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Algebraic invariants for crystallographic defects in cellular automata

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Abstract. Let $\mathcal{A}^{\mathbb{Z}^D}$ be the Cantor space of $\mathbb{Z}^D$-indexed configurations in a finite alphabet $\mathcal{A}$, and let σ be the $\mathbb{Z}^D$-action of shifts on $\mathcal{A}^{\mathbb{Z}^D}$. A *cellular automaton* is a continuous, σ-commuting self-map Φ of $\mathcal{A}^{\mathbb{Z}^D}$, and a Φ-*invariant subshift* is a closed, (Φ, σ)-invariant subset $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$. Suppose $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ is $\mathfrak{A}$-admissible everywhere except for some small region we call a *defect*. It has been empirically observed that such defects persist under iteration of Φ, and often propagate like 'particles' which coalesce or annihilate on contact. We construct algebraic invariants for these defects, which explain their persistence under Φ, and partly explain the outcomes of their collisions. Some invariants are based on the cocycles of multidimensional subshifts; others arise from the higher-dimensional (co)homology/homotopy groups for subshifts, obtained by generalizing the Conway-Lagarias tiling groups and the Geller-Propp fundamental group.

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A striking phenomenon in cellular automata is the emergence of homogeneous 'domains' (each exhibiting a particular spatial pattern), punctuated by *defects* (analogous to 'domain boundaries' or 'kinks' in a crystalline solid) which evolve and propagate over time, and occasionally collide. This phenomenon has been studied empirically in [**Gra84b, Gra84a, KS88, BR91, BNR91, Han93, CH92, CH93a, CH93b, CH97, CHM98**] and theoretically in [**Lin84, EN92, Elo93a, Elo93b, Elo94, CHS01, KM00, Kůr03, KM02, Kůr03, Kůr05**]; see [**Piv05, Piv06**] for a summary. The mathematical theory of cellular automaton defect dynamics is still in its infancy. Even the term 'defect' does not yet have a unanimous definition. Other open questions include:

1. Why do defects persist under the action of cellular automata, rather than disappearing? Are there 'topological' constraints imposed by the structure of the underlying domain, which make defects indestructible?

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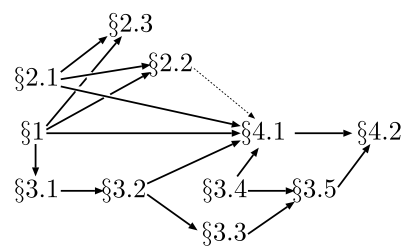
2. When defects collide, they often coalesce into a new type of defect, or mutually annihilate. Is there a 'chemistry' governing these defect collisions?
3. Can we assign algebraic invariants to defects, which reflect **(a)** the 'topological constraints' of question #1 or **(b)** the 'defect chemistry' of question #2?

In a companion paper [**Piv06**], we developed a new framework for describing defects, and used spectral theory to get invariants (as in question #3) for codimension-one ('domain boundary') defects. Unfortunately, these spectral invariants were not applicable to defects of codimension two or higher (e.g. 'holes' in $\mathbb{Z}^2$, 'strings' in $\mathbb{Z}^3$, etc.). In this paper, we will answer question #3 for such defects, using methods inspired by algebraic topology.

This paper is organized as follows: in §1 we review the framework developed in [**Piv06**]. We also define defect *codimension*, and introduce many examples which recur throughout the paper. In §2, we address question #3 using dynamical cohomology, while in §3, we address #3 using tiling homotopy/(co)homology groups. In §4 we relate the dynamical cohomology of §2 to the tiling cohomology of §3. In all cases, we are able to use these algebraic invariants to answer question #1, and partially answer question #2.

The diagram at right portrays the logical dependency of these sections. In particular, notice that §2 and §3 are logically independent of one another, although §4 depends upon both. Our main results are in sections 2.2, 2.3, 3.5, and 4.2.



Preliminaries & Notation: Let $\mathcal{A}$ be a finite alphabet. Let $D \geq 1$, let $\mathbb{Z}^D$ be the D-dimensional lattice, and let $\mathcal{A}^{\mathbb{Z}^D}$ be the set of all $\mathbb{Z}^D$-indexed *configurations* of the form $\mathbf{a} = [a_\mathsf{z}]_{\mathsf{z}\in\mathbb{Z}^D}$, where $a_\mathsf{z} \in \mathcal{A}$ for all $\mathsf{z} \in \mathbb{Z}^D$. The *Cantor metric* on $\mathcal{A}^{\mathbb{Z}^D}$ is defined by $d(\mathbf{a},\mathbf{b}) = 2^{-\Delta(\mathbf{a},\mathbf{b})}$, where $\Delta(\mathbf{a},\mathbf{b}) := \min\{|\mathsf{z}| \,;\, a_\mathsf{z} \neq b_\mathsf{z}\}$. It follows that $(\mathcal{A}^{\mathbb{Z}^D}, d)$ is a Cantor space (i.e. a compact, totally disconnected, perfect metric space). If $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ and $\mathbb{U} \subset \mathbb{Z}^D$, then we define $\mathbf{a}_\mathbb{U} \in \mathcal{A}^\mathbb{U}$ by $\mathbf{a}_\mathbb{U} := [a_\mathsf{u}]_{\mathsf{u}\in\mathbb{U}}$. If $\mathsf{z} \in \mathbb{Z}^D$, then strictly speaking, $\mathbf{a}_{\mathsf{z}+\mathbb{U}} \in \mathcal{A}^{\mathsf{z}+\mathbb{U}}$; however, it will often be convenient to 'abuse notation' and treat $\mathbf{a}_{\mathsf{z}+\mathbb{U}}$ as an element of $\mathcal{A}^\mathbb{U}$ in the obvious way.

For any $\mathsf{v} \in \mathbb{Z}^D$, we define the *shift* $\sigma^\mathsf{v} : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ by $\sigma^\mathsf{v}(\mathbf{a})_\mathsf{z} = a_{\mathsf{z}+\mathsf{v}}$ for all $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ and $\mathsf{z} \in \mathbb{Z}^D$. A *cellular automaton* is a transformation $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ that is continuous and commutes with all shifts. Equivalently, Φ is determined by a *local rule* $\phi : \mathcal{A}^\mathbb{H} \longrightarrow \mathcal{A}$ such that $\Phi(\mathbf{a})_\mathsf{z} = \phi(\mathbf{a}_{\mathsf{z}+\mathbb{H}})$ for all $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ and $\mathsf{z} \in \mathbb{Z}^D$ [**Hed69**]. Here, $\mathbb{H} \subset \mathbb{Z}^D$ is a finite set which we normally imagine as a 'neighbourhood of the origin'. If $\mathbb{H} \subseteq \mathbb{B}(r) := [-r...r]^D$, we say that Φ has *radius* r.

A subset $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ is a *subshift* [**LM95, Kit98**] if $\mathfrak{A}$ is closed in the Cantor topology, and if $\sigma^\mathsf{z}(\mathfrak{A}) = \mathfrak{A}$ for all $\mathsf{z} \in \mathbb{Z}^D$. For any $\mathbb{U} \subset \mathbb{Z}^D$, we define $\mathfrak{A}_\mathbb{U} := \{\mathbf{a}_\mathbb{U} \,;\, \mathbf{a} \in \mathfrak{A}\}$. In particular, for any $r > 0$, let $\mathfrak{A}_{(r)} := \mathfrak{A}_{\mathbb{B}(r)}$ be the set of *admissible r-blocks* for $\mathfrak{A}$. We say $\mathfrak{A}$ is a *subshift of finite type* (SFT) if there is some $r > 0$ (the *radius* of $\mathfrak{A}$) such that $\mathfrak{A}$ is entirely described by $\mathfrak{A}_{(r)}$, in the sense that $\mathfrak{A} = \left\{\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D} \,;\, \mathbf{a}_{\mathbb{B}(\mathsf{z},r)} \in \mathfrak{A}_{(r)},\ \forall \mathsf{z} \in \mathbb{Z}^D\right\}$. If $D = 1$, then a *Markov subshift* is an SFT $\mathfrak{A} \subset \mathcal{A}^\mathbb{Z}$ determined by a set $\mathfrak{A}_{\{0,1\}} \subset \mathcal{A}^{\{0,1\}}$ of *admissible transitions*; equivalently, $\mathfrak{A}$ is the set of all bi-infinite directed paths in a digraph whose vertices are the elements of $\mathfrak{A}$, with an edge $a \rightsquigarrow b$ iff $(a,b) \in \mathfrak{A}_{\{0,1\}}$. If $D = 2$, then let $\mathbb{E}_1 := \{(0,0),(1,0)\}$ and $\mathbb{E}_2 := \{(0,0),(0,1)\}$. A *Wang subshift* is an SFT $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^2}$ determined by sets $\mathfrak{A}_{\mathbb{E}_1} \subset \mathcal{A}^{\mathbb{E}_1}$ and $\mathfrak{A}_{\mathbb{E}_2} \subset \mathcal{A}^{\mathbb{E}_2}$ of *edge-matching conditions*. Equivalently, $\mathfrak{A}$ is the set of all *tilings* of the plane $\mathbb{R}^2$ by unit square tiles (corresponding to the elements of $\mathcal{A}$) with notched edges representing the edge-matching conditions [**GS87**, Ch.11]. More generally, if $D \geq 3$, then for all $d \in [1...D]$, let $\mathbb{E}_d := \{0\}^{d-1} \times \{0,1\} \times \{0\}^{D-d}$. A *Wang subshift* is an SFT $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ determined by sets $\mathfrak{A}_{\mathbb{E}_d} \subset \mathcal{A}^{\mathbb{E}_d}$ of *face-matching conditions* for $d \in [1...D]$. Equivalently, $\mathfrak{A}$ is the set of tesselations of $\mathbb{R}^D$ by unit (hyper)cubes with 'notched' faces.

If $\mathcal{X}$ is any set and $F : \mathfrak{A} \longrightarrow \mathcal{X}$ is a function, then F is *locally determined* if there is some *radius* $r \in \mathbb{N}$ and some *local rule* $f : \mathfrak{A}_{(r)} \longrightarrow \mathcal{X}$ such that $F(\mathbf{a}) = f(\mathbf{a}_{\mathbb{B}(r)})$ for any $\mathbf{a} \in \mathfrak{A}$. If $\mathcal{X}$ is any discrete space, then $F : \mathfrak{A} \longrightarrow \mathcal{X}$ is continuous iff F is locally determined. For example, if $\mathcal{A}$ and $\mathcal{B}$ are finite sets, then a (subshift) *homomorphism* is a continuous, σ-commuting function $\Phi : \mathcal{B}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ (e.g. a CA is a homomorphism with $\mathcal{A} = \mathcal{B}$); it follows that $F(\mathbf{a}) = \Phi(\mathbf{a})_0$ is locally determined. If $\mathfrak{B} \subset \mathcal{B}^{\mathbb{Z}^D}$ is a subshift of finite type, and $\Psi : \mathcal{B}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ is a homomorphism, then $\mathfrak{A} := \Psi(\mathfrak{B}) \subset \mathcal{A}^{\mathbb{Z}^D}$ is called a *sofic shift*.

If $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ is a cellular automaton, then we say $\mathfrak{A}$ is (weakly) Φ-*invariant* if $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$ (i.e. Φ is an *endomorphism* of $\mathfrak{A}$). For example, if $p \in \mathbb{N}$ and $\mathsf{v} \in \mathbb{Z}^D$, then the set $\mathsf{Fix}\,[\Phi^p]$ of (Φ, p)-*periodic points* and the set $\mathsf{Fix}\,[\Phi^p \circ \sigma^{-p\mathsf{v}}]$ of (Φ, p, v)-*travelling waves* are Φ-invariant SFTs. If $\Phi^\infty(\mathcal{A}^{\mathbb{Z}^D}) := \bigcap_{t=1}^\infty \Phi^t(\mathcal{A}^{\mathbb{Z}^D})$ is the *eventual image* of Φ, then $\Phi^\infty(\mathcal{A}^{\mathbb{Z}^D})$ is a Φ-invariant subshift (possibly non-sofic), which contains $\mathsf{Fix}\,[\Phi^p \circ \sigma^{-p\mathsf{v}}]$ for any $p \in \mathbb{N}$ and $\mathsf{v} \in \mathbb{Z}^D$.

If $\mathsf{y}, \mathsf{z} \in \mathbb{Z}^D$, then we write "$\mathsf{y} \rightsquigarrow \mathsf{z}$" if $|\mathsf{z} - \mathsf{y}| = 1$. A *trail* is a sequence $\zeta = (\mathsf{z}_1 \rightsquigarrow \mathsf{z}_2 \rightsquigarrow \cdots \rightsquigarrow \mathsf{z}_n)$. A subset $\mathbb{Y} \subset \mathbb{Z}^D$ is *trail-connected* if, for any $\mathsf{x}, \mathsf{y} \in \mathbb{Y}$, there is a trail $\mathsf{x} = \mathsf{z}_0 \rightsquigarrow \mathsf{z}_1 \rightsquigarrow \cdots \rightsquigarrow \mathsf{z}_n = \mathsf{y}$ in $\mathbb{Y}$.

Font conventions: Upper case calligraphic letters $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots)$ denote alphabets or groups. Upper-case Gothic letters $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots)$ denote subsets of $\mathcal{A}^{\mathbb{Z}^D}$ (e.g. subshifts), lowercase bold-faced letters $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots)$ denote elements of $\mathcal{A}^{\mathbb{Z}^D}$, and Roman letters $(a, b, c, \ldots)$ are elements of $\mathcal{A}$ or ordinary numbers. Lower-case sans-serif $(\ldots, \mathsf{x}, \mathsf{y}, \mathsf{z})$ are elements of $\mathbb{Z}^D$, upper-case hollow font $(\mathbb{U}, \mathbb{V}, \mathbb{W}, \ldots)$ are subsets of $\mathbb{Z}^D$, and upper-case bold $(\mathbf{U}, \mathbf{V}, \mathbf{W}, \ldots)$ are subsets of $\mathbb{R}^D$. Upper-case Greek letters $(\Phi, \Psi, \ldots)$ are functions on $\mathcal{A}^{\mathbb{Z}^D}$ (e.g. CA), and lower-case Greek letters $(\phi, \psi, \ldots)$ are other functions (e.g. local rules.)

1. *Defects and Codimension*

Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be any subshift. If $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$, then the *defect field* $\mathcal{F}_\mathbf{a} : \mathbb{Z}^D \longrightarrow \mathbb{N} \cup \{\infty\}$ is defined by $\mathcal{F}_\mathbf{a}(\mathsf{z}) := \max\left\{r \in \mathbb{N} \; ; \; \mathbf{a}_{\mathbb{B}(\mathsf{z},r)} \in \mathfrak{A}_{(r)}\right\}$, for all $\mathsf{z} \in \mathbb{Z}^D$. Clearly, $\mathcal{F}_\mathbf{a}$ is 'Lipschitz' in the sense that $|\mathcal{F}_\mathbf{a}(\mathsf{y}) - \mathcal{F}_\mathbf{a}(\mathsf{z})| \leq |\mathsf{y} - \mathsf{z}|$. The *defect set* of $\mathbf{a}$ is the set $\mathbb{D}(\mathbf{a}) \subset \mathbb{Z}^D$ of local minima of $\mathcal{F}_\mathbf{a}$. See [**Piv06**, §1] for further discussion.

