. The invariant measures of algebraic $\mathbb{Z}^D$-actions have been studied by Schmidt (1995)[§29], Silberger (2005)[§7], and Einsiedler (2004, 2005).

If $\Phi \in \mathsf{CA}(\mathbf{G})$, then a *complete history* for Φ is a sequence $(\mathbf{g}_t)_{t \in \mathbb{Z}} \in \mathbf{G}^{\mathbb{Z}}$ such that $\Phi(\mathbf{g}_t) = \mathbf{g}_{t+1}$ for all $t \in \mathbb{Z}$. Let $\Phi^{\mathbb{Z}}(\mathbf{G}) \subset \mathbf{G}^{\mathbb{Z}} \subsetneq (\mathcal{A}^{\mathbb{Z}^D})^{\mathbb{Z}} \cong \mathcal{A}^{\mathbb{Z}^{D+1}}$ be the set of all complete histories for Φ; then $\Phi^{\mathbb{Z}}(\mathbf{G})$ is a subshift of $\mathcal{A}^{\mathbb{Z}^{D+1}}$. If $\Phi \in \mathsf{ECA}(\mathbf{G})$, then $\Phi^{\mathbb{Z}}(\mathbf{G})$ is itself an abelian subgroup shift, and the shift action of $\mathbb{Z}^{D+1}$ on $\Phi^{\mathbb{Z}}(\mathbf{G})$ is thus an algebraic $\mathbb{Z}^{D+1}$-action. Any (Φ, σ)-invariant measure on $\mathbf{G}$ extends in the obvious way to a σ-invariant measure on $\Phi^{\mathbb{Z}}(\mathbf{G})$. Thus, any result about the invariant measures (or rigidity) of algebraic $\mathbb{Z}^{D+1}$-actions can be translated immediately into a result about the invariant measures (or rigidity) of endomorphic cellular automata.

Proposition 2E.9 *Let* $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ *be an abelian subgroup shift and let* $\Phi \in \mathsf{ECA}(\mathbf{G})$. *Suppose* $\mu \in \mathfrak{M}_{eas}(\mathbf{G}; \Phi, \sigma)$ *is* (Φ, σ)*-totally ergodic, and has entropy dimension* $d \in [1...D]$ (see §5C). *If the system* $(\mathbf{G}, \mu; \Phi, \sigma)$ *admits no factors whose d-dimensional measurable entropy is zero, then there is a* Φ*-invariant subgroup shift* $\mathbf{G}' \subseteq \mathbf{G}$ *and some element* $\mathbf{x} \in \mathbf{G}$ *such that* μ *is the translated Haar measure on the 'affine' subset* $\mathbf{G}' + \mathbf{x}$.

Proof: This follows from (Einsiedler, 2005, Corollary 2.3). □

If we remove the requirement of 'no zero-entropy factors', and instead require $\mathbf{G}$ and Φ to satisfy certain technical algebraic conditions, then μ must be the Haar measure on $\mathbf{G}$ (Einsiedler, 2005, Theorem 1.2). These strong hypotheses are probably necessary, because in general, the system $(\mathbf{G}, \sigma, \Phi)$ admits uncountably many distinct nontrivial invariant measures, even if $(\mathbf{G}, \sigma, \Phi)$ is ***irreducible***, meaning that $\mathbf{G}$ contains no proper, infinite, Φ-invariant subgroup shifts:

Proposition 2E.10 *Let* $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ *be an abelian subgroup shift, let* $\Phi \in \mathsf{ECA}(\mathbf{G})$, *and suppose* $(\mathbf{G}, \sigma, \Phi)$ *is irreducible. For any* $s \in [0, 1)$, *there exists a* (Φ, σ)*-ergodic measure* $\mu \in \mathfrak{M}_{eas}(\mathbf{G}; \Phi, \sigma)$ *such that* $h(\mu, \Phi^n \circ \sigma^{\mathsf{z}}) = s \cdot h_{\text{top}}(\mathbf{G}, \Phi^n \circ \sigma^{\mathsf{z}})$ *for every* $n \in \mathbb{N}$ *and* $\mathsf{z} \in \mathbb{Z}^D$.

Proof: This follows from (Einsiedler, 2004, Corollary 1.4). □

Let $\mu \in \mathfrak{M}_{eas}(\mathcal{A}^{\mathbb{M}}; \sigma)$ and let $\mathbb{H} \subset \mathbb{M}$ be a finite subset. We say that μ is $\mathbb{H}$-***mixing*** if, for any $\mathbb{H}$-indexed collection $\{\mathbf{U}_{\mathsf{h}}\}_{\mathsf{h}\in\mathbb{H}}$ of measurable subsets of $\mathcal{A}^{\mathbb{M}}$,

$$\lim_{n\to\infty} \mu \left[\bigcap_{\mathsf{h}\in\mathbb{H}} \sigma^{n\mathsf{h}}(\mathbf{U}_{\mathsf{h}}) \right] \quad = \quad \prod_{\mathsf{h}\in\mathbb{H}} \mu\left[\mathbf{U}_{\mathsf{h}}\right].$$

For example, if $|\mathbb{H}| = H$, then any H-multiply σ-mixing measure (see §4A) is $\mathbb{H}$-mixing.

Proposition 2E.11 *Let* $\mathbf{G} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ *be an abelian subgroup shift and let* $\Phi \in \mathsf{ECA}(\mathbf{G})$ *have neighbourhood* $\mathbb{H}$ *(with* $|\mathbb{H}| \geq 2$*). Suppose* $(\mathbf{G}, \sigma, \Phi)$ *is irreducible, and let* $\mu \in \mathfrak{M}_{eas}(\mathcal{A}^{\mathbb{Z}^D}; \Phi, \sigma)$. *Then* μ *is* $\mathbb{H}$*-mixing if and only if* μ *is the Haar measure of* $\mathbf{G}$.

Proof: This follows from (Schmidt, 1995, Corollary 29.5, p.289) (note that Schmidt uses 'almost minimal' to mean 'irreducible'). □

2F The Furstenberg conjecture

Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ be the circle group, which we identify with the interval $[0, 1)$. Define the functions $\times_2, \times_3 : \mathbb{T}^1 \longrightarrow \mathbb{T}^1$ by $\times_2(t) = 2t \pmod 1$ and $\times_3(t) = 3t \pmod 1$. Clearly, these maps commute, and preserve the Lebesgue measure on $\mathbb{T}^1$. Furstenberg (1967) speculated that the *only* nonatomic $\times_2$- and $\times_3$-invariant measure on $\mathbb{T}^1$ was the Lebesgue measure. Rudolph (1990) showed that, if ρ is $(\times_2, \times_3)$-invariant measure and *not* Lebesgue, then the systems $(\mathbb{T}^1, \rho, \times_2)$ and $(\mathbb{T}^1, \rho, \times_2)$ have zero entropy; this was later generalized by Johnson (1992) and Host (1995). It is not known whether any nonatomic measures exist on $\mathbb{T}^1$ which satisfy Rudolph's conditions; this is considered an outstanding problem in abstract ergodic theory.

To see the connection between Furstenberg's Conjecture and cellular automata, let $\mathcal{A} = \{0, 1, 2, 3, 4, 5\}$, and define the surjection $\Psi : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathbb{T}^1$ by mapping each $\mathbf{a} \in \mathcal{A}^{\mathbb{N}}$ to the element of $[0, 1)$ having $\mathbf{a}$ as its base-6 expansion. That is:

$$\Psi(a_0, a_1, a_2, \ldots) \quad := \quad \sum_{n=0}^{\infty} \frac{a^n}{6^n}.$$

The map Ψ is injective everywhere except on the countable set of sequences ending in $[000\ldots]$ or $[555\ldots]$ (on this set, Ψ is 2-to-1). Furthermore, Ψ defines a semiconjugacy from $\times_2$ and $\times_3$ into two CA on $\mathcal{A}^{\mathbb{N}}$. Let $\mathbb{H} := \{0, 1\}$, and define local maps $\xi_2, \xi_3 : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$ as follows:

$$\xi_2(a_0, a_1) \;=\; \Big[2a_0\Big]_6 + \left\lceil \frac{a_1}{3} \right\rceil \quad \text{and} \quad \xi_3(a_0, a_1) \;=\; \Big[3a_0\Big]_6 + \left\lceil \frac{a_1}{2} \right\rceil,$$

where, $[a]_6$ is the least residue of a, mod 6. If $\Xi_p \in \mathsf{CA}(\mathcal{A}^{\mathbb{N}})$ has local map ξ_p (for $p = 2, 3$), then it is easy to check that Ξ_p corresponds to multiplication by p in base-6 notation. In other words, $\Psi \circ \times_p = \Xi_p \circ \Psi$ for $p = 2, 3$.

If λ is the Lebesgue measure on $\mathbb{T}^1$, then $\Psi(\lambda) = \eta$, where η is the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$. Thus, η is Ξ_2- and Ξ_3-invariant, and Furstenberg's Conjecture asserts that η is the *only* nonatomic measure on $\mathcal{A}^{\mathbb{N}}$ which is both Ξ_2- and Ξ_3-invariant. The shift map $\sigma : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{A}^{\mathbb{N}}$ corresponds to multiplication by 6 in base-6 notation. Hence, $\Xi_2 \circ \Xi_3 = \sigma$. From this it follows that a measure μ is (Ξ_2, Ξ_3)-invariant if and only if μ is (Ξ_2, σ)-invariant if and only if μ is (σ, Ξ_3)-invariant. Thus, Furstenberg's Conjecture equivalently asserts that η is the only stationary, Ξ_3-invariant nonatomic measure on $\mathcal{A}^{\mathbb{N}}$, and Rudolph's result asserts that η is the only such nonatomic measure with nonzero entropy; this is analogous to the 'measure rigidity' results of §2E. The existence of zero-entropy, (σ, Ξ_3)-invariant, nonatomic measures remains an open question.

Remarks 2F.1: (a) There is nothing special about 2 and 3; the same results hold for any pair of prime numbers.

(b) Lyons (1988) and Johnson and Rudolph (1995) have also established that a wide variety of $\times_2$-invariant probability measures on $\mathbb{T}^1$ will weak* converge, under the iteration of $\times_3$, to the Lebesgue measure (and vice-versa). In the terminology of §3A, these results immediately translate into equivalent statements about the 'asymptotic randomization' of initial probability measures on $\mathcal{A}^{\mathbb{N}}$ under the iteration of Ξ_2 or Ξ_3. ♢

2G Domains, defects, and particles

Suppose $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$, and there is a collection of Φ-invariant subshifts $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_N \subset \mathcal{A}^{\mathbb{Z}}$ (called *phases*). Any sequence $\mathbf{a}$ can be expressed a finite or infinite concatenation

$$\mathbf{a} \quad = \quad [\ldots \mathbf{a}_{-2}\, \mathbf{d}_{-2}\, \mathbf{a}_{-1}\, \mathbf{d}_{-1}\, \mathbf{a}_0\, \mathbf{d}_0\, \mathbf{a}_1\, \mathbf{d}_1\, \mathbf{a}_2 \cdots],$$

where each *domain* $\mathbf{a}_k$ is a finite word (or half-infinite sequence) which is admissible to phase $\mathbf{P}_n$ for some $n \in [1...N]$, and where each *defect* $\mathbf{d}_k$ is a (possibly empty) finite word (note that this decomposition may not be unique). Thus, $\Phi(\mathbf{a}) = \mathbf{a}'$, where

$$\mathbf{a}' \quad = \quad [\ldots \mathbf{a}'_{-2}\, \mathbf{d}'_{-2}\, \mathbf{a}'_{-1}\, \mathbf{d}'_{-1}\, \mathbf{a}'_0\, \mathbf{d}'_0\, \mathbf{a}'_1\, \mathbf{d}'_1\, \mathbf{a}'_2 \cdots],$$

and, for every $k \in \mathbb{Z}$, $\mathbf{a}'_k$ belongs to the same phase as $\mathbf{a}_k$. We say that Φ has *stable phases* if, for any such $\mathbf{a}$ and $\mathbf{a}'$ in $\mathcal{A}^{\mathbb{Z}}$, it is the case that, for all $k \in \mathbb{Z}$, $|\mathbf{d}'_k| \le |\mathbf{d}_k|$. In other words, the defects do not grow over time. However, they may propagate sideways; for example, $\mathbf{d}'_k$ may be slightly to the right of $\mathbf{d}_k$, if the domain $\mathbf{a}'_k$ is larger than $\mathbf{a}_k$, while the domain $\mathbf{a}'_{k+1}$ is slightly smaller than $\mathbf{a}_{k+1}$. If $\mathbf{a}_k$ and $\mathbf{a}_{k+1}$ belong to different phases, then the defect $\mathbf{d}_k$ is sometimes called a *domain boundary* (or 'wall', or 'edge particle'). If $\mathbf{a}_k$ and $\mathbf{a}_{k+1}$ belong to the same phase, then the defect $\mathbf{d}_k$ is sometimes called a *dislocation* (or 'kink'). (See also COMPUTATIONAL MECHANICS IN CA.)

Often $\mathbf{P}_n = \{\mathbf{p}\}$ where $\mathbf{p} = [\ldots ppp \ldots]$ is a constant sequence, or each $\mathbf{P}_n$ consists of the σ-orbit of a single periodic sequence. More generally, the phases $\mathbf{P}_1, \ldots, \mathbf{P}_N$ may be subshifts of finite type. In this case, most sequences in $\mathcal{A}^{\mathbb{Z}}$ can be fairly easily and unambiguously decomposed into domains separated by defects. However, if the phases are more complex (e.g. sofic shifts), then the exact definition of a 'defect' is actually fairly complicated —see Pivato (2007) for a rigorous discussion.

Example 2G.1: Let $\mathcal{A} = \{0,1\}$ and let $\mathbb{H} = \{-1,0,1\}$. *Elementary cellular automaton* (ECA) #184 is the CA $\Phi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ with local rule $\phi : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$ given as follows: $\phi(a_{-1}, a_0, a_1) = 1$ if $a_0 = a_1 = 1$, or if $a_{-1} = 1$ and $a_0 = 0$. On the other hand, $\phi(a_{-1}, a_0, a_1) = 0$ if $a_{-1} = a_0 = 0$, or if $a_1 = 0$ and $a_0 = 1$. Heuristically, each '1' represents a 'car' moving cautiously to the right on a single-lane road. During each iteration, each car will advance to the site in front of it, unless that site is already occupied, in which case the car will remain stationary. ECA #184 exhibits one stable phase $\mathbf{P}$, given by the 2-periodic sequence $[\ldots 0101.0101 \ldots]$ and its translate $[\ldots 1010.1010 \ldots]$ (here the decimal point indicates the zeroth coordinate), and Φ acts on $\mathbf{P}$ like the shift. The phase $\mathbf{P}$ admits two dislocations of width 2. The dislocation $\mathbf{d}_0 = [00]$ moves uniformly to the right, while the dislocation $\mathbf{d}_1 = [11]$ moves uniformly to the left. In the traffic interpretation, $\mathbf{P}$ represents freely flowing traffic, $\mathbf{d}_0$ represents a stretch of empty road, and $\mathbf{d}_1$ represents a traffic jam. ♢

Example 2G.2: Let $\mathcal{A} := \mathbb{Z}_{/N}$, and let $\mathbb{H} := [-1...1]$. The one-dimensional, N-colour *cyclic cellular automaton* (CCA_N) $\Phi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ has local rule $\phi : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$ defined:

$$\phi(\mathbf{a}) \quad := \quad \begin{cases} a_0 + 1 & \text{if there is some } \mathsf{h} \in \mathbb{H} \text{ with } a_{\mathsf{h}} = a_0 + 1; \\ a_0 & \text{otherwise.} \end{cases}$$

(here, addition is mod N). The CCA has phases $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_{N-1}$, where $\mathbf{P}_a = \{[\ldots aaa \ldots]\}$ for each $a \in \mathcal{A}$. A domain boundary between $\mathbf{P}_a$ and $\mathbf{P}_{a-1}$ moves with constant velocity towards the $\mathbf{P}_{a-1}$ side. All other domain boundaries are stationary. ♢

In a *particle cellular automaton* (PCA), $\mathcal{A} = \{\emptyset\} \sqcup \mathcal{P}$, where $\mathcal{P}$ is a set of 'particle types' and $\emptyset$ represents a vacant site. Each particle $p \in \mathcal{P}$ is assigned some (constant) velocity vector $\mathsf{v}(p) \in (-\mathbb{H})$ (where $\mathbb{H}$ is the neighbourhood of the automaton). Particles propagate with constant velocity through $\mathbb{M}$ until two particles try to simultaneously enter the same site in the lattice, at which point the outcome is determined by a *collision rule*: a stylized 'chemical reaction equation'. For example, an equation "$p_1 + p_2 \rightsquigarrow p_3$" means that, if particle types p_1 and p_2 collide, they coalesce to produce a particle of type p_3. On the other hand, "$p_1 + p_2 \rightsquigarrow \emptyset$" means that the two particles annihilate on contact. Formally, given a set of velocities and collision rules, the local rule $\phi : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$ is defined

$$\phi(\mathbf{a}) \quad := \quad \begin{cases} p & \text{if there is a unique } \mathsf{h} \in \mathbb{H} \text{ and } p \in \mathcal{P} \text{ with } a_{\mathsf{h}} = p \text{ and } \mathsf{v}(p) = -\mathsf{h}; \\ q & \text{if } \left\{p \in \mathcal{P} \,;\, a_{-\mathsf{v}(p)} = p\right\} \;=\; \{p_1, p_2, \ldots, p_n\}, \text{ and } p_1 + \cdots + p_n \;\rightsquigarrow\; q. \end{cases}$$

Example 2G.3: The one-dimensional *ballistic annihilation model* (BAM) contains two particle types: $\mathcal{P} = \{\pm 1\}$, with the following rules:

$$\mathsf{v}(1) \;=\; 1, \qquad \mathsf{v}(-1) \;=\; -1, \qquad \text{and} \qquad -1 + 1 \;\rightsquigarrow\; \emptyset.$$

(This CA is sometimes also called *Just Gliders*.) Thus, $a_{\mathsf{z}} = 1$ if the cell z contains a particle moving to the right with velocity 1, whereas $a_{\mathsf{z}} = -1$ if the cell z contains a particle moving left with velocity -1, and $a_{\mathsf{z}} = \emptyset$ if cell z is vacant. Particles move with constant velocity until they collide with oncoming particles, at which point both particles are annihilated. If $\mathcal{A} := \{\pm 1, \emptyset\}$ and $\mathbb{H} = [-1...1] \subset \mathbb{Z}$, then we can represent the BAM using $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ with local rule $\phi : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$ defined:

$$\phi(a_{-1}, a_0, a_1) \quad := \quad \begin{cases} -1 & \text{if} \quad a_1 = -1 \text{ and } \; a_{-1}, a_0 \in \{-1, \emptyset\}; \\ 1 & \text{if} \quad a_{-1} = 1 \text{ and } \; a_0, a_1 \in \{1, \emptyset\}; \\ \emptyset & \quad\;\; \text{otherwise.} \end{cases}$$

◇

Particle CA can be seen as 'toy models' of particle physics or microscale chemistry. More interestingly, however, one-dimensional PCA often arise as factors of coalescent-domain CA, with the 'particles' tracking the motion of the defects.

Example 2G.4: (a) Let $\mathcal{A} := \{0, 1\}$ and let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ be ECA #184. Let $\mathcal{B} := \{\pm 1, 0\}$, and let $\Psi \in \mathsf{CA}(\mathcal{B}^{\mathbb{Z}})$ be the BAM. Let $\mathbb{G} := \{0, 1\}$, and let $\Gamma : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{B}^{\mathbb{Z}}$ be the block map with local rule $\gamma : \mathcal{A}^{\mathbb{G}} \longrightarrow \mathcal{B}$ defined

$$\gamma(a_0, a_1) \quad := \quad 1 - a_0 - a_1 \quad = \quad \begin{cases} 1 & \text{if} \quad [a_0, a_1] = [0, 0] = \mathbf{d}_0; \\ -1 & \text{if} \quad [a_0, a_1] = [1, 1] = \mathbf{d}_1; \\ 0 & \quad\;\; \text{otherwise.} \end{cases}$$

Then $\Gamma \circ \Phi = \Psi \circ \Gamma$; in other words, the BAM is a factor of ECA #184, and tracks the motion of the dislocations.

(b) Again, let $\Psi \in \mathsf{CA}(\mathcal{B}^{\mathbb{Z}})$ be the BAM. Let $\mathcal{A} = \mathbb{Z}_{/3}$, and let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ be the 3-color CCA. Let $\mathbb{G} := \{0, 1\}$, and let $\Gamma : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{B}^{\mathbb{Z}}$ be the block map with local rule $\gamma : \mathcal{A}^{\mathbb{G}} \longrightarrow \mathcal{B}$ defined

$$\gamma(a_0, a_1) \quad := \quad (a_0 - a_1) \bmod 3.$$

Then $\Gamma \circ \Phi = \Psi \circ \Gamma$; in other words, the BAM is a factor of CCA_3, and tracks the motion of the domain boundaries. ♢

Thus, it is often possible to translate questions about coalescent domain CA into questions about particle CA, which are generally easier to study. For example, the invariant measures of the BAM have been completely characterized.

Proposition 2G.5 *Let $\mathcal{B} = \{\pm 1, 0\}$, and let $\Psi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ be the BAM.*

(a) *The sets $\mathbf{R} := \{0, 1\}^{\mathbb{Z}}$ and $\mathbf{L} := \{0, -1\}^{\mathbb{Z}}$ are Φ-invariant, and Ψ acts as a right-shift on $\mathbf{R}$ and as a left-shift on $\mathbf{L}$.*

(b) *Let $\mathbf{L}^+ := \{0, -1\}^{\mathbb{N}}$ and $\mathbf{R}^- := \{0, 1\}^{-\mathbb{N}}$, and let*

$$\mathbf{X} \;:=\; \left\{ \mathbf{a} \in \mathcal{A}^{\mathbb{Z}} \;;\; \exists \mathsf{z} \in \mathbb{Z} \text{ such that } \mathbf{a}_{(-\infty \ldots \mathsf{z}]} \in \mathbf{R}^- \text{ and } \mathbf{a}_{[\mathsf{z} \ldots \infty)} \in \mathbf{L}^+ \right\}.$$

Then $\mathbf{X}$ is Φ-invariant. For any $\mathbf{x} \in \mathbf{X}$, Φ acts as a right shift on $\mathbf{a}_{(-\infty \ldots \mathsf{z})}$, and as a left-shift on $\mathbf{x}_{(\mathsf{z} \ldots \infty)}$. (The boundary point z executes some kind of random walk.)

(c) *Any Ψ-invariant measure on $\mathcal{A}^{\mathbb{Z}}$ can be written in a unique way as a convex combination of four measures δ_0, ρ, λ, and μ, where: δ_0 is the point mass on the 'vacuum' configuration $[\ldots 0\,0\,0 \ldots]$, ρ is any shift-invariant measure on $\mathbf{R}$, λ is any shift-invariant measure on $\mathbf{L}$, and μ is a measure on $\mathbf{X}$.*

Furthermore, there exist shift-invariant measures μ_- and μ_+ on $\mathbf{R}^-$ and $\mathbf{L}^+$, respectively, such that, for μ-almost all $\mathbf{x} \in \mathbf{X}$, $\mathbf{x}_{(-\infty \ldots \mathsf{z}]}$ is μ_--distributed and $\mathbf{x}_{[\mathsf{z} \ldots \infty)}$ is μ_+-distributed.

Proof: **(a)** and **(b)** are obvious; **(c)** is (Belitsky and Ferrari, 2005, Theorem 1). □

Remark 2G.6: (a) Proposition 2G.5(c) can be immediately translated into a complete characterization of the invariant measures of ECA #184, via the factor map Γ in Example 2G.4(a); see (Belitsky and Ferrari, 2005, Theorem 2). Likewise, using the factor map in Example 2G.4(b) we get a complete characterization of the invariant measures for CCA_3.

(b) Proposition 3B.7 and Corollaries 3B.8 and 3B.9 describe the limit measures of the BAM, CCA_3, and ECA #184. Also, Blank (2003) has characterized invariant measures for a broad class of multilane, multi-speed traffic models (including ECA#184); see Remark 3B.10(b).

(c) Kůrka (2005) has defined, for any $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$, a construction similar to the set $\mathbf{X}$ in Proposition 2G.5(b). For any $n \in \mathbb{N}$ and $z \in \mathbb{Z}$, let $\mathbf{S}_{z,n}$ be the set of fixed points of $\Phi^n \circ \sigma^z$; then $\mathbf{S}_{z,n}$ is a subshift of finite type, which Kůrka calls a *signal subshift* with *velocity* $v = z/n$. (For example, if Φ is the BAM, then $\mathbf{R} = \mathbf{S}_{1,1}$ and $\mathbf{L} = \mathbf{S}_{-1,1}$.)