Example 1.1: (a) Suppose $\mathfrak{A}$ is an SFT determined by a set $\mathfrak{A}_{(r)} \subset \mathcal{A}^{\mathbb{B}(r)}$ of admissible r-blocks, and let $\mathbb{X} := \left\{\mathsf{z} \in \mathbb{Z}^D \; ; \; \mathbf{a}_{\mathbb{B}(\mathsf{z},r)} \notin \mathfrak{A}_{(r)}\right\}$. Assume for simplicity that $\mathfrak{A}_{(r-1)} = \mathcal{A}^{\mathbb{B}(r-1)}$. Then $\mathcal{F}_\mathbf{a}(\mathsf{z}) = r + d(\mathsf{z}, \mathbb{X})$, where $d(\mathsf{z}, \mathbb{X}) := \min_{\mathsf{x} \in \mathbb{X}} |\mathsf{z} - \mathsf{x}|$. In particular, $\mathcal{F}_\mathbf{a}(\mathsf{z}) = r$ if and only if $\mathsf{z} \in \mathbb{X}$, and this is the smallest possible value for $\mathcal{F}_\mathbf{a}(\mathsf{z})$. Thus, $\mathbb{D}(\mathbf{a}) = \mathbb{X}$.

(b) Let $\mathcal{A} = \mathcal{B} \cup \mathcal{D}$ and let $\mathfrak{A} := \mathcal{B}^{\mathbb{Z}^D} \cup \mathcal{D}^{\mathbb{Z}^D}$. Let $\mathcal{C} := \mathcal{B} \cap \mathcal{D}$, and let $\mathcal{B}^* := \mathcal{B} \setminus \mathcal{C}$ and $\mathcal{D}^* := \mathcal{D} \setminus \mathcal{C}$. Any $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ is a mixture of $\mathcal{B}^*$-symbols, $\mathcal{C}$-symbols, and $\mathcal{D}^*$-symbols. If $\mathsf{z} \in \mathbb{Z}^D$ and $a_\mathsf{z} \in \mathcal{B}^*$, then $\mathcal{F}_\mathbf{a}(\mathsf{z}) = \min\{|\mathsf{y}-\mathsf{z}| \; ; \; a_\mathsf{y} \in \mathcal{D}^*\}$. If $a_\mathsf{z} \in \mathcal{D}^*$, then $\mathcal{F}_\mathbf{a}(\mathsf{z}) = \min\{|\mathsf{y}-\mathsf{z}| \; ; \; a_\mathsf{y} \in \mathcal{B}^*\}$. If $a_\mathsf{z} \in \mathcal{C}$, then $\mathcal{F}_\mathbf{a}(\mathsf{z}) = \min\{r \; ; \; a_\mathsf{x} \in \mathcal{B}^*$ and $a_\mathsf{y} \in \mathcal{B}^*$ for some $\mathsf{x},\mathsf{y} \in \mathbb{B}(\mathsf{z},r)\}$. Thus, $\mathbb{D}(\mathbf{a})$ is the set of all points which are either on a 'boundary' between a $\mathcal{B}^*$-domain and a $\mathcal{D}^*$-domain, or roughly in the middle of a $\mathcal{C}$-domain. ♢

Let $\widetilde{\mathfrak{A}} := \left\{\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D} \; ; \; \sup_{\mathsf{z}\in\mathbb{Z}^D} \mathcal{F}_\mathbf{a}(\mathsf{z}) = \infty\right\}$ be the set of 'slightly defective' configurations. If $\mathbf{a} \in \widetilde{\mathfrak{A}} \setminus \mathfrak{A}$, then we say $\mathbf{a}$ is *defective*. Elements of $\widetilde{\mathfrak{A}}$ may have infinitely large defects, but also have arbitrarily large non-defective regions. Clearly $\mathfrak{A} \subset \widetilde{\mathfrak{A}}$, and $\widetilde{\mathfrak{A}}$ is a σ-invariant, dense subset of $\mathcal{A}^{\mathbb{Z}^D}$ (but not a subshift).

For any $R > 0$, let $\mathbb{G}_R(\mathbf{a}) := \{\mathsf{z} \in \mathbb{Z}^D \; ; \; \mathcal{F}_\mathbf{a}(\mathsf{z}) \geq R\}$. Thus, $\mathbf{a} \in \widetilde{\mathfrak{A}}$ iff $\mathbb{G}_R(\mathbf{a}) \neq \emptyset$ for all $R > 0$. For example, if $\mathfrak{A}$ is an SFT determined by a set $\mathfrak{A}_{(r)}$ of admissible r-blocks, and $\mathbb{D} = \{\mathsf{z} \in \mathbb{Z}^D \; ; \; \mathbf{a}_{\mathbb{B}(\mathsf{z},r)} \notin \mathfrak{A}_{(r)}\}$ as in Example 1.1(a), then $\mathbb{G}_R(\mathbf{a}) = \{\mathsf{z} \in \mathbb{Z}^D \; ; \; d(\mathsf{z},\mathbb{D}) \geq R - r\} = \mathbb{Z}^D \setminus \mathbb{B}(\mathbb{D}, R-r)$. Thus, $\mathbb{D}(\mathbf{a})$ encodes all information about the 'defect structure' of $\mathbf{a}$. However, if $\mathfrak{A}$ is *not* an SFT [e.g. Example 1.1(b)], then in general $\mathbb{G}_R(\mathbf{a}) \neq \mathbb{Z}^D \setminus \mathbb{B}(\mathbb{D}, R')$ for any $R' > 0$. In this case, $\mathbb{D}(\mathbf{a})$ is an inadequate description of the larger-scale 'defect structures' of $\mathbf{a}$. Thus, instead of treating the defect as a precisely defined subset of $\mathbb{Z}^D$, it is better to think of it as a 'fuzzy' object residing in the low areas in the defect field $\mathcal{F}_\mathbf{a}$. The advantage of this approach is its applicability to any kind of subshift (finite type, sofic, or otherwise). Nevertheless, most of our examples will be SFTs, and we may then refer to the specific region $\mathbb{D} \subset \mathbb{Z}^D$ as 'the defect'.

PROPOSITION 1.2. *Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA with radius $r > 0$.*

(a) *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a weakly Φ-invariant subshift. Then $\Phi(\widetilde{\mathfrak{A}}) \subseteq \widetilde{\mathfrak{A}}$.*

(b) *If $\mathbf{a} \in \widetilde{\mathfrak{A}}$, and $\mathbf{a}' = \Phi(\mathbf{a})$, then $\mathcal{F}_{\mathbf{a}'} \geq \mathcal{F}_\mathbf{a} - r$. Thus, for all $R \in \mathbb{N}$, $\mathbb{G}_{R+r}(\mathbf{a}) \subseteq \mathbb{G}_R(\mathbf{a}')$.*

Proof: **(b)** Let $\mathsf{z} \in \mathbb{Z}^D$ and suppose $\mathcal{F}_\mathbf{a}(\mathsf{z}) = R$. Thus, $\mathbf{a}_{\mathbb{B}(\mathsf{z},R)} \in \mathfrak{A}_R$. But $\mathfrak{A}$ is Φ-invariant; hence $\mathbf{a}'_{\mathbb{B}(\mathsf{z},R-r)} \in \mathfrak{A}_{(R-r)}$. Hence $\mathcal{F}_{\mathbf{a}'}(\mathsf{z}) \geq R - r$. Then **(a)** follows from **(b)**. □

Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a cellular automaton, and suppose that $\phi(\mathfrak{A}) = \mathfrak{A}$. If $\mathbf{a} \in \widetilde{\mathfrak{A}}$ then $\mathbf{a}$ has a Φ-*persistent* defect if, for all $t \in \mathbb{N}$, $\mathbf{a}' = \Phi^t(\mathbf{a})$ is also defective. Otherwise $\mathbf{a}$ has a *transient* defect —i.e. one which eventually disappears. We say $\mathbf{a}$ has a *removable* defect if there is some $r > 0$ and some $\mathbf{a}' \in \mathfrak{A}$ such that $a'_\mathsf{z} = a_\mathsf{z}$ for all $\mathsf{z} \in \mathbb{G}_r(\mathbf{a})$ (i.e. the defect can be erased by modifying $\mathbf{a}$ in a finite radius of the defective region). Otherwise $\mathbf{a}$ has an *essential* defect.

EXAMPLE 1.3: The defect in $\mathbf{a}$ is *finite* if $\mathbb{D}(\mathbf{a})$ is finite (or equivalently $\lim_{|\mathsf{z}|\to\infty} \mathcal{F}_\mathbf{a}(\mathsf{z}) = \infty$).

(a) Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$. Then $(\mathfrak{A}, \sigma)$ is topologically mixing if and only if no finite defect is essential.

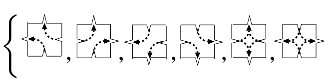
(b) $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ satisfies the *hole-filling property* if no finite defect is essential (i.e. every configuration with a finite defect is 'weakly extensible' in the sense of [**Sch98**, §5]). ♢

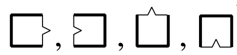
PROPOSITION 1.4. *Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA and let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a Φ-invariant subshift. If $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}$ is bijective, then any essential defect is Φ-persistent.*

In particular, if $\mathfrak{A} \subseteq \mathsf{Fix}[\Phi]$ or $\mathfrak{A} \subseteq \mathsf{Fix}[\Phi^p \circ \sigma^{p\mathsf{v}}]$ (for some $p \in \mathbb{N}$ and $\mathsf{v} \in \mathbb{Z}^D$), *then any essential defect in $\mathbf{a}$ is Φ-persistent.* □[**Piv06**]

1.1. *Codimension:* Our main goal in the present paper is to develop algebraic invariants (as described by question #3 from the introduction) which provide sufficient conditions for the persistence of defects, even when Φ is not bijective. To do this, we must first assign a 'codimension' to defects, but in a somewhat indirect fashion. Strictly speaking, the defect set $\mathbb{D}(\mathbf{a}) \subset \mathbb{Z}^D$ is discrete, hence of codimension D in $\mathbb{R}^D$. We could 'thicken' $\mathbb{D}$ by replacing each point $\mathsf{d} \in \mathbb{D}$ with a unit cube around d. However, the cellular automaton Φ, the subshift $\mathfrak{A}$, and other gadgets we require (e.g. eigenfunctions, cocycles) may have interaction ranges greater than one (and possibly unbounded), so a unit cube isn't big enough. Furthermore, the action of Φ may locally change the geometry of the defect, and we are mainly interested in properties that are invariant under such change (as in the definition of 'essential' defects, above). Loosely speaking, we will use the word 'projective' to describe 'large scale' geometric properties which remain visible when seen from 'far away' (precise definitions will appear below).

For any $r > 0$ and $\mathbf{a} \in \widetilde{\mathfrak{A}}$, we say $\mathbf{a}$ has a *range r domain boundary* (or a *range r codimension-one defect*) if $\mathbb{G}_r(\mathbf{a})$ is trail-disconnected. Domain boundaries divide $\mathbb{Z}^D$ into different 'domains', which may correspond to different transitive components of $\mathfrak{A}$ [**Piv06**, §2], different eigenfunction phases [**Piv06**, §3], or different cocycle asymptotics (§2.3). A connected component $\mathbb{Y}$ of $\mathbb{G}_r(\mathbf{a})$ is called *projective* if $\mathbb{Y} \cap \mathbb{G}_R(\mathbf{a}) \neq \emptyset$ for all $R \geq r$. (This implies that for any $R \geq 0$, there exists $\mathsf{y} \in \mathbb{Y}$ with $\mathbb{B}(\mathsf{y}, R) \subset \mathbb{Y}$. If $\mathfrak{A}$ is of finite type, then the two conditions are equivalent.) We say that $\mathbf{a}$ has a *projective domain boundary* (or a *projective codimension-one defect*) if there is some $R \geq 0$ such that $\mathbb{G}_R(\mathbf{a})$ has at least two projective components. (Hence $\mathbb{G}_r(\mathbf{a})$ is disconnected for all $r \geq R$.)

Example 1.5: (a) (*Square ice*) Let $\mathcal{I} = \Big\{$  $\Big\}$, and let $\mathfrak{Ice} \subset \mathcal{I}^{\mathbb{Z}^2}$ be the Wang subshift defined by the obvious edge-matching conditions. Figure 1(A) shows a domain boundary in $\mathfrak{Ice}$. See also Example 2.14(a).

(b) (*Domino Tiling*) Let $\mathcal{D} := \Big\{$  $\Big\}$, and let $\mathfrak{Dom} \subset \mathcal{D}^{\mathbb{Z}^2}$ be the Wang subshift defined by the obvious edge-matching conditions. Figure 1(B,C) shows two domain boundaries in $\mathfrak{Dom}$; see also Example 2.14(b,c).