Now, suppose that $z_1/n_1 > z_2/n_2 > \cdots > z_J/n_J$. The *join* of the signal subshifts $\mathbf{S}_{z_1,n_1}, \mathbf{S}_{z_2,n_2}, \ldots, \mathbf{S}_{z_J,n_J}$ is the set $\mathbf{S}$ of all infinite sequences $[\mathbf{a}_1\, \mathbf{a}_2\, \ldots\, \mathbf{a}_J]$, where for all $j \in [1..J]$, $\mathbf{a}_j$ is a (possibly empty) finite word or (half-)infinite sequence admissible to the subshift $\mathbf{S}_{z_j,n_j}$. (For example, if $\mathbf{S}$ is the join of $\mathbf{S}_{1,1} = \mathbf{R}$ and $\mathbf{S}_{-1,1} = \mathbf{L}$ from Proposition 2G.5(a), then $\mathbf{S} = \mathbf{L} \cup \mathbf{X} \cup \mathbf{R}$.) It follows that $\mathbf{S} \subseteq \Phi(\mathbf{S}) \subseteq \Phi^2(\mathbf{S}) \subseteq \cdots$. If we define $\Phi^\infty(\mathbf{S}) := \bigcup_{t=0}^\infty \Phi^t(\mathbf{S})$, then $\Phi^\infty(\mathbf{S}) \subseteq \Phi^\infty(\mathcal{A}^\mathbb{Z})$, where $\Phi^\infty(\mathcal{A}^\mathbb{Z}) := \bigcap_{t=0}^\infty \Phi^t(\mathcal{A}^\mathbb{Z})$ is the omega limit set of Φ (Kůrka, 2005, Proposition 5). The support of any Φ-invariant measure must be contained in $\Phi^\infty(\mathcal{A}^\mathbb{Z})$, so invariant measures may be closely related to the joins of signal subshifts.

In the case of the BAM, it is not hard to check that $\Phi^\infty(\mathbf{S}) = \mathbf{S} = \Phi^\infty(\mathcal{A}^\mathbb{Z})$; this suggests an alternate proof of Proposition 2G.5(c). It would be interesting to know whether a conclusion analogous to Proposition 2G.5(c) holds for other $\Phi \in \mathsf{CA}(\mathcal{A}^\mathbb{Z})$ such that $\Phi^\infty(\mathcal{A}^\mathbb{Z})$ is a join of signal subshifts. ◊

3 Limit measures and other asymptotics

3A Asymptotic randomization by linear cellular automata

The results of §2E suggest that the uniform Bernoulli measure η is the 'natural' measure for algebraic CA, because η is the unique invariant measure satisfying any one of several collections of reasonable criteria. In this section, we will see that η is 'natural' in quite another way: it is the unique limit measure for linear CA from a large set of initial conditions.

If $\{\mu_n\}_{n=1}^\infty$ is a sequence of measures on $\mathcal{A}^\mathbb{M}$, then this sequence *weak* converges* to the measure μ_∞ ("$\operatorname{wk*lim}_{n\to\infty} \mu_n = \mu_\infty$") if, for all cylinder sets $\mathbf{B} \subset \mathcal{A}^\mathbb{M}$, $\lim_{n\to\infty} \mu_n[\mathbf{B}] = \mu_\infty[\mathbf{B}]$. Equivalently, for all continuous functions $f : \mathcal{A}^\mathbb{M} \longrightarrow \mathbb{C}$, we have

$$\lim_{n\to\infty} \int_{\mathcal{A}^\mathbb{M}} f \, d\mu_n \;=\; \int_{\mathcal{A}^\mathbb{M}} f \, d\mu_\infty.$$

The *Cesàro average* (or *Cesàro limit*) of $\{\mu_n\}_{n=1}^\infty$ is $\operatorname{wk*lim}_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu_n$, if this limit exists.

Let $\mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^\mathbb{M})$ and let $\Phi \in \mathsf{CA}(\mathcal{A}^\mathbb{M})$. For any $t \in \mathbb{N}$, the measure $\Phi^t\mu$ is defined by $\Phi^t\mu(\mathbf{B}) = \mu(\phi^{-t}(\mathbf{B}))$, for any measurable subset $\mathbf{B} \subset \mathcal{A}^\mathbb{M}$. We say that Φ *asymptotically randomizes* μ if the Cesàro average of the sequence $\{\phi^n\mu\}_{n=1}^\infty$ is η. Equivalently, there is a subset $\mathbb{J} \subset \mathbb{N}$ of density 1, such that

$$\operatorname*{wk*lim}_{\substack{j\to\infty \\ j\in\mathbb{J}}} \Phi^j\mu \quad = \quad \eta.$$

The uniform measure η is the measure of maximal entropy on $\mathcal{A}^\mathbb{M}$. Thus, asymptotic randomization is kind of 'Second Law of Thermodynamics' for CA.

Let $(\mathcal{A}, +)$ be a finite abelian group, and let Φ be a linear cellular automaton (LCA) on $\mathcal{A}^{\mathbb{M}}$. Recall that Φ has *scalar coefficients* if there is some finite $\mathbb{H} \subset \mathbb{M}$, and integer coefficients $\{c_{\mathsf{h}}\}_{\mathsf{h} \in \mathbb{H}}$ so that Φ has a local rule of the form

$$\phi(\mathbf{a}_{\mathbb{H}}) \quad := \quad \sum_{\mathsf{h} \in \mathbb{H}} c_{\mathsf{h}} a_{\mathsf{h}}, \tag{3.1}$$

An LCA Φ is *proper* if Φ has scalar coefficients as in eqn.(3.1), and if, furthermore, for any prime divisor p of $|\mathcal{A}|$, there are at least two $\mathsf{h}, \mathsf{h}' \in \mathbb{H}$ such that $c_{\mathsf{h}} \not\equiv 0 \not\equiv c_{\mathsf{h}'} \bmod p$. For example, if $\mathcal{A} = \mathbb{Z}_{/n}$ for some $n \in \mathbb{N}$, then every LCA on $\mathcal{A}^{\mathbb{M}}$ has scalar coefficients; in this case, Φ is proper if, for every prime p dividing n, at least two of these coefficients are coprime to p. In particular, if $\mathcal{A} = \mathbb{Z}_{/p}$ for some prime p, then Φ is proper as long as $|\mathbb{H}| \geq 2$.

Let $\mathsf{PLCA}\left(\mathcal{A}^{\mathbb{M}}\right)$ be the set of proper linear CA on $\mathcal{A}^{\mathbb{M}}$. If $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{M}})$, recall that μ has *full support* if $\mu[\mathbf{B}] > 0$ for every cylinder set $\mathbf{B} \subset \mathcal{A}^{\mathbb{M}}$.

Theorem 3A.1 *Let $(\mathcal{A}, +)$ be a finite abelian group, let $\mathbb{M} := \mathbb{Z}^D \times \mathbb{N}^E$ for some $D, E \geq 0$, and let $\Phi \in \mathsf{PLCA}\left(\mathcal{A}^{\mathbb{M}}\right)$. Let μ be any Bernoulli measure or Markov random field on $\mathcal{A}^{\mathbb{M}}$ having full support. Then Φ asymptotically randomizes μ.*

History: Theorem 3A.1 was first proved for simple one-dimensional LCA randomizing Bernoulli measures on $\mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A}$ was a cyclic group. In the case $\mathcal{A} = \mathbb{Z}_{/2}$, Theorem 3A.1 was independently proved for the *nearest-neighbour XOR* CA (having local rule $\phi(a_{-1}, a_0, a_1) = a_{-1} + a_1 \bmod 2$) by Miyamoto (1979) and Lind (1984). This result was then generalized to $\mathcal{A} = \mathbb{Z}_{/p}$ for any prime p by Cai and Luo (1993). Next, Maass and Martínez (1998) duplicated the Miyamoto/Lind result for the *binary Ledrappier* CA (local rule $\phi(a_0, a_1) = a_0 + a_1 \mod 2$). Soon after, Ferrari et al. (2000) considered the case when $\mathcal{A}$ was an abelian group of order p^k (p prime), and proved Theorem 3A.1 for any *Ledrappier* CA (local rule $\phi(a_0, a_1) = c_0 a_0 + c_1 a_1$, where $c_0, c_1 \not\equiv 0 \bmod p$) acting on any measure on $\mathcal{A}^{\mathbb{Z}}$ having full support and 'rapidly decaying correlations' (see Part II(a) below). For example, this includes any Markov measure on $\mathcal{A}^{\mathbb{Z}}$ with full support. Next, Pivato and Yassawi (2002) generalized Theorem 3A.1 to *any* PLCA acting on any fully supported N-step Markov chain on $\mathcal{A}^{\mathbb{Z}}$ or any nontrivial Bernoulli measure on $\mathcal{A}^{\mathbb{Z}^D \times \mathbb{N}^E}$, where $\mathcal{A} = \mathbb{Z}_{/p^k}$ (p prime). Finally, Pivato and Yassawi (2004) proved Theorem 3A.1 in full generality, as stated above. $\diamondsuit$.

The proofs of Theorem 3A.1 and its variations all involve two parts:

Part I. A careful analysis of the local rule of Φ^t (for all $t \in \mathbb{N}$), showing that the neighbourhood of Φ^t grows large as $t \to \infty$ (and in some cases, contains large 'gaps').

Part II. A demonstration that the measure μ exhibits 'rapidly decaying correlations' between widely separated elements of $\mathbb{M}$; hence, when these elements are combined using Φ^t, it is as if we are summing independent random variables.

Part I: Any linear CA with scalar coefficients can be written as a 'Laurent polynomial of shifts'. That is, if Φ has local rule (3.1), then for any $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$,

$$\Phi(\mathbf{a}) \quad := \quad \sum_{\mathsf{h}\in\mathbb{H}} c_{\mathsf{h}}\sigma^{\mathsf{h}}(\mathbf{a}) \quad \text{(where we add configurations componentwise).}$$

We indicate this by writing "$\Phi = F(\sigma)$", where $F \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_D^{\pm 1}]$ is the D-variable Laurent polynomial defined:

$$F(x_1, \ldots, x_D) \quad := \quad \sum_{(h_1,\ldots,h_D)\in\mathbb{H}} c_{\mathsf{h}} x_1^{h_1} x_2^{h_1} \ldots x_D^{h_D}.$$

For example, if Φ is the *nearest-neighbour XOR* CA, then $\Phi = \sigma^{-1} + \sigma^1 = F(\sigma)$, where $F(x) = x^{-1} + x$. If Φ is a *Ledrappier* CA, then $\Phi = c_0\mathbf{Id} + c_1\sigma^1 = F(\sigma)$, where $F(x) = c_0 + c_1 x$.

It is easy to verify that, if F and G are two such polynomials, and $\Phi = F(\sigma)$ while $\Gamma = G(\sigma)$, then $\Phi \circ \Gamma = (F \cdot G)(\sigma)$, where $F \cdot G$ is the product of F and G in the polynomial ring $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_D^{\pm 1}]$. In particular, this means that $\Phi^t = F^t(\sigma)$ for all $t \in \mathbb{N}$. Thus, iterating an LCA is equivalent to computing the powers of a polynomial.

If $\mathcal{A} = \mathbb{Z}_{/p}$, then we can compute the coefficients of F^t modulo p. If p is prime, then this can be done using a result of Lucas (1878), which provides a formula for the binomial coefficient $\binom{a}{b}$ in terms of the base-p expansions of a and b. For example, if $p = 2$, then Lucas' theorem says that Pascal's triangle, modulo 2, looks like a 'discrete Sierpinski triangle', made out of 0's and 1's.[1] Thus, Lucas' Theorem, along with some combinatorial lemmas about the structure of base-p expansions, provides the machinery for Part I.

Part II: There are two approaches to analyzing probability measures on $\mathcal{A}^{\mathbb{M}}$; one using renewal theory, and the other using harmonic analysis.

II(a) Renewal theory: This approach was developed by Maass, Martínez and their collaborators. Loosely speaking, if $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}}, \sigma)$ has sufficiently large support and sufficiently rapid decay of correlations (e.g. a Markov chain), and $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$ is a μ-random sequence, then we can treat $\mathbf{a}$ as if there is a sparse, randomly distributed set of 'renewal times' when the normal stochastic evolution of $\mathbf{a}$ is interupted by independent, random 'errors'. By judicious use of Part I described above, one can use this 'renewal process' to make it seem as though Φ^t is summing independent random variables.

For example, if $(\mathcal{A}, +)$ be an abelian group of order p^k where p is prime, and $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}}; \sigma)$ has *complete connections* [see Example 2E.4(a)] and *summable decay* [which means that a certain sequence of coefficients (measuring long-range correlation) decays

[1] This is why fragments of the Sierpinski triangle appear frequently in the spacetime diagrams of linear CA on $\mathcal{A} = \mathbb{Z}_{/2}$, a phenomenon which has inspired much literature on 'fractals and automatic sequences in cellular automata'; see Willson (1984a,b, 1986, 1987a,b); Takahashi (1990, 1992, 1993); von Haeseler et al. (1992, 1993, 1995a,b, 2001a,b); Allouche et al. (1996, 1997); Allouche (1999); Barbé et al. (1995, 2003); and Mauldin and Skordev (2000).

fast enough that its sum is finite], and $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ is a Ledrappier CA, then Ferrari et al. (2000)[Theorem 1.3] showed that Φ asymptotically randomizes μ. (For example, this applies to any N-step Markov chain with full support on on $\mathcal{A}^{\mathbb{Z}}$.) Furthermore, if $\mathcal{A} = \mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$, and $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ has linear local rule $\phi\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = (y_0,\ x_0 + y_1)$, then Maass and Martínez (1999) showed that Φ randomizes any Markov measure with full support on $\mathcal{A}^{\mathbb{Z}}$. Maass and Martínez again handled Part II using renewal theory. However, in this case, Part I involves some delicate analysis of the (noncommutative) algebra of the matrix-valued coefficients; unfortunately, their argument does not generalize to other LCA with noncommuting, matrix-valued coefficients. (However, Proposition 8 of Pivato and Yassawi (2004) suggests a general strategy for dealing with such LCA).

II(b) Harmonic analysis: This approach to Part II was implicit in the early work of Lind (1984) and Cai and Luo (1993), but was developed in full generality by Pivato and Yassawi (2002, 2004, 2006). We regard $\mathcal{A}^{\mathbb{M}}$ as a direct product of copies of the group $(\mathcal{A}, +)$, and endow it with the product group structure; then $(\mathcal{A}^{\mathbb{M}}, +)$ a compact abelian topological group. A *character* on $(\mathcal{A}^{\mathbb{M}}, +)$ is a continuous group homomorphism $\boldsymbol{\chi} : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathbb{T}$, where $\mathbb{T} := \{c \in \mathbb{C}\ ;\ |c| = 1\}$ is the unit circle group. If μ is a measure on $\mathcal{A}^{\mathbb{M}}$, then the *Fourier coefficients* of μ are defined: $\widehat{\mu}[\boldsymbol{\chi}] = \int_{\mathcal{A}^{\mathbb{M}}} \boldsymbol{\chi}\ d\mu$, for every character $\boldsymbol{\chi}$.

If $\boldsymbol{\chi} : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathbb{T}$ is any character, then there is a unique finite subset $\mathbb{K} \subset \mathbb{M}$ (called the *support* of $\boldsymbol{\chi}$) and a unique collection of nontrivial characters $\chi_{\mathsf{k}} : \mathcal{A} \longrightarrow \mathbb{T}$ for all $\mathsf{k} \in \mathbb{K}$, such that,

$$\boldsymbol{\chi}(\mathbf{a}) \quad = \quad \prod_{\mathsf{k} \in \mathbb{K}} \chi_{\mathsf{k}}(a_{\mathsf{k}}), \qquad \forall\ \mathbf{a} \in \mathcal{A}^{\mathbb{M}}. \tag{3.2}$$

We define $\mathsf{rank}\,[\boldsymbol{\chi}] := |\mathbb{K}|$. The measure μ is called *harmonically mixing* if, for all $\epsilon > 0$, there is some R such that for all characters $\boldsymbol{\chi}$, $\Big(\mathsf{rank}\,[\boldsymbol{\chi}] \geq R\Big) \implies \Big(|\widehat{\mu}[\boldsymbol{\chi}]| < \epsilon\Big)$.

The set $\mathfrak{Hm}\left(\mathcal{A}^{\mathbb{M}}\right)$ of harmonically mixing measures on $\mathcal{A}^{\mathbb{M}}$ is quite inclusive. For example, if μ is any (N-step) Markov chain with full support on $\mathcal{A}^{\mathbb{Z}}$, then $\mu \in \mathfrak{Hm}\left(\mathcal{A}^{\mathbb{Z}}\right)$ (Pivato and Yassawi, 2002, Propositions 8 and 10), and if $\nu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{Z}})$ is absolutely continuous with respect to this μ, then $\nu \in \mathfrak{Hm}\left(\mathcal{A}^{\mathbb{Z}}\right)$ also (Pivato and Yassawi, 2002, Corollary 9). If $\mathcal{A} = \mathbb{Z}_{/p}$ (p prime) then any nontrivial Bernoulli measure on $\mathcal{A}^{\mathbb{M}}$ is harmonically mixing (Pivato and Yassawi, 2002, Proposition 6). Furthermore, if $\mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{Z}}; \sigma)$ has complete connections and summable decay, then $\mu \in \mathfrak{Hm}\left(\mathcal{A}^{\mathbb{Z}}\right)$ (Host et al., 2003, Theorem 23). If $\mathfrak{M} := \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{M}}; \mathbb{C})$ is the set of all complex-valued measures on $\mathcal{A}^{\mathbb{M}}$, then $\mathfrak{M}$ is *Banach algebra* (i.e. it is a vector space under the obvious definition of addition and scalar multiplication for measures, and a Banach space under the total variation norm, and finally, since $\mathcal{A}^{\mathbb{M}}$ is a topological group, $\mathfrak{M}$ is a ring under convolution). Then $\mathfrak{Hm}\left(\mathcal{A}^{\mathbb{M}}\right)$ is an ideal in $\mathfrak{M}$, is closed under the total variation norm, and is dense in the weak* topology on $\mathfrak{M}$ (Pivato and Yassawi, 2002, Propositions 4 and 7).

Finally, if μ is any Markov random field on $\mathcal{A}^{\mathbb{M}}$ which is *locally free* (which roughly means that the boundary of any finite region does not totally determine the interior of that region), then $\mu \in \mathfrak{Hm}\left(\mathcal{A}^{\mathbb{M}}\right)$ (Pivato and Yassawi, 2006, Theorem 1.3). In particular, this implies:

Proposition 3A.2 *If $(\mathcal{A}, +)$ is any finite group, and $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{M}})$ is any Markov random field with full support, then μ is harmonically mixing.*

Proof: This follows from (Pivato and Yassawi, 2006, Theorem 1.3). It is also a special case of (Pivato and Yassawi, 2004, Theorem 15). □

If $\boldsymbol{\chi}$ is a character, and Φ is a LCA, then $\boldsymbol{\chi} \circ \Phi^t$ is also a character, for any $t \in \mathbb{N}$ (because it is a composition of two continuous group homomorphisms). We say Φ is ***diffusive*** if there is a subset $\mathbb{J} \subset \mathbb{N}$ of density 1, such that, for every character $\boldsymbol{\chi}$ of $\mathcal{A}^{\mathbb{M}}$,

$$\lim_{\mathbb{J} \ni j \to \infty} \mathsf{rank}\left[\boldsymbol{\chi} \circ \Phi^j\right] \quad = \quad \infty.$$

Proposition 3A.3 *Let $(\mathcal{A}, +)$ be any finite abelian group and let $\mathbb{M}$ be any monoid. If μ is harmonically mixing and Φ is diffusive, then Φ asymptotically randomizes μ.*

Proof: See (Pivato and Yassawi, 2004, Theorem 12). □

Proposition 3A.4 *Let $(\mathcal{A}, +)$ be any abelian group and let $\mathbb{M} := \mathbb{Z}^D \times \mathbb{N}^E$ for some $D, E \geq 0$. If $\Phi \in \mathsf{PLCA}\left(\mathcal{A}^{\mathbb{M}}\right)$, then Φ is diffusive.*

Proof: The proof uses Lucas' theorem, as described in Part I above. See (Pivato and Yassawi, 2002, Theorem 15) for the case $\mathcal{A} = \mathbb{Z}_{/p}$ when p prime. See (Pivato and Yassawi, 2004, Theorem 6) for the case when $\mathcal{A}$ is any cyclic group. That proof easily extends to any finite abelian group $\mathcal{A}$: write $\mathcal{A}$ as a product of cyclic groups and decompose Φ into separate automata over these cyclic factors. □

Proof of Theorem 3A.1: Combine Propositions 3A.2, 3A.3, and 3A.4. □

Remarks 3A.5: (a) Proposition 3A.4. can be generalized: we do not need the coefficients of Φ to be integers, but merely to be a collection of automorphisms of $\mathcal{A}$ which commute with one another (so that Lucas' theorem from Part I is still applicable). See (Pivato and Yassawi, 2004, Theorem 9).

(b) For simplicity, we stated Theorem 3A.1 for measures with full support; however, Proposition 3A.3 actually applies to many Markov random fields *without* full support, because harmonic mixing only requires 'local freedom' (Pivato and Yassawi, 2006, Theorem 1.3). For example, the support of a Markov chain on $\mathcal{A}^{\mathbb{Z}}$ is Markov subshift. If $\mathcal{A} = \mathbb{Z}_{/p}$ (p prime), then Proposition 3A.3 yields asymptotic randomization of the Markov chain as long as the transition digraph of the underlying Markov subshift admits at least *two* distinct paths of length 2 between any pair of vertices in $\mathcal{A}$. More generally, if $\mathbb{M} = \mathbb{Z}^D$, then the support of any Markov random field on $\mathcal{A}^{\mathbb{Z}^D}$ is an SFT, which we can regard as the set of all tilings of $\mathbb{R}^D$ by a certain collection of Wang tiles. If $\mathcal{A} = \mathbb{Z}_{/p}$ (p prime), then Proposition 3A.3 yields asymptotic randomization of the Markov random field as long as the underlying Wang tiling is flexible enough that any hole can always be filled in at least two ways; see (Pivato and Yassawi, 2006, §1). ♢

Remarks 3A.6: *Generalizations and Extensions.*

(a) Pivato and Yassawi (2006)[Thm 3.1] proved a variation of Theorem 3A.3 where diffusion (of Φ) is replaced with a slightly stronger condition called *dispersion*, so that harmonic mixing (of μ) can be replaced with a slightly weaker condition called *dispursion mixing* (DM). It is unknown whether all proper linear CA are dispersive, but a very large class are (including, for example, $\Phi = \mathbf{Id}+\sigma$). Any uniformly mixing measure with positive entropy is DM (Pivato and Yassawi, 2006, Theorem 5.2); this includes, for example, any mixing *quasimarkov measure* (i.e. the image of a Markov measure under a block map; these are the natural measures supported on sofic shifts). Quasimarkov measures are not, in general, harmonically mixing (Pivato and Yassawi, 2006, §2), but this result shows they are still asymptotically randomized by most linear CA.

(b) Suppose $\mathbf{G} \subset \mathcal{A}^{\mathbb{Z}^D}$ is a σ-transitive *subgroup shift* (see §2E for definition), and let $\Phi \in \mathsf{PLCA}(\mathbf{G})$. If $\mathbf{G}$ satisfies an algebraic condition called the *follower lifting property* (FLP) and μ is any Markov random field with $\mathsf{supp}(\mu) = \mathbf{G}$, then Maass et al. (2006a) have shown that Φ asymptotically randomizes μ to a maxentropy measure on $\mathbf{G}$. Furthermore, if $D = 1$, then this maxentropy measure is the Haar measure on $\mathbf{G}$. In particular, if $\mathcal{A}$ is an abelian group of prime-power order, then *any* transitive Markov subgroup $\mathbf{G} \subset \mathcal{A}^{\mathbb{Z}}$ satisfies the FLP, so this result holds for any multistep Markov measure on $\mathbf{G}$. See also Maass et al. (2006b) for the special case when $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ has local rule $\phi(x_0, x_1) = x_0 + x_1$. In the special case when Φ has local rule $\phi(x_0, x_1) = c_0x_0 + c_1x_1 + a$, the result has been extended to measures with complete connections and summable decay; see (Sobottka, 2005, Teorema III.2.1, p.71) or see Maass et al. (2006c).

(c) All the aforementioned results concern asymptotic randomization of initial measures with nonzero entropy. Is nonzero entropy either necessary or sufficient for asymptotic randomization? First let $\mathbf{X}_N \subset \mathcal{A}^{\mathbb{Z}}$ be the set of N-periodic points (see §2C) and suppose $\mathsf{supp}(\mu) \subseteq \mathbf{X}_N$. Then the Cesàro limit of $\{\Phi^t(\mu)\}_{t\in\mathbb{N}}$ will also be a measure supported on $\mathbf{X}_N$, so μ_∞ cannot be the uniform measure on $\mathcal{A}^{\mathbb{Z}}$. Nor, in general, will μ_∞ be the uniform measure on $\mathbf{X}_N$; this follows from Jen's (1988) exact characterization of the limit cycles of linear CA acting on $\mathbf{X}_N$.

What if μ is a *quasiperiodic* measure, such as the unique σ-invariant measure on a Sturmian shift? There exist quasiperiodic measures on $(\mathbb{Z}_{/2})^{\mathbb{Z}}$ which are *not* asymptotically randomized by the Ledrappier CA (Pivato, 2005a, §15). But it is unknown whether this extends to all quasiperiodic measures or all linear CA.

There is also a measure μ on $\mathcal{A}^{\mathbb{Z}}$ which has zero σ-entropy, yet is still asymptotically randomized by Φ (Pivato and Yassawi, 2006, §8). Loosely speaking, μ is a Toeplitz measure with a very low density of 'bit errors'. Thus, μ is 'almost' deterministic (so it has zero entropy), but by sufficiently increasing the density of 'bit errors', we can introduce just enough randomness to allow asymptotic randomization to occur.

(d) Suppose $(\mathcal{G}, \cdot)$ is a *nonabelian* group and $\Phi : \mathcal{G}^{\mathbb{Z}} \longrightarrow \mathcal{G}^{\mathbb{Z}}$ has *multiplicative* local rule $\phi(\mathsf{g}) := g_{\mathsf{h}_1}^{n_1} g_{\mathsf{h}_2}^{n_2} \cdots g_{\mathsf{h}_J}^{n_J}$, for some $\{\mathsf{h}_1, \ldots, \mathsf{h}_J\} \subset \mathbb{Z}$ (possibly not distinct) and $n_1, \ldots, n_J \in \mathbb{N}$. If $\mathcal{G}$ is nilpotent, then $\mathcal{G}$ can be decomposed into a tower of abelian group extensions; this induces a structural decomposition of Φ into a tower of skew products of 'relative' linear CA. This strategy was first suggested by Moore (1998), and was developed by Pivato

(2003)[Theorem 21], who proved a version of Theorem 3A.1 in this setting.

(e) Suppose $(\mathcal{Q}, \star)$ is a *quasigroup* —that is, $\star$ is a binary operation such that for any $q, r, s \in \mathcal{Q}$, $(q \star r = q \star s) \iff (r = s) \iff (r \star q = s \star q)$. Any finite *associative* quasigroup has an identity, and any associative quasigroup with an identity is a group. However there are also many nonassociative finite quasigroups. If we define a 'multiplicative' CA $\Phi : \mathcal{Q}^{\mathbb{Z}} \longrightarrow \mathcal{Q}^{\mathbb{Z}}$ with local rule $\phi : \mathcal{Q}^{\{0,1\}} \longrightarrow \mathcal{Q}$ given by $\phi(q_0, q_1) = q_0 \star q_1$, then it is easy to see that Φ is bipermutative if and only if $(\mathcal{Q}, \star)$ is a quasigroup. Thus, quasigroups seem to provide the natural algebraic framework for studying bipermutative CA; this was first proposed by Moore (1997), and later explored by Host et al. (2003)[§3], Pivato (2005b)[§2], and Sobottka (2005, 2007a,b).

Note that $\mathcal{Q}^{\mathbb{Z}}$ is a quasigroup under componentwise $\star$-multiplication. A *quasigroup shift* is a subshift $\mathbf{X} \subset \mathcal{Q}^{\mathbb{Z}}$ which is also a subquasigroup; it follows that $\Phi(\mathbf{X}) \subseteq \mathbf{X}$. If $\mathbf{X}$ and Φ satisfy certain strong algebraic conditions, and $\mu \in \mathfrak{M}_{\text{eas}}(\mathbf{X}; \sigma)$ has complete connections and summable decay, then the sequence $\{\Phi^t \mu\}_{t=1}^{\infty}$ Cesàro -converges to some limit μ_∞; see (Sobottka, 2007a, Theorem 6.3), or (Sobottka, 2005, Teorema IV.5.3, p.107). However, it is unknown whether μ_∞ is equal to the Parry measure on $\mathbf{X}$ (which would be the appropriate notion of 'asymptotic randomization' in this context).

◊

3B Hybrid modes of self-organization

Most cellular automata do not asymptotically randomize; instead they seem to converge to limit measures concentrated on small (i.e. low-entropy) subsets of the statespace $\mathcal{A}^{\mathbb{M}}$ —a phenomenon which can be interpreted as a form of 'self-organization'. Exact limit measures have been computed for a few CA. For example, let $\mathcal{A} = \{0, 1, 2\}$ and let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$ be the *Greenberg-Hastings* model (a simple model of an an excitable medium). Durrett and Steif (1991) showed that, if $D \geq 2$ and μ is any Bernoulli measure on $\mathcal{A}^{\mathbb{Z}^D}$, then $\mu_\infty := \operatorname{wk^*lim}_{t \to \infty} \Phi^t \mu$ exists; μ_∞-almost all points are 3-periodic for Φ, and although μ_∞ is not a Bernoulli measure, the system $(\mathcal{A}^{\mathbb{Z}^D}, \mu_\infty, \sigma)$ is measurably isomorphic to a Bernoulli system.

In other cases, the limit measure cannot be exactly computed, but can still be estimated. For example, let $\mathcal{A} = \{\pm 1\}$, $\theta \in (0, 1)$, and $R > 0$, and let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ be the (R, θ)*-threshold voter CA* (where each cell computes the fraction of its radius-R neighbours which disagree with its current sign, and negates its sign if this fraction is at least θ). Durrett and Steif (1993) and Fisch and Gravner (1995) have described the long-term behaviour of Φ in the limit as $R \to \infty$. If $\theta < 1/2$, then every initial condition falls into a two-periodic orbit (and if $\theta < 1/4$, then every cell simply alternates its sign). Let η be the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$; if $1/2 < \theta$, then for any finite subset $\mathbb{B} \subset \mathbb{Z}$, if R is large enough, then 'most' initial conditions (relative to η) converge to orbits that are fixed inside $\mathbb{B}$. Indeed, there is a critical value $\theta_c \approx 0.6469076$ such that, if $\theta_c < \theta$, and R is large enough, then 'most' initial conditions (for η) are already fixed inside $\mathbb{B}$.

However, for most CA, it is difficult to determine the limit measures. Except for the linear CA of §3A, there is no large class of CA whose limit measures have been exactly characterized. Often, it is much easier to study the dynamical asymptotics of CA at a purely topological level.

If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$, then $\mathcal{A}^{\mathbb{M}} \supseteq \Phi(\mathcal{A}^{\mathbb{M}}) \supseteq \Phi^2(\mathcal{A}^{\mathbb{M}}) \supseteq \cdots$. The *limit set* of Φ is the nonempty subshift $\Phi^\infty(\mathcal{A}^{\mathbb{M}}) := \bigcap_{t=1}^{\infty} \Phi^t(\mathcal{A}^{\mathbb{M}})$. For any $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$, the *omega-limit set* of $\mathbf{a}$ is the set $\omega(\mathbf{a}, \Phi)$ of all cluster points of the Φ-orbit $\{\Phi^t(\mathbf{a})\}_{t=1}^{\infty}$. A closed subset $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ is a (Conley) *attractor* if there exists a clopen subset $\mathbf{U} \supseteq \mathbf{X}$ such that $\Phi(\mathbf{U}) \subseteq \mathbf{U}$ and $\mathbf{X} = \bigcap_{t=1}^{\infty} \Phi^t(\mathbf{U})$. It follows that $\omega(\Phi, \mathbf{u}) \subseteq \mathbf{X}$ for all $\mathbf{u} \in \mathbf{U}$. For example, $\Phi^\infty(\mathcal{A}^{\mathbb{M}})$ is an attractor (let $\mathbf{U} := \mathcal{A}^{\mathbb{M}}$). The topological attractors of CA were analyzed by Hurley (1990a, 1991, 1992), who discovered severe constraints on the possible attractor structures a CA could exhibit (see TOPOLOGICAL DYNAMICS OF CA and also CA, CLASSIFICATION OF).

Within pure topological dynamics, attractors and (omega) limit sets are the natural formalizations of the heuristic notion of 'self-organization'. The corresponding formalization in pure ergodic theory is the weak* limit measure. However, both weak* limit measures and topological attractors fail to adequately describe the sort of self-organization exhibited by many CA. Thus, several 'hybrid' notions self-organization have been developed, which combine topological and measurable criteria. These hybrid notions are more flexible and inclusive than purely topological notions. However, they do not require the explicit computation (or even the existence) of weak* limit measures, so in practice they are much easier to verify than purely ergodic notions.

Milnor-Hurley μ-attractors: If $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ is a closed subset, then for any $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$, we define $d(\mathbf{a}, \mathbf{X}) := \inf_{\mathbf{x} \in \mathbf{X}} d(\mathbf{a}, \mathbf{x})$. If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$, then the *basin* (or *realm*) of $\mathbf{X}$ is the set

$$\mathsf{B}_{\mathsf{asin}}(\mathbf{X}) \quad := \quad \left\{\mathbf{a} \in \mathcal{A}^{\mathbb{M}} \; ; \; \lim_{t\to\infty} d\left(\Phi^t(\mathbf{a}), \mathbf{X}\right) \;=\; 0\right\} \quad = \quad \left\{\mathbf{a} \in \mathcal{A}^{\mathbb{M}} \; ; \; \omega(\mathbf{a}, \Phi) \;\subseteq\; \mathbf{X}\right\}.$$

If $\mu \in \mathfrak{M}_{\mathsf{eas}}(\mathcal{A}^{\mathbb{M}})$, then $\mathbf{X}$ is a *μ-attractor* if $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{X})] > 0$; we call $\mathbf{X}$ a *lean μ-attractor* if in addition, $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{X})] > \mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{Y})]$ for any proper closed subset $\mathbf{Y} \subsetneq \mathbf{X}$. Finally, a μ-attractor $\mathbf{X}$ is *minimal* if $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{Y})] = 0$ for any proper closed subset $\mathbf{Y} \subsetneq \mathbf{X}$. For example, if $\mathbf{X}$ is a μ-attractor, and $(\mathbf{X}, \Phi)$ is minimal as a dynamical system, then $\mathbf{X}$ is a minimal μ-attractor. This concept was introduced by Milnor (1985a,b) in the context of smooth dynamical systems; its ramifications for CA were first explored by Hurley (1990b, 1991).

If $\mu \in \mathfrak{M}_{\mathsf{eas}}(\mathcal{A}^{\mathbb{Z}^D}, \sigma)$, then μ is *weakly σ-mixing* if, for any measurable sets $\mathbf{U}, \mathbf{V} \subset \mathcal{A}^{\mathbb{Z}^D}$, there is a subset $\mathbb{J} \subset \mathbb{Z}^D$ of density 1 such that $\lim_{\mathbb{J} \ni \mathsf{j} \to \infty} \mu[\sigma^{\mathsf{j}}(\mathbf{U}) \cap \mathbf{V}] \;=\; \mu[\mathbf{U}] \cdot \mu[\mathbf{V}]$ (see §4A). For example, any Bernoulli measure is weakly mixing. A subshift $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^D}$ is *σ-minimal* if $\mathbf{X}$ contains no proper nonempty subshifts. For example, if $\mathbf{X}$ is just the σ-orbit of some σ-periodic point, then $\mathbf{X}$ is σ-minimal.

Proposition 3B.1 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$, let $\mu \in \mathfrak{M}_{\mathsf{eas}}(\mathcal{A}^{\mathbb{M}}, \sigma)$, and let $\mathbf{X}$ be a μ-attractor.*

(a) *If μ is σ-ergodic, and $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ is a subshift, then $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{X})] = 1$.*

(b) *If $\mathbb{M}$ is countable, and $\mathbf{X}$ is σ-minimal subshift with $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{X})] = 1$, then $\mathbf{X}$ is lean.*

(c) *Suppose* $\mathbb{M} = \mathbb{Z}^D$ *and* μ *is weakly* σ*-mixing.*

[i] *If* $\mathbf{X}$ *is a minimal* μ*-attractor, then* $\mathbf{X}$ *is a subshift, so* $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{X})] = 1$*, and thus* $\mathbf{X}$ *is the only lean* μ*-attractor of* Φ*.*

[ii] *If* $\mathbf{X}$ *is a* Φ*-periodic orbit which is also a lean* μ*-attractor, then* $\mathbf{X}$ *is minimal,* $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{X})] = 1$*, and* $\mathbf{X}$ *contains only constant configurations.*

Proof: **(a)** If $\mathbf{X}$ is σ-invariant, then $\mathsf{B}_{\mathsf{asin}}(\mathbf{X})$ is also σ-invariant; hence $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{X})] = 1$ because μ is σ-ergodic.

(b) Suppose $\mathbf{Y} \subsetneq \mathbf{X}$ was a proper closed subset with $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{Y})] = 1$. For any $\mathsf{m} \in \mathbb{M}$, it is easy to check that $\mathsf{B}_{\mathsf{asin}}(\sigma^{\mathsf{m}}[\mathbf{Y}]) = \sigma^{\mathsf{m}}[\mathsf{B}_{\mathsf{asin}}(\mathbf{Y})]$. Thus, if $\widetilde{\mathbf{Y}} := \bigcap_{\mathsf{m}\in\mathbb{M}} \sigma^{\mathsf{m}}(\mathbf{Y})$, then $\mathsf{B}_{\mathsf{asin}}(\widetilde{\mathbf{Y}}) = \bigcap_{\mathsf{m}\in\mathbb{M}} \sigma^{\mathsf{m}}[\mathsf{B}_{\mathsf{asin}}(\mathbf{Y})]$, so $\mu\left[\mathsf{B}_{\mathsf{asin}}(\widetilde{\mathbf{Y}})\right] = 1$ (because $\mathbb{M}$ is countable). Thus, $\widetilde{\mathbf{Y}}$ is nonempty, and is a subshift of $\mathbf{X}$. But $\mathbf{X}$ is σ-minimal, so $\widetilde{\mathbf{Y}} = \mathbf{X}$, which means $\mathbf{Y} = \mathbf{X}$. Thus, $\mathbf{X}$ is a lean μ-attractor.

(c) In the case when μ is a Bernoulli measure, **(c)[i]** is (Hurley, 1990b, Theorem B) or (Hurley, 1991, Proposition 2.7), while **(c)[ii]** is (Hurley, 1991, Theorem A). Hurley's proofs easily extend to the case when μ is weakly σ-mixing. The only property we require of μ is this: for any nontrivial measurable sets $\mathbf{U}, \mathbf{V} \subset \mathcal{A}^{\mathbb{Z}^D}$, and any $\mathsf{z} \in \mathbb{Z}^D$, there is some $\mathsf{x}, \mathsf{y} \in \mathbb{Z}^D$ with $\mathsf{z} = \mathsf{x} - \mathsf{y}$, such that $\mu[\sigma^{\mathsf{y}}(\mathbf{U}) \cap \mathbf{V}] > 0$ and $\mu[\sigma^{\mathsf{x}}(\mathbf{U}) \cap \mathbf{V}] > 0$. This is clearly true if μ is weakly mixing (because if $\mathbb{J} \subset \mathbb{Z}^D$ has density 1, then $\mathbb{J} \cap (\mathsf{z} + \mathbb{J}) \neq \emptyset$ for any $\mathsf{z} \in \mathbb{Z}^D$).

Proof sketch for **(c)[i]:** If $\mathbf{X}$ is a (minimal) μ-attractor, then so is $\sigma^{\mathsf{y}}(\mathbf{X})$, and $\mathsf{B}_{\mathsf{asin}}[\sigma^{\mathsf{y}}(\mathbf{X})] = \sigma^{\mathsf{y}}(\mathsf{B}_{\mathsf{asin}}[\mathbf{X}])$. Thus, weak mixing yields $\mathsf{x}, \mathsf{y} \in \mathbb{Z}^D$ such that $\mathsf{B}_{\mathsf{asin}}[\sigma^{\mathsf{x}}(\mathbf{X})] \cap \mathsf{B}_{\mathsf{asin}}[\mathbf{X}]$ and $\mathsf{B}_{\mathsf{asin}}[\sigma^{\mathsf{y}}(\mathbf{X})] \cap \mathsf{B}_{\mathsf{asin}}[\mathbf{X}]$ are both nontrivial. But the basins of distinct minimal μ-attractors must be disjoint; thus $\sigma^{\mathsf{x}}(\mathbf{X}) = \mathbf{X} = \sigma^{\mathsf{y}}(\mathbf{X})$. But $\mathsf{x} - \mathsf{y} = \mathsf{z}$, so this means $\sigma^{\mathsf{z}}(\mathbf{X}) = \mathbf{X}$. This holds for all $\mathsf{z} \in \mathbb{Z}^D$, so $\mathbf{X}$ is a subshift, so (a) implies $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{X})] = 1$. □

Section 4 of Hurley (1990b) contains several examples showing that the minimal topological attractor of Φ can be different from its minimal μ-attractor. For example, a CA can have different minimal μ-attractors for different choices of μ. On the other hand, there is a CA possessing a minimal topological attractor but with no minimal μ-attractors for any Bernoulli measure μ.

Hilmy-Hurley Centers: Let $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$. For any closed subset $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$, we define

$$\mu_{\mathbf{a}}[\mathbf{X}] \quad := \quad \liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\mathbf{X}}(\Phi^t(\mathbf{a})).$$

(Thus, if μ is a Φ-ergodic measure on $\mathcal{A}^{\mathbb{M}}$, then Birkhoff's Ergodic Theorem asserts that $\mu_{\mathbf{a}}[\mathbf{X}] = \mu[\mathbf{X}]$ for μ-almost all $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$). The *center* of $\mathbf{a}$ is the set:

$$\mathsf{C}_{\mathsf{ent}}(\mathbf{a}, \Phi) \quad := \quad \bigcap \left\{\text{closed subsets } \mathbf{X} \subseteq \mathcal{A}^{\mathbb{M}} \ ; \ \mu_{\mathbf{a}}[\mathbf{X}] = 1\right\}.$$

Thus, $\mathsf{Cent}(\mathbf{a},\Phi)$ is the smallest closed subset such that $\mu_{\mathbf{a}}[\mathsf{Cent}(\mathbf{a},\Phi)] = 1$. If $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ is closed, then the ***well*** of $\mathbf{X}$ is the set

$$\mathsf{Well}(\mathbf{X}) \quad := \quad \left\{\mathbf{a} \in \mathcal{A}^{\mathbb{M}} \ ; \ \mathsf{Cent}(\mathbf{a},\Phi) \subseteq \mathbf{X}\right\}.$$

If $\mu \in \mathfrak{M}_{\mathrm{eas}}(AM)$, then $\mathbf{X}$ is a ***μ-center*** if $\mu[\mathsf{Well}(\mathbf{X})] > 0$; we call $\mathbf{X}$ a ***lean μ-center*** if in addition, $\mu[\mathsf{Well}(\mathbf{X})] > \mu[\mathsf{Well}(\mathbf{Y})]$ for any proper closed subset $\mathbf{Y} \subsetneq \mathbf{X}$. Finally, a μ-center $\mathbf{X}$ is ***minimal*** if $\mu[\mathsf{Well}(\mathbf{Y})] = 0$ for any proper closed subset $\mathbf{Y} \subsetneq \mathbf{X}$. This concept was introduced by Hilmy (1936) in the context of smooth dynamical systems; its ramifications for CA were first explored by Hurley (1991).

Proposition 3B.2 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$, let $\mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{M}},\sigma)$, and let $\mathbf{X}$ be a μ-center.*

- **(a)** *If μ is σ-ergodic, and $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ is a subshift, then $\mu[\mathsf{Well}(\mathbf{X})] = 1$.*
- **(b)** *If $\mathbb{M}$ is countable, and $\mathbf{X}$ is σ-minimal subshift with $\mu[\mathsf{Well}(\mathbf{X})] = 1$, then $\mathbf{X}$ is lean.*
- **(c)** *Suppose $\mathbb{M} = \mathbb{Z}^D$ and μ is weakly σ-mixing. If $\mathbf{X}$ is a minimal μ-center, then $\mathbf{X}$ is a subshift, $\mathbf{X}$ is the only lean μ-center, and $\mu[\mathsf{Well}(\mathbf{X})] = 1$.*

Proof: **(a)** and **(b)** are very similar to the proofs of Proposition 3B.1(a,b).

(c) is proved for Bernoulli measures as (Hurley, 1991, Theorem B). The proof is quite similar to Proposition 3B.1(c)[i], and again, we only need μ to be weakly mixing. □

Section 4 of Hurley (1991) contains several examples of minimal μ-centers which are not μ-attractors. In particular, the analogue of Proposition 3B.1(c)[ii] is false for μ-centers.

Kůrka-Maass μ-limit sets: If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$ and $\mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{M}},\sigma)$, then Kůrka and Maass define the ***μ-limit set*** of Φ:

$$\Lambda(\Phi,\mu) \quad := \quad \bigcap \left\{\text{closed subsets } \mathbf{X} \subset \mathcal{A}^{\mathbb{M}} \ ; \ \lim_{t\to\infty} \Phi^t\mu(\mathbf{X}) = 1\right\}.$$

It suffices to take this intersection only over all *cylinder* sets $\mathbf{X}$. By doing this, we see that $\Lambda(\mu,\Phi)$ is a subshift of $\mathcal{A}^{\mathbb{M}}$, and is defined by the following property: for any finite $\mathbb{B} \subset \mathbb{M}$ and any word $\mathbf{b} \in \mathcal{A}^{\mathbb{B}}$, $\mathbf{b}$ is admissible to $\Lambda(\Phi,\mu)$ if and only if $\liminf_{t\to\infty} \Phi^t\mu[\mathbf{b}] > 0$.

Proposition 3B.3 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$ and $\mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{M}},\sigma)$.*

- **(a)** *If $\text{wk*}\lim_{t\to\infty} \Phi^t\mu = \nu$, then $\Lambda(\Phi,\mu) = \mathsf{supp}(\nu)$.*

Suppose $\mathbb{M} = \mathbb{Z}$.

- **(b)** *If Φ is surjective and has an equicontinuous point, and μ has full support on $\mathcal{A}^{\mathbb{Z}}$, then $\Lambda(\Phi,\mu) = \mathcal{A}^{\mathbb{Z}}$.*

(c) *If* Φ *is left- or right-permutative and* μ *is connected* (see below), *then* $\Lambda(\Phi, \mu) = \mathcal{A}^{\mathbb{Z}}$.

Proof: For **(a)**, see (Kůrka and Maass, 2000, Proposition 2). For **(b,c)**, see (Kůrka, 2003, Theorems 2 and 3); for earlier special cases of these results, see also (Kůrka and Maass, 2000, Propositions 4 and 5). □

Remarks 3B.4: (a) In Proposition 3B.3(c), the measure μ is *connected* if there is some constant $C > 0$ such that, for any finite word $\mathbf{b} \in \mathcal{A}^*$, and any $a \in \mathcal{A}$, we have $\mu[\mathbf{b}\, a] \geq C \cdot \mu[\mathbf{b}]$ and $\mu[a\, \mathbf{b}] \geq C \cdot \mu[\mathbf{b}]$. For example, any Bernoulli, Markov, or N-step Markov measure with full support is connected. Also, any measure with 'complete connections' [see Example 2E.4(a)] is connected.

(b) Proposition 3B.3(a) shows that μ-limit sets are closely related to the weak* limits of measures. Recall from §3A that the uniform Bernoulli measure η is the weak* limit of a large class of initial measures under the action of linear CA. Presumably the same result should hold for a much larger class of *permutative* CA, but so far this is unproven [see Remarks 3A.6(d,e)]. Proposition 3B.3(a,c) implies that the limit measure of a permutative CA (if it exists) must have full support —hence it can't be 'too far' from η. ♢

Kůrka's measure attractors: Let $\mathfrak{M}^{\sigma}_{\text{inv}} := \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{M}}, \sigma)$ have the weak* topology, and define $\Phi_* : \mathfrak{M}^{\sigma}_{\text{inv}} \longrightarrow \mathfrak{M}^{\sigma}_{\text{inv}}$ by $\Phi_*(\mu) = \mu \circ \Phi^{-1}$. Then Φ_* is continuous, so we can treat $(\mathfrak{M}^{\sigma}_{\text{inv}}, \Phi_*)$ itself as a compact topological dynamical system. The "weak* limit measures" of Φ are simply the attracting fixed points of $(\mathfrak{M}^{\sigma}_{\text{inv}}, \Phi_*)$. However, even if the Φ_*-orbit of a measure μ does not weak* converge to a fixed point, we can still consider the omega-limit set of μ. In particular, the limit set $\Phi_*^{\infty}(\mathfrak{M}^{\sigma}_{\text{inv}})$ is the union of the omega-limit sets of all σ-invariant initial measures under Φ_*. Kůrka defines the *measure attractor* of Φ:

$$\mathsf{M}_{\text{eas}}\mathsf{A}_{\text{ttr}}(\Phi) \quad := \quad \overline{\bigcup \{\mathsf{supp}\,(\mu) \ ; \ \mu \in \Phi_*^{\infty}(\mathfrak{M}^{\sigma}_{\text{inv}})\}} \quad \subseteq \quad \mathcal{A}^{\mathbb{M}}.$$

(The bar denotes topological closure.) A configuration $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ is *densely recurrent* if any word which occurs in $\mathbf{a}$ does so with nonzero frequency. Formally, for any finite $\mathbb{B} \subset \mathbb{Z}^D$

$$\limsup_{N \to \infty} \frac{\#\left\{\mathsf{z} \in [-N...N]^D \ ; \ \mathbf{a}_{\mathbb{B}+\mathsf{z}} = \mathbf{a}_{\mathbb{B}}\right\}}{(2N+1)^D} \quad > \quad 0.$$

If $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^D}$ is a subshift, then the *densely recurrent subshift* of $\mathbf{X}$ is the closure $\mathbf{D}$ of the set of all densely recurrent points in $\mathbf{X}$. If $\mu \in \mathfrak{M}^{\sigma}_{\text{inv}}(\mathbf{X})$, then the Birkhoff Ergodic Theorem implies that $\mathsf{supp}\,(\mu) \subseteq \mathbf{D}$; see (Akin, 1993, Proposition 8.8, p.164). From this it follows that $\mathfrak{M}^{\sigma}_{\text{inv}}(\mathbf{X}) = \mathfrak{M}^{\sigma}_{\text{inv}}(\mathbf{D})$. On the other hand, $\mathbf{D} = \bigcup \{\mathsf{supp}\,(\mu) \ ; \ \mu \in \mathfrak{M}^{\sigma}_{\text{inv}}(\mathbf{D})\}$. In other words, densely recurrent subshifts are the only subshifts which are 'covered' by their own set of shift-invariant measures.