(c) (*Ice cubes*) Let $\mathcal{Q}$ be the set of twenty 'ball-and-pin' structures in Figure 2(A), and let $\mathfrak{Q} \subset \mathcal{Q}^{\mathbb{Z}^3}$ be the Wang subshift defined by the obvious matching conditions (this is a three-dimensional version of 'square ice'). Figure 2(B) shows a domain boundary in $\mathfrak{Q}$. $\diamondsuit$

For any $\mathsf{z} \in \mathbb{Z}^D$, let $\ulcorner_{\mathsf{z}}\urcorner := \mathsf{z} + [0,1]^D$; hence $\ulcorner_{\mathsf{z}}\urcorner$ is a unit cube with one corner at z, and the other corners at adjacent points in $\mathbb{Z}^D$. Let $\mathbb{K}_{\mathsf{z}} \subset \mathbb{Z}^D$ be the set of corner points of $\ulcorner_{\mathsf{z}}\urcorner$. We adopt the following notational convention: if $\mathbb{Y} \subset \mathbb{Z}^D$ is any subset, then let $\mathbf{Y}$ be the minimal closed subset of $\mathbb{R}^D$ containing $\mathbb{Y}$ and all unit cubes whose corners are in $\mathbb{Y}$. Formally:

$$\mathbf{Y} \quad := \bigcup_{\mathsf{z} \in \mathbb{Z}^D \ \& \ \mathbb{K}_{\mathsf{z}} \subset \mathbb{Y}} \ulcorner_{\mathsf{z}}\urcorner$$

It follows that $\mathbb{Y}$ is trail-connected iff $\mathbf{Y}$ is path-connected. In this case, for any $k \in [2...D]$, we define the *kth homotopy group* $\pi_k(\mathbb{Y}, \mathsf{y}) := \pi_k(\mathbf{Y}, \mathsf{y})$, for some fixed basepoint $\mathsf{y} \in \mathbb{Y}$ (different choices of y yield isomorphic groups); see [**Hat02**, §4.1]. If $\mathbf{a} \in \widetilde{\mathfrak{A}}$, and $r > 0$, then $\mathbf{a}$ has a *range r codimension-k* defect if $\pi_{k-1}(\mathbb{G}_r(\mathbf{a}), \mathsf{y})$ is nontrivial for some $\mathsf{y} \in \mathbb{G}_r(\mathbf{a})$. If

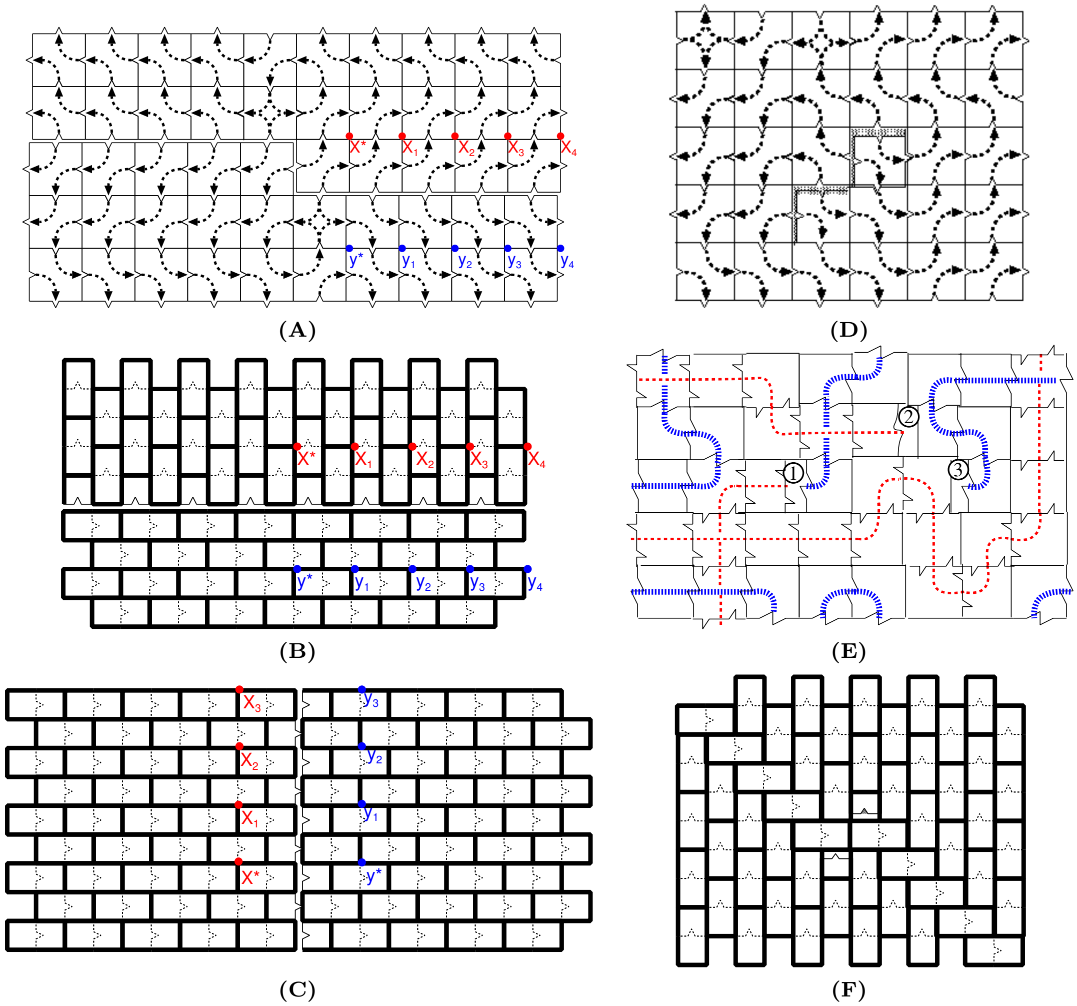


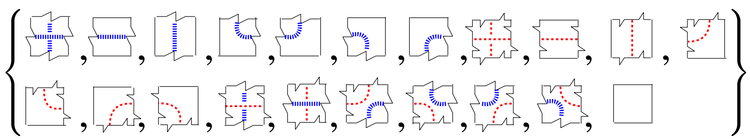
FIGURE 1. **(A)** A gap in $\mathfrak{Ice}$; see Examples 1.5(a) and 2.14(a). **(B,C)** Gaps in $\mathfrak{Dom}$; see Examples 1.5(b) and 2.14(b,c). **(D)** A 'pole' in $\mathfrak{Ice}$; see Examples 1.6(a) and 2.9(a). **(E)** Three 'poles' in $\mathfrak{Pth}$; see Examples 1.6(d) and 2.9(b). **(F)** A non-pole in $\mathfrak{Dom}$; see Examples 1.6(b) and 2.9(c).

$\mathbb{G}_r(\mathbf{a})$ is disconnected (e.g. by a domain boundary) then different connected components may have different homotopy groups; we only require one of these to be nontrivial.

EXAMPLE 1.6: (a) Let $\mathfrak{Ice}$ be as in Example 1.5(a). Then Figure 1(D) shows a codimension-two defect in $\mathfrak{Ice}$. See also Example 2.9(a).

(b) Let $\mathfrak{Dom}$ as in Example 1.5(b). Then Figure 1(F) shows a codimension-two defect in $\mathfrak{Dom}$. See also Example 2.9(c).

(c) Let $\mathfrak{Q}$ be as in Example 1.5(c). Figure 2(C) shows a codimension-two defect in $\mathfrak{Q}$, and Figure 2(D) shows a codimension-three defect.

$\mathcal{P} :=$  .

(d) (*Two-coloured, undirected, crossing path tiling*) Let $\mathcal{P}$ be the set of 21 tiles shown above, and let $\mathfrak{Pth} \subset \mathcal{P}^{\mathbb{Z}^2}$ be the Wang subshift defined by the obvious edge-matching conditions. Then $\mathfrak{Pth}$-admissible configurations are tangles of undirected, freely crossing paths in two colours [**Ein01**, §3]. Figure 1(E) shows three codimension-two defects in $\mathfrak{Pth}$. See also Example 2.9(b). $\diamondsuit$

1.2. *Proper homotopy and projective codimension:* Let $\mathbf{X}$ be a topological space and let $x \in \mathbf{X}$. Let $\mathbb{S}^k \subset \mathbb{R}^{k+1}$ be the unit k-sphere, and let $s \in \mathbb{S}^k$ be some distinguished point. We write $\alpha : (\mathbb{S}^k, s) \longrightarrow (\mathbf{X}, x)$ to mean α is a continuous function from $\mathbb{S}^k$ into $\mathbf{X}$ and $f(s) = x$. If $\alpha, \beta : (\mathbb{S}^k, s) \longrightarrow (\mathbf{X}, x)$ then we write $\alpha \approx \beta$ to mean that α is homotopic to β in a manner which always maps s to x; we call this a ***basepoint-fixing homotopy*** (where x is the ***basepoint***). We then use $\underline{\alpha}$ to refer to the (basepoint-fixing) homotopy class of α.

If $\beta : [0,1] \longrightarrow \mathbf{X}$, then $\overleftarrow{\beta} : [0,1] \longrightarrow \mathbf{X}$ is defined by $\overleftarrow{\beta}(t) = \beta(1-t)$. If $\alpha : [0,1] \longrightarrow \mathbf{X}$, and $\beta(0) = \alpha(1)$, then let $\alpha \star \beta : [0,1] \longrightarrow \mathbf{X}$ be the concatenation of α and β (i.e. $\alpha \star \beta(t) := \alpha(2t)$ if $t \in \left[0, \frac{1}{2}\right]$ and $\alpha \star \beta(t) := \beta(2t-1)$ if $t \in \left[\frac{1}{2}, 1\right]$). Thus, $\pi_1(\mathbf{X}, x)$ is the group of all homotopy classes of loops $\alpha : [0,1] \longrightarrow \mathbf{X}$ with $\alpha(0) = x = \alpha(1)$, with operation $\underline{\alpha} \cdot \underline{\beta} := \underline{\alpha \star \beta}$ [**Hat02**, §1.1]. We can also treat the elements of $\pi_1(\mathbf{X}, x)$ as homotopy classes of functions $\alpha : (\mathbb{S}^1, s) \longrightarrow (\mathbf{X}, x)$. By generalizing this construction, we can define an abelian group $\pi_k(\mathbf{X}, x)$ of homotopy classes of functions $\alpha : (\mathbb{S}^k, s) \longrightarrow (\mathbf{X}, x)$. See [**Hat02**, §4.1] for details.

Let $x, y \in \mathbf{X}$. If $\beta : [0,1] \longrightarrow \mathbf{X}$ is any path with with $\beta(0) = x$ and $\beta(1) = y$, then β yields an isomorphism $\beta_* : \pi_1(\mathbf{X}, x) \longrightarrow \pi_1(\mathbf{X}, y)$ by $\beta_*(\underline{\alpha}) := \underline{\beta \star \alpha \star \overleftarrow{\beta}}$. We can likewise use β to define isomorphisms $\beta_* : \pi_k(\mathbf{X}, x) \longrightarrow \pi_k(\mathbf{X}, y)$ for all $k \geq 2$. If $\gamma : [0,1] \longrightarrow \mathbf{X}$ is another path from x to y, and $\gamma \approx \beta$, then $\gamma_* = \beta_*$. However, if γ is *not* homotopic to β, then γ_* and β_* may be different. Hence, although $\pi_k(\mathbf{X}, x) \cong \pi_k(\mathbf{X}, y)$, this isomorphism is not 'canonical'.

Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift and let $\mathbf{a} \in \widetilde{\mathfrak{A}}$, and suppose, for some $r_0 \in \mathbb{N}$, that $\mathbb{G}_{r_0}(\mathbf{a})$ contains a unique projective connected component $\mathbb{Y}$ (it is necessary, but not sufficient, to assume that $\mathbf{a}$ has no projective domain boundaries). Thus, for all $r > r_0$, $\mathbb{Y}_r := \mathbb{Y} \cap \mathbb{G}_r(\mathbf{a})$ is the unique projective component of $\mathbb{G}_r(\mathbf{a})$. A ***proper base ray*** is a continuous path $\omega : [0, \infty) \longrightarrow \mathbf{Y}_{r_0}$ with $\lim_{t \to \infty} |\omega(t)| = \infty$. For each $r > r_0$, we define $\pi_k(\mathbb{Y}_r, \omega) := \pi_k(\mathbb{Y}_r, \mathsf{y})$, where $\mathsf{y} \in \mathbb{Y}_r \cap \omega[0, \infty)$ is any point. This definition is independent of the choice of y in the following sense: if $\mathsf{y}' \in \mathbb{Y}_r \cap \omega[0, \infty)$ is another point, then there is a canonical isomorphism $\pi_k(\mathbb{Y}_r, \mathsf{y}) \cong \pi_k(\mathbb{Y}_r, \mathsf{y}')$ given by the segment of ω between y and y'.

Recall that $\mathbb{Y}_{r+1} \subset \mathbb{Y}_r$; the inclusion map $\iota_r : \mathbb{Y}_{r+1} \hookrightarrow \mathbb{Y}_r$ yields a (canonical) homomorphism $\iota_r^* : \pi_k(\mathbb{Y}_{r+1}, \omega) \longrightarrow \pi_k(\mathbb{Y}_r, \omega)$. We define the ***$k$th proper homotopy group*** to be the inverse limit:

$$\pi_k(\mathbb{G}_\infty(\mathbf{a}), \omega) \quad := \quad \varprojlim \left(\pi_k(\mathbb{Y}_1, \omega) \xleftarrow{\iota_1^*} \pi_k(\mathbb{Y}_2, \omega) \xleftarrow{\iota_2^*} \pi_k(\mathbb{Y}_3, \omega) \xleftarrow{\iota_3^*} \cdots \right) \qquad (1)$$

(Of course "$\pi_k(\mathbb{G}_\infty(\mathbf{a}))$" is an abuse of notation, because technically, $\mathbb{G}_\infty(\mathbf{a}) = \emptyset$. See [**Hat02**, §3.F] or [**Lan84**, §III.9] for background on inverse limits. The group $\pi_k(\mathbb{G}_\infty(\mathbf{a}), \omega)$

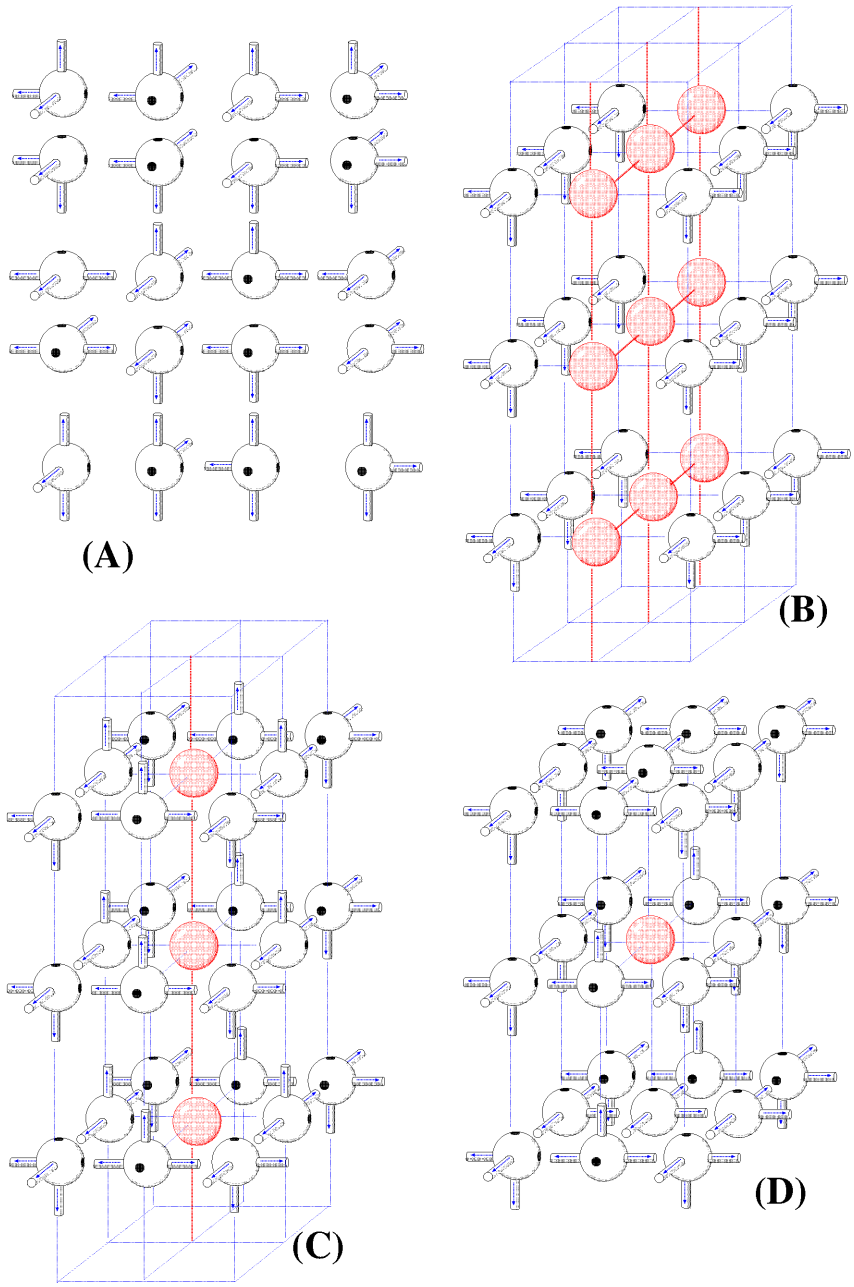


FIGURE 2. **(A)** The twenty tiles of the 'ice cube' shift $\mathfrak{Q} \subset \mathcal{Q}^{\mathbb{Z}^3}$ from Examples 1.5(c) and 1.6(c). **(B)** A domain boundary in $\mathfrak{Q}$; we assume that the same pattern is continued up, down, north, and south. (We have 'stretched out' the configuration for visibility. The balls in the defect region are shaded, but left unspecified, because they don't matter.) **(C)** A codimension-two defect in $\mathfrak{Q}$; we assume that the same pattern is continued upwards and downwards. **(D)** A codimension-three ('pole') defect in $\mathfrak{Q}$; see also Example 4.8(b).

is analogous (but not identical) to the proper homotopy group of a noncompact topological space; see [**Bro74**], [**BT74**] or [**Pes90**, §2].) We say that $\mathbf{a}$ has *projective codimension* $(k+1)$ *defect* if $\pi_k[\mathbb{G}_\infty(\mathbf{a}), \omega]$ is nontrivial (it follows that $\pi_k[\mathbb{G}_r(\mathbf{a}), \omega]$ is nontrivial for all large enough $r \in \mathbb{N}$). Heuristically, elements of $\pi_k(\mathbb{G}_\infty(\mathbf{a}))$ are homotopy classes of 'extremely large' k-sphere embeddings in the unflawed part of $\mathbf{a}$. Technically, this definition depends upon the homotopy class of the proper base ray ω; different rays may yield nonisomorphic groups.

2. *Cohomological Defects*

The main results of this section are Theorems 2.8 and 2.15 and Proposition 2.11.

2.1. *Dynamical Cocycles*

Let $\mathfrak{A} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ be a subshift, and let $(\mathcal{G}, \cdot)$ be a topological group (usually discrete). A $\mathcal{G}$-valued continuous (*dynamical*) *cocycle* for $\mathfrak{A}$ is a continuous function $C : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ satisfying the *cocycle equation*

$$C(\mathsf{y}+\mathsf{z}, \mathbf{a}) \quad = \quad C(\mathsf{y}, \sigma^{\mathsf{z}}(\mathbf{a})) \cdot C(\mathsf{z}, \mathbf{a}), \qquad \forall\, \mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D} \text{ and } \forall\, \mathsf{y}, \mathsf{z} \in \mathbb{Z}^D. \tag{2}$$

Example 2.1: (a) If $b : \mathfrak{A} \longrightarrow \mathcal{G}$ is any continuous function, then the function $C(\mathsf{z}, \mathbf{a}) := b(\sigma^{\mathsf{z}}(\mathbf{a})) \cdot b(\mathbf{a})^{-1}$ is a cocycle, and is called a *coboundary* with *cobounding function* b.

(b) If $h : \mathbb{Z}^D \longrightarrow \mathcal{G}$ is a homomorphism, then the function $C(\mathsf{z}, \mathbf{a}) := h(\mathsf{z})$ is a cocycle. Conversely, if C is any cocycle such that $C(\mathsf{z}, \_)$ is constant for all $\mathsf{z} \in \mathbb{Z}^D$, then C arises from a homomorphism in this manner. In particular, if $e_{\mathcal{G}} \in \mathcal{G}$ is the identity, then the constant function $C_e(\mathsf{z}, \mathbf{a}) \equiv e_{\mathcal{G}}$ is a cocycle.

(c) Let $\mathfrak{Ice} \subset \mathcal{I}^{\mathbb{Z}^2}$ be as in Example 1.5(a). Define $c_1, c_2 : \mathcal{I} \longrightarrow \{\pm 1\}$ by $c_1(\;) := +1 =: c_2(\;)$ and $c_1(\;) := -1 =: c_2(\;)$ ('$*$' means 'anything'). We define cocycle $C : \mathbb{Z}^2 \times \mathfrak{Ice} \longrightarrow \mathbb{Z}$ as follows: If $\mathbf{i} \in \mathfrak{Ice}$ and $\mathsf{z} = (z_1, z_2) \in \mathbb{Z}^2$, then

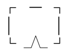
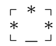
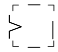
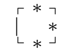
$$C(\mathsf{z}, \mathbf{i}) \quad := \quad \sum_{x=0}^{z_1 - 1} c_1(i_{x,0}) + \sum_{y=0}^{z_2 - 1} c_2(i_{z_1,y}). \tag{3}$$

(d) Let $\mathfrak{Pth} \subset \mathcal{P}^{\mathbb{Z}^2}$ be as in Example 1.6(d). Define $c_1, c_2 : \mathcal{I} \longrightarrow \mathbb{Z}_{/2} \oplus \mathbb{Z}_{/2}$ by $c_1(\;) := (1,0) =: c_2(\;)$ and $c_1(\;) := (0,1) =: c_2(\;)$ (where '$*$' means 'anything'), and extend this to a cocycle $C : \mathbb{Z}^2 \times \mathfrak{Pth} \longrightarrow \mathbb{Z}_{/2} \oplus \mathbb{Z}_{/2}$ exactly as in eqn.(3).

(e) More generally, a *height function* is any integer-valued† cocycle $H : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathbb{Z}$ defined by functions $h_1, \ldots, h_D : \mathcal{A} \longrightarrow \mathbb{Z}$ via the obvious generalization of eqn.(3). Height functions appear in tiling systems like dominos [**CEP96, CKP01**] and 'ice' tiles [**Elo99, Elo03, Elo05**], and in many lattice models of statistical physics [**Bax89**].

(f) Let $\mathfrak{Dom} \subset \mathcal{D}^{\mathbb{Z}^2}$ be as in Example 1.5(b). Let $\mathcal{G} := \mathbb{Z}_{/2} * \mathbb{Z}_{/2}$ be the group of finite products like $vhvhv \cdots vhv$, where v and h are noncommuting generators with $v^2 = e = h^2$. Define $c_1, c_2 : \mathcal{I} \longrightarrow \mathcal{G}$ by

† Sometimes H maps into $\mathbb{Z}^n$ [**She02**] or other groups [**KK93**], and the 'height' metaphor is somewhat strained.

$c_1($  $) := vhv$; $c_1($  $) := h$; $c_2($  $) := hvh$; and $c_2($  $) := v$.

and extend this to a cocycle $C : \mathbb{Z}^2 \times \mathfrak{Dom} \longrightarrow \mathcal{G}$ through the multiplicative analogy of eqn.(3).

(g) If $\mathcal{X}$ is a topological space, then an *$\mathcal{X}$-extension* of $\mathfrak{A}$ is a continuous $\mathbb{Z}^D$-action $\Xi : \mathbb{Z}^D \times \mathcal{X} \times \mathfrak{A} \longrightarrow \mathcal{X} \times \mathfrak{A}$ such that $(\mathfrak{A}, \sigma)$ is a factor of $(\mathcal{X} \times \mathfrak{A}, \Xi)$ via the projection $\pi_{\mathfrak{A}} : \mathcal{X} \times \mathfrak{A} \twoheadrightarrow \mathfrak{A}$. Let $\mathcal{G} := \mathrm{Homeo}(\mathcal{X})$ be the self-homeomorphism group of $\mathcal{X}$, topologized as a subspace of the Tychonoff product $\mathcal{X}^{\mathcal{X}}$ (e.g. if $\mathcal{X} := [1...n]$, then $\mathcal{G} = \mathbf{S}_n$ is a (discrete) permutation group; this is called an *n-point extension*). For each $\mathbf{a} \in \mathfrak{A}$ and $\mathsf{z} \in \mathbb{Z}^D$, let $c(\mathsf{z}, \mathbf{a}) := \pi_{\mathcal{X}} \circ \Xi^{\mathsf{z}}(\_, \mathbf{a}) : \mathcal{X} \longrightarrow \mathcal{X}$. Then $c : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ is a continuous cocycle. Conversely, any continuous cocycle $c : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ defines an $\mathcal{X}$-extension of $\mathfrak{A}$ in the obvious way; see e.g. [**Zim76, Zim77, Zim80, Kam90, Kam92, Kam93**]. $\diamondsuit$

Two continuous cocycles C and C' are *cohomologous* ($C \approx C'$) if there is a continuous *transfer function* $b : \mathfrak{A} \longrightarrow \mathcal{G}$ such that $C'(\mathsf{z}, \mathbf{a}) = b(\sigma^{\mathsf{z}}(\mathbf{a})) \cdot C(\mathsf{z}, \mathbf{a}) \cdot b(\mathbf{a})^{-1}$, for all $\mathsf{z} \in \mathbb{Z}^D$ and $\mathbf{a} \in \mathfrak{A}$. A cocycle C is *trivial* if C is cohomologous to a homomorphism. We will use $\underline{C}$ to denote the cohomology equivalence class of the cocycle C.

Example 2.2: (a) Any coboundary [Example 2.1(a)] is trivial, because it is cohomologous to the homomorphism C_e [Example 2.1(b)].

(b) Fix $\mathsf{y} \in \mathbb{Z}^D$ and define cocycle $C'(\mathsf{z}, \mathbf{a}) := C(\mathsf{z}, \sigma^{\mathsf{y}}(\mathbf{a}))$. Then $C \approx C'$ via the transfer function $b(\mathbf{a}) := C(\mathsf{y}, \mathbf{a})$.

(c) Let $\Xi, \Xi' : \mathbb{Z}^D \times \mathcal{X} \times \mathfrak{A} \longrightarrow \mathcal{X} \times \mathfrak{A}$ be two $\mathcal{X}$-extensions of $\mathfrak{A}$, with cocycles $C, C' : \mathbb{Z}^D \times \mathfrak{X} \longrightarrow \mathcal{G} := \mathrm{Homeo}(\mathcal{X})$ as in Example 2.1(g). Then $C \approx C'$ via the continuous transfer function $b : \mathfrak{A} \longrightarrow \mathcal{G}$ iff the systems $(\mathcal{X} \times \mathfrak{A}, \Xi)$ and $(\mathcal{X} \times \mathfrak{A}, \Xi')$ are conjugate via the function $(x, \mathbf{a}) \mapsto (b(\mathbf{a})(x), \mathbf{a})$. Also, C is a homomorphism iff there is a continuous $\mathbb{Z}^D$-action ξ on $\mathcal{X}$ such that $(\mathcal{X} \times \mathfrak{A}, \Xi) = (\mathcal{X}, \xi) \times (\mathfrak{A}, \sigma)$ [i.e. $C(\mathsf{z}, \mathbf{a}) = \xi^{\mathsf{z}}$, for all $\mathbf{a} \in \mathfrak{A}$]. Hence, C is trivial iff $(\mathcal{X} \times \mathfrak{A}, \Xi)$ is isomorphic to such a Cartesian product. $\diamondsuit$

If $(\mathcal{G}, \cdot)$ is an abelian group, then the set $\mathcal{Z} = \mathcal{Z}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$ of all $\mathcal{G}$-valued continuous cocycles is a group under pointwise multiplication. The set of trivial cocycles is a subgroup $\mathcal{B} = \mathcal{B}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$. The quotient group $\mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G}) := \mathcal{Z}/\mathcal{B}$ is the (first dynamical) *cohomology group* of $\mathfrak{A}$ (with *coefficients* in $\mathcal{G}$). If $\mathcal{G}$ is not abelian, then $\mathcal{Z}$ is not a group, but we still use $\mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$ to denote the set of cohomology equivalence classes of cocycles in $\mathcal{Z}$. (Sadly, some SFTs (e.g. dominoes) admit nontrivial cocycles only in nonabelian groups [**Sch98**, Thm.6.6].) The cohomology of multidimensional SFTs is closely related [**Sch98**, Thm.4.2(b)] to tiling homotopy groups (see Example 3.2). Nontrivial cocycles represent an algebraic obstruction to the 'hole-filling problem'. For example, in the full shift $\mathcal{A}^{\mathbb{Z}^D}$, the hole-filling problem is trivial, and indeed, $\mathcal{H}^1_{\mathrm{dy}}(\mathcal{A}^{\mathbb{Z}^D}, \mathbf{S}_n)$ is trivial [**Kam90, Kam92, Kam93**]. More generally, if $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ has certain mixing properties, then $\mathcal{H}^1_{\mathrm{dy}}(\mathcal{A}^{\mathbb{Z}^D}, \mathcal{G})$ is trivial [**Sch95**, Thm.3.2, Cor.3.3-3.4].

Cocycles along trails: Let $\mathbb{E} := \left\{\mathsf{z} \in \mathbb{Z}^D \; ; \; \mathsf{z} = (0, ..., 0, \pm 1, 0, ..., 0)\right\}$. Recall that a sequence $\zeta = (\mathsf{z}_0, \mathsf{z}_1, \ldots, \mathsf{z}_N) \subset \mathbb{Z}^D$ is a *trail* if $\mathsf{z}'_n \in \mathbb{E}$ for all $n \in [1...N]$, where $\mathsf{z}'_n := \mathsf{z}_n - \mathsf{z}_{n-1}$. Let $r > 0$ and let $c : \mathbb{E} \times \mathfrak{A}_{(r)} \longrightarrow \mathcal{G}$ be some function. We define

$$c(\zeta, \mathbf{a}) \quad := \quad \prod_{n=1}^{N} c(\mathsf{z}'_n, \mathbf{a}_{\mathbb{B}(\mathsf{z}_{n-1}, r)}). \tag{4}$$

Suppose that, for all $\mathsf{e}, \mathsf{e}' \in \mathbb{E}$, and $\mathbf{a} \in \mathfrak{A}$,

$$\begin{array}{lrcl} (\mathbf{a}) & c(\mathsf{e}', \mathbf{a}_{\mathbb{B}(\mathsf{e},r)}) \cdot c(\mathsf{e}, \mathbf{a}_{\mathbb{B}(r)}) & = & c(\mathsf{e}, \mathbf{a}_{\mathbb{B}(\mathsf{e}',r)}) \cdot c(\mathsf{e}', \mathbf{a}_{\mathbb{B}(r)}). \\ (\mathbf{b}) & c(-\mathsf{e}, \mathbf{a}_{\mathbb{B}(\mathsf{e},r)}) & = & c(\mathsf{e}, \mathbf{a}_{\mathbb{B}(r)})^{-1}. \end{array} \tag{5}$$

Then the value of eqn.(4) depends only on z_0 and z_N, and is independent of the particular trail ζ from z_0 to z_N. In particular, if ζ is any *closed trail* (i.e. $\mathsf{z}_N = 0 = \mathsf{z}_0$) then $c(\zeta, \mathbf{a}) = C(0, \mathbf{a}) = e_{\mathcal{G}}$. For any $\mathbf{a} \in \mathfrak{A}$ and $\mathsf{z} \in \mathbb{Z}^D$, we define $C(\mathsf{z}, \mathbf{a}) := c(\zeta, \mathbf{a})$, where ζ is any trail from 0 to z. The resulting function $C : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ is a continuous cocycle; we say that C is a *locally determined* cocycle with *local rule* c of *radius* r. If $\mathcal{G}$ is discrete, then every continuous $\mathcal{G}$-valued cocycle is locally determined in this way. For instance, the cocycles in Examples 2.1(c,d,e,f) had radius $r = 0$, so that $\mathfrak{A}_{(0)} = \mathcal{A}$ and the local rule was a function $c : \mathbb{E} \times \mathcal{A} \longrightarrow \mathcal{G}$.