Proposition 3B.5 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$. Let $\mathbf{D}$ be the densely recurrent subshift of $\Phi^\infty(\mathcal{A}^{\mathbb{Z}^D})$. Then $\mathbf{D} = \mathsf{MeasAttr}(\Phi)$, and $\Phi^\infty(\mathfrak{M}_{\mathrm{inv}}^{\sigma}) = \mathfrak{M}_{\mathrm{eas}}(\mathbf{D}, \sigma)$.*

Proof: Case $D = 1$ is (Kůrka, 2005, Proposition 13). The same proof works for $D \geq 2$. □

Synthesis: The various hybrid modes of self-organization are related as follows:

Proposition 3B.6 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$.*

(a) *Let $\mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{M}}, \sigma)$ and let $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ be any closed set.*

- **[i]** *If $\mathbf{X}$ is a topological attractor and μ has full support, then $\mathbf{X}$ is a μ-attractor.*
- **[ii]** *If $\mathbf{X}$ is a μ-attractor, then $\mathbf{X}$ is a μ-center.*
- **[iii]** *Suppose $\mathbb{M} = \mathbb{Z}^D$, and that μ is weakly σ-mixing. Let $\mathbf{Y}$ be the intersection of all topological attractors of Φ. If Φ has a minimal μ-attractor $\mathbf{X}$, then $\mathbf{X} \subseteq \mathbf{Y}$.*
- **[iv]** *If μ is σ-ergodic, then $\Lambda(\Phi, \mu) \subseteq \bigcap \{\mathbf{X} \subseteq \mathcal{A}^{\mathbb{M}} ;\ \mathbf{X}$ a subshift and μ-attractor$\} \subseteq \Phi^\infty(\mathcal{A}^{\mathbb{Z}^D})$.*
- **[v]** *Thus, if μ is σ-ergodic and has full support, then*

$$\Lambda(\Phi, \mu) \quad \subseteq \quad \bigcap \left\{\mathbf{X} \subseteq \mathcal{A}^{\mathbb{M}} ;\ \mathbf{X} \text{ a subshift and a topological attractor}\right\}.$$

- **[vi]** *If $\mathbf{X}$ is a subshift, then $\Big(\Lambda(\Phi, \mu) \subseteq \mathbf{X}\Big) \iff \Big(\omega(\Phi_*, \mu) \subseteq \mathfrak{M}_{\mathrm{inv}}^{\sigma}(\mathbf{X})\Big)$.*

(b) *Let $\mathbb{M} = \mathbb{Z}^D$. Let $\mathfrak{B}$ be the set of all Bernoulli measures on $\mathcal{A}^{\mathbb{Z}^D}$, and for any $\beta \in \mathfrak{B}$, let $\mathbf{X}_\beta$ be the minimal β-attractor for Φ (if it exists).*

There is a comeager subset $\mathbf{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ such that $\overline{\bigcup_{\beta \in \mathfrak{B}} \mathbf{X}_\beta} \subseteq \bigcap_{\mathbf{a} \in \mathbf{A}} \omega(\mathbf{a}, \Phi)$.

(c) $\mathsf{MeasAttr}(\Phi) \;=\; \bigcup \left\{\Lambda(\Phi, \mu) ;\ \mu \in \mathfrak{M}_{\mathrm{inv}}^{\sigma}(\mathcal{A}^{\mathbb{M}})\right\}$.

(d) *If $\mathbb{M} = \mathbb{Z}^D$, then $\mathsf{MeasAttr}(\Phi) \subseteq \Phi^\infty(\mathcal{A}^{\mathbb{Z}^D})$.*

Proof: **(a)[i]:** If $\mathbf{U}$ is a clopen subset and $\Phi^\infty(\mathbf{U}) = \mathbf{X}$, then $\mathbf{U} \subseteq \mathsf{Basin}(\mathbf{X})$; thus, $0 < \mu[\mathbf{U}] \leq \mu[\mathsf{Basin}(\mathbf{X})]$, where the "$<$" is because μ has full support.

(a)[ii]: For any $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$, it is easy to see that $\mathsf{Cent}(\mathbf{a}, \Phi) \subseteq \omega(\mathbf{a}, \Phi)$. Thus, $\mathsf{Well}(\mathbf{X}) \supseteq \mathsf{Basin}(\mathbf{X})$. Thus, $\mu[\mathsf{Well}(\mathbf{X})] \geq \mu[\mathsf{Basin}(\mathbf{X})] > 0$.

(a)[iii] is (Hurley, 1990b, Proposition 3.3). (Again, Hurley states and proves this in the case when μ is a Bernoulli measure, but his proof only requires weak mixing).

(a)[iv]: Let $\mathbf{X}$ be a subshift and a μ-attractor; we claim that $\Lambda(\Phi, \mu) \subseteq \mathbf{X}$. Proposition 3B.1(a) says $\mu[\mathsf{Basin}(\mathbf{X})] = 1$. Let $\mathbb{B} \subset \mathbb{M}$ be any finite set. If $\mathbf{b} \in \mathcal{A}^{\mathbb{B}} \setminus \mathbf{X}_{\mathbb{B}}$, then

$$\left\{\mathbf{a} \in \mathcal{A}^{\mathbb{M}} ;\ \exists T \in \mathbb{N} \text{ such that } \forall t \geq T,\ \Phi^t(\mathbf{a})_{\mathbb{B}} \neq \mathbf{b}\right\} \supseteq \mathsf{Basin}(\mathbf{X}).$$

It follows that the left-hand set has μ-measure 1, which implies that $\lim_{t\to\infty} \Phi^t\mu\langle \mathbf{b}\rangle = 0$ —hence $\mathbf{b}$ is a forbidden word in $\Lambda(\Phi, \mu)$.

Thus, all the words forbidden in $\mathbf{X}$ are also forbidden in $\Lambda(\Phi, \mu)$. Thus $\Lambda(\Phi, \mu) \subseteq \mathbf{X}$. [The case $\mathbb{M} = \mathbb{Z}$ of (a)[iv] appears as (Kůrka and Maass, 2000, Proposition 1).]

(a)[v] follows from (a)[iv] and (a)[i].

(a)[vi] is (Kůrka, 2003, Proposition 1) or (Kůrka, 2005, Proposition 10); the argument is fairly similar to (a)[iv]. (Kůrka assumes $\mathbb{M} = \mathbb{Z}$, but this is not necessary.)

(b) is (Hurley, 1990b, Proposition 5.2).

(c) Let $\mathbf{X} \subset \mathcal{A}^{\mathbb{M}}$ be a subshift and let $\mathfrak{M}^{\sigma}_{\rm inv} = \mathfrak{M}^{\sigma}_{\rm inv}(\mathcal{A}^{\mathbb{M}})$. Then

$$\begin{aligned}
&\Big(\mathsf{MeasAttr}(\Phi) \subseteq \mathbf{X}\Big)\\
\iff\ &\Big(\mathsf{supp}\,(\nu) \subseteq \mathbf{X},\ \ \forall\, \nu \in \Phi_*^{\infty}(\mathfrak{M}^{\sigma}_{\rm inv})\Big) \iff \Big(\nu \in \mathfrak{M}^{\sigma}_{\rm inv}(\mathbf{X}),\ \ \forall\, \nu \in \Phi_*^{\infty}(\mathfrak{M}^{\sigma}_{\rm inv})\Big)\\
\iff\ &\Big(\Phi_*^{\infty}(\mathfrak{M}^{\sigma}_{\rm inv}) \subseteq \mathfrak{M}^{\sigma}_{\rm inv}(\mathbf{X})\Big) \iff \Big(\omega(\Phi_*, \mu) \subseteq \mathfrak{M}^{\sigma}_{\rm inv}(\mathbf{X}),\ \ \forall\, \mu \in \mathfrak{M}^{\sigma}_{\rm inv}\Big)\\
\underset{(*)}{\iff}\ &\Big(\Lambda(\Phi, \mu) \subseteq \mathbf{X},\ \ \forall\, \mu \in \mathfrak{M}^{\sigma}_{\rm inv}\Big) \iff \Big(\bigcup \{\Lambda(\Phi, \mu)\ ;\ \mu \in \mathfrak{M}^{\sigma}_{\rm inv}\} \ \subseteq\ \mathbf{X}\Big).
\end{aligned}$$

where $(*)$ is by (a)[vi]. It follows that $\mathsf{MeasAttr}(\Phi) \ = \ \bigcup \big\{\Lambda(\Phi, \mu)\ ;\ \mu \in \mathfrak{M}^{\sigma}_{\rm inv}(\mathcal{A}^{\mathbb{M}})\big\}$.

(d) follows immediately from Proposition 3B.5. □

Examples and Applications: The most natural examples of these hybrid modes of self-organization arise in the *particle cellular automata* (PCA) introduced in §2G. The long-term dynamics of a PCA involves a steady reduction in particle density, as particles coalesce or annihilate one another in collisions. Thus, presumably, for almost any initial configuration $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$, the sequence $\{\Phi^t(\mathbf{a})\}_{t=1}^{\infty}$ should converge to the subshift $\mathbf{Z}$ of configurations containing no particles (or at least, no particles of certain types), as $t\to\infty$. Unfortunately, this presumption is generally *false* if we interpret 'convergence' in the strict topological dynamical sense: the occasional particles will continue to wander near the origin at arbitrarily large times in the future orbit of $\mathbf{a}$ (albeit with diminishing frequency), so $\omega(\mathbf{a}, \Phi)$ will *not* be contained in $\mathbf{Z}$. However, the presumption becomes true if we instead employ one of the more flexible hybrid notions introduced above. For example, most initial probability measures μ should converge, under iteration of Φ to a measure concentrated on configurations with few or no particles; hence we expect that $\Lambda(\mu, \Phi) \subseteq \mathbf{Z}$. As discussed in §2G, a result about self-organization in a PCA can sometimes be translated into an analogous result about self-organization in associated coalescent-domain CA.

Proposition 3B.7 *Let* $\mathcal{A} = \{0, \pm 1\}$ *and let* $\Psi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ *be the Ballistic Annihilation Model* (BAM) *from* Example 2G.3. *Let* $\mathbf{R} := \{0, 1\}^{\mathbb{Z}}$ *and* $\mathbf{L} := \{0, -1\}^{\mathbb{Z}}$.

(a) *If $\mu \in \mathfrak{M}_{eas}(\mathcal{A}^{\mathbb{Z}}, \sigma)$, then $\nu = \text{wk*lim}_{t\to\infty} \Psi^t(\mu)$ exists, and has one of three forms: either $\nu \in \mathfrak{M}_{eas}(\mathbf{R}, \sigma)$, or $\nu \in \mathfrak{M}_{eas}(\mathbf{L}, \sigma)$, or $\nu = \delta_0$, the point mass on the sequence* $\mathbf{0} = [\ldots 000 \ldots]$.

(b) *Thus, the measure attractor of Φ is $\mathbf{R} \cup \mathbf{L}$* (note that $\mathbf{R} \cap \mathbf{L} = \{\mathbf{0}\}$).

(c) *In particular, if μ is a Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$ with $\mu[+1] = \mu[-1]$, then $\nu = \delta_0$.*

(d) *Let μ be a Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$.*

[i] *If $\mu[+1] > \mu[-1]$, then $\mathbf{R}$ is a μ-attractor —i.e. $\mu[\mathsf{B}_{asin}(\mathbf{R})] > 0$.*

[ii] *If $\mu[+1] < \mu[-1]$, then $\mathbf{L}$ is a μ-attractor.*

[iii] *If $\mu[+1] = \mu[-1]$, then $\{\mathbf{0}\}$ is not a μ-attractor, because $\mu[\mathsf{B}_{asin}\{\mathbf{0}\}] = 0$. However, $\Lambda(\Phi, \mu) = \{\mathbf{0}\}$.*

Proof: (a) is Theorem 6 of Belitsky and Ferrari (2005), and (b) follows from (a). (c) follows from Theorem 2 of Fisch (1992). (d)[i,ii] were first observed by Gilman (1987)[§3, pp.111-112], and later by Kůrka and Maass (2002)[Example 4]. (d)[iii] follows immediately from (c): the statement $\Lambda(\Phi, \mu) = \{\mathbf{0}\}$ is equivalent to asserting that $\lim_{t\to\infty} \Phi^t\mu[\pm 1] = 0$, which a consequence of (c). Another proof of (d)[iii] is (Kůrka and Maass, 2002, Proposition 11); see also (Kůrka and Maass, 2000, Example 3). □

Corollary 3B.8 *Let $\mathcal{A} = \mathbb{Z}_{/3}$, let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ be the CCA$_3$* [see Example 2G.2], *and let η be the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$. Then $\underset{t\to\infty}{\text{wk*lim}}\, \Phi^t(\eta) = \frac{1}{3}(\delta_0 + \delta_1 + \delta_2)$, where δ_a is the point mass on the sequence $[\ldots aaa \ldots]$ for each $a \in \mathcal{A}$.*

Proof: Combine Proposition 3B.7(c) with the factor map Γ in Example 2G.4(b). See Theorem 1 of Fisch (1992) for details. □

Corollary 3B.9 *Let $\mathcal{A} = \{0, 1\}$, let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ be ECA#184* [see Example 2G.1].

(a) $\mathsf{M}_{eas}\mathsf{A}_{ttr}(\Phi) = \mathbf{R} \cup \mathbf{L}$, *where $\mathbf{R} \subset \mathcal{A}^{\mathbb{Z}}$ is the set of sequences not containing* [11], *and $\mathbf{L} \subset \mathcal{A}^{\mathbb{Z}}$ is the set of sequences not containing* [00].

(b) *If η is the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$, then $\underset{t\to\infty}{\text{wk*lim}}\, \Phi^t(\eta) = \frac{1}{2}(\delta_0 + \delta_1)$, where δ_0 and δ_1 are the point masses on* $[\ldots 010.101 \ldots]$ *and* $[\ldots 101.010 \ldots]$.

(c) *Let μ be a Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$.*

[i] *If $\mu[0] > \mu[1]$, then $\mathbf{R}$ is a μ-attractor —i.e. $\mu[\mathsf{B}_{asin}(\mathbf{R})] > 0$.*

[ii] *If $\mu[0] < \mu[1]$, then $\mathbf{L}$ is a μ-attractor.*

Proof sketch: Let Γ be the factor map from Example 2G.4(a). To prove (a), apply Γ to Proposition 3B.7(b); see (Kůrka, 2005, Example 5, §9) for details. To prove (b), apply Γ to Proposition 3B.7(c); see (Kůrka and Maass, 2002, Proposition 12) for details. To prove (c), apply Γ to Proposition 3B.7(d)[i,ii]; □

Remarks 3B.10: (a) The other parts of Proposition 3B.7 can likewise be translated into equivalent statements about the measure attractors and μ-attractors of CCA_3 and ECA#184.

(b) Recall that ECA#184 is a model of single-lane, traffic, where each car is either stopped or moving rightwards at unit speed. Blank (2003) has extended Corollary 3B.9(c) to a much broader class of CA models of multi-lane, multi-speed traffic. For any such model, let $\mathbf{R} \subset \mathcal{A}^{\mathbb{Z}}$ be the set of 'free flowing' configurations where each car has enough space to move rightwards at its maximum possible speed. Let $\mathbf{L} \subset \mathcal{A}^{\mathbb{Z}}$ be the set of 'jammed' configurations where the cars are so tightly packed that the jammed clusters can propagate (leftwards) through the cars at maximum speed. If μ is any Bernoulli measure, then $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{R})] = 1$ if the μ-average density of cars is greater than 1/2, whereas $\mu[\mathsf{B}_{\mathsf{asin}}(\mathbf{L})] = 1$ if the density is less than 1/2 (Blank, 2003, Theorems 1.2 and 1.3). Thus, $\mathbf{L}\sqcup\mathbf{R}$ is a (non-lean) μ-attractor, although not a topological atractor (Blank, 2003, Lemma 2.13). ♢

Example 3B.11: A *cyclic addition and ballistic annihilation model* (CABAM) contains the same 'moving' particles ± 1 as the BAM [Example 2G.3], but also has one or more 'stationary' particle types. Let $3 \leq N \in \mathbb{N}$, and let $\mathcal{P} = \{1, 2, \ldots, N-1\} \subset \mathbb{Z}_{/N}$, where we identify $N-1$ with -1, modulo N. It will be convenient to represent the 'vacant' state $\emptyset$ as 0; thus, $\mathcal{A} = \mathbb{Z}_{/N}$. The particles 1 and -1 have velocities and collisions as in the BAM, namely:

$$\mathsf{v}(1) \;=\; 1, \qquad \mathsf{v}(-1) \;=\; -1, \qquad \text{and} \qquad -1+1 \;\rightsquigarrow\; \emptyset.$$

We set $\mathsf{v}(p) \;=\; 0$, for all $p \in [2...N-2]$, and employ the following collision rule:

$$\text{If } p_{-1} + p_0 + p_1 \;\equiv\; q, \quad (\text{mod } N), \text{ then } p_{-1} + p_0 + p_1 \;\rightsquigarrow\; q. \tag{3.3}$$

(here, any one of p_{-1}, p_0, p_1, or q could be 0, signifying vacancy). For example, if $N = 5$ and a (rightward moving) type $+1$ particle strikes a (stationary) type 3 particle, then the $+1$ particle is annihilated and the 3 particle turns into a (stationary) 4 particle. If another $+1$ particle hits the 4 particle, then both are annihilated, leaving a vacancy (0).

Let $\mathcal{B} = \mathbb{Z}_{/N}$, and let $\Psi \in \mathsf{CA}(\mathcal{B}^{\mathbb{Z}})$ be the CABAM. Then the set of fixed points of Ψ is $\mathbf{F} = \left\{\mathbf{f} \in \mathcal{B}^{\mathbb{Z}} \; ; \; f_{\mathsf{z}} \neq \pm 1, \; \forall \; \mathsf{z} \in \mathbb{Z}\right\}$. Note that, if $\mathbf{b} \in \mathsf{B}_{\mathsf{asin}}[\mathbf{F}]$ —that is, if $\omega(\mathbf{b}, \Psi) \subseteq \mathbf{F}$ —then in fact $\lim_{t\to\infty} \Psi^t(\mathbf{b})$ exists and is a Ψ-fixed point.

Proposition 3B.12 *Let $\mathcal{B} = \mathbb{Z}_{/N}$, let $\Psi \in \mathsf{CA}(\mathcal{B}^{\mathbb{Z}})$ be the CABAM, and let η be the uniform Bernoulli measure on $\mathcal{B}^{\mathbb{Z}}$. If $N \geq 5$, then $\mathbf{F}$ is a 'global' η-attractor —that is, $\eta[\mathsf{B}_{\mathsf{asin}}(\mathbf{F})] = 1$. However, if $N \leq 4$, then $\eta[\mathsf{B}_{\mathsf{asin}}(\mathbf{F})] = 0$.*

Proof: See Theorem 1 of Fisch (1990). □

Let $\mathcal{A} = \mathbb{Z}_{/N}$ and let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ be the N-colour CCA from Example 2G.2. Then the set of fixed points of Φ is $\mathbf{F} = \left\{\mathbf{f} \in \mathcal{A}^{\mathbb{Z}} \; ; \; f_{\mathsf{z}} - f_{\mathsf{z}+1} \neq \pm 1, \; \forall \; \mathsf{z} \in \mathbb{Z}\right\}$. Note that, if $\mathbf{a} \in \mathsf{B}_{\mathsf{asin}}[\mathbf{F}]$, then in fact $\lim_{t\to\infty} \Phi^t(\mathbf{a})$ exists and is a Φ-fixed point.

Corollary 3B.13 *Let $\mathcal{A} = \mathbb{Z}_{/N}$, let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ be the N-colour CCA, and let η be the uniform Bernoulli measure on $\mathcal{A}^{\mathbb{Z}}$ If $N \geq 5$, then $\mathbf{F}$ is a 'global' η-attractor —that is, $\eta[\mathsf{B}_{\mathsf{asin}}(\mathbf{F})] = 1$. However, if $N \leq 4$, then $\eta[\mathsf{B}_{\mathsf{asin}}(\mathbf{F})] = 0$.*

Proof sketch: Let $\mathcal{B} = \mathbb{Z}_{/N}$ and let $\Psi \in \mathsf{CA}(\mathcal{B}^{\mathbb{Z}})$ be the N-particle CABAM. Construct a factor map $\Gamma : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{B}^{\mathbb{Z}}$ with local rule $\gamma(a_0, a_1) := (a_0 - a_1) \bmod N$, similar to Example 2G.4(b). Then $\Gamma \circ \Phi = \Psi \circ \Gamma$, and the Ψ-particles track the Φ-domain boundaries. Now apply Γ to Proposition 3B.12. □

Example 3B.14: Let $\mathcal{A} = \{0, 1\}$ and let $\mathbb{H} = \{-1, 0, 1\}$. *Elementary Cellular Automaton #18* is the one-dimensional CA with local rule $\phi : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$ given: $\phi[100] = 1 = \phi[001]$, and $\phi(\mathbf{a}) = 0$ for all other $\mathbf{a} \in \mathcal{A}^{\mathbb{H}}$.

Empirically, ECA #18 has one stable phase: the *odd sofic shift* $\mathbf{S}$, defined by the $\mathcal{A}$-labelled digraph ① ⇆ ⓪ ⇆ ⓪. In other words, a sequence is admissible to $\mathbf{S}$ as long as an pair of consecutive ones are separated by an *odd* number of zeroes. Thus, a *defect* is any word of the form $10^{2m}1$ (where 0^{2m} represents $2m$ zeroes) for any $m \in \mathbb{N}$. Thus, defects can be arbitrarily large, they can grow and move arbitrarily quickly, and they can coalesce across arbitrarily large distances. Thus, it is impossible to construct a particle CA which tracks the motion of these defects. Nevertheless, in computer simulations, one can visually follow the moving defects through time, and they appear to perform random walks. Over time, the density of defects decreases as they randomly collide and annihilate. This was empirically observed by Grassberger (1984a,b) and Boccara et al. (1991). Lind (1984)[§5] conjectured that this gradual elimination of defects caused almost all initial conditions to converge, in some sense, to $\mathbf{S}$ under application of Φ.

Eloranta and Nummelin (1992) proved that the defects of Φ individually perform random walks. However, the motions of neighbouring defects are highly correlated. They are not *independent* random walks, so one cannot use standard results about stochastic interacting particle systems to conclude that the defect density converges to zero. To solve problems like this, Kůrka (2003) developed a theory of 'particle weight functions' for CA.

Let $\mathcal{A}^*$ be the set of all finite words in the alphabet $\mathcal{A}$. A *particle weight function* is a bounded function $p : \mathcal{A}^* \longrightarrow \mathbb{N}$, so that, for any $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$, we interpret

$$\#_p(\mathbf{a}) \quad := \quad \sum_{r=0}^{\infty} \sum_{z \in \mathbb{Z}} p(\mathbf{a}_{[z \ldots z+r]}) \quad \text{and} \quad \delta_p(\mathbf{a}) \quad := \quad \sum_{r=0}^{\infty} \lim_{N \to \infty} \frac{1}{2N} \sum_{z=-N}^{N} p(\mathbf{a}_{[z \ldots z+r]})$$

to be, respectively the 'number of particles' and 'density of particles' in configuration $\mathbf{a}$ (clearly $\#_p(\mathbf{a})$ is finite if and only if $\delta_p(\mathbf{a}) = 0$). The function p can count the single-letter 'particles' of a PCA, or the short-length 'domain boundaries' found in ECA#184 and the CCA of Examples 2G.1 and 2G.2. However, p can also track the arbitrarily large defects of ECA#18. For example, define $p_{18}(10^{2m}1) = 1$ (for any $m \in \mathbb{N}$), and define $p_{18}(\mathbf{a}) = 0$ for all other $\mathbf{a} \in \mathcal{A}^*$.

Let $\mathbf{Z}_p := \left\{\mathbf{a} \in \mathcal{A}^{\mathbb{Z}} \; ; \; \#_p(\mathbf{a}) = 0\right\}$ be the set of *vacuum configurations*. (For example, if $p = p_{18}$ as above, then $\mathbf{Z}_p$ is just the odd sofic shift $\mathbf{S}$.) If the iteration of a CA Φ

decreases the number (or density) of particles, then one expects $\mathbf{Z}_p$ to be a limit set for Φ in some sense. Indeed, if $\mu \in \mathfrak{M}_{inv}^{\sigma} := \mathfrak{M}_{eas}(\mathcal{A}^{\mathbb{Z}}, \sigma)$, then we define $\Delta_p(\mu) := \int_{\mathcal{A}^{\mathbb{Z}}} \delta_p \, d\mu$. If Φ is 'p-decreasing' in a certain sense, then Δ_p acts as a Lyapunov function for the dynamical system $(\mathfrak{M}_{inv}^{\sigma}, \Phi_*)$. Thus, with certain technical assumptions, we can show that, if $\mu \in \mathfrak{M}_{inv}^{\sigma}$ is connected, then $\Lambda(\mu, \Phi) \subseteq \mathbf{Z}_p$ (Kůrka, 2003, Theorem 8). Furthermore, under certain conditions, $\mathsf{MeasAttr}(\Phi) \subseteq \mathbf{Z}_p$ (Kůrka, 2003, Theorem 7). Using this machinery, Kůrka proved:

Proposition 3B.15 *Let $\Phi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ be ECA#18, and let $\mathbf{S} \subset \mathcal{A}^{\mathbb{Z}}$ be the odd sofic shift. If $\mu \in \mathfrak{M}_{eas}(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is connected, then $\Lambda(\mu, \Phi) \subseteq \mathbf{S}$.*

Proof: See Example 6.3 of Kůrka (2003). □

4 Measurable Dynamics

If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$ and $\mu \in \mathfrak{M}_{eas}(\mathcal{A}^{\mathbb{M}}, \Phi)$, then the triple $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is a *measure-preserving dynamical system* (MPDS), and thus, amenable to the methods of classical ergodic theory.