Example 2.3: (a) Let $C : \mathbb{Z}^2 \times \mathfrak{Ice} \longrightarrow \mathbb{Z}$ be as in Example 2.1(c). Any $\mathbf{i} \in \mathfrak{Ice}$ defines a set of directed 'paths' through the plane, each without beginning or end. If ζ is a trail from y to z in $\mathbb{Z}^2$, then $C(\zeta, \mathbf{i}) = \#\{$paths which cut across ζ going left$\} - \#\{$paths which cut across ζ going right$\}$. In particular, if ζ is the counterclockwise boundary of a region $\mathbb{U} \subset \mathbb{Z}^2$, then $C(\zeta, \mathbf{i}) = \#\{$paths entering $\mathbb{U}\} - \#\{$paths leaving $\mathbb{U}\} = 0$ (because every path which enters $\mathbb{U}$ must leave).

(b) Let $C : \mathbb{Z}^2 \times \mathfrak{Pth} \longrightarrow (\mathbb{Z}_{/2})^2$ be as in Example 2.1(d). Any $\mathbf{p} \in \mathfrak{Pth}$ defines a set of undirected paths in two colours, say 'blue' and 'red'. If ζ is a trail from y to z in $\mathbb{Z}^2$, then $C(\zeta, \mathbf{p}) = (b, r) \in (\mathbb{Z}_{/2})^2$, where b is the parity of blue paths crossing ζ, and r is the parity of red paths. ♢

If C_1 and C_2 are have local rules $c_1, c_2 : \mathbb{E} \times \mathfrak{A}_{(R)} \longrightarrow \mathcal{G}$, then $C_1 \approx C_2$ iff there is some *local transfer function* $b : \mathfrak{A}_{(r)} \longrightarrow \mathcal{G}$ (for some $r \leq R - 1$) such that:

$$\text{For any } \mathsf{e} \in \mathbb{E} \text{ and } \mathbf{a} \in \mathfrak{A}_{(R)}, \quad c_2(\mathsf{e}, \mathbf{a}) = b(\mathbf{a}_{\mathbb{B}(\mathsf{e},r)}) \cdot c_1(\mathsf{e}, \mathbf{a}) \cdot b(\mathbf{a}_{\mathbb{B}(r)})^{-1}. \tag{6}$$

Fundamental cocycles: Fix a cocycle $C^* : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$. If $\psi : (\mathcal{G}, \cdot) \longrightarrow (\mathcal{H}, \cdot)$ is any group homomorphism, then $\psi \circ C^*$ is also a cocycle. The cocycle C^* is called *fundamental* [**Sch98**] if, for any group $(\mathcal{H}, \cdot)$ and any cocycle $C \in \mathcal{Z}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{H})$, there is a homomorphism $\psi : \mathcal{G} \longrightarrow \mathcal{H}$ such that C is cohomologous to $\psi \circ C^*$. It is unknown whether every multidimensional subshift possesses a fundamental cocycle, but fundamental cocycles have been identified for many specific $\mathbb{Z}^2$-shifts, including dominoes [**Sch98**, Thm.6.7], rectangular polyominoes [**Ein01**, Thm.2.7], L-shaped triominoes [**Ein01**, Thm.4.8], three-coloured chessboards [**Sch98**, Thm.7.1], lozenge tilings [**Sch98**, Thm.9.1], coloured path systems [**Ein01**, Thm.3.3], and certain factors of cohomologically trivial subshifts [**Sch98**, Thm.11.1]. If a fundamental cocycle exists, then it encodes essentially the same information [**Sch98**, Thm.5.5] as the projective fundamental group of [**GP95**] (see §3.5).

Example 2.4: If $C : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ is any cocycle, then the $\mathbb{Z}^D$-*extension* of C is the cocycle $C' : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathbb{Z}^D \times \mathcal{G}$ defined by $C'(\mathsf{z}; \mathbf{a}) := (\mathsf{z}, C(\mathsf{z}, \mathbf{a}))$, for any $\mathsf{z} \in \mathbb{Z}^D$ and $\mathbf{a} \in \mathfrak{A}$.

(a) The $\mathbb{Z}^D$-extension of Example 2.1(c) is a fundamental cocycle for $\mathfrak{Ice}$ [**Sch98**, Thm.8.1].

(b) The $\mathbb{Z}^D$-extension of Example 2.1(d) is a fundamental cocycle for $\mathfrak{Pth}$ [**Ein01**, Thm.3.3].

(c) The $\mathbb{Z}^D$-extension of Example 2.1(f) is a fundamental cocycle for $\mathfrak{Dom}$ [**Ein01**, Thm.2.7]. ♢

CA vs. Cocycles: The following can be checked through straightforward calculation:

PROPOSITION 2.5. *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ and $\mathfrak{B} \subset \mathcal{B}^{\mathbb{Z}^D}$ be subshifts. Let $\Phi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a subshift homomorphism.*

(a) *Suppose $C : \mathbb{Z}^D \times \mathfrak{B} \longrightarrow \mathcal{G}$ is cocycle on $\mathfrak{B}$, and we define $\Phi_* C : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ by $\Phi_* C(\mathsf{z}, \mathbf{a}) = C(\mathsf{z}, \Phi(\mathbf{a}))$. Then $\Phi_* C$ is a cocycle on $\mathfrak{A}$.*
If Φ has radius R, and C is locally determined with radius r, then $\Phi_* C$ is locally determined with radius $r + R$.

(b) *Let $C, C' \in \mathcal{Z}^1_{\mathrm{dy}}(\mathfrak{B}, \mathcal{G})$. If $C \approx C'$, then $\Phi^* C \approx \Phi^* C'$. Thus, Φ induces a function $\Phi_* : \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{B}, \mathcal{G}) \longrightarrow \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$.*

(c) *If $(\mathcal{G}, \cdot)$ is abelian, then $\Phi_* : \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{B}, \mathcal{G}) \longrightarrow \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$ is a group homomorphism.* □

In particular, if $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ is a cellular automaton, and $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$, then Proposition 2.5(c) yields a group endomorphism $\Phi_* : \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$. [For instance, if $\mathsf{y} \in \mathbb{Z}^D$, then $\sigma^{\mathsf{y}}_* : \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$ is the identity, by Example 2.2(b).] If $\mathfrak{A}$ has an abelian fundamental cocycle, then this cohomological endomorphism Φ_* takes a simple form. To see this, let $\mathsf{End}\,(\mathcal{G})$ be the set of endomorphisms of $\mathcal{G}$. If $(\mathcal{G}, \cdot)$ is an abelian group, then $\mathsf{End}\,(\mathcal{G})$ is an abelian group under pointwise multiplication.

PROPOSITION 2.6. *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ have abelian fundamental cocycle $C^* \in \mathcal{Z}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G}^*)$.*

(a) *There is a group epimorphism $\mathsf{End}\,(\mathcal{G}^*) \ni \epsilon \mapsto C_\epsilon \in \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G}^*)$, defined by $C_\epsilon := \epsilon \circ C^*$.*

(b) *Let $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}$ be a cellular automaton. Then there is some $\varphi \in \mathsf{End}\,(\mathcal{G}^*)$ such that $\Phi_*(C_\epsilon) \approx C_{\epsilon \circ \varphi}$ for all $\epsilon \in \mathsf{End}\,(\mathcal{G}^*)$.*

Proof: **(a)** For any $\epsilon \in \mathsf{End}\,(\mathcal{G}^*)$, the function $C_\epsilon = \epsilon \circ C^*$ is a cocycle. The map $(\epsilon \mapsto C_\epsilon)$ is a group homomorphism because $\mathcal{G}$-multiplication is commutative. The map $(\epsilon \mapsto C_\epsilon)$ is surjective onto $\mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G}^*)$ because C^* is fundamental.

(b) $\Phi_* C^*$ is a cocycle, so there is some $\varphi \in \mathsf{End}\,(\mathcal{G}^*)$ so that $\Phi_* C^* \approx \varphi \circ C^*$ (because C^* is fundamental). For any $\epsilon \in \mathsf{End}\,(\mathcal{G}^*)$, note that $\Phi_*(C_\epsilon) = \epsilon \circ (\Phi_* C^*)$, because, for any $\mathsf{z} \in \mathbb{Z}^D$ and $\mathbf{a} \in \mathfrak{A}$, $\Phi_*(C_\epsilon)(\mathsf{z}, \mathbf{a}) = (C_\epsilon)(\mathsf{z}, \Phi(\mathbf{a})) = \epsilon \circ C^*(\mathsf{z}, \Phi(\mathbf{a})) = \epsilon \circ (\Phi_* C^*)(\mathsf{z}, \mathbf{a})$. But then $\epsilon \circ (\Phi_* C^*) \underset{(*)}{\approx} \epsilon \circ (\varphi \circ C^*) \underset{(\dagger)}{=} C_{\epsilon \circ \varphi}$. Here, $(*)$ is because $\Phi_* C^* \approx \varphi \circ C^*$ (say with transfer function b), so $\epsilon \circ (\Phi_* C^*) \approx \epsilon \circ (\varphi \circ C^*)$ (with transfer function $\epsilon \circ b$). $(\dagger)$ is by definition of $C_{\epsilon \circ \varphi}$. □

Trail homotopy: If $\mathbb{Y} \subset \mathbb{Z}^D$, and $\zeta = (\mathsf{z}_1 \rightsquigarrow \cdots \rightsquigarrow \mathsf{z}_N)$ and $\zeta' = (\mathsf{z}'_1 \rightsquigarrow \cdots \rightsquigarrow \mathsf{z}'_{N'})$ are trails in $\mathbb{Y}$, then ζ' is an *elementary* $\mathbb{Y}$*-homotope* of ζ (notation: $\zeta \underset{\mathbb{Y}}{\sim} \zeta'$) if there is some $n \in [1...N]$ such that $\mathsf{z}'_i = \mathsf{z}_i$ for all $i \in [1...n)$, and one of the following is true:

(EH1) $N' = N$ and $\mathsf{z}'_i = \mathsf{z}_i$ for all $i \in (n...N]$, as follows:

$$\begin{array}{ccccc} \cdots \rightsquigarrow \mathsf{z}'_{n-1} = \mathsf{z}_{n-1} & \rightsquigarrow & \mathsf{z}_n & & \\ \wr & & \wr & & \\ \mathsf{z}'_n & \rightsquigarrow & \mathsf{z}_{n+1} = \mathsf{z}'_{n+1} \rightsquigarrow \cdots & & \end{array}$$

(EH2) $\mathsf{z}_{n+1} = \mathsf{z}_{n-1}$, $N' = N - 1$, and $\mathsf{z}'_{i-1} = \mathsf{z}_i$ for all $i \in (n...N]$, as follows:

$$\begin{array}{ccccccc} & & & \mathsf{z}_n & & & \\ & & \nearrow & & \searrow & & \\ \cdots \rightsquigarrow \mathsf{z}_{n-2} \rightsquigarrow & & \mathsf{z}_{n-1} & = & \mathsf{z}_{n+1} & \rightsquigarrow \mathsf{z}_{n+2} \rightsquigarrow \cdots \\ \| & & \| & & \| & \| \\ \cdots \rightsquigarrow \mathsf{z}'_{n-2} \rightsquigarrow & & \mathsf{z}'_{n-1} & = & \mathsf{z}'_n & \rightsquigarrow \mathsf{z}'_{n+1} \rightsquigarrow \cdots \end{array}$$

(EH3) Same as **(EH2)**, but with ζ and ζ' switched.

Two trails ζ and ζ' are *homotopic in* $\mathbb{Y}$ (notation: $\zeta \underset{\mathbb{Y}}{\approx} \zeta'$) if there is a sequence of elementary $\mathbb{Y}$-homotopes $\zeta = \zeta_0 \underset{\mathbb{Y}}{\approx} \zeta_1 \underset{\mathbb{Y}}{\approx} \cdots \underset{\mathbb{Y}}{\approx} \zeta_M = \zeta'$. This is clearly an equivalence relation. Assume $\mathbb{Y}$ is connected; then every homotopy class of $\pi_1(\mathbb{Y})$ can be represented as a trail in $\mathbb{Y}$, and two such trails are $\mathbb{Y}$-homotopic iff they belong in the same class of $\pi_1(\mathbb{Y})$. Hence we can treat $\pi_1(\mathbb{Y})$ as a group of $\mathbb{Y}$-homotopy classes of $\mathbb{Y}$-trails.

2.2. *Poles and Residues* If $\mathbf{a} \in \widetilde{\mathfrak{A}}$, and $\zeta = (\mathsf{z}_0, \mathsf{z}_1, \ldots, \mathsf{z}_N) \subset \mathbb{G}_r(\mathbf{a})$ is a closed trail in $\mathbb{G}_r(\mathbf{a})$, then we can define $c(\zeta, \mathbf{a})$ as in eqn.(4) (see also [**Sch98**, p.1489]). This yields a natural algebraic invariant for range-r codimension-two defects:

Proposition 2.7. *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift. Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$ have a range r codimension-two defect. Let $C \in \mathcal{Z}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$ be locally determined with radius r. Then:*

(a) *There is a group homomorphism $\mathrm{Res}^r_{\mathbf{a}} C \colon \pi_1[\mathbb{G}_r(\mathbf{a})] \to \mathcal{G}$ defined $\mathrm{Res}^r_{\mathbf{a}} C(\underline{\zeta}) := C(\zeta, \mathbf{a})$.*

(b) *If $(\mathcal{G}, \cdot)$ is abelian, and $C \approx C'$, then $\mathrm{Res}^r_{\mathbf{a}} C \equiv \mathrm{Res}^r_{\mathbf{a}} C'$.*

The corresponding result for projective codimension-two defects is as follows:

Theorem 2.8. *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift, and let $(\mathcal{G}, \cdot)$ be a discrete group. Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$ have a projective codimension-two defect. Let $\mathbb{G}_\infty := \mathbb{G}_\infty(\mathbf{a})$ and let $\omega : [0, \infty) \longrightarrow \mathbb{G}_\infty$ be a proper base ray. Define $\mathrm{Res}_{\mathbf{a}} : \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G}) \times \pi_1(\mathbb{G}_\infty, \omega) \longrightarrow \mathcal{G}$ by $\mathrm{Res}_{\mathbf{a}}(\underline{C}, \underline{\zeta}) := C(\zeta, \mathbf{a})$.*

(a) *If $\mathrm{Res}_{\mathbf{a}}$ is nontrivial, then $\mathbf{a}$ has an essential codimension-two defect.*

(b) *If $(\mathcal{G}, \cdot)$ is abelian, then $\mathrm{Res}_{\mathbf{a}}$ is a group homomorphism.*

If the homomorphism $\mathrm{Res}^r_{\mathbf{a}} C$ in Proposition 2.7 is nontrivial, we say that $\mathbf{a}$ has a C-*pole* (of range r), and $\mathrm{Res}^r_{\mathbf{a}} C$ is called the C-*residue* of $\mathbf{a}$, by analogy with complex analysis. In this analogy, elements of $\mathfrak{A}$ are like entire functions, elements of $\widetilde{\mathfrak{A}}$ with codimension-two defects are like meromorphic functions, and $C(\zeta, \mathbf{a})$ is like a contour integral. If the function $\mathrm{Res}_{\mathbf{a}}$ in Theorem 2.8 is nontrivial, we say that $\mathbf{a}$ has a *(projective)* $\mathcal{G}$-*pole*, and $\mathrm{Res}_{\mathbf{a}}$ is called the $\mathcal{G}$-*residue* of $\mathbf{a}$.