4A Mixing and Ergodicity

If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$, then the topological dynamical system $(\mathcal{A}^{\mathbb{M}}, \Phi)$ is *topologically transitive* (or *topologically ergodic*) if, if, for any open subsets $\mathbf{U}, \mathbf{V} \subseteq \mathcal{A}^{\mathbb{M}}$, there exists $t \in \mathbb{N}$ such that $\mathbf{U} \cap \Phi^{-t}(\mathbf{V}) \neq \emptyset$. Equivalently, there exists some $\mathbf{a} \in \mathcal{A}^{\mathbb{M}}$ whose orbit $\mathcal{O}(\mathbf{a}) := \{\Phi^t(\mathbf{a})\}_{t=0}^{\infty}$ is dense in $\mathcal{A}^{\mathbb{M}}$. If $\mu \in \mathfrak{M}_{eas}(\mathcal{A}^{\mathbb{M}}, \Phi)$, then the system $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is *ergodic* if, for any measurable $\mathbf{U}, \mathbf{V} \subseteq \mathcal{A}^{\mathbb{M}}$, there exists some $t \in \mathbb{N}$ such that $\mu[\mathbf{U} \cap \Phi^{-t}(\mathbf{V})] > 0$. The system $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is *totally ergodic* if $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi^n)$ is ergodic for every $n \in \mathbb{N}$. The system $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is *(strongly) mixing* if, for any measurable $\mathbf{U}, \mathbf{V} \subseteq \mathcal{A}^{\mathbb{M}}$.

$$\lim_{t \to \infty} \mu\left[\mathbf{U} \cap \Phi^{-t}(\mathbf{V})\right] \quad = \quad \mu[\mathbf{U}] \cdot \mu[\mathbf{V}]. \tag{4.1}$$

The system $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is *weakly mixing* if the limit (4.1) holds as $n \to \infty$ along an increasing subsequence $\{t_n\}_{n=1}^{\infty}$ of *density one* —i.e. such that $\lim_{n\to\infty} t_n/n = 1$. For any $M \in \mathbb{N}$, we say $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is *M-mixing* if, for any measurable $\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_M \subseteq \mathcal{A}^{\mathbb{M}}$.

$$\lim_{\substack{|t_n - t_m| \to \infty \\ \forall\ n \neq m \in [0\ldots M]}} \mu\left[\bigcap_{m=0}^{M} \Phi^{-t_m}(\mathbf{U}_m)\right] \quad = \quad \prod_{m=0}^{M} \mu[\mathbf{U}_m] \tag{4.2}$$

(thus, 'strong' mixing is 1-mixing). We say $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is *multimixing* (or *mixing of all orders*) if $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is M-mixing for all $M \in \mathbb{N}$.

We say $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is a *Kolmogorov endomorphism* if its natural extension is a Kolmogorov automorphism (see Ergodic Theory: Basic Examples and Constructions for the definition of *natural extension*; see ERGODICITY AND MIXING PROPERTIES for the definition of the *Kolmogorov* (or "*K*") property.) We say $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is a *Bernoulli endomorphism* if its natural extension is measurably isomorphic to a system $(\mathcal{B}^{\mathbb{Z}}, \beta; \sigma)$, where $\beta \in \mathfrak{M}_{eas}(\mathcal{B}^{\mathbb{Z}}; \sigma)$ is a Bernoulli measure.

The following chain of implications is well-known

Theorem 4A.1 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$, let $\mu \in \mathfrak{M}_{eas}(\mathcal{A}^{\mathbb{M}}; \Phi)$, and let $\mathbf{X} = \mathsf{supp}(\mu)$. Then $\mathbf{X}$ is a compact, Φ-invariant set. Furthermore:*

(μ, Φ) is Bernoulli $\Longrightarrow$ (μ, Φ) is Kolmogorov $\Longrightarrow$ (μ, Φ) is multimixing $\Longrightarrow$ (μ, Φ) is mixing $\Longrightarrow$ (μ, Φ) is weakly mixing $\Longrightarrow$ (μ, Φ) is totally ergodic $\Longrightarrow$ (μ, Φ) is ergodic $\Longrightarrow$ The system $(\mathbf{X}, \Phi)$ is topologically transitive $\Longrightarrow$ $\Phi : \mathbf{X} \longrightarrow \mathbf{X}$ is surjective.

Proof: See see ERGODICITY AND MIXING PROPERTIES. □

Theorem 4A.2 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{N}})$ be posexpansive* (see §2D). *Then $(\mathcal{A}^{\mathbb{N}}, \Phi)$ has topological entropy $\log_2(k)$ for some $k \in \mathbb{N}$, Φ preserves the uniform measure η, and $(\mathcal{A}^{\mathbb{N}}, \eta; \Phi)$ is a uniformly distributed Bernoulli endomorphism on an alphabet of cardinality k.*

Proof: Extend the argument of Theorem 2D.8. See (Blanchard and Maass, 1997, Corollary 3.10) or (Maass, 1996, Theorem 4.8(5)). □

Example 4A.3: Suppose $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{N}})$ is right-permutative, with neighbourhood $[r...R]$, where $0 \leq r < R$. Then $h_{\text{top}}(\Phi) = \log_2(|\mathcal{A}|^R)$, so Theorem 4A.2 says that $(\mathcal{A}^{\mathbb{N}}, \eta; \Phi)$ is a uniformly distributed Bernoulli endomorphism on the alphabet $\mathcal{B} := \mathcal{A}^R$.

In this case, it is easy to see this directly. If $\Phi_{\mathbb{B}}^{\mathbb{N}} : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{B}^{\mathbb{N}}$ is as in eqn.(2.1), then $\beta := \Phi_{\mathbb{B}}^{\mathbb{N}}(\eta)$ is the uniform Bernoulli measure on $\mathcal{B}^{\mathbb{N}}$, and $\Phi_{\mathbb{B}}^{\mathbb{N}}$ is an isomorphism from $(\mathcal{A}^{\mathbb{N}}, \mu; \Phi)$ to $(\mathcal{B}^{\mathbb{N}}, \beta; \sigma)$. ◇

Theorem 4A.4 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ have neighbourhood $[L...R]$. Suppose that*

either (a) *$0 \leq L < R$ and Φ is right-permutative;*
or (b) *$L < R \leq 0$ and Φ is left-permutative;*
or (c) *$L < R$ and Φ is bipermutative;*
or (d) *Φ is posexpansive.*

Then Φ preserves the uniform measure η, and $(\mathcal{A}^{\mathbb{Z}}, \eta; \Phi)$ is a Bernoulli endomorphism.

Proof: For cases **(a)** and **(b)**, see (Shereshevsky, 1992a, Theorem 2.2). For case **(c)**, see (Shereshevsky, 1992a, Theorem 2.7) or (Kleveland, 1997, Corollary 7.3). For **(d)**, extend the argument of Theorem 2D.4; see (Maass, 1996, Theorem 4.9). □

Theorem 4A.5 *Let* $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ *have neighbourhood* $[L...R]$*. Suppose that*

either (a) Φ *is surjective and* $0 < L \leq R$*;*

or (b) Φ *is surjective and* $L \leq R < 0$*;*

or (c) Φ *is right-permutative and* $R \neq 0$*;*

or (d) Φ *is left-permutative and* $L \neq 0$*.*

Then Φ *preserves* η*, and* $(\mathcal{A}^{\mathbb{Z}}, \eta; \Phi)$ *is a Kolmogorov endomorphism.*

Proof: Cases **(a)** and **(b)** are (Shereshevsky, 1992a, Theorem 2.4). Cases **(c)** and **(d)** are Shereshevsky (1997). □

Corollary 4A.6 *Any CA satisfying the hypotheses of Theorem 4A.5 is multimixing.*

Proof: This follows from Theorems 4A.1 and 4A.5. See also (Shirvani and Rogers, 1991, Theorem 3.2) for a direct proof that any CA satisfying hypotheses (a) or (b) is 1-mixing. See (Kleveland, 1997, Theorem 6.6) for a proof that any CA satisfying hypotheses (c) or (d) is multimixing. □

Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$ have neighbourhood $\mathbb{H}$. An element $\mathsf{x} \in \mathbb{H}$ is *extremal* if $\langle \mathsf{x}, \mathsf{x} \rangle > \langle \mathsf{x}, \mathsf{h} \rangle$ for all $\mathsf{h} \in \mathbb{H} \setminus \{\mathsf{x}\}$. We say Φ is *extremally permutative* if Φ is permutative in some extremal coordinate.

Theorem 4A.7 *Let* $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$ *and let* η *be the uniform measure. If* Φ *is extremally permutative, then* $(\mathcal{A}^{\mathbb{Z}^D}, \eta; \Phi)$ *is mixing.*

Proof: See (Willson, 1975, Theorem A) for the case $D = 2$ and $\mathcal{A} = \mathbb{Z}_{/2}$. Willson described Φ as 'linear' in an extremal coordinate (which is equivalent to permutative when $\mathcal{A} = \mathbb{Z}_{/2}$), and then concluded that Φ was 'ergodic' —however, he did this by explicitly showing that Φ was mixing. His proof technique easily generalizes to any extremally permutative CA on any alphabet, and any $D \geq 1$. □

Theorem 4A.8 *Let* $\mathcal{A} = \mathbb{Z}_{/m}$*. Let* $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$ *have linear local rule* $\phi : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$ *given by* $\phi(\mathbf{a}_{\mathbb{H}}) = \sum_{\mathsf{h} \in \mathbb{H}} c_{\mathsf{h}} \cdot a_{\mathsf{h}}$*, where* $c_{\mathsf{h}} \in \mathbb{Z}$ *for all* $\mathsf{h} \in \mathbb{H}$*. Let* η *be the uniform measure on* $\mathcal{A}^{\mathbb{Z}^D}$*. The following are equivalent:*

(a) Φ *preserves* η *and* $(\mathcal{A}^{\mathbb{Z}^D}, \eta, \Phi)$ *is ergodic.*

(b) $(\mathcal{A}^{\mathbb{Z}^D}, \Phi)$ *is topologically transitive.*

(c) $\gcd\{c_{\mathsf{h}}\}_{0 \neq \mathsf{h} \in \mathbb{H}}$ *is coprime to* m*.*

(d) *For all prime divisors* p *of* m*, there is some nonzero* $\mathsf{h} \in \mathbb{H}$ *such that* c_{h} *is not divisible by* p*.*

Proof: (Cattaneo et al., 2000, Theorem 3.2); see also Cattaneo et al. (1997). For a different proof in the case $D = 2$, see (Sato, 1997, Theorem 6). □

4B Spectral Properties

If $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{M}})$, then let $\mathbf{L}^2_\mu = \mathbf{L}^2(\mathcal{A}^{\mathbb{M}}, \mu)$ be the set of measurable functions $f : \mathcal{A}^{\mathbb{M}} \longrightarrow \mathbb{C}$ such that $\|f\|_2 := (\int_{\mathcal{A}^{\mathbb{M}}} |f|^2 \, d\mu)^{1/2}$ is finite. If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$ and $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{M}}, \Phi)$, then Φ defines a unitary linear operator $\Phi_* : \mathbf{L}^2_\mu \longrightarrow \mathbf{L}^2_\mu$ by $\Phi_*(f) = f \circ \Phi$ for all $f \in \mathbf{L}^2_\mu$. If $f \in \mathbf{L}^2_\mu$, then f is an ***eigenfunction*** of Φ, with ***eigenvalue*** $c \in \mathbb{C}$, if $\Phi_*(f) = c \cdot f$. By definition of Φ_*, any eigenvalue must be an element of the unit circle $\mathbb{T} := \{c \in \mathbb{C} \,;\, |c| = 1\}$. Let $\mathbb{S}_\Phi \subset \mathbb{T}$ be the set of all eigenvalues of Φ, and for any $s \in \mathbb{S}_\Phi$, let $\mathbf{E}_s(\Phi) := \left\{f \in \mathbf{L}^2_\mu \,;\, \Phi_* f = sf\right\}$ be the corresponding eigenspace. For example, if f is constant μ-almost everywhere, then $f \in \mathbf{E}_1(\Phi)$. Let $\mathbf{E}(\Phi) := \bigsqcup_{s \in \mathbb{S}_\Phi} \mathbf{E}_s(\Phi)$. Note that $\mathbb{S}_\Phi$ is a group. Indeed, if $s_1, s_2 \in \mathbb{S}_\Phi$, and $f_1 \in \mathbf{E}_{s_1}$ and $f_2 \in \mathbf{E}_{s_2}$, then $(f_1 f_2) \in \mathbf{E}_{s_1 s_2}$ and $(1/f_1) \in \mathbf{E}_{1/s_1}$. Thus, $\mathbb{S}_\Phi$ is called the ***spectral group*** of Φ.

If $s \in \mathbb{S}_\Phi$, then heuristically, an s-eigenfunction is an 'observable' of the dynamical system $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ which exhibits quasiperiodically recurrent behaviour. Thus, the spectral properties of Φ characterize the 'recurrent aspect' of its dynamics (or the lack thereof). For example:

- $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is ergodic $\iff$ $\mathbf{E}_1(\Phi)$ contains only constant functions $\iff$ $\dim[\mathbf{E}_s(\Phi)] = 1$ for all $s \in \mathbb{S}_\Phi$.
- $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is weakly mixing (see §4A) $\iff$ $\mathbf{E}(\Phi)$ contains only constant functions $\iff$ $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is ergodic and $\mathbb{S}_\Phi = \{1\}$.

We say $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ has ***discrete spectrum*** if $\mathbf{L}^2_\mu$ is spanned by $\mathbf{E}(\Phi)$. In this case, $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is measurably isomorphic to an MPDS defined by translation on a compact abelian group (e.g. an irrational rotation of a torus, an odometer, etc.). Please refer to the article SPECTRAL PROPERTIES for more information.

If $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{M}}, \sigma)$, then there is a natural unitary $\mathbb{M}$-action on $\mathbf{L}^2_\mu$, where $\sigma^{\mathsf{m}}_*(f) = f \circ \sigma^{\mathsf{m}}_*$. A ***character*** of $\mathbb{M}$ is a monoid homomorphism $\chi : \mathbb{M} \longrightarrow \mathbb{T}$. The set $\widehat{\mathbb{M}}$ of all characters is a group under pointwise multiplication, called the ***dual group*** of $\mathbb{M}$. If $f \in \mathbf{L}^2_\mu$ and $\chi \in \widehat{\mathbb{M}}$, then f is a χ***-eigenfunction*** of $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ if $\sigma^{\mathsf{m}}_*(f) = \chi(\mathsf{m}) \cdot f$ for all $\mathsf{m} \in \mathbb{M}$; then χ is called a ***eigencharacter***. The ***spectral group*** of $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ is then the subgroup $\mathbb{S}_\sigma \subset \widehat{\mathbb{M}}$ of all eigencharacters. For any $\chi \in \mathbb{S}_\sigma$, let $\mathbf{E}_\chi(\sigma)$ be the corresponding eigenspace, and let $\mathbf{E}(\sigma) := \bigsqcup_{\chi \in \mathbb{S}_\Phi} \mathbf{E}_\chi(\sigma)$.

- $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ is ergodic $\iff$ $\mathbf{E}_1(\sigma)$ contains only constant functions $\iff$ $\dim[\mathbf{E}_\chi(\sigma)] = 1$ for all $\chi \in \mathbb{S}_\sigma$.
- $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ is weakly mixing $\iff$ $\mathbf{E}(\sigma)$ contains only constant functions $\iff$ $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ is ergodic and $\mathbb{S}_\sigma = \{\mathbb{1}\}$.

$(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ has ***discrete spectrum*** if $\mathbf{L}^2_\mu$ is spanned by $\mathbf{E}(\sigma)$. In this case, the system $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ is measurably isomorphic to an action of $\mathbb{M}$ by translations on a compact abelian group.

Example 4B.1: Let $\mathbb{M} = \mathbb{Z}$; then any character $\chi : \mathbb{Z} \longrightarrow \mathbb{T}$ has the form $\chi(n) = c^n$ for some $c \in \mathbb{T}$, so a χ-eigenfunction is just a eigenfunction with eigenvalue c. In this case, the aforementioned spectral properties for the $\mathbb{Z}$-action by shifts are equivalent to the corresponding spectral properties of the CA $\Phi = \sigma^1$. Bernoulli measures and irreducible Markov chains are weakly mixing. On other hand, several important classes of symbolic dynamical systems have discrete spectrum, including Sturmian shifts, constant-length substitution shifts, and regular Toeplitz shifts. See SYMBOLIC DYNAMICS. ♢

Proposition 4B.2 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$, and let $\mu \in \mathfrak{M}_{\rm eas}(\mathcal{A}^{\mathbb{M}}; \Phi, \sigma)$ be σ-ergodic.*

(a) $\mathbf{E}(\sigma) \subseteq \mathbf{E}(\Phi)$.

(b) *If $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ has discrete spectrum, then so does $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$.*

(c) *Suppose μ is Φ-ergodic. If $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ is weakly mixing, then so is $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$.*

Proof: **(a)** Suppose $\chi \in \widehat{\mathbb{M}}$ and $f \in \mathbf{E}_\chi$. Then $f \circ \Phi \in \mathbf{E}_\chi$ also, because for all $\mathsf{m} \in \mathbb{M}$, $f \circ \Phi \circ \sigma^{\mathsf{m}} = f \circ \sigma^{\mathsf{m}} \circ \Phi = \chi(\mathsf{m}) \cdot f \circ \Phi$. But if $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ is ergodic, then $\dim[\mathbf{E}_\chi(\sigma)] = 1$; hence $f \circ \Phi$ must be a scalar multiple of f. Thus, f is also an eigenfunction for Φ. **(b)** follows from (a).

(c) By reversing the roles of Φ and σ in (a), we see that $\mathbf{E}(\Phi) \subseteq \mathbf{E}(\sigma)$. But if $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ is weakly mixing, then $\mathbf{E}(\sigma) = \{\text{constant functions}\}$. Thus, $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is also weakly mixing. □

Example 4B.3: (a) Let μ be any Bernoulli measure on $\mathcal{A}^{\mathbb{M}}$. If μ is Φ-invariant and Φ-ergodic, then $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$ is weakly mixing (because $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ is weakly mixing).

(b) Let $P \in \mathbb{N}$ and suppose μ is a Φ-invariant measure supported on the set $\mathbf{X}_P$ of P-periodic sequences (see Proposition 2C.1). Then $(\mathcal{A}^{\mathbb{Z}}, \mu; \sigma)$ has discrete spectrum (with rational eigenvalues). But $\mathbf{X}_P$ is finite, so the system $(\mathbf{X}_P, \Phi)$ is also periodic; hence $(\mathcal{A}^{\mathbb{Z}}, \mu; \Phi)$ also has discrete spectrum (with rational eigenvalues).

(c) Downarowicz (1997) has constructed an example of a regular Toeplitz shift $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}}$ and $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ (not the shift) such that $\Phi(\mathbf{X}) \subseteq \mathbf{X}$. Any regular Toeplitz shift is uniquely ergodic, and the unique shift-invariant measure μ has discrete spectrum; thus, $(\mathcal{A}^{\mathbb{Z}}, \mu; \Phi)$ also has discrete spectrum. ♢

Aside from Examples 4B.3(b,c), the literature contains no examples of discrete-spectrum, invariant measures for CA; this is an interesting area for future research.

5 Entropy

Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{M}})$. For any finite $\mathbb{B} \subset \mathbb{M}$, let $\mathcal{B} := \mathcal{A}^{\mathbb{B}}$, let $\Phi_{\mathbb{B}}^{\mathbb{N}} : \mathcal{A}^{\mathbb{N}} \longrightarrow \mathcal{B}^{\mathbb{N}}$ be as in eqn.(2.1), and let $\mathbf{X} := \Phi_{\mathbb{B}}^{\mathbb{N}}(\mathcal{A}^{\mathbb{M}}) \subseteq \mathcal{A}^{\mathbb{N}}$; then define

$$H_{\mathrm{top}}(\mathbb{B};\Phi) \quad := \quad h_{\mathrm{top}}(\mathbf{X}) \quad = \quad \lim_{T\to\infty} \frac{1}{T} \log_2(\#\mathbf{X}_{[0\ldots T)}).$$

If $\mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{M}}, \Phi)$, let $\nu := \Phi_{\mathbb{B}}^{\mathbb{N}}(\mu)$; then ν is a σ-invariant measure on $\mathcal{B}^{\mathbb{N}}$. Define

$$H_\mu(\mathbb{B};\Phi) \quad := \quad h_\nu(\sigma) \quad = \quad -\lim_{T\to\infty} \frac{1}{T} \sum_{\mathbf{b}\in\mathcal{B}^{[0\ldots T)}} \nu[\mathbf{b}] \log_2(\nu[\mathbf{b}]).$$

The *topological entropy* of $(\mathcal{A}^{\mathbb{M}}, \Phi)$ and the *measurable entropy* of $(\mathcal{A}^{\mathbb{M}}, \Phi, \mu)$ are then defined

$$h_{\mathrm{top}}(\Phi) \quad := \quad \sup_{\substack{\mathbb{B}\subset\mathbb{M}\\ \text{finite}}} H_{\mathrm{top}}(\mathbb{B};\Phi) \quad \text{and} \quad h_\mu(\Phi) \quad := \quad \sup_{\substack{\mathbb{B}\subset\mathbb{M}\\ \text{finite}}} H_\mu(\mathbb{B};\Phi).$$

The famous *Variational Principle* states that $h_{\mathrm{top}}(\Phi) = \sup\left\{h_\mu(\Phi) \; ; \; \mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{M}};\Phi)\right\}$. (Please see Entropy in Ergodic Theory for more information).

If $\mathbb{M}$ has more than one dimension (e.g. $\mathbb{M} = \mathbb{Z}^D$ or $\mathbb{N}^D$ for $D \geq 2$) then most CA on $\mathcal{A}^{\mathbb{M}}$ have infinite entropy. Thus, entropy is mainly of interest in the case $\mathbb{M} = \mathbb{Z}$ or $\mathbb{N}$. Coven (1980) was the first to compute the topological entropy of a CA; he showed that $h_{\mathrm{top}}(\Phi) = 1$ for a large class of left-permutative, one-sided CA on $\{0,1\}^{\mathbb{N}}$ (which have since been called *Coven CA*). Later, Lind (1987) showed how to construct a CA whose topological entropy was any element of an uncountable dense subset of $\mathbb{R}_+$, defined using Perron numbers. Theorems 2D.4 and 2D.8(b) above characterize the topological entropy of posexpansive CA. However, Hurd et al. (1992) showed that there is no algorithm which can compute the topological entropy of an arbitrary CA.

Measurable entropy has also been computed for a few special classes of CA. For example, if $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ is bipermutative with neighbourhood $\{0,1\}$ and $\mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{Z}}, \Phi; \sigma)$ is σ-ergodic, then $h_\mu(\Phi) = \log_2(K)$ for some integer $K \leq |\mathcal{A}|$ (Pivato, 2005b, Thm 4.1). If η is the uniform measure, and Φ is posexpansive, then Theorems 4A.2 and 4A.4 above characterize $h_\eta(\Phi)$. Also, if Φ satisfies the conditions of Theorem 4A.5, then $h_\eta(\Phi) > 0$, and furthermore, all factors of the MPDS $(\mathcal{A}^{\mathbb{Z}}, \mu; \Phi)$ also have positive entropy.

However, unlike abstract dynamical systems, CA come with an explicit spatial 'geometry'. The most fruitful investigations of CA entropy are those which have interpreted entropy in terms of how information propagates through this geometry.

5A Lyapunov Exponents

Wolfram (1985) suggested that the propagation speed of 'perturbations' in a one-dimensional CA Φ could transform 'spatial' entropy [i.e. $h(\sigma)$] into 'temporal' entropy [i.e. $h(\Phi)$]. He compared this propagation speed to the 'Lyapunov exponent' of a smooth dynamical system: it determines the exponential rate of divergence between two initially close Φ-orbits

(Wolfram, 1986, pp.172, 261 and 514). Shereshevsky (1992b) formalized Wolfram's intuition and proved the conjectured entropy relationship; his results were later improved by Tisseur (2000). Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$, let $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$, and let $\mathsf{z} \in \mathbb{Z}$. Define

$$\begin{aligned} \mathbf{W}^+_{\mathsf{z}}(\mathbf{a}) &:= \left\{\mathbf{w} \in \mathcal{A}^{\mathbb{Z}} \; ; \; \mathbf{w}_{[z\ldots\infty)} = \mathbf{a}_{[z\ldots\infty)}\right\}, \\ \text{and } \mathbf{W}^-_{\mathsf{z}}(\mathbf{a}) &:= \left\{\mathbf{w} \in \mathcal{A}^{\mathbb{Z}} \; ; \; \mathbf{w}_{(-\infty\ldots z]} = \mathbf{a}_{(-\infty\ldots z]}\right\}. \end{aligned}$$

Thus, we obtain each $\mathbf{w} \in \mathbf{W}^+_{\mathsf{z}}(\mathbf{a})$ [respectively $\mathbf{W}^-_{\mathsf{z}}(\mathbf{a})$] by 'perturbing' $\mathbf{a}$ somewhere to the left [resp. right] of coordinate z. Next, for any $t \in \mathbb{N}$, define

$$\begin{aligned} \widetilde{\Lambda}^+_t(\mathbf{a}) &:= \min\left\{\mathsf{z} \in \mathbb{N} \; ; \; \Phi^t\left[\mathbf{W}^+_0(\mathbf{a})\right] \subseteq \mathbf{W}^+_{\mathsf{z}}\left(\Phi^t[\mathbf{a}]\right)\right\}, \\ \text{and } \widetilde{\Lambda}^-_t(\mathbf{a}) &:= \min\left\{\mathsf{z} \in \mathbb{N} \; ; \; \Phi^t\left[\mathbf{W}^-_0(\mathbf{a})\right] \subseteq \mathbf{W}^-_{-\mathsf{z}}\left(\Phi^t[\mathbf{a}]\right)\right\}. \end{aligned}$$

Thus, $\widetilde{\Lambda}^\pm_t$ measures the farthest distance which any perturbation of $\mathbf{a}$ at coordinate 0 could have propagated by time t. Next, define $\Lambda^\pm_t(\mathbf{a}) := \max_{\mathsf{z}\in\mathbb{Z}} \widetilde{\Lambda}^\pm_t(\mathbf{a})$. Then Shereshevsky (1992b) defined the *(maximum) Lyapunov exponents*

$$\lambda^+(\Phi, \mathbf{a}) \quad := \quad \lim_{t\to\infty} \frac{1}{t}\Lambda^+_t(\mathbf{a}), \qquad \text{and} \qquad \lambda^-(\Phi, \mathbf{a}) \quad := \quad \lim_{t\to\infty} \frac{1}{t}\Lambda^-_t(\mathbf{a}),$$

whenever these limits exist. Let $\mathbf{G}(\Phi) := \left\{\mathbf{g} \in \mathcal{A}^{\mathbb{Z}} \; ; \; \lambda^\pm(\Phi, \mathbf{g}) \text{ both exist}\right\}$. The subset $\mathbf{G}(\Phi)$ is 'generic' within $\mathcal{A}^{\mathbb{Z}}$ in a very strong sense, and the Lyapunov exponents detect 'chaotic' topological dynamics.

Proposition 5A.1 *Let* $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$.

- **(a)** *Let* $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}}; \sigma)$. *Suppose that either:* [i] μ *is also* Φ*-invariant; or:* [ii] μ *is* σ*-ergodic and* $\mathsf{supp}\,(\mu)$ *is a* Φ*-invariant subset. Then* $\mu(\mathbf{G}) = 1$.

- **(b)** *The set* $\mathbf{G}$ *and the functions* $\lambda^\pm(\Phi, \bullet)$ *are* (Φ, σ)*-invariant. Thus, if* μ *is either* Φ*-ergodic or* σ*-ergodic, then there exist constants* $\lambda^\pm_\mu(\Phi) \geq 0$ *such that* $\lambda^\pm(\Phi, \mathbf{g}) = \lambda^\pm_\mu(\Phi)$ *for* μ*-almost all* $\mathbf{g} \in \mathbf{G}$.

- **(c)** *If* Φ *is posexpansive, then there is a constant* $c > 0$ *such that* $\lambda^\pm(\Phi, \mathbf{g}) \geq c$ *for all* $\mathbf{g} \in \mathbf{G}$.

- **(d)** *Let* η *be the uniform Bernoulli measure. If* Φ *is surjective, then* $h_{\text{top}}(\Phi) \leq \left(\lambda^+_\eta(\Phi) + \lambda^-_\eta(\Phi)\right) \cdot \log|\mathcal{A}|$.

Proof: **(a)** follows from the fact that, for any $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$, the sequence $[\Lambda^\pm_t(\mathbf{a})]_{t\in\mathbb{N}}$ is subadditive in t. Condition [i] is (Shereshevsky, 1992b, Theorem 1), and follows from Kingman's subadditive ergodic theorem. Condition [ii] is (Tisseur, 2000, Proposition 3.1).

(b) is clear by definition of $\lambda^\pm$. **(c)** is (Finelli et al., 1998, Theorem 5.2). **(d)** is (Tisseur, 2000, Proposition 5.3). □

For any Φ-ergodic $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}};\Phi,\sigma)$, Shereshevsky (1992b)[Theorem 2] showed that $h_\mu(\Phi) \leq \left(\lambda^+_\mu(\Phi) + \lambda^-_\mu(\Phi)\right) \cdot h_\mu(\sigma)$. Tisseur later improved this estimate. For any $T \in \mathbb{N}$, let

$$\begin{aligned} \widetilde{I}^+_T(\mathbf{a}) &:= \min\left\{\mathsf{z} \in \mathbb{N} \;;\; \forall\, t \in [1...T],\; \Phi^t\left[\mathbf{W}^+_{-\mathsf{z}}(\mathbf{a})\right] \subseteq \mathbf{W}^+_0\left(\Phi^t[\mathbf{a}]\right)\right\} \\ \text{and}\;\; \widetilde{I}^-_T(\mathbf{a}) &:= \min\left\{\mathsf{z} \in \mathbb{N} \;;\; \forall\, t \in [1...T],\; \Phi^t\left[\mathbf{W}^-_{\mathsf{z}}(\mathbf{a})\right] \subseteq \mathbf{W}^-_0\left(\Phi^t[\mathbf{a}]\right)\right\}. \end{aligned}$$

Next, for any $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}},\sigma)$, define $\widehat{I}^\pm_T(\mu) := \displaystyle\int_{\mathcal{A}^{\mathbb{Z}}} \widetilde{I}^\pm_T(\mathbf{a})\, d\mu[\mathbf{a}]$.
Tisseur then defined the ***average Lyapunov exponents***: $I^\pm_\mu(\Phi) := \liminf_{T\to\infty} \widehat{I}^\pm_T(\mu)/T$.

Theorem 5A.2 *Let* $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ *and let* $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}};\sigma)$.

(a) *If* $\mathsf{supp}(\mu)$ *is Φ-invariant, then* $I^+_\mu(\Phi) \leq \lambda^+_\mu(\Phi)$ *and* $I^-_\mu(\Phi) \leq \lambda^-_\mu(\Phi)$*, and one or both inequalities are sometimes strict.*

(b) *If μ is σ-ergodic and Φ-invariant, then* $h_\mu(\Phi) \leq \left(I^+_\mu(\Phi) + I^-_\mu(\Phi)\right) \cdot h_\mu(\sigma)$*, and this inequality is sometimes strict.*

(c) *If* $\mathsf{supp}(\mu)$ *contains Φ-equicontinuous points, then* $I^+_\mu(\Phi) = I^-_\mu(\Phi) = h_\mu(\Phi) = 0$.

Proof: See Tisseur (2000): **(a)** is Proposition 3.2 and Example 6.1; **(b)** is Theorem 5.1 and Example 6.2; and **(c)** is Proposition 5.2. □

5B Directional Entropy

Milnor (1986, 1988) introduced directional entropy to capture the intuition that information in a CA propagates in particular directions with particular 'velocities', and that different CA 'mix' information in different ways. Classical entropy is unable to detect this informational anisotropy. For example, if $\mathcal{A} = \{0,1\}$ and $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ has local rule $\phi(a_0,a_1) = a_0 + a_1 \pmod 2$, then $h_{\text{top}}(\Phi) = 1 = h_{\text{top}}(\sigma)$, despite the fact that Φ vigorously 'mixes' information together and propagates any 'perturbation' outwards in an expanding cone, whereas σ merely shifts information to the left in a rigid and essentially trivial fashion.

If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$, then a ***complete history*** for Φ is a sequence $(\mathbf{a}_t)_{t\in\mathbb{Z}} \in (\mathcal{A}^{\mathbb{Z}^D})^{\mathbb{Z}} \cong \mathcal{A}^{\mathbb{Z}^{D+1}}$ such that $\Phi(\mathbf{a}_t) = \mathbf{a}_{t+1}$ for all $t \in \mathbb{Z}$. Let $\mathbf{X}^{\text{Hist}} := \mathbf{X}^{\text{Hist}}(\Phi) \subset \mathcal{A}^{\mathbb{Z}^{D+1}}$ be the subshift of all complete histories for Φ, and let σ be the $\mathbb{Z}^{D+1}$ shift action on $\mathbf{X}^{\text{Hist}}$; then $(\mathbf{X}^{\text{Hist}};\sigma)$ is conjugate to the natural extension of the system $(\mathbf{Y};\Phi,\sigma)$, where $\mathbf{Y} := \Phi^\infty(\mathcal{A}^{\mathbb{M}}) := \bigcap_{t=1}^\infty \Phi^t(\mathcal{A}^{\mathbb{Z}^D})$ is the omega-limit set of Φ. If $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}^D};\Phi,\sigma)$, then $\mathsf{supp}(\mu) \subseteq \mathbf{Y}$, and μ extends to a σ-invariant measure $\widetilde{\mu}$ on $\mathbf{X}^{\text{Hist}}$ in the obvious way.

Let $\vec{\mathsf{v}} = (v_0;v_1,\ldots,v_D) \in \mathbb{R}\times\mathbb{R}^D \cong \mathbb{R}^{D+1}$. For any bounded open subset $\mathsf{B} \subset \mathbb{R}^{D+1}$ and $T > 0$, let $\mathsf{B}(T\vec{\mathsf{v}}) := \{\mathsf{b} + t\vec{\mathsf{v}} \;;\; \mathsf{b} \in \mathsf{B} \text{ and } t \in [0,T]\}$ be the 'sheared cylinder' in $\mathbb{R}^{D+1}$

with cross-section B and length $T|\vec{\mathrm{v}}|$ in the direction $\vec{\mathrm{v}}$, and let $\mathbb{B}(T\vec{\mathrm{v}}) := \mathsf{B}(T\vec{\mathrm{v}}) \cap \mathbb{Z}^{D+1}$. Let $\mathbf{X}^{\mathrm{Hist}}_{\mathbb{B}(T\vec{\mathrm{v}})} := \left\{\mathbf{x}_{\mathbb{B}(T\vec{\mathrm{v}})} \ ; \ \mathbf{x} \in \mathbf{X}^{\mathrm{Hist}}(\Phi)\right\}$. We define

$$H_{\mathrm{top}}(\Phi; \mathsf{B}, \vec{\mathrm{v}}) \;:=\; \limsup_{T\to\infty} \frac{1}{T} \log_2[\#\mathbf{X}^{\mathrm{Hist}}_{\mathbb{B}(T\vec{\mathrm{v}})}];$$

$$\text{and} \quad H_{\mu}(\Phi; \mathsf{B}, \vec{\mathrm{v}}) \;:=\; -\limsup_{T\to\infty} \frac{1}{T} \sum_{\mathbf{x}\in\mathbf{X}^{\mathrm{Hist}}_{\mathbb{B}(T\vec{\mathrm{v}})}} \widetilde{\mu}[\mathbf{x}] \log_2(\widetilde{\mu}[\mathbf{x}]).$$

We then define the $\vec{\mathrm{v}}$-*directional topological entropy* and $\vec{\mathrm{v}}$-*directional μ-entropy* of Φ by

$$h_{\mathrm{top}}(\Phi; \vec{\mathrm{v}}) \;:=\; \sup_{\substack{\mathsf{B}\subset\mathbb{R}^{D+1}\\ \text{open \& bounded}}} h_{\mathrm{top}}(\Phi; \mathsf{B}, \vec{\mathrm{v}}); \tag{5.1}$$

$$\text{and} \quad h_{\mu}(\Phi; \vec{\mathrm{v}}) \;:=\; \sup_{\substack{\mathsf{B}\subset\mathbb{R}^{D+1}\\ \text{open \& bounded}}} h_{\mu}(\Phi; \mathsf{B}, \vec{\mathrm{v}}). \tag{5.2}$$

Proposition 5B.1 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$ and let $\mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{Z}^D}; \Phi, \sigma)$.*

(a) *Directional entropy is homogeneous. That is, for any $\vec{\mathrm{v}} \in \mathbb{R}^{D+1}$ and $r > 0$, $h_{\mathrm{top}}(\Phi, r\vec{\mathrm{v}}) = r \cdot h_{\mathrm{top}}(\Phi, \vec{\mathrm{v}})$ and $h_\mu(\Phi, r\vec{\mathrm{v}}) = r \cdot h_\mu(\Phi, \vec{\mathrm{v}})$.*

(b) *If $\vec{\mathrm{v}} = (t; \mathbf{z}) \in \mathbb{Z} \times \mathbb{Z}^D$, then $h_{\mathrm{top}}(\Phi, \vec{\mathrm{v}}) = h_{\mathrm{top}}(\Phi^t \circ \sigma^{\mathbf{z}})$ and $h_\mu(\Phi, \vec{\mathrm{v}}) = h_\mu(\Phi^t \circ \sigma^{\mathbf{z}})$.*

(c) *There is an extension of the $\mathbb{Z} \times \mathbb{Z}^D$-system $(\mathbf{X}^{\mathrm{Hist}}, \Phi; \sigma)$ to an $\mathbb{R} \times \mathbb{R}^D$-system $(\widetilde{\mathbf{X}}, \widetilde{\Phi}, \widetilde{\sigma})$ such that, for any $\vec{\mathrm{v}} = (t; \vec{\mathrm{u}}) \in \mathbb{R} \times \mathbb{R}^D$ we have $h_{\mathrm{top}}(\Phi, \vec{\mathrm{v}}) = h_{\mathrm{top}}(\widetilde{\Phi}^t \circ \widetilde{\sigma}^{\vec{\mathrm{u}}})$.*

For any $\mu \in \mathfrak{M}_{\mathrm{eas}}(\mathcal{A}^{\mathbb{Z}^D}; \Phi, \sigma)$, there is an extension $\widetilde{\mu} \in \mathfrak{M}_{\mathrm{eas}}(\widetilde{\mathbf{X}}; \widetilde{\Phi}, \widetilde{\sigma})$ such that for any $\vec{\mathrm{v}} = (t; \vec{\mathrm{u}}) \in \mathbb{R} \times \mathbb{R}^D$ we have $h_\mu(\Phi, \vec{\mathrm{v}}) = h_{\widetilde{\mu}}(\widetilde{\Phi}^t \circ \widetilde{\sigma}^{\vec{\mathrm{u}}})$.

Proof: **(a,b)** follow from the definition. **(c)** is (Park, 1999, Proposition 2.1). □

Remark 5B.2: Directional entropy can actually be defined for any continuous $\mathbb{Z}^{D+1}$-action on a compact metric space, and in particular, for any subshift of $\mathcal{A}^{\mathbb{Z}^{D+1}}$. The directional entropy of a CA Φ is then just the directional entropy of the subshift $\mathbf{X}^{\mathrm{Hist}}(\Phi)$. Proposition 5B.1 holds for any subshift. ◊

Directional entropy is usually infinite for multidimensional CA (for the same reason that classical entropy is usually infinite). Thus, most of the analysis has been for one-dimensional CA. For example, Kitchens and Schmidt (1992)[§1] studied the directional topological entropy of one-dimensional linear CA, while Smillie (1988)[Proposition 1.1] computed the directional topological entropy for ECA#184. If Φ is linear, then the function $\vec{\mathrm{v}} \mapsto h_{\mathrm{top}}(\Phi, \vec{\mathrm{v}})$ is piecewise linear and convex, but if Φ is ECA#184, it is neither.

If $\vec{\mathrm{v}}$ has rational entries, then Proposition 5B.1(a,b) shows that $h(\Phi, \vec{\mathrm{v}})$ is a rational multiple of the classical entropy of some composite CA, which can be computed through classical methods. However, if $\vec{\mathrm{v}}$ is irrational, then $h(\Phi, \vec{\mathrm{v}})$ is quite difficult to compute using the formulae (5.1) and (5.2), and Proposition 5B.1(c), while theoretically interesting, is not very computationally useful. Can we compute $h(\Phi, \vec{v})$ as the limit of $h(\Phi, \vec{v}_k)$ where $\{\vec{v}_k\}_{k=1}^{\infty}$ is a sequence of *rational* vectors tending to $\vec{v}$? In other words, is directional entropy *continuous* as a function of $\vec{v}$? What other properties has $h(\Phi, \vec{v})$ as a function of $\vec{v}$?

Theorem 5B.3 *Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ and let $\mu \in \mathfrak{M}_{\mathrm{cas}}(\mathcal{A}^{\mathbb{Z}}; \Phi)$.*

(a) *The function $\mathbb{R}^2 \ni \vec{v} \mapsto h_\mu(\Phi, \vec{v}) \in \mathbb{R}$ is continuous.*

(b) *Suppose there is some $(t, z) \in \mathbb{N} \times \mathbb{Z}$ with $t \geq 1$, such that $\Phi^t \circ \sigma^z$ is posexpansive. Then the function $\mathbb{R}^2 \ni \vec{v} \mapsto h_{\mathrm{top}}(\Phi, \vec{v}) \in \mathbb{R}$ is convex, and thus, Lipschitz-continuous.*

(c) *However, there exist other $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$ for which the function $\mathbb{R}^2 \ni \vec{v} \mapsto h_{\mathrm{top}}(\Phi, \vec{v}) \in \mathbb{R}$ is not continuous.*

(d) *Suppose Φ has neighbourhood $[-\ell...r] \subset \mathbb{Z}$. If $\vec{v} = (t; x) \in \mathbb{R}^2$, then let $z_\ell := x - \ell t$ and $z_r := x + rt$. Let $L := \log |\mathcal{A}|$.*

[i] *Suppose $z_\ell \cdot z_r \geq 0$. Then $h_\mu(\Phi; \vec{v}) \leq \max\{|z_\ell|, |z_r|\} \cdot L$. Furthermore:*

- *If Φ is right-permutative, and $|z_\ell| \leq |z_r|$, then $h_\mu(\Phi; \vec{v}) = |z_r| \cdot L$.*
- *If Φ is left-permutative, and $|z_r| \leq |z_\ell|$, then $h_\mu(\Phi; \vec{v}) = |z_r| \cdot L$.*

[ii] *Suppose $z_\ell \cdot z_r \leq 0$. Then $h_\mu(\Phi; \vec{v}) \leq |z_r - z_\ell| \cdot L$.*
Furthermore, if Φ is bipermutative in this case, then $h_\mu(\Phi; \vec{v}) = |z_r - z_\ell| \cdot L$.

Proof: **(a)** is (Park, 1999, Corollary 3.3), while **(b)** is (Sablik, 2006, Théorème III.11 and Corollaire III.12, pp.79-80). **(c)** is (Smillie, 1988, Proposition 1.2).

(d) summarizes the main results of Courbage and Kamiński (2002). See also (Milnor, 1988, Example 6.2) for an earlier analysis of permutative CA in the case $r = \ell = 1$; see also (Boyle and Lind, 1997, Example 6.4) and (Kitchens and Schmidt, 1992, §1) for the special case when Φ is linear. □

Remarks 5B.4: (a) In fact, the conclusion of Theorem 5B.3(b) holds as long as Φ has any *posexpansive directions* (even irrational ones). A posexpansive direction is analogous to an *expansive subspace* (see §5C), and is part of Sablik's theory of 'directional dynamics' for one-dimensional CA; see Remark 5C.10(b) below. Using this theory, Sablik has also shown that $h_\mu(\Phi; \vec{v}) = 0 = h_{\mathrm{top}}(\Phi, \vec{v})$ whenever $\vec{v}$ is an *equicontinuous direction* for Φ, whereas $h_\mu(\Phi; \vec{v}) \neq 0 \neq h_{\mathrm{top}}(\Phi, \vec{v})$ whenever $\vec{v}$ is a *right-* or *left posexpansive direction* for Φ. See (Sablik, 2006, §III.4.5-§III.4.6, pp.86-88).

(b) Courbage and Kamiński have defined a 'directional' version of the Lyapunov exponents introduced in §5A. If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$, $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$ and $\vec{v} = (t; z) \in \mathbb{N} \times \mathbb{Z}$, then $\lambda^{\pm}_{\vec{v}}(\Phi, \mathbf{a}) := \lambda^{\pm}(\Phi^t \circ \sigma^z, \mathbf{a})$, where $\lambda^{\pm}$ are defined as in §5A. If $\vec{v} \in \mathbb{R}^2$ is irrational, then the definition of $\lambda^{\pm}_{\vec{v}}(\Phi, \mathbf{a})$ is somewhat more subtle. For any Φ and $\mathbf{a}$, the function $\mathbb{R}^2 \ni \vec{v} \mapsto \lambda^{\pm}_{\vec{v}}(\Phi, \mathbf{a}) \in \mathbb{R}$ is homogeneous and continuous (Courbage and Kamiński, 2006, Lemma 2 and Proposition 3). If $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}}; \Phi, \sigma)$ is σ-ergodic, then $\lambda^{\pm}_{\vec{v}}(\Phi, \bullet)$ is constant μ-almost everywhere, and is related to $h_\mu(\Phi; \vec{v})$ through an inequality exactly analogous to Theorem 5A.2(b); see (Courbage and Kamiński, 2006, Theorem 1). ♢

Cone Entropy: For any $\vec{v} \in \mathbb{R}^{D+1}$, any angle $\theta > 0$, and any $N > 0$, we define

$$\mathbb{K}(N\vec{v}, \theta) \quad := \quad \left\{ \mathsf{z} \in \mathbb{Z}^{D+1} \; ; \; |\mathsf{z}| \leq N|\vec{v}| \text{ and } \mathsf{z} \bullet \vec{v}/|\mathsf{z}||\vec{v}| \geq \cos(\theta) \right\}.$$

Geometrically, this is the set of all $\mathbb{Z}^{D+1}$-lattice points in a cone of length $N|\vec{v}|$ which subtends an angle of 2θ around an axis parallel to $\vec{v}$, and which has its apex at the origin. If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$, then let $\mathbf{X}^{\text{Hist}}(N\vec{v}, \theta) := \left\{ \mathbf{x}_{\mathbb{K}(N\vec{v},\theta)} \; ; \; \mathbf{x} \in \mathbf{X}^{\text{Hist}}(\Phi) \right\}$. If $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}^D}; \Phi)$, and $\widetilde{\mu}$ is the extension of μ to $\mathbf{X}^{\text{Hist}}$, then the *cone entropy* of (Φ, μ) in direction $\vec{v}$ is defined

$$h^{\text{cone}}_\mu(\Phi, \vec{v}) \quad := \quad -\lim_{\theta \searrow 0} \lim_{N \to \infty} \frac{1}{N} \sum_{\mathbf{x} \in \mathbf{X}^{\text{Hist}}(N\vec{v},\theta)} \widetilde{\mu}[\mathbf{x}] \log_2(\widetilde{\mu}[\mathbf{x}]).$$

Park (1995, 1996) attributes this concept to Doug Lind. Like directional entropy, cone entropy can be defined for any continuous $\mathbb{Z}^{D+1}$-action, and is generally infinite for multi-dimensional CA. However, for one-dimensional CA, Park has proved:

Theorem 5B.5 *If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$, $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}}; \Phi)$ and $\vec{v} \in \mathbb{R}^2$, then $h^{\text{cone}}_\mu(\Phi, \vec{v}) = h_\mu(\Phi, \vec{v})$.*

Proof: See (Park, 1996, Theorem 1). □

5C Entropy Geometry and Expansive Subdynamics

Directional entropy is the one-dimensional version of a multidimensional 'entropy density' function, which was introduced by Milnor (1988) to address the fact that classical and directional entropy are generally infinite for multidimensional CA. Milnor's ideas were then extended by Boyle and Lind (1997), using their theory of *expansive subdynamics*.

Let $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^{D+1}}$ be a subshift, and let $\mu \in \mathfrak{M}_{\text{eas}}(\mathbf{X}; \sigma)$. For any bounded $\mathsf{B} \subset \mathbb{R}^{D+1}$, let $\mathbb{B} := \mathsf{B} \cap \mathbb{Z}^{D+1}$, let $\mathbf{X}_{\mathsf{B}} := \mathbf{X}_{\mathbb{B}}$, and then define

$$H_{\mathbf{X}}(\mathsf{B}) \quad := \quad \log_2 |\mathbf{X}_{\mathsf{B}}| \quad \text{and} \quad H_\mu(\mathsf{B}) \quad := \quad -\sum_{\mathbf{x} \in \mathbf{X}_{\mathsf{B}}} \mu[\mathbf{x}] \log_2(\mu[\mathbf{x}]).$$

The *topological entropy dimension* $\dim(\mathbf{X})$ is the smallest $d \in [0, D+1]$ having some constant $c > 0$ such that, for any finite $\mathsf{B} \subset \mathbb{R}^{D+1}$, $H_{\mathbf{X}}(\mathsf{B}) \leq c \cdot \mathsf{diam}\,[\mathsf{B}]^d$. The *measurable entropy dimension* $\dim(\mu)$ is defined similarly, only with H_μ in place of $H_{\mathbf{X}}$. Note that $\dim(\mu) \leq \dim(\mathbf{X})$, because $H_\mu(\mathsf{B}) \leq H_{\mathbf{X}}(\mathsf{B})$ for all $\mathsf{B} \subset \mathbb{R}^{D+1}$.