Example 2.9: (a) Let $C : \mathbb{Z}^2 \times \mathfrak{Ice} \longrightarrow \mathbb{Z}$ be as in Example 2.1(c), and let $\mathbf{i} \in \widetilde{\mathfrak{Ice}}$ be the configuration shown in Figure 1(D), having a codimension-two defect. If ζ is any simple, closed clockwise trail around this defect, then $C(\zeta, \mathbf{i}) = 8$. Observe that $\pi_1(\mathbb{G}_\infty(\mathbf{i})) \cong \mathbb{Z}$ is the cyclic group generated by $\underline{\zeta}$. For any $\underline{\zeta}^n \in \pi_1(\mathbb{G}_\infty)$, we have $\mathrm{Res}_{\mathbf{i}}(\underline{C}, \underline{\zeta}^n) = 8n$. Thus, $\mathbf{i}$ has a projective pole, and hence, an essential defect.

(b) Theorem 2.8(a) is false if we replace $\pi_1[\mathbb{G}_\infty(\mathbf{a})]$ with $\pi_1[\mathbb{G}_r(\mathbf{a})]$ for some finite $r > 0$. For example, let $C : \mathbb{Z}^2 \times \mathfrak{Pth} \longrightarrow (\mathbb{Z}_{/2})^2$ be as in Example 2.1(d), and let $\mathbf{p} \in \widetilde{\mathfrak{Pth}}$ be the configuration shown in Figure 1(E), having a codimension-two defect with three components, labelled ①, ②, and ③. For $k = 1, 2, 3$, let ζ_k be a simple clockwise loop going around ⓚ, and not around the other two defects. Then $C(\zeta_1, \mathbf{p}) = (1, 1)$, $C(\zeta_2, \mathbf{p}) = (0, 1)$, and $C(\zeta_3, \mathbf{p}) = (1, 0)$. Hence, each of the defects ①, ②, and ③ individually is a nontrivial range-1 pole. However, $\pi_1(\mathbb{G}_\infty(\mathbf{p})) \cong \mathbb{Z}$ is the cyclic group generated by a simple closed curve ζ that goes around all three defects, and $\mathrm{Res}_{\mathbf{p}}(\underline{C}, \underline{\zeta}) = C(\zeta, \mathbf{p}) = C(\zeta_1, \mathbf{p}) + C(\zeta_2, \mathbf{p}) + C(\zeta_3, \mathbf{p}) = (0, 0)$. Hence, $\mathrm{Res}_{\mathbf{p}}(\underline{C}, \_)$ is trivial. Indeed, inspection of of Figure 1(E) shows that the defect can be removed through a local change; hence $\mathrm{Res}_{\mathbf{p}}$ must be trivial by Theorem 2.8(a).

(c) The converse to Theorem 2.8(a) is false. Triviality of $\mathrm{Res}_{\mathbf{a}}$ is necessary, but *not sufficient* to conclude that the defect in $\mathbf{a}$ is removable. For example, the codimension-two defect in the domino tiling of Figure 1(F) is essential, but $\mathrm{Res}_{\mathbf{a}} C \equiv e$ for every cocycle C. (This follows from [**Ein01**, Example 2.3], attributed to Sam Lightwood). $\diamondsuit$

The proofs of Proposition 2.7 and Theorem 2.8 depend on the following:

LEMMA 2.10. *Let* $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$, $\mathbf{a} \in \widetilde{\mathfrak{A}}$, *and* $C \in \mathcal{Z}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$ *be as in* Proposition 2.7. *Let* ζ *be a closed trail in* $\mathbb{G}_r := \mathbb{G}_r(\mathbf{a})$.

(a) *If* ζ *is homotopic to* ζ' *in* $\mathbb{G}_r$, *then* $C(\zeta, \mathbf{a}) = C(\zeta', \mathbf{a})$. *In particular, if* ζ *is nullhomotopic in* $\mathbb{G}_r$, *then* $C(\zeta, \mathbf{a}) = e_{\mathcal{G}}$.

(b) *If* $C \approx C'$ *via transfer function* b, *then* $C(\zeta, \mathbf{a}) = b(\mathbf{a}) C'(\zeta, \mathbf{a}) b(\mathbf{a})^{-1}$ (assuming ζ begins and ends at 0). *Hence, if* $(\mathcal{G}, \cdot)$ *is abelian, then* $C(\zeta, \mathbf{a}) = C'(\zeta, \mathbf{a})$.

Proof: It suffices to check **(a)** this when ζ and ζ' differ by an elementary homotopy, which can be done by combining eqn.(5) with **(EH1)-(EH3)**. To see **(b)**, substitute eqn.(6) into eqn.(4) to get $C(\zeta, \mathbf{a}) = b(\mathbf{a}) C'(\zeta, \mathbf{a}) b(\mathbf{a})^{-1}$. $\square$

Proof of Proposition 2.7: **(a)** Lemma 2.10(a) implies that $\mathrm{Res}^r_{\mathbf{a}} C(\underline{\zeta})$ is well-defined, because $C(\mathbf{a}, \zeta)$ is determined by the homotopy class $\underline{\zeta}$. If $\zeta = \zeta_1 \star \zeta_2$, then eqn.(4) implies that that $C(\zeta, \mathbf{a}) = C(\zeta_1, \mathbf{a}) \cdot C(\zeta_2, \mathbf{a})$; hence $\mathrm{Res}^r_{\mathbf{a}} C(\underline{\zeta}) = \mathrm{Res}^r_{\mathbf{a}} C(\underline{\zeta}_1) \cdot \mathrm{Res}^r_{\mathbf{a}} C(\underline{\zeta}_2)$; hence $\mathrm{Res}^r_{\mathbf{a}} C$ is a homomorphism. **(b)** follows from Lemma 2.10(b). $\square$

Proof of Theorem 2.8: **(a)** By contradiction, suppose there exist $R \geq r$ and $\mathbf{a}' \in \mathfrak{A}$ with $\mathbf{a}'_{\mathbb{G}_R} = \mathbf{a}_{\mathbb{G}_R}$, where $\mathbb{G}_R := \mathbb{G}_R(\mathbf{a})$. Let $\underline{\zeta} \in \pi_1(\mathbb{G}_\infty, \omega)$; then $\underline{\zeta}$ has a representative trail ζ' in $\mathbb{G}_R$. Thus $\mathrm{Res}_{\mathbf{a}} C(\underline{\zeta}) \underset{(*)}{=} \mathrm{Res}_{\mathbf{a}} C(\underline{\zeta}') := C(\zeta', \mathbf{a}) \underset{(\dagger)}{=} C(\zeta', \mathbf{a}') \underset{(\ddagger)}{=} e_{\mathcal{G}}$. Here, $(*)$ is because $\zeta \approx \zeta'$; $(\dagger)$ is because $\mathbf{a}'_{\mathbb{G}_R} = \mathbf{a}_{\mathbb{G}_R}$; and $(\ddagger)$ is by Lemma 2.10(a), because ζ' is nullhomotopic in $\mathbb{G}_R(\mathbf{a}') = \mathbb{Z}^D$.

(b) Fix $\underline{\zeta} \in \pi_1(\mathbb{G}_\infty, \omega)$. We claim the function $\mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G}) \ni \underline{C} \mapsto C(\zeta, \mathbf{a}) \in \mathcal{G}$ is a homomorphism. First, note that, for any $\underline{C} \in \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$, with any radius $R > 0$, we can find some representative of $\underline{\zeta}$ in $\pi_1(\mathbb{G}_R, \omega)$ [because $\underline{\zeta} \in \pi_1(\mathbb{G}_\infty, \omega)$]. Hence $C(\zeta, \mathbf{a})$ is always well-defined. Furthermore, the value of $C(\zeta, \mathbf{a})$ depends only on the cohomology class of C, by Lemma 2.10(b). Now, let $\underline{C}_1, \underline{C}_2 \in \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$, and $\underline{C} := \underline{C}_1 \cdot \underline{C}_2$; it follows from eqn.(4) that $C(\zeta, \mathbf{a}) = C_1(\zeta, \mathbf{a}) \cdot C_2(\zeta, \mathbf{a})$.

Now fix $\underline{C} \in \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$. We claim the function $\mathrm{Res}^\infty_{\mathbf{a}} C : \pi_1(\mathbb{G}_\infty, \omega) \ni \underline{\zeta} \mapsto C(\zeta, \mathbf{a}) \in \mathcal{G}$ is a homomorphism. Now, C is locally determined (say, with radius r), because $\mathcal{G}$ is discrete. Thus, Proposition 2.7 yields a homomorphism $\mathrm{Res}^R_{\mathbf{a}} C : \pi_1(\mathbb{G}_R, \omega) \ni \underline{\zeta} \mapsto C(\zeta, \mathbf{a}) \in \mathcal{G}$, for all $R \geq r$. This yields a commuting cone of homomorphisms:

$$\begin{tikzcd}
\pi_1(\mathbb{G}_r, \omega) \arrow[d, "\mathrm{Res}^r_{\mathbf{a}} C"'] & \pi_1(\mathbb{G}_{r+1}, \omega) \arrow[l, "\iota^*_r"'] \arrow[dl, "\mathrm{Res}^{r+1}_{\mathbf{a}} C"'] & \pi_1(\mathbb{G}_{r+2}, \omega) \arrow[l, "\iota^*_{r+1}"'] \arrow[dll, "\mathrm{Res}^{r+2}_{\mathbf{a}} C"'] & \cdots\cdots \arrow[l, "\iota^*_{r+2}"'] & \pi_1(\mathbb{G}_\infty, \omega) \arrow[dllll, dotted, "\cdots \ \mathrm{Res}^\infty_{\mathbf{a}} C"'] \\
\mathcal{G} & & & &
\end{tikzcd}$$

Each triangle in this cone commutes because of Lemma 2.10(a). The cone converges to $\mathrm{Res}^\infty_{\mathbf{a}} C : \pi_1(\mathbb{G}_\infty, \omega) \longrightarrow \mathcal{G}$, so $\mathrm{Res}^\infty_{\mathbf{a}} C$ is also a homomorphism. $\square$

When are poles persistent under cellular automata? Recall from Proposition 2.5 that any CA $\Phi : \mathfrak{A}\longrightarrow\mathfrak{A}$ defines an endomorphism $\Phi_* : \mathcal{H}^1_{\rm dy}(\mathfrak{A},\mathcal{G})\longrightarrow\mathcal{H}^1_{\rm dy}(\mathfrak{A},\mathcal{G})$.

PROPOSITION 2.11. *Let* $\Phi : \mathcal{A}^{\mathbb{Z}^D}\longrightarrow\mathcal{A}^{\mathbb{Z}^D}$ *be a CA* (with radius $R > 0$) *and let* $\mathfrak{A}\subset\mathcal{A}^{\mathbb{Z}^D}$ *be a* Φ*-invariant subshift. Let* $\mathbf{a}\in\widetilde{\mathfrak{A}}$ *and let* $\mathbf{b} = \Phi(\mathbf{a})$.

- **(a)** *If* $C \in \mathcal{Z}^1_{\rm dy}(\mathfrak{A},\mathcal{G})$ *is locally determined with radius* $r > 0$*, then* $\mathrm{Res}^{R+r}_{\mathbf{b}}C \equiv \mathrm{Res}^{R+r}_{\mathbf{a}}(\Phi_*C)$.
- **(b)** *If* $\Phi_* : \mathcal{H}^1_{\rm dy}(\mathfrak{A},\mathcal{G})\longrightarrow\mathcal{H}^1_{\rm dy}(\mathfrak{A},\mathcal{G})$ *is surjective, then every* $\mathcal{G}$*-pole is* Φ*-persistent.*

Suppose $\mathfrak{A}$ *has abelian fundamental cocycle* $C^* \in \mathcal{Z}^1_{\rm dy}(\mathfrak{A},\mathcal{G}^*)$*. Define* $\mathrm{Res}^*_{\mathbf{a}} : \pi_1(\mathbb{G}_\infty(\mathbf{a}))\longrightarrow\mathcal{G}^*$ *by* $\mathrm{Res}^*_{\mathbf{a}}(\underline{\zeta}) = C^*(\zeta,\mathbf{a})$ *for any* $\underline{\zeta}\in\pi_1(\mathbb{G}_\infty(\mathbf{a}))$.

- **(c)** *If* $(\mathcal{H},\cdot)$ *is any abelian group, and* $\mathbf{a}$ *has an* $\mathcal{H}$*-pole, then* $\mathrm{Res}^*_{\mathbf{a}}$ *is nontrivial.*
- **(d)** *Let* Φ *and* φ *be as in* Proposition 2.6(b)*. If* $\mathbf{b} = \Phi(\mathbf{a})$*, then* $\mathrm{Res}^*_{\mathbf{b}} = \varphi\circ\mathrm{Res}^*_{\mathbf{a}}$*.*
  Hence, if $\varphi^n\circ\mathrm{Res}^*_{\mathbf{a}}$ *is nontrivial for all* $n\in\mathbb{N}$*, then the defect of* $\mathbf{a}$ *is* Φ*-persistent.*
- **(e)** *If* φ *is a monomorphism, then every pole is* Φ*-persistent.*

Proof: **(a)** If ζ is any trail in $\mathbb{G}_{R+r}(\mathbf{a})$, then $C(\mathbf{b},\zeta) = \Phi_*C(\mathbf{a},\zeta)$. **(b)** follows.

(c) Suppose there exists $C\in\mathcal{Z}^1_{\rm dy}(\mathfrak{A},\mathcal{H})$, and $\underline{\zeta}\in\pi_1(\mathbb{G}_\infty(\mathbf{a}))$ such that $\mathrm{Res}_{\mathbf{a}}(\underline{C},\underline{\zeta})\neq e_{\mathcal{H}}$. There is a homomorphism $\psi : \mathcal{G}^*\longrightarrow\mathcal{H}$ such that $\psi\circ C^*$ is cohomologous to C (because C^* is fundamental). Thus

$$\psi\left[C^*(\zeta,\mathbf{a})\right] \underset{(\dagger)}{=} (\psi\circ C^*)(\zeta,\mathbf{a}) \underset{(*)}{=} C(\zeta,\mathbf{a}) =: \mathrm{Res}_{\mathbf{a}}(\underline{C},\underline{\zeta}) \neq e_{\mathcal{H}}.$$

Here, $(\dagger)$ is by applying homomorphism ψ to eqn.(4), and $(*)$ is by Lemma 2.10(b) (because $\psi\circ C^*\approx C$). Thus $\mathrm{Res}^*_{\mathbf{a}}(\underline{\zeta}) := C^*(\zeta,\mathbf{a})$ is nontrivial in $\mathcal{G}^*$; hence $\mathrm{Res}^*_{\mathbf{a}}$ is nontrivial.