For any bounded $\mathsf{B} \subset \mathbb{R}^{D+1}$ and 'scale factor' $s > 0$, let $s\mathsf{B} := \{s\mathsf{b} \; ; \; \mathsf{b} \in \mathsf{B}\}$. For any radius $r > 0$, let $(s\mathsf{B})^r := \left\{ \mathsf{x} \in \mathbb{R}^{D+1} \; ; \; d(\mathsf{x}, s\mathsf{B}) \leq r \right\}$. Define the *$d$-dimensional topological entropy density* of B by

$$h^d_{\mathbf{X}}(\mathsf{B}) \quad := \quad \sup_{r>0} \limsup_{s \to \infty} H_{\mathbf{X}}[(s\mathsf{B})^r]/s^d. \tag{5.3}$$

Define d-*dimensional measurable entropy density* $h^d_\mu(\mathsf{B})$ similarly, only using H_μ instead of $H_{\mathbf{X}}$. Note that, for any $d < \dim(\mathbf{X})$ [respectively, $d < \dim(\mu)$], $h^d_{\mathbf{X}}(\mathsf{B})$ [resp. $h^d_\mu(\mathsf{B})$] will be infinite, whereas for for any $d > \dim(\mathbf{X})$ [resp. $d > \dim(\mu)$], $h^d_{\mathbf{X}}(\mathsf{B})$ [resp. $h^d_\mu(\mathsf{B})$] will be zero; hence $\dim(\mathbf{X})$ [resp. $\dim(\mu)$] is the unique value of d for which the function $h^d_{\mathbf{X}}$ [resp. h^d_μ] defined in eqn.(5.3) is nontrivial.

Example 5C.1: (a) If $d = D+1$, and B is a unit cube centred at the origin, then $h^{D+1}_{\mathbf{X}}(\mathsf{B})$ [resp. $h^{D+1}_\mu(\mathsf{B})$] is just the classical $(D+1)$-dimensional topological [resp. measurable] entropy of $\mathbf{X}$ [resp. μ] as a $(D+1)$-dimensional subshift [resp. random field]; see ENTROPY IN ERGODIC THEORY.

(b) However, the most important case for Milnor (1988) (and us) is when $\mathbf{X} = \mathbf{X}^{\text{Hist}}(\Phi)$ for some $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$. In this case, $\dim(\mu) \le \dim(\mathbf{X}) \le D < D+1$. In particular, if $d = 1$, then for any $\vec{\mathsf{v}} \in \mathbb{R}^{D+1}$, if $\mathsf{B} := \{r\vec{\mathsf{v}}\ ;\ r \in [0,1]\}$, then $h^1_{\mathbf{X}}(\mathsf{B}) = h_{\text{top}}(\Phi;\vec{\mathsf{v}})$ and $h^1_\mu(\mathsf{B}) = h_\mu(\Phi;\vec{\mathsf{v}})$ are *directional entropies* of §5B. ◊

In general, $\dim(\mu)$ and $\dim(\mathbf{X})$ may not be integers, but the theory is best developed in the case when they are. For any $d \in [0,\ D+1]$, let λ^d be a d-dimensional Hausdorff measure on $\mathbb{R}^{D+1}$. For example, if $d \in [1 \ldots D+1]$, and $\mathsf{P} \subset \mathbb{R}^{D+1}$ is a d-*plane* (i.e. a d-dimensional linear subspace of $\mathbb{R}^{D+1}$), then λ^d restricts to a d-dimensional Lebesgue measure on P.

Theorem 5C.2 *Let* $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^{D+1}}$ *be a subshift, and let* $\mu \in \mathfrak{M}_{\text{eas}}(\mathbf{X};\sigma)$*. Let* $d = \dim(\mathbf{X})$ [or $\dim(\mu)$] *and let* h^d *be* $h^d_{\mathbf{X}}$ [or h^d_μ]*. Let* $\mathsf{B}, \mathsf{C} \subset \mathbb{R}^{D+1}$ *be compact sets. Then*

(a) $h^d(\mathsf{B})$ *is well-defined and finite.*

(b) *If* $\mathsf{B} \subseteq \mathsf{C}$ *then* $h^d(\mathsf{B}) \le h^d(\mathsf{C})$.

(c) $h^d(\mathsf{B} \cup \mathsf{C}) \le h^d(\mathsf{B}) + h^d(\mathsf{C})$.

(d) $h^d(\mathsf{B} + \vec{\mathsf{v}}) = h^d(\mathsf{B})$ *for any* $\vec{\mathsf{v}} \in \mathbb{R}^{D+1}$.

(e) $h^d(s\mathsf{B}) = s^d \cdot h^d(\mathsf{B})$ *for any* $s > 0$.

(f) *There is some constant* c *such that* $h_d(\mathsf{B}) \le c\lambda^d(\mathsf{B})$ *for all compact* $\mathsf{B} \subset \mathbb{R}^{D+1}$.

(g) *If* $d \in \mathbb{N}$*, then for any* d*-plane* $\mathsf{P} \subset \mathbb{R}^{D+1}$*, there is some* $\mathfrak{H}^d(\mathsf{P}) \ge 0$ *such that* $h^d(\mathsf{B}) = \mathfrak{H}^d(\mathsf{P}) \cdot \lambda^d(\mathsf{B})$ *for any compact subset* $\mathsf{B} \subset \mathsf{P}$ *with* $\lambda^d(\partial\mathsf{B}) = 0$.

(h) *There is a constant* $\overline{H}^d_{\mathbf{X}} < \infty$ *such that* $\mathfrak{H}^d_{\mathbf{X}}(\mathsf{P}) \le \overline{H}^d_{\mathbf{X}}$ *for all* d*-planes* P.

Proof: See (Milnor, 1988, Theorems 1 and 2 and Corollary 1), or see (Boyle and Lind, 1997, Theorems 6.2, 6.3, and 6.13). □

Example 5C.3: Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$ and let $\mathbf{X} := \mathbf{X}^{\text{Hist}}(\Phi)$. If $\mathsf{P} := \{0\} \times \mathbb{R}^D$, then $\mathfrak{H}^D(\mathsf{P})$ is the classical D-dimensional entropy of the omega limit set $\mathbf{Y} := \Phi^\infty(\mathcal{A}^{\mathbb{Z}^D})$; heuristically, this measures the asymptotic level of 'spatial disorder' in $\mathbf{Y}$. If $\mathsf{P} \subset \mathbb{R}^{D+1}$ is some other D-plane, then $\mathfrak{H}^D(\mathsf{P})$ measures some combination of the 'spatial disorder' of $\mathbf{Y}$ with the dynamical entropy of Φ. ♢

Let $d \in [1 \ldots D{+}1]$, and let $\mathsf{P} \subset \mathbb{R}^{D+1}$ be a d-plane. For any $r > 0$, let $\mathsf{P}(r) := \{\mathsf{z} \in \mathbb{Z}^{D+1} \; ; \; d(\mathsf{z}, \mathsf{P}) < r\}$. We say P is *expansive* for $\mathbf{X}$ if there is some $r > 0$ such that, for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\left(\mathbf{x}_{\mathsf{P}(r)} = \mathbf{y}_{\mathsf{P}(r)}\right) \iff (\mathbf{x} = \mathbf{y})$. If P is spanned by d rational vectors, then $\mathsf{P} \cap \mathbb{Z}^{D+1}$ is a rank-d sublattice $\mathbb{L} \subset \mathbb{Z}^{D+1}$, and P is expansive if and only if the induced $\mathbb{L}$-action on $\mathbf{X}$ is expansive. However, if P is 'irrational', then expansiveness is a more subtle concept; see (Boyle and Lind, 1997, §2) for more information.

If $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$ and $\mathbf{X} = \mathbf{X}^{\text{Hist}}(\Phi)$, then Φ is *quasi-invertible* if $\mathbf{X}$ admits an expansive D-plane P (this is a natural extension of Milnor's (1988; §7) definition in terms of 'causal cones'). Heuristically, if we regard $\mathbb{Z}^{D+1}$ as 'spacetime' (in the spirit of special relativity), then P can be seen as 'space', and any direction transversal to P can be interpreted as the flow of 'time'.

Example 5C.4: (a) If Φ is invertible, then it is quasi-invertible, because $\{0\} \times \mathbb{R}^D$ is an expansive D-plane (recall that the zeroth coordinate is time).

(b) Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}})$, so that $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^2}$. Let Φ have neighbourhood $[-\ell...r]$, with $-\ell \leq 0 \leq r$, and let $\mathsf{L} \subset \mathbb{R}^2$ be a line with slope S through the origin.

[**i**] If Φ is right-permutative, and $0 < S < 1/\ell$, then L is expansive for $\mathbf{X}$.

[**ii**] If Φ is left-permutative, and $-1/r < S < 0$, then L is expansive for $\mathbf{X}$.

[**iii**] If Φ is bipermutative, and $-1/r < S < 0$ or $0 < S < 1/\ell$, then L is expansive for $\mathbf{X}$.

[**iv**] If Φ is posexpansive (see §2D) then the 'time' axis $\mathsf{L} = \mathbb{R} \times \{0\}$ is expansive for $\mathbf{X}$.

Hence, in any of these cases, Φ is quasi-invertible. (Presumably, something similar is true for multidimensional permutative CA.) ♢

Proposition 5C.5 *Let $\Phi \in \mathcal{A}^{\mathbb{Z}^D}$, let $\mathbf{X} = \mathbf{X}^{\text{Hist}}(\Phi)$, let $\mu \in \mathfrak{M}_{\text{eas}}(\mathbf{X}; \sigma)$, and let $\mathfrak{H}^d$ and $\overline{H}^d_{\mathbf{X}}$ be as in Theorem* 5C.2(g,h).

(a) *If $\mathfrak{H}^D_{\mathbf{X}}(\{0\} \times \mathbb{R}^D) = 0$, then $\overline{H}^D_{\mathbf{X}} = 0$.*

(b) *Let $d \in [1...D]$, and suppose that $\mathbf{X}$ admits an expansive d-plane. Then:*

[**i**] $\dim(\mathbf{X}) \leq d$;

[**ii**] *There is a constant $\overline{H}^d_\mu < \infty$ such that $\mathfrak{H}^d_\mu(\mathsf{P}) \leq \overline{H}^d_\mu$ for all d-planes P;*

[**iii**] *If $\mathfrak{H}^d(\mathsf{P}) = 0$ for some expansive d-plane P, then $\overline{H}^d = 0$.*

Proof: **(a)** is (Milnor, 1988, Corollary 3), **(b)[i]** is (Shereshevsky, 1996, Corollary 1.4), and **(b)[ii]** is (Boyle and Lind, 1997, Theorem 6.19(2)).

(b)[iii]: See (Boyle and Lind, 1997, Theorem 6.3(4)) for "$\overline{H}^d_{\mathbf{X}} = 0$". See (Boyle and Lind, 1997, Theorem 6.19(1)) for "$\overline{H}^d_{\mu} = 0$". □

If $d \in [1 \ldots D{+}1]$, then a d-*frame* in $\mathbb{R}^{D+1}$ is a d-tuple $\mathsf{F} := (\vec{v}_1, \ldots, \vec{v}_d)$, where $\vec{v}_1, \ldots, \vec{v}_d \in \mathbb{R}^{D+1}$ are linearly independent. Let $\mathfrak{F}_{\text{rame}}(D{+}1, d)$ be the set of all d-frames in $\mathbb{R}^{D+1}$; then $\mathfrak{F}_{\text{rame}}(D{+}1, d)$ is an open subset of $\mathbb{R}^{D+1} \times \cdots \times \mathbb{R}^{D+1} =: \mathbb{R}^{(D+1)\times d}$. Let

$$\mathfrak{E}_{\text{xpans}}(\mathbf{X}, d) \quad := \quad \{\mathsf{F} \in \mathfrak{F}_{\text{rame}}(D{+}1, d) \; ; \; \text{span}(\mathsf{F}) \text{ is expansive for } \mathbf{X}\}.$$

Then $\mathfrak{E}_{\text{xpans}}(\mathbf{X}, d)$ is an open subset of $\mathfrak{F}_{\text{rame}}(D+1, d)$, by (Boyle and Lind, 1997, Lemma 3.4). A connected component of $\mathfrak{E}_{\text{xpans}}(\mathbf{X}, d)$ is called an *expansive component* for $\mathbf{X}$. For any $\mathsf{F} \in \mathfrak{F}_{\text{rame}}(D{+}1, d)$, let $[\mathsf{F}]$ be the d-dimensional parallellipiped spanned by F, and let $\mathfrak{h}^d_{\mathbf{X}}(\mathsf{F}) := h^d_{\mathbf{X}}([\mathsf{F}]) = \mathfrak{H}^d_{\mathbf{X}}(\text{span}(\mathsf{F})) \cdot \lambda^d([\mathsf{F}])$, where the last equality is by Theorem 5C.2(g). The next result is a partial extension of Theorem 5B.3(b).

Proposition 5C.6 *Let* $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^{D+1}}$ *be a subshift, suppose* $d := \dim(\mathbf{X}) \in \mathbb{N}$, *and let* $\mathfrak{C} \subset \mathfrak{E}_{\text{xpans}}(\mathbf{X}, d)$ *be an expansive component. Then the function* $\mathfrak{h}^d_{\mathbf{X}} : \mathfrak{C} \longrightarrow \mathbb{R}$ *is convex in each of its* d *distinct* $\mathbb{R}^{D+1}$*-valued arguments. Thus,* $\mathfrak{h}^d_{\mathbf{X}}$ *is Lipschitz-continuous on* $\mathfrak{C}$.

Proof: See (Boyle and Lind, 1997, Theorem 6.9(1,4)). □

For measurable entropy, we can say much more. Recall that a d-*linear form* is a function $\omega : \mathbb{R}^{(D+1)\times d} \longrightarrow \mathbb{R}$ which is linear in each of its d distinct $\mathbb{R}^{D+1}$-valued arguments.

Theorem 5C.7 *Let* $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^{D+1}}$ *be a subshift and let* $\mu \in \mathfrak{M}_{\text{eas}}(\mathbf{X}; \sigma)$. *Suppose* $d := \dim(\mu) \in \mathbb{N}$, *and let* $\mathfrak{C} \subset \mathfrak{E}_{\text{xpans}}(\mathbf{X}, d)$ *be an expansive component for* $\mathbf{X}$. *Then there is a* d*-linear form* $\omega : \mathbb{R}^{(D+1)\times d} \longrightarrow \mathbb{R}$ *such that* $\mathfrak{h}^d_{\mu}$ *agrees with* ω *on* $\mathfrak{C}$.

Proof: (Boyle and Lind, 1997, Theorem 6.16). □

Theorem 5C.7 means that there is an orthonormal $(D{+}1{-}d)$-frame $\mathtt{W} := (\vec{w}_{d+1}, \ldots, \vec{w}_{D+1})$ (transversal to all frames in $\mathfrak{C}$) such that, for any d-frame $\mathtt{V} := (\vec{v}_1, \ldots, \vec{v}_d) \in \mathfrak{C}$,

$$\mathfrak{h}^d_{\mu}(\mathtt{V}) \quad = \quad |\det(\vec{v}_1, \ldots, \vec{v}_d; \vec{w}_{d+1}, \ldots, \vec{w}_{D+1})| \, .$$

Thus, the d-plane orthogonal to $\{\vec{w}_{d+1}, \ldots, \vec{w}_{D+1}\}$ is the d-plane which maximizes $\mathfrak{H}^d_{\mu}$ —this is the d-plane manifesting the most rapid decay of correlation with distance. On the other hand, span($\mathtt{W}$) is the $(D + 1 - d)$-plane along which correlations decay the most slowly.

Example 5C.8: Let $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$ be quasi-invertible, and let P be an expansive D-plane for $\mathbf{X} := \mathbf{X}^{\text{Hist}}(\Phi)$ [see Example 5C.4]. The D-frames spanning P fall into two expansive components (related by orientation-reversal); let $\mathfrak{C}$ be union of these two components. Let $\mu \in \mathfrak{M}_{\text{eas}}(\mathcal{A}^{\mathbb{Z}^D};\Phi)$, and extend μ to a σ-invariant measure on $\mathbf{X}$. In this case, Theorem 5C.7 is equivalent to (Milnor, 1988, Theorem 4), which says there a unit vector $\vec{\mathtt{w}} \in \mathbb{R}^{D+1}$ such that, for any D-frame $(\vec{\mathtt{v}}_1,\ldots,\vec{\mathtt{v}}_D) \in \mathfrak{C}$, $\mathfrak{h}^d_\mu(\mathsf{F}) = |\det(\vec{\mathtt{v}}_1,\ldots,\vec{\mathtt{v}}_D;\vec{\mathtt{w}})|$. Thus, $\mathfrak{H}^d_\mu(\mathsf{P})$ is maximized when P is the hyperplane orthogonal to $\vec{\mathtt{w}}$. Heuristically, $\vec{\mathtt{w}}$ points in the direction of minimum correlation decay (or maximum 'causality') —the direction which could most properly be called 'time' for the MPDS (Φ,μ). ♢

Theorem 5C.7 yields the following generalization the Variational Principle:

Theorem 5C.9 *Let $\mathbf{X} \subset \mathcal{A}^{\mathbb{Z}^{D+1}}$ be a subshift and suppose $d := \dim(\mathbf{X}) \in \mathbb{N}$.*

(a) *If $\mathsf{F} \in \mathfrak{E}_{\text{xpans}}(\mathbf{X},d)$, then there exists $\mu \in \mathfrak{M}_{\text{eas}}(\mathbf{X};\sigma)$ such that $\mathfrak{h}^d_{\mathbf{X}}(\mathsf{F}) = \mathfrak{h}^d_\mu(\mathsf{F})$.*

(b) *Let $\mathfrak{C} \subset \mathfrak{E}_{\text{xpans}}(\mathbf{X},d)$ be an expansive component for $\mathbf{X}$. There exists some $\mu \in \mathfrak{M}_{\text{eas}}(\mathbf{X};\sigma)$ such that $\mathfrak{h}^d_{\mathbf{X}} = \mathfrak{h}^d_\mu$ on $\mathfrak{C}$ if and only if $\mathfrak{h}^d_{\mathbf{X}}$ is a d-linear form on $\mathfrak{C}$.*

Proof: (Boyle and Lind, 1997, Proposition 6.24 and Theorem 6.25). □

Remark 5C.10: (a) If $\mathbf{G} \subset \mathcal{A}^{\mathbb{Z}^D}$ is an abelian subgroup shift and $\Phi \in \mathsf{ECA}(\mathbf{G})$, then $\mathbf{X}^{\text{Hist}}(\Phi)$ is a subgroup shift of $\mathcal{A}^{\mathbb{Z}^{D+1}}$, which can be viewed as an algebraic $\mathbb{Z}^{D+1}$-action (see discussion prior to Proposition 2E.9). In this context, the expansive subspaces of $\mathbf{X}^{\text{Hist}}(\Phi)$ have been completely characterized by Einsiedler et al. (2001)[Theorem 8.4]. Furthermore, certain dynamical properties (such as positive entropy, completely positive entropy, or Bernoullicity) are common amongst all elements of each expansive component of $\mathbf{X}^{\text{Hist}}(\Phi)$ (Einsiedler et al., 2001, Theorem 9.8). If $\mathbf{G}_1$ and $\mathbf{G}_2$ are subgroup shifts, and $\Phi_k \in \mathsf{ECA}(\mathbf{G}_k)$ and $\mu_k \in \mathfrak{M}_{\text{eas}}(\mathbf{G};\Phi,\sigma)$ for $k=1,2$, with $\dim(\mu_1) = \dim(\mu_2) = 1$, then Einsiedler and Ward (2005) have given conditions for the measure-preserving systems $(\mathbf{G}_1,\mu_1;\Phi_1,\sigma)$ and $(\mathbf{G}_2,\mu_2;\Phi_2,\sigma)$ to be disjoint.

(b) Boyle and Lind's 'expansive subdynamics' concerns expansiveness along certain directions in the space-time diagram of a CA. Recently, M. Sablik has developed a theory of *directional dynamics*, which explores other topological dynamical properties (such as equicontinuity and sensitivity to initial conditions) along spatiotemporal directions in a CA; see (Sablik, 2006, Chapitre II) or Sablik (2007a). ♢

6 Future directions and open problems

1. We now have a fairly good understanding of the ergodic theory of linear and/or 'abelian' CA. The next step is to extend these results to CA with nonlinear and/or nonabelian algebraic structures. In particular:

(a) Almost all the measure rigidity results of §2E are for endomorphic CA on abelian group shifts, except for Propositions 2E.3 and 2E.5. Can we extend these results to CA on nonabelian group shifts or other permutative CA?

(b) Likewise, the asymptotic randomization results of §3A are almost exclusively for linear CA with scalar coefficients. Can we extend these results to LCA with noncommuting, matrix-valued coefficients? (The problem is: if the coefficients do not commute, then the 'polynomial representation' and Lucas' theorem become inapplicable.) Also, can we obtain similar results for multiplicative CA on *nonabelian* groups? [See Remark 3A.6(d).] What about other permutative CA? [See Remark 3A.6(e).]

2. Cellular automata are often seen as models of spatially distributed computation. Meaningful 'computation' could possibly occur when a CA interacts with a highly structured initial configuration (e.g. a substitution sequence), whereas such computation is probably impossible in the roiling cauldron of noise arising from a mixing, positive entropy measure (e.g. a Bernoulli measure or Markov random field). Yet almost all the results in this article concern the interaction of CA with such mixing, positive-entropy measures. Almost nothing is known about the interaction of CA with non-mixing and/or zero-entropy measures, such as the unique stationary measures on substitution shifts, automatic shifts, regular Toeplitz shifts, quasisturmian shifts, and many other 'finite rank' systems. In particular:

   (a) The invariant measures discussed in §2 all have nonzero entropy [see, however, Example 4B.3(c)]. Are there any nontrivial zero-entropy measures for interesting CA?

   (b) The results of §3A all concern the asymptotic randomization of initial measures with nonzero entropy, except for Remark 3A.6(c). Are there similar results for zero-entropy measures?

   (c) Zero-entropy systems often have an appealing combinatorial description via cutting-and-stacking constructions, Bratteli diagrams, or finite state machines. Likewise, CA admit a combinatorial description (via local rules). How do these combinatorial descriptions interact?

3. As we saw in §2G, and also in Propositions 3B.7-3B.15, emergent defect dynamics can be a powerful tool for analyzing the measurable dynamics of CA. Defects in one-dimensional CA generally act like 'particles', and their 'kinematics' is fairly well-understood. However, in higher dimensions, defects can be much more topologically complicated (e.g. they can look like curves or surfaces), and their evolution in time is totally mysterious. Can we develop a theory of multidimensional defect dynamics?

4. Almost all the results about mixing and ergodicity in §4A are for one-dimensional (mostly permutative) CA and for the uniform measure on $\mathcal{A}^{\mathbb{Z}}$. Can similar results be obtaned for other CA and/or measures on $\mathcal{A}^{\mathbb{Z}}$? What about CA in $\mathcal{A}^{\mathbb{Z}^D}$ for $D \geq 2$?

5. Let μ be a (Φ, σ)-invariant measure on $\mathcal{A}^{\mathbb{M}}$. Proposition 4B.2 suggests an intriguing correspondence between certain spectral properties (namely, weak mixing and discrete spectrum) for the system $(\mathcal{A}^{\mathbb{M}}, \mu; \sigma)$ and those for the system $(\mathcal{A}^{\mathbb{M}}, \mu; \Phi)$. Does a similar correspondence hold for other spectral properties, such as continuous spectrum, Lebesgue spectral type, spectral multiplicity, rigidity, or mild mixing?

6. Let $\mathbf{X} \in \mathcal{A}^{\mathbb{Z}^{D+1}}$ be a subshift admitting an expansive D-plane $\mathsf{P} \subset \mathbb{R}^{D+1}$. As discussed in §5C, if we regard $\mathbb{Z}^{D+1}$ as 'spacetime', then we can treat P as a 'space', and a transversal direction as 'time'. Indeed, if P is spanned by rational vectors, then the Curtis-Hedlund-Lyndon theorem implies that $\mathbf{X}$ is isomorphic to the history shift of some invertible $\Phi \in \mathsf{CA}(\mathcal{A}^{\mathbb{Z}^D})$ acting on some Φ-invariant subshift $\mathbf{Y} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ (where we embed $\mathbb{Z}^D$ in P). If P is irrational, then this is not the case; however, $\mathbf{X}$ still seems very much like the history shift of a spatially distributed symbolic dynamical system, closely analogous to a CA, except with a continually fluctuating 'spatial distribution' of state information, and perhaps with occasional nonlocal interactions. For example, Proposition 5C.5(b)[i] implies that $\dim(\mathbf{X}) \leq D$, just as for a CA. How much of the theory of invertible CA can be generalized to such systems?

I will finish with the hardest problem of all. Cellular automata are tractable mainly because of their *homogeneity*: CA are embedded in a highly regular spatial geometry (i.e. a lattice or other Cayley digraph) with the same local rule everywhere. However, many of the most interesting spatially distributed symbolic dynamical systems are not nearly this homogeneous. For example:

- CA are often proposed as models of spatially distributed physical systems. Yet in many such systems (e.g. living tissues, quantum 'foams'), the underlying geometry is not a flat Euclidean space, but a curved manifold. A good discrete model of such a manifold can be obtained through a Voronoi tesselation of sufficient density; a realistic symbolic dynamical model would be a CA-like system defined on the dual graph of this Voronoi tesselation.

- As mentioned in question #3, defects in multidimensional CA may have the geometry of curves, surfaces, or other embedded submanifolds (possibly with varying nonzero thickness). To model the evolution of such a defect, we could treat it as a CA-like object whose underlying geometry is an (evolving) manifold, and whose local rules (although partly determined by the local rule of the original CA) are spatially heterogenous (because they are also influenced by incoming information from the ambient 'nondefective' space).

- The CA-like system arising in question #6 has a D-dimensional planar geometry, but the distribution of 'cells' within this plane (and, presumably, the local rules between them) are constantly fluctuating.

More generally, any topological dynamical system on a Cantor space can be represented as a *cellular network*: a CA-like system defined on an infinite digraph, with different local rules at different nodes. Gromov (1999) has generalized the Garden of Eden Theorem 2A.3 to this setting [see Remark 2A.5(a)]. However, other than Gromov's work, basically nothing is known about such systems. Can we generalize any of the theory of cellular automata to cellular networks? Is it possible to develop a nontrivial ergodic theory for such systems?

References

Akin, E., 1993. The general topology of dynamical systems. Vol. 1 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI.

Allouche, J.-P., 1999. Cellular automata, finite automata, and number theory. In: Cellular automata (Saissac, 1996). Vol. 460 of Math. Appl. Kluwer Acad. Publ., Dordrecht, pp. 321–330.

Allouche, J.-P., Skordev, G., 2003. Remarks on permutive cellular automata. J. Comput. System Sci. 67 (1), 174–182.

Allouche, J.-P., von Haeseler, F., Peitgen, H.-O., Petersen, A., Skordev, G., 1997. Automaticity of double sequences generated by one-dimensional linear cellular automata. Theoret. Comput. Sci. 188 (1-2), 195–209.

Allouche, J.-P., von Haeseler, F., Peitgen, H.-O., Skordev, G., 1996. Linear cellular automata, finite automata and Pascal's triangle. Discrete Appl. Math. 66 (1), 1–22.

Barbé, A., von Haeseler, F., Peitgen, H.-O., Skordev, G., 1995. Coarse-graining invariant patterns of one-dimensional two-state linear cellular automata. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 5 (6), 1611–1631.

Barbé, A., von Haeseler, F., Peitgen, H.-O., Skordev, G., 2003. Rescaled evolution sets of linear cellular automata on a cylinder. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 13 (4), 815–842.

Belitsky, V., Ferrari, P. A., 2005. Invariant measures and convergence properties for cellular automaton 184 and related processes. J. Stat. Phys. 118 (3-4), 589–623.

Blanchard, F., Kůrka, P., Maass, A., 1997. Topological and measure-theoretic properties of one-dimensional cellular automata. Phys. D 103 (1-4), 86–99, lattice dynamics (Paris, 1995).

Blanchard, F., Maass, A., 1997. Dynamical properties of expansive one-sided cellular automata. Israel J. Math. 99, 149–174.

Blank, M., 2003. Ergodic properties of a simple deterministic traffic flow model. J. Statist. Phys. 111 (3-4), 903–930.

Boccara, N., Naser, J., Roger, M., 1991. Particle-like structures and their interactions in spatiotemporal patterns generated by one-dimensional deterministic cellular automata. Physical Review A 44 (2), 866–875.

Boyle, M., Fiebig, D., Fiebig, U.-R., 1997. A dimension group for local homeomorphisms and endomorphisms of onesided shifts of finite type. J. Reine Angew. Math. 487, 27–59.

Boyle, M., Lind, D., 1997. Expansive subdynamics. Trans. Amer. Math. Soc. 349 (1), 55–102.

Burton, R., Steif, J. E., 1994. Non-uniqueness of measures of maximal entropy for subshifts of finite type. Ergodic Theory Dynam. Systems 14 (2), 213–235.

Burton, R., Steif, J. E., 1995. New results on measures of maximal entropy. Israel J. Math. 89 (1-3), 275–300.

Cai, H., Luo, X., 1993. Laws of large numbers for a cellular automaton. Ann. Probab. 21 (3), 1413–1426.

Cattaneo, G., Formenti, E., Manzini, G., Margara, L., 1997. On ergodic linear cellular automata over Z_m. In: STACS 97 (Lübeck). Vol. 1200 of Lecture Notes in Comput. Sci. Springer, Berlin, pp. 427–438.

Cattaneo, G., Formenti, E., Manzini, G., Margara, L., 2000. Ergodicity, transitivity, and regularity for linear cellular automata over Z_m. Theoret. Comput. Sci. 233 (1-2), 147–164.

Ceccherini-Silberstein, T., Fiorenzi, F., Scarabotti, F., 2004. The Garden of Eden theorem for cellular automata and for symbolic dynamical systems. In: Random walks and geometry. Walter de Gruyter GmbH & Co. KG, Berlin, pp. 73–108.

Ceccherini-Silberstein, T. G., Machì, A., Scarabotti, F., 1999. Amenable groups and cellular automata. Ann. Inst. Fourier (Grenoble) 49 (2), 673–685.

Courbage, M., Kamiński, B., 2002. On the directional entropy of $\mathbb{Z}^2$-actions generated by cellular automata. Studia Math. 153 (3), 285–295.

Courbage, M., Kamiński, B., 2006. Space-time directional Lyapunov exponents for cellular automata. J. Stat. Phys. 124 (6), 1499–1509.

Coven, E. M., 1980. Topological entropy of block maps. Proc. Amer. Math. Soc. 78 (4), 590–594.

Coven, E. M., Paul, M. E., 1974. Endomorphisms of irreducible subshifts of finite type. Math. Systems Theory 8 (2), 167–175.

Downarowicz, T., 1997. The royal couple conceals their mutual relationship: a noncoalescent Toeplitz flow. Israel J. Math. 97, 239–251.

Durrett, R., Steif, J. E., 1991. Some rigorous results for the Greenberg-Hastings model. J. Theoret. Probab. 4 (4), 669–690.

Durrett, R., Steif, J. E., 1993. Fixation results for threshold voter systems. Ann. Probab. 21 (1), 232–247.

Einsiedler, M., 2004. Invariant subsets and invariant measures for irreducible actions on zero-dimensional groups. Bull. London Math. Soc. 36 (3), 321–331.

Einsiedler, M., 2005. Isomorphism and measure rigidity for algebraic actions on zero-dimensional groups. Monatsh. Math. 144 (1), 39–69.

Einsiedler, M., Lind, D., Miles, R., Ward, T., 2001. Expansive subdynamics for algebraic $\mathbb{Z}^d$-actions. Ergodic Theory Dynam. Systems 21 (6), 1695–1729.

Einsiedler, M., Ward, T., 2005. Entropy geometry and disjointness for zero-dimensional algebraic actions. J. Reine Angew. Math. 584, 195–214.

Eloranta, K., Nummelin, E., 1992. The kink of cellular automaton Rule 18 performs a random walk. J. Statist. Phys. 69 (5-6), 1131–1136.

Fagnani, F., Margara, L., 1998. Expansivity, permutivity, and chaos for cellular automata. Theory Comput. Syst. 31 (6), 663–677.

Ferrari, P. A., Maass, A., Martínez, S., Ney, P., 2000. Cesàro mean distribution of group automata starting from measures with summable decay. Ergodic Theory Dynam. Systems 20 (6), 1657–1670.

Finelli, M., Manzini, G., Margara, L., 1998. Lyapunov exponents versus expansivity and sensitivity in cellular automata. J. Complexity 14 (2), 210–233.

Fiorenzi, F., 2000. The Garden of Eden theorem for sofic shifts. Pure Math. Appl. 11 (3), 471–484.

Fiorenzi, F., 2003. Cellular automata and strongly irreducible shifts of finite type. Theoret. Comput. Sci. 299 (1-3), 477–493.

Fiorenzi, F., 2004. Semi-strongly irreducible shifts. Adv. in Appl. Math. 32 (3), 421–438.

Fisch, R., 1990. The one-dimensional cyclic cellular automaton: a system with deterministic dynamics that emulates an interacting particle system with stochastic dynamics. J. Theoret. Probab. 3 (2), 311–338.

Fisch, R., 1992. Clustering in the one-dimensional three-color cyclic cellular automaton. Ann. Probab. 20 (3), 1528–1548.

Fisch, R., Gravner, J., 1995. One-dimensional deterministic Greenberg-Hastings models. Complex Systems 9 (5), 329–348.

Furstenberg, H., 1967. Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. Math. Systems Theory 1, 1–49.

Gilman, R. H., 1987. Classes of linear automata. Ergodic Theory Dynam. Systems 7 (1), 105–118.

Gottschalk, W., 1973. Some general dynamical notions. In: Recent advances in topological dynamics (Proc. Conf. Topological Dynamics, Yale Univ., New Haven, Conn., 1972; in honor of Gustav Arnold Hedlund). Springer, Berlin, pp. 120–125. Lecture Notes in Math., Vol. 318.

Grassberger, P., 1984a. Chaos and diffusion in deterministic cellular automata. Phys. D 10 (1-2), 52–58, cellular automata (Los Alamos, N.M., 1983).

Grassberger, P., 1984b. New mechanism for deterministic diffusion. Phys. Rev. A 28 (6), 3666–3667.

Gromov, M., 1999. Endomorphisms of symbolic algebraic varieties. J. Eur. Math. Soc. (JEMS) 1 (2), 109–197.

Hedlund, G. A., 1969. Endormorphisms and automorphisms of the shift dynamical system. Math. Systems Theory 3, 320–375.

Hilmy, H., 1936. Sur les centres d'attraction minimaux des systeémes dynamiques. Compositio Mathematica 3, 227–238.

Host, B., 1995. Nombres normaux, entropie, translations. Israel J. Math. 91 (1-3), 419–428.

Host, B., Maass, A., Martínez, S., 2003. Uniform Bernoulli measure in dynamics of permutative cellular automata with algebraic local rules. Discrete Contin. Dyn. Syst. 9 (6), 1423–1446.

Hurd, L. P., Kari, J., Culik, K., 1992. The topological entropy of cellular automata is uncomputable. Ergodic Theory Dynam. Systems 12 (2), 255–265.

Hurley, M., 1990a. Attractors in cellular automata. Ergodic Theory Dynam. Systems 10 (1), 131–140.

Hurley, M., 1990b. Ergodic aspects of cellular automata. Ergodic Theory Dynam. Systems 10 (4), 671–685.

Hurley, M., 1991. Varieties of periodic attractor in cellular automata. Trans. Amer. Math. Soc. 326 (2), 701–726.

Hurley, M., 1992. Attractors in restricted cellular automata. Proc. Amer. Math. Soc. 115 (2), 563–571.

Jen, E., 1988. Linear cellular automata and recurring sequences in finite fields. Comm. Math. Phys. 119 (1), 13–28.

Johnson, A., Rudolph, D. J., 1995. Convergence under $\times_q$ of $\times_p$ invariant measures on the circle. Adv. Math. 115 (1), 117–140.

Johnson, A. S. A., 1992. Measures on the circle invariant under multiplication by a nonlacunary subsemigroup of the integers. Israel J. Math. 77 (1-2), 211–240.

Kitchens, B., 2000. Dynamics of $\mathbf{Z}^d$ actions on Markov subgroups. In: Topics in symbolic dynamics and applications (Temuco, 1997). Vol. 279 of London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, pp. 89–122.

Kitchens, B., Schmidt, K., 1989. Automorphisms of compact groups. Ergodic Theory Dynam. Systems 9 (4), 691–735.

Kitchens, B., Schmidt, K., 1992. Markov subgroups of $(\mathbf{Z}/2\mathbf{Z})^{\mathbf{Z}^2}$. In: Symbolic dynamics and its applications (New Haven, CT, 1991). Vol. 135 of Contemp. Math. Amer. Math. Soc., Providence, RI, pp. 265–283.

Kitchens, B. P., 1987. Expansive dynamics on zero-dimensional groups. Ergodic Theory Dynam. Systems 7 (2), 249–261.

Kleveland, R., 1997. Mixing properties of one-dimensional cellular automata. Proc. Amer. Math. Soc. 125 (6), 1755–1766.

Kůrka, P., 1997. Languages, equicontinuity and attractors in cellular automata. Ergodic Theory Dynam. Systems 17 (2), 417–433.

Kůrka, P., 2001. Topological dynamics of cellular automata. In: Codes, systems, and graphical models (Minneapolis, MN, 1999). Vol. 123 of IMA Vol. Math. Appl. Springer, New York, pp. 447–485.

Kůrka, P., 2003. Cellular automata with vanishing particles. Fund. Inform. 58 (3-4), 203–221.

Kůrka, P., 2005. On the measure attractor of a cellular automaton. Discrete Contin. Dyn. Syst. (suppl.), 524–535.

Kůrka, P., Maass, A., 2000. Limit sets of cellular automata associated to probability measures. J. Statist. Phys. 100 (5-6), 1031–1047.

Kůrka, P., Maass, A., 2002. Stability of subshifts in cellular automata. Fund. Inform. 52 (1-3), 143–155, special issue on cellular automata.

Lind, D., Marcus, B., 1995. An introduction to symbolic dynamics and coding. Cambridge University Press, Cambridge.

Lind, D. A., 1984. Applications of ergodic theory and sofic systems to cellular automata. Phys. D 10 (1-2), 36–44, cellular automata (Los Alamos, N.M., 1983).

Lind, D. A., 1987. Entropies of automorphisms of a topological Markov shift. Proc. Amer. Math. Soc. 99 (3), 589–595.

Lucas, E., 1878. Sur les congruences des nombres eulériens et les coefficients différentiels des functions trigonométriques suivant un module premier. Bull. Soc. Math. France 6, 49–54.

Lyons, R., 1988. On measures simultaneously 2- and 3-invariant. Israel J. Math. 61 (2), 219–224.

Maass, A., 1996. Some dynamical properties of one-dimensional cellular automata. In: Dynamics of complex interacting systems (Santiago, 1994). Vol. 2 of Nonlinear Phenom. Complex Systems. Kluwer Acad. Publ., Dordrecht, pp. 35–80.

Maass, A., Martínez, S., 1998. On Cesàro limit distribution of a class of permutative cellular automata. J. Statist. Phys. 90 (1-2), 435–452.

Maass, A., Martínez, S., 1999. Time averages for some classes of expansive one-dimensional cellular automata. In: Cellular automata and complex systems (Santiago, 1996). Vol. 3 of Nonlinear Phenom. Complex Systems. Kluwer Acad. Publ., Dordrecht, pp. 37–54.

Maass, A., Martínez, S., Pivato, M., Yassawi, R., 2006a. Asymptotic randomization of subgroup shifts by linear cellular automata. Ergodic Theory Dynam. Systems 26 (4), 1203–1224.

Maass, A., Martínez, S., Pivato, M., Yassawi, R., 2006b. Attractiveness of the Haar measure for the action of linear cellular automata in abelian topological Markov chains. In: Dynamics and Stochastics: Festschrift in honour of Michael Keane volume 48 of. Vol. 48 of Lecture Notes Monograph Series of the IMS. Institute for Mathematical Statistics, pp. 100–108.

Maass, A., Martínez, S., Sobottka, M., 2006c. Limit measures for affine cellular automata on topological markov subgroups. Nonlinearity 19 (9), 2137–2147.
URL `http://stacks.iop.org/0951-7715/19/2137`

Machì, A., Mignosi, F., 1993. Garden of Eden configurations for cellular automata on Cayley graphs of groups. SIAM J. Discrete Math. 6 (1), 44–56.

Maruoka, A., Kimura, M., 1976. Condition for injectivity of global maps for tessellation automata. Information and Control 32 (2), 158–162.

Mauldin, R. D., Skordev, G., 2000. Random linear cellular automata: fractals associated with random multiplication of polynomials. Japan. J. Math. (N.S.) 26 (2), 381–406.

Meester, R., Steif, J. E., 2001. Higher-dimensional subshifts of finite type, factor maps and measures of maximal entropy. Pacific J. Math. 200 (2), 497–510.

Milnor, J., 1985a. Correction and remarks: "On the concept of attractor". Comm. Math. Phys. 102 (3), 517–519.

Milnor, J., 1985b. On the concept of attractor. Comm. Math. Phys. 99 (2), 177–195.

Milnor, J., 1986. Directional entropies of cellular automaton-maps. In: Disordered systems and biological organization (Les Houches, 1985). Vol. 20 of NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci. Springer, Berlin, pp. 113–115.

Milnor, J., 1988. On the entropy geometry of cellular automata. Complex Systems 2 (3), 357–385.

Miyamoto, M., 1979. An equilibrium state for a one-dimensional life game. J. Math. Kyoto Univ. 19 (3), 525–540.

Moore, C., 1997. Quasilinear cellular automata. Phys. D 103 (1-4), 100–132, lattice dynamics (Paris, 1995).

Moore, C., 1998. Predicting nonlinear cellular automata quickly by decomposing them into linear ones. Phys. D 111 (1-4), 27–41.

Moore, E. F., 1963. Machine models of self reproduction. Proc. Symp. Appl. Math. 14, 17–34.

Myhill, J., 1963. The converse of Moore's Garden-of-Eden theorem. Proc. Amer. Math. Soc. 14, 685–686.

Nasu, M., 1995. Textile systems for endomorphisms and automorphisms of the shift. Mem. Amer. Math. Soc. 114 (546), viii+215.

Nasu, M., 2002. The dynamics of expansive invertible onesided cellular automata. Trans. Amer. Math. Soc. 354 (10), 4067–4084 (electronic).

Park, K. K., 1995. Continuity of directional entropy for a class of $\mathbf{Z}^2$-actions. J. Korean Math. Soc. 32 (3), 573–582.

Park, K. K., 1996. Entropy of a skew product with a $\mathbf{Z}^2$-action. Pacific J. Math. 172 (1), 227–241.

Park, K. K., 1999. On directional entropy functions. Israel J. Math. 113, 243–267.

Parry, W., 1964. Intrinsic Markov chains. Trans. Amer. Math. Soc. 112, 55–66.

Pivato, M., 2003. Multiplicative cellular automata on nilpotent groups: structure, entropy, and asymptotics. J. Statist. Phys. 110 (1-2), 247–267.

Pivato, M., 2005a. Cellular automata versus quasisturmian shifts. Ergodic Theory Dynam. Systems 25 (5), 1583–1632.

Pivato, M., 2005b. Invariant measures for bipermutative cellular automata. Discrete Contin. Dyn. Syst. 12 (4), 723–736.

Pivato, M., 2007. Spectral domain boundaries cellular automata. Fundamenta Informaticae 77 (special issue), available at: `http://arxiv.org/abs/math.DS/0507091`.

Pivato, M., Yassawi, R., 2002. Limit measures for affine cellular automata. Ergodic Theory Dynam. Systems 22 (4), 1269–1287.

Pivato, M., Yassawi, R., 2004. Limit measures for affine cellular automata. II. Ergodic Theory Dynam. Systems 24 (6), 1961–1980.

Pivato, M., Yassawi, R., 2006. Asymptotic randomization of sofic shifts by linear cellular automata. Ergodic Theory Dynam. Systems 26 (4), 1177–1201.

Rudolph, D. J., 1990. $\times 2$ and $\times 3$ invariant measures and entropy. Ergodic Theory Dynam. Systems 10 (2), 395–406.

Sablik, M., July 2006. Étude de l'action conjointe d'un automate cellulaire et du décalage: Une approche topologique et ergodique. Ph.D. thesis, Université de la Méditerranée, Faculté des science de Luminy, Marseille.

Sablik, M., 2007a. Directional dynamics for cellular automata: A sensitivity to initial conditions approach. submitted to Theoretical Computer Science.

Sablik, M., 2007b. Measure rigidity for algebraic bipermutative cellular automata. to appear in Ergodic Theory Dynam. Systems 27.

Sato, T., 1997. Ergodicity of linear cellular automata over $\mathbf{Z}_m$. Inform. Process. Lett. 61 (3), 169–172.

Schmidt, K., 1995. Dynamical systems of algebraic origin. Vol. 128 of Progress in Mathematics. Birkhäuser Verlag, Basel.

Shereshevsky, M. A., 1992a. Ergodic properties of certain surjective cellular automata. Monatsh. Math. 114 (3-4), 305–316.

Shereshevsky, M. A., 1992b. Lyapunov exponents for one-dimensional cellular automata. J. Nonlinear Sci. 2 (1), 1–8.

Shereshevsky, M. A., 1993. Expansiveness, entropy and polynomial growth for groups acting on subshifts by automorphisms. Indag. Math. (N.S.) 4 (2), 203–210.

Shereshevsky, M. A., 1996. On continuous actions commuting with actions of positive entropy. Colloq. Math. 70 (2), 265–269.

Shereshevsky, M. A., 1997. K-property of permutative cellular automata. Indag. Math. (N.S.) 8 (3), 411–416.

Shereshevsky, M. A., Afraĭmovich, V. S., 1992/93. Bipermutative cellular automata are topologically conjugate to the one-sided Bernoulli shift. Random Comput. Dynam. 1 (1), 91–98.

Shirvani, M., Rogers, T. D., 1991. On ergodic one-dimensional cellular automata. Comm. Math. Phys. 136 (3), 599–605.

Silberger, S., 2005. Subshifts of the three dot system. Ergodic Theory Dynam. Systems 25 (5), 1673–1687.

Smillie, J., 1988. Properties of the directional entropy function for cellular automata. In: Dynamical systems (College Park, MD, 1986–87). Vol. 1342 of Lecture Notes in Math. Springer, Berlin, pp. 689–705.

Sobottka, M., November 2005. Representación y aleatorización en sistemas dynámicos de tipo algebraico. Ph.D. thesis, Universidad de Chile, Facultad de ciencias fisicas y matematicas, Santiago.

Sobottka, M., 2007a. Right-permutative cellular automata on topological Markov chains. (submitted).

Sobottka, M., January 2007b. Topological quasi-group shifts. Discrete and Continuous Dynamical Systems 17 (1), 77–93.

Takahashi, S., 1990. Cellular automata and multifractals: dimension spectra of linear cellular automata. Phys. D 45 (1-3), 36–48, cellular automata: theory and experiment (Los Alamos, NM, 1989).

Takahashi, S., 1992. Self-similarity of linear cellular automata. J. Comput. System Sci. 44 (1), 114–140.

Takahashi, S., 1993. Cellular automata, fractals and multifractals: space-time patterns and dimension spectra of linear cellular automata. In: Chaos in Australia (Sydney, 1990). World Sci. Publishing, River Edge, NJ, pp. 173–195.

Tisseur, P., 2000. Cellular automata and Lyapunov exponents. Nonlinearity 13 (5), 1547–1560.

von Haeseler, F., Peitgen, H.-O., Skordev, G., 1992. Pascal's triangle, dynamical systems and attractors. Ergodic Theory Dynam. Systems 12 (3), 479–486.

von Haeseler, F., Peitgen, H.-O., Skordev, G., 1993. Cellular automata, matrix substitutions and fractals. Ann. Math. Artificial Intelligence 8 (3-4), 345–362, theorem proving and logic programming (1992).

von Haeseler, F., Peitgen, H.-O., Skordev, G., 1995a. Global analysis of self-similarity features of cellular automata: selected examples. Phys. D 86 (1-2), 64–80, chaos, order and patterns: aspects of nonlinearity—the "gran finale" (Como, 1993).

von Haeseler, F., Peitgen, H.-O., Skordev, G., 1995b. Multifractal decompositions of rescaled evolution sets of equivariant cellular automata. Random Comput. Dynam. 3 (1-2), 93–119.

von Haeseler, F., Peitgen, H.-O., Skordev, G., 2001a. Self-similar structure of rescaled evolution sets of cellular automata. I. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 11 (4), 913–926.

von Haeseler, F., Peitgen, H.-O., Skordev, G., 2001b. Self-similar structure of rescaled evolution sets of cellular automata. II. Internat. J. Bifur. Chaos Appl. Sci. Engrg. 11 (4), 927–941.

Walters, P., 1982. An introduction to ergodic theory. Vol. 79 of Graduate Texts in Mathematics. Springer-Verlag, New York.

Weiss, B., 2000. Sofic groups and dynamical systems. Sankhyā Ser. A 62 (3), 350–359.

Willson, S. J., 1975. On the ergodic theory of cellular automata. Math. Systems Theory 9 (2), 132–141.

Willson, S. J., 1984a. Cellular automata can generate fractals. Discrete Appl. Math. 8 (1), 91–99.

Willson, S. J., 1984b. Growth rates and fractional dimensions in cellular automata. Phys. D 10 (1-2), 69–74, cellular automata (Los Alamos, N.M., 1983).

Willson, S. J., 1986. A use of cellular automata to obtain families of fractals. In: Chaotic dynamics and fractals (Atlanta, Ga., 1985). Vol. 2 of Notes Rep. Math. Sci. Engrg. Academic Press, Orlando, FL, pp. 123–140.

Willson, S. J., 1987a. Computing fractal dimensions for additive cellular automata. Phys. D 24 (1-3), 190–206.

Willson, S. J., 1987b. The equality of fractional dimensions for certain cellular automata. Phys. D 24 (1-3), 179–189.

Wolfram, S., 1985. Twenty problems in the theory of cellular automata. Physica Scripta 9, 1–35.

Wolfram, S., 1986. Theory and Applications of Cellular Automata. World Scientific, Singapore.