(d) Let $\mathbf{b} = \Phi(\mathbf{a})$. Then for any $\underline{\zeta}\in\pi_1(\mathbb{G}_\infty(\mathbf{a}))$, $\mathrm{Res}^*_{\mathbf{b}}(\underline{\zeta}) = C^*(\zeta,\mathbf{b}) = C^*(\zeta,\Phi(\mathbf{a})) = \Phi_*C^*(\zeta,\mathbf{a}) \underset{(\dagger)}{=} \varphi\circ C^*(\zeta,\mathbf{a}) = \varphi\circ\mathrm{Res}^*_{\mathbf{a}}(\underline{\zeta})$, where $(\dagger)$ is by Proposition 2.6(b) and Lemma 2.10(b). Then **(e)** follows from **(d)**. □

2.3. *Gaps and Tilt* The domain boundary in Examples 1.5(c,d) are not detected by the spectral invariants of [**Piv06**]. However, they can be detected using cohomology. Let $C : \mathbb{Z}^D\times\mathfrak{A}\longrightarrow\mathcal{G}$ be a locally determined cocycle with radius $r > 0$ and local rule $c : \mathbb{E}\times\mathfrak{A}_{(r)}\longrightarrow\mathcal{G}$. For any $\mathbf{a}\in\mathfrak{A}$, we define $C_{\mathbf{a}} : \mathbb{Z}^D\times\mathbb{Z}^D\longrightarrow\mathbb{N}$ by

$$C_{\mathbf{a}}(\mathsf{y},\mathsf{z}) := C(\mathsf{y},\mathbf{a})\cdot C(\mathsf{z},\mathbf{a})^{-1} = c(\zeta,\mathbf{a}), \tag{7}$$

where ζ is any trail from z to y, and the expression $c(\zeta,\mathbf{a})$ is interpreted as in eqn.(4). For example, if $C : \mathbb{Z}^D\times\mathfrak{A}\longrightarrow\mathbb{Z}$ is a height function [Example 2.1(e)] and $\mathbf{a}\in\mathfrak{A}$, then C assigns a 'height' $h(\mathsf{z}) := C(\mathsf{z},\mathbf{a})$ to every point in $\mathbb{Z}^D$, so $C_{\mathbf{a}}(\mathsf{y},\mathsf{z}) = h(\mathsf{y}) - h(\mathsf{z})$ is the 'altitude difference' between y and z.

Now suppose $\mathbf{a}\in\widetilde{\mathfrak{A}}$ has a range r domain boundary and let $\mathbb{Y} := \mathbb{G}_r(\mathbf{a})$ have projective connected components $\mathbb{Y}_1,\ldots,\mathbb{Y}_N$. Assume that $\mathbf{a}$ has no codimension-two defects –hence $\pi_1(\mathbb{Y}_n) = 0$ for all $n\in[1...N]$. Thus, if $\mathsf{y}_1,\mathsf{y}_2\in\mathbb{Y}$ are in the same projective component of $\mathbb{Y}$, then we can define $C_{\mathbf{a}}(\mathsf{y}_1,\mathsf{y}_2)$ by the right-hand expression in (7) (this is path-independent because $\pi_1(\mathbb{Y}_n) = 0$). However, if $\mathsf{y}_1,\mathsf{y}_2$ are in *different* connected components of $\mathbb{Y}$, then $C_{\mathbf{a}}(\mathsf{y}_1,\mathsf{y}_2)$ is not well-defined by eqn.(7), because there is no trail in $\mathbb{Y}$ connecting y_1 to y_2.

Instead, we will define $C_{\mathbf{a}}$ up to a constant as follows: first, for each $n \in [1...N]$, choose a reference point $\mathsf{y}_n^* \in \mathbb{Y}_n$ and decree that $C_{\mathbf{a}}(\mathsf{y}_n^*, \mathsf{y}_m^*) := e_{\mathcal{G}}$ for all $n, m \in [1...N]$. Then for any $\mathsf{y}_n \in \mathbb{Y}_n$ and $\mathsf{y}_m \in \mathbb{Y}_m$, define

$$C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m) \quad := \quad C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_n^*) \cdot C_{\mathbf{a}}(\mathsf{y}_m^*, \mathsf{y}_m). \tag{8}$$

Let $\mathcal{C} := c[\mathbb{E} \times \mathfrak{A}_{(r)}] \subset \mathcal{G}$; then $\mathcal{C}$ is a finite subset of $\mathcal{G}$, and for any $\mathsf{z} \in \mathbb{Z}^D$ and $\mathbf{a} \in \mathfrak{A}$, $C(\mathsf{z}, \mathbf{a})$ is an element of the subgroup generated by $\mathcal{C}$. Hence we can assume without loss of generality that $\mathcal{C}$ generates $\mathcal{G}$. A function $|\_| : \mathcal{G} \longrightarrow [0, \infty]$ is a *pseudonorm* on $\mathcal{G}$ if:

$$\text{For all } g, h \in \mathcal{G}, \qquad (\mathbf{a}) \ \ |g \cdot h| \ = \ |h \cdot g|, \qquad \text{and} \qquad (\mathbf{b}) \ \ |g \cdot h| \ \leq \ |g| + |h|. \tag{9}$$

(Hence $|\_|$ is constant on each conjugacy class of $\mathcal{G}$).

Example 2.12: (a) If $\mathcal{G} = \mathbb{Z}$, let $|\_|$ be the Euclidean norm.

(b) If $\mathcal{G}$ is abelian, then $\mathcal{G}$ is finitely generated (by $\mathcal{C}$), so $\mathcal{G} \cong \mathbb{Z}^R \oplus \mathbb{Z}_{/n_1} \oplus \cdots \oplus \mathbb{Z}_{/n_K}$ for some $R, n_1, \ldots, n_K \in \mathbb{N}$; see [**DF91**, §5.2] or [**Lan84**, §I.10]. For any $(z_1, \ldots, z_R) \in \mathbb{Z}^R$ and $(y_1, \ldots, y_K) \in \bigoplus_{k=1}^K \mathbb{Z}_{/n_k}$, define $|(z_1, \ldots, z_R; y_1, \ldots, y_K) := |z_1| + \cdots + |z_R| + |y_1|_{n_1} + \cdots |y_k|_{n_K}$, where $|y|_{n_k} := \min\{|y|, |n_k - y|\}$ for all $k \in [1...K]$.

(c) If $\mathcal{G}$ is nonabelian, then let $q : \mathcal{G} \longrightarrow \widetilde{\mathcal{G}}$ be the abelianization map. Then $\widetilde{\mathcal{G}}$ is finitely generated (by $\widetilde{\mathcal{C}} := q[\mathcal{C}]$), so let $|\_|_* : \widetilde{\mathcal{G}} \longrightarrow \mathbb{N}$ be as in (b). Then define $|g| = |q(g)|_*$ for any $g \in \mathcal{G}$. $\diamondsuit$

Remarks: A pseudonorm on $\mathcal{G}$ is equivalent to a pseudometric $d : \mathcal{G} \times \mathcal{G} \longrightarrow [0, \infty]$ which is *bilaterally invariant* i.e. $d(fg, fh) = d(g, h) = d(gf, hf)$ for all $f, g, h \in \mathcal{G}$. Here $|g| = d(g, e)$ and $d(g, h) = |gh^{-1}|$. We do *not* require that d be compatible with the topology of $\mathcal{G}$ (although this can always be arranged if $\mathcal{G}$ is unimodular; i.e. the left- and right- Haar measures are the same). However, the following theory is trivial unless $|\_|$ is unbounded (so if $\mathcal{G}$ is compact then d *shouldn't* be topologically compatible with $\mathcal{G}$). $\diamondsuit$

We allow that $|g| = 0$ or $g = \infty$ for some $g \neq e$. However, we require that (i) $\forall\ c \in \mathcal{C}$, $|c| \leq 1$ and (ii) $\exists\ c \in \mathcal{C}$ with $|c| > 0$. (This can always be obtained through renormalization, if $|\_|$ is nontrivial.) It follows that $C_{\mathbf{a}}$ satisfies a Lipschitz-type condition:

Lemma 2.13. **(a)** *If $\mathbf{a} \in \mathfrak{A}$, and $\mathsf{y}, \mathsf{z} \in \mathbb{Z}^D$, then $|C_{\mathbf{a}}(\mathsf{y}, \mathsf{z})| \leq |\mathsf{y} - \mathsf{z}|$.*

(b) *If $\mathbf{a} \in \widetilde{\mathfrak{A}}$, and y, z are in the same connected component of $\mathbb{G}_r(\mathbf{a})$, then $|C_{\mathbf{a}}(\mathsf{y}, \mathsf{z})| \leq d_{r,\mathbf{a}}(\mathsf{y}, \mathsf{z})$, where $d_{r,\mathbf{a}}(\mathsf{y}, \mathsf{z})$ is the length of the shortest trail from y to z in $\mathbb{G}_r(\mathbf{a})$.* $\Box$

Suppose $\mathbf{a} \in \widetilde{\mathfrak{A}}$ has a range r domain boundary and let $\mathbb{Y}_1, \ldots, \mathbb{Y}_N$ be as above. The *tilt* of $\mathbb{Y}_n$ relative to $\mathbb{Y}_m$ is then defined:

$$\measuredangle_{\mathbf{a}}^{C}(\mathbb{Y}_n, \mathbb{Y}_m) \quad := \quad \sup_{\mathsf{y}_n \in \mathbb{Y}_n,\ \mathsf{y}_n \in \mathbb{Y}_m} \frac{|C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m)|}{|\mathsf{y}_n - \mathsf{y}_m|}. \tag{10}$$

If $\measuredangle_{\mathbf{a}}^{C}(\mathbb{Y}_n, \mathbb{Y}_m) = \infty$, then we say the domain boundary is a *C-gap*.

Example 2.14: (a) Let $C : \mathbb{Z}^2 \times \mathfrak{J}_{\mathfrak{ce}} \longrightarrow \mathbb{Z}$ be as in Example 2.1(c), and let $\mathbf{i} \in \widetilde{\mathfrak{J}_{\mathfrak{ce}}}$ be the domain boundary configuration shown in Figure 1(A). Suppose for simplicity that the domain boundary straddles the x axis. Let $\mathbb{X}$ and $\mathbb{Y}$ be the north and south connected components, respectively. Let $\mathsf{x}^* := (0, 1) \in \mathbb{X}$ and $\mathsf{y}^* := (0, -1) \in \mathbb{Y}$, and for all $n \in \mathbb{N}$, let $\mathsf{x}_n := (n,1) \in \mathbb{X}$ and $\mathsf{y}_n := (n,-1) \in \mathbb{Y}$, as shown in Figure 1(A). Then $C_{\mathbf{i}}(\mathsf{x}_n, \mathsf{x}^*) = n$, while $C_{\mathbf{i}}(\mathsf{y}_n, \mathsf{y}^*) = -n$; hence $C_{\mathbf{i}}(\mathsf{x}_n, \mathsf{y}_n) = 2n$, However, $|\mathsf{x}_n - \mathsf{y}_n| = 2$ for all n; hence $\measuredangle_{\mathbf{i}}^{C}(\mathbb{X}, \mathbb{Y}) = \infty$, so this defect is a gap.

(b) Let $C : \mathbb{Z}^2 \times \mathfrak{Dom} \longrightarrow \mathcal{G} := \mathbb{Z}_{/2} * \mathbb{Z}_{/2}$ be the cocycle from Example 2.1(f). Unfortunately, $\mathcal{G}$ does not admit any nontrivial pseudonorms (because every nonidentity element belongs to the same conjugacy class). However, if $\mathcal{Z} \subset \mathcal{G}$ is the cyclic subgroup generated by vh, then $(\mathcal{Z}, \cdot) \cong (\mathbb{Z}, +)$, and for any $\mathbf{d} \in \mathfrak{Dom}$ and $2\mathsf{z} \in 2\mathbb{Z}^2$, $C(2\mathsf{z}, \mathbf{d}) \in \mathcal{Z}$. Let $\mathcal{D}_2 \subset \mathcal{D}^{2\times 2}$ be the alphabet of $\mathfrak{Dom}$-admissible 2×2 blocks, and let $\mathfrak{D}_2 \subset \mathcal{D}_2^{\mathbb{Z}^2}$ be the 'recoding' of $\mathfrak{Dom}$ in this alphabet. Then $2\mathbb{Z}^2$ acts on $\mathfrak{D}_2$ by shifts in the obvious way, and C yields a cocycle $C' : 2\mathbb{Z}^2 \times \mathfrak{D}_2 \longrightarrow \mathcal{Z} \cong \mathbb{Z}$.

Let $\mathbf{d} \in \widetilde{\mathfrak{Dom}}$ be the domain boundary configuration in Figure 1(B) and let $\mathbf{d}_2$ be its recoding as an element of $\widetilde{\mathfrak{D}}_2$. Let $\mathsf{x}^* := (0,2) \in \mathbb{X} \cap (2\mathbb{Z}^2)$ and $\mathsf{y}^* := (0,-2) \in \mathbb{Y} \cap (2\mathbb{Z}^2)$, and for all $n \in \mathbb{N}$, let $\mathsf{x}_n := (2n, 2) \in \mathbb{X} \cap (2\mathbb{Z}^2)$ and $\mathsf{y}_n := (2n,-2) \in \mathbb{Y} \cap (2\mathbb{Z}^2)$, as shown in Figure 1(B). Then $C'_{\mathbf{d}_2}(\mathsf{x}_n, \mathsf{x}^*) = (vhvh)^n$, whereas $C'_{\mathbf{d}_2}(\mathsf{y}_n, \mathsf{y}^*) = h^{2n} = e_{\mathcal{G}} = (vh)^0$. Hence $C'_{\mathbf{d}_2}(\mathsf{x}_n, \mathsf{y}_n) = (vhvh)^n \cong 2n \in \mathbb{Z}$. However, $|\mathsf{x}_n - \mathsf{y}_n| = 2$ for all n; hence $\measuredangle_{\mathbf{a}}^{C}(\mathbb{X}, \mathbb{Y}) = \infty$, so this defect is a gap.

(c) Let $\mathbf{d}' \in \widetilde{\mathfrak{Dom}}$ be the domain boundary configuration in Figure 1(C), and let $\mathbf{d}'_2$ be its recoding as an element of $\widetilde{\mathfrak{D}}_2$, as in (b). Let $\mathbb{X}$ and $\mathbb{Y}$ denote the eastern and western domains (assume the boundary straddles the y axis). Let $\mathsf{x}^* := (-2,0) \in \mathbb{X} \cap (2\mathbb{Z}^2)$ and $\mathsf{y}^* := (2,0) \in \mathbb{Y} \cap (2\mathbb{Z}^2)$, and for all $n \in \mathbb{N}$, let $\mathsf{x}_n := (-2, 2n) \in \mathbb{X} \cap (2\mathbb{Z}^2)$ and $\mathsf{y}_n := (2, 2n) \in \mathbb{Y} \cap (2\mathbb{Z}^2)$, as shown in Figure 1(C). Then $C'_{\mathbf{d}'_2}(\mathsf{x}_n, \mathsf{x}^*) = (hvhv)^n$, whereas $C'_{\mathbf{d}'_2}(\mathsf{y}_n, \mathsf{y}^*) = (vhvh)^n$. Hence $C'_{\mathbf{d}'_2}(\mathsf{x}_n, \mathsf{y}_n) = (hvhv)^n (vhvh)^{-n} = (vhvh)^{-2n} \cong -4n \in \mathbb{Z}$. However, $|\mathsf{x}_n - \mathsf{y}_n| = 2$ for all n; hence $\measuredangle_{\mathbf{a}}^{C}(\mathbb{X}, \mathbb{Y}) = \infty$, so this defect is a gap.

(d) If C is a cocycle into a finite group, then there can be no C-gaps. For example, the cocycle $C : \mathbb{Z}^D \times \mathfrak{Pth} \longrightarrow (\mathbb{Z}_{/2})^2$ of Example 2.1(d) admits no gaps. ♢

If $(\mathcal{G}, \cdot)$ is a group, then a $\mathcal{G}$-***gap*** is a C-gap for some $\underline{C} \in \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$. We say the gap is ***sharp*** if, for all $R \geq r \geq 0$, there is some constant $K = K(R, r) \in \mathbb{N}$ such that, for any $\mathsf{y} \in \mathbb{G}_r(\mathbf{a})$, there exists some $\mathsf{x} \in \mathbb{G}_R(\mathbf{a})$ which is trail-connected to y in $\mathbb{G}_r(\mathbf{a})$, with $d_{r,\mathbf{a}}(\mathsf{x}, \mathsf{y}) \leq K$. Heuristically, this means that the defect field $\mathcal{F}_{\mathbf{a}}$ does not have arbitrarily large 'flat' areas, and that the gap does not ramify into lots of 'tributaries'. For example, if $\mathfrak{A}$ is a subshift of finite type, and the defect set $\mathbb{D}(\mathbf{a})$ is confined to a thickened hyperplane [as in Examples 2.14(a,b,c)], then the gap is sharp.

THEOREM 2.15. **(a)** *Sharp $\mathcal{G}$-Gaps are essential defects.*

(b) *If $\Phi\colon \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ is a CA with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$, and $\Phi_* : \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$ is surjective, then any $\mathcal{G}$-gap is Φ-persistent.*

To prove Theorem 2.15, we use:

LEMMA 2.16. **(a)** *The existence of a gap does not depend on the choice of* $\{\mathsf{y}_1^*, \ldots, \mathsf{y}_N^*\}$.

(b) *Gaps depend only on cohomology classes. If $C \approx C'$, then any C-gap is also a C'-gap.*

Proof: **(a)** Suppose we defined $C_{\mathbf{a}}^{\dagger}$ as in eqn.(8), but using a different set $\{\mathsf{y}_1^{\dagger}, \ldots, \mathsf{y}_N^{\dagger}\}$. For all $n \in [1...N]$, let $c_n := C_{\mathbf{a}}(\mathsf{y}_n^*, \mathsf{y}_n^{\dagger})$ (this well-defined by eqn.(7) because they are in the same connected component $\mathbb{Y}_n$). If $\mathsf{y}_n \in \mathbb{Y}_n$ and $\mathsf{y}_m \in \mathbb{Y}_m$, then $C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m) \leq C^\dagger_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m) + c_n + c_m$, because

$$\begin{array}{rcl}
C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m) & \underset{(\diamond)}{=\!=} & C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}^*_n) \cdot C_{\mathbf{a}}(\mathsf{y}^*_m, \mathsf{y}_m) \\
& \underset{(\dagger)}{=\!=} & C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}^\dagger_n) \cdot C_{\mathbf{a}}(\mathsf{y}^\dagger_n, \mathsf{y}^*_n) \cdot C_{\mathbf{a}}(\mathsf{y}^*_m, \mathsf{y}^\dagger_m) \cdot C_{\mathbf{a}}(\mathsf{y}^\dagger_m, \mathsf{y}_m), \\
\text{so } \ |C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m)| & = & \left|C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}^\dagger_n) \cdot C_{\mathbf{a}}(\mathsf{y}^\dagger_n, \mathsf{y}^*_n) \cdot C_{\mathbf{a}}(\mathsf{y}^*_m, \mathsf{y}^\dagger_m) \cdot C_{\mathbf{a}}(\mathsf{y}^\dagger_m, \mathsf{y}_m)\right| \\
& \underset{(\ddagger)}{=\!=} & \left|C_{\mathbf{a}}(\mathsf{y}^\dagger_n, \mathsf{y}^*_n) \cdot C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}^\dagger_n) \cdot C_{\mathbf{a}}(\mathsf{y}^\dagger_m, \mathsf{y}_m) \cdot C_{\mathbf{a}}(\mathsf{y}^*_m, \mathsf{y}^\dagger_m)\right| \\
& \underset{(*)}{\leq} & \left|C_{\mathbf{a}}(\mathsf{y}^\dagger_n, \mathsf{y}^*_n)\right| + \left|C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}^\dagger_n) \cdot C_{\mathbf{a}}(\mathsf{y}^\dagger_m, \mathsf{y}_m)\right| + \left|C_{\mathbf{a}}(\mathsf{y}^*_m, \mathsf{y}^\dagger_m)\right| \\
& \underset{(\diamond)}{=\!=} & c_n + \left|C^\dagger_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m)\right| + c_m.
\end{array}$$

Here, $(\diamond)$ is by eqn.(8), $(\dagger)$ is by eqn.(7), $(\ddagger)$ is by eqn.(9a), and $(*)$ is by eqn.(9b).

Likewise, $C^\dagger_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m) \leq C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m) + c_n + c_m$ (by symmetric reasoning). Thus,

$$\frac{C^\dagger_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m) - c_n - c_m}{|\mathsf{y}_n - \mathsf{y}_m|} \quad \leq \quad \frac{C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m)}{|\mathsf{y}_n - \mathsf{y}_m|} \quad \leq \quad \frac{C^\dagger_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m) + c_n + c_m}{|\mathsf{y}_n - \mathsf{y}_m|}.$$

Substitute into eqn.(10) to see that $\measuredangle^C_{\mathbf{a}}(\mathbb{Y}_n, \mathbb{Y}_m) = \infty$ if and only if $\measuredangle^\dagger_{\mathbf{a}}(\mathbb{Y}_n, \mathbb{Y}_m) = \infty$.

(b) If $C' \approx C$, then there is a local transfer function $b : \mathfrak{A}_{(r)} \longrightarrow \mathcal{G}$ satisfying eqn.(6). Thus, for any $\mathsf{y}_n \in \mathbb{Y}_n$ and $\mathsf{y}_m \in \mathbb{Y}_m$, $C'_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}^*_n) = b(\mathbf{a}_{\mathbb{B}(\mathsf{y}_n, r)}) \cdot C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}^*_n) \cdot b(\mathbf{a}_{\mathbb{B}(\mathsf{y}^*_n, r)})^{-1}$. Now, $b[\mathfrak{A}_{(r)}]$ is finite because $\mathfrak{A}_{(r)}$ is finite. Thus $B := \max \left\{|b(\mathbf{a})| \ ; \ \mathbf{a} \in \mathfrak{A}_{(r)}\right\} \cup \left\{|b(\mathbf{a})^{-1}| \ ; \ \mathbf{a} \in \mathfrak{A}_{(r)}\right\}$ is finite. Furthermore, part (a) says that we can assume without loss of generality that y^*_n and y^*_m are chosen such that $b(\mathbf{a}_{\mathbb{B}(\mathsf{y}^*_n, r)}) = b(\mathbf{a}_{\mathbb{B}(\mathsf{y}^*_m, r)})$. Thus, for any $\mathsf{y}_n \in \mathbb{Y}_n$ and $\mathsf{y}_m \in \mathbb{Y}_m$,

$$\begin{array}{l}
C'_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m) \ \underset{(\diamond)}{=\!=} \ C'_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}^*_n) \cdot C'_{\mathbf{a}}(\mathsf{y}^*_m, \mathsf{y}_m) \\
\quad = \ b(\mathbf{a}_{\mathbb{B}(\mathsf{y}_n, r)}) \cdot C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}^*_n) \cdot b(\mathbf{a}_{\mathbb{B}(\mathsf{y}^*_n, r)})^{-1} \cdot b(\mathbf{a}_{\mathbb{B}(\mathsf{y}^*_m, r)}) \cdot C_{\mathbf{a}}(\mathsf{y}^*_m, \mathsf{y}^m) \cdot b(\mathbf{a}_{\mathbb{B}(\mathsf{y}_m, r)})^{-1} \\
\quad = \ b(\mathbf{a}_{\mathbb{B}(\mathsf{y}_n, r)}) \cdot C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}^*_n) \cdot C_{\mathbf{a}}(\mathsf{y}^*_m, \mathsf{y}^m) \cdot b(\mathbf{a}_{\mathbb{B}(\mathsf{y}_m, r)})^{-1} \underset{(\diamond)}{=\!=} b(\mathbf{a}_{\mathbb{B}(\mathsf{y}_n, r)}) \cdot C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m) \cdot b(\mathbf{a}_{\mathbb{B}(\mathsf{y}_m, r)})^{-1}
\end{array}$$

$$\begin{array}{l}
\text{Thus,} \ \ |C'_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m)| \quad = \quad \left|b(\mathbf{a}_{\mathbb{B}(\mathsf{y}_n, r)}) \cdot C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m) \cdot b(\mathbf{a}_{\mathbb{B}(\mathsf{y}_m, r)})^{-1}\right| \\
\quad \underset{(*)}{\leq} \quad \left|b(\mathbf{a}_{\mathbb{B}(\mathsf{y}_n, r)})\right| + |C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m)| + \left|b(\mathbf{a}_{\mathbb{B}(\mathsf{y}_m, r)})^{-1}\right| \quad \leq \quad |C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m)| + 2B.
\end{array}$$

Here, $(\diamond)$ is by eqn.(8) and $(*)$ is by eqn.(9b). By symmetric reasoning, we can show that $|C'_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m)| \ \leq \ |C_{\mathbf{a}}(\mathsf{y}_n, \mathsf{y}_m)| + 2B$. Hence, just as in part (a), we conclude that $\measuredangle^C_{\mathbf{a}}(\mathbb{Y}_n, \mathbb{Y}_m) = \infty$ if and only if $\measuredangle^{C'}_{\mathbf{a}}(\mathbb{Y}_n, \mathbb{Y}_m) = \infty$. □

Proof of Theorem 2.15: **(a)** Let $\mathbb{Y} := \mathbb{G}_r(\mathbf{a})$ have a sharp gap, but suppose (by contradiction) that the defect in $\mathbf{a}$ is removable. Thus, there is some $R \geq r$ and $\mathbf{b} \in \mathfrak{A}$ such that $\mathbf{b}_{\mathbb{X}} = \mathbf{a}_{\mathbb{X}}$, where $\mathbb{X} := \mathbb{G}_R(\mathbf{a})$. By Lemma 2.16(a), we can assume without loss of generality that $\{\mathsf{y}^*_1, \ldots, \mathsf{y}^*_N\}$ are in $\mathbb{X}$, and were chosen so that $C_{\mathbf{b}}(\mathsf{y}^*_n, \mathsf{y}^*_m) = 0$ for all $n, m \in [1...M]$. It follows that $C_{\mathbf{a}}(\mathsf{x}_1, \mathsf{x}_2) = C_{\mathbf{b}}(\mathsf{x}_1, \mathsf{x}_2)$ for all $\mathsf{x}_1, \mathsf{x}_2 \in \mathbb{X}$. Let $K = K(R, r)$ be the constant arising from the sharpness of the gap.

Claim 1: *For all* $\mathsf{y}_1, \mathsf{y}_2 \in \mathbb{Y}$, $\ C_{\mathbf{a}}(\mathsf{y}_1, \mathsf{y}_2) \leq |\mathsf{y}_1 - \mathsf{y}_2| + 4K$.

Proof: Suppose y_1 (resp. y_2) is in projective component $\mathbb{Y}_1$ (resp. $\mathbb{Y}_2$) of $\mathbb{Y}$. For $n = 1, 2$, there exists $\mathsf{x}_n \in \mathbb{X} \cap \mathbb{Y}_n$ such that $d_{r,\mathbf{a}}(\mathsf{x}_n, \mathsf{y}_n) \leq K$ (by definition of 'sharpness'). Then

$$\begin{aligned} |C_{\mathbf{a}}(\mathsf{y}_1,\mathsf{y}_2)| &\underset{(\dagger)}{\leq} |C_{\mathbf{a}}(\mathsf{y}_1,\mathsf{x}_1)| + |C_{\mathbf{a}}(\mathsf{x}_1,\mathsf{x}_2)| + |C_{\mathbf{a}}(\mathsf{x}_2,\mathsf{y}_2)| \underset{(*)}{\leq} K + |C_{\mathbf{b}}(\mathsf{x}_1,\mathsf{x}_2)| + K \\ &\underset{(\diamond)}{\leq} K + |\mathsf{x}_1 - \mathsf{x}_2| + K \underset{(\triangle)}{\leq} K + K + |\mathsf{y}_1 - \mathsf{y}_2| + K + K. \end{aligned}$$

($\dagger$) is eqn.(9b), ($*$) is Lemma 2.13(b) applied to $\mathsf{x}_1, \mathsf{y}_1 \in \mathbb{Y}_1$ and $\mathsf{x}_2, \mathsf{y}_2 \in \mathbb{Y}_2$ in $\mathbf{a}$, and ($\diamond$) is Lemma 2.13(a) applied to $\mathsf{x}_n, \mathsf{x}_m$ in $\mathbf{b}$. ($\triangle$) is the triangle inequality. ◇ `Claim 1`

For any $n, m \in [1...M]$, Claim 1 and eqn.(10) imply that $\measuredangle^C_{\mathbf{a}}(\mathbb{Y}_n, \mathbb{Y}_m) \leq 1$, which means that $\mathbf{a}$ has no gap between $\mathbb{Y}_n$ and $\mathbb{Y}_m$. Contradiction.

(b) Suppose $\mathbf{a} \in \widetilde{\mathfrak{A}}$ has a C-gap for some $C \in \mathcal{Z}^1_{\rm dy}(\mathfrak{A}, \mathcal{G})$, and let $\mathbf{a}' := \Phi(\mathbf{a})$. There exists $C' \in \mathcal{Z}^1_{\rm dy}(\mathfrak{A}, \mathcal{G})$ with $C_1 := \Phi_*(C') \approx C$ (because $\Phi_* : \mathcal{H}^1_{\rm dy}(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^1_{\rm dy}(\mathfrak{A}, \mathcal{G})$ is surjective). Thus, Lemma 2.16(b) says that $\mathbf{a}$ also has a C_1-gap. But for any $\mathsf{z} \in \mathbb{Z}^D$, $C_1(\mathsf{z}, \mathbf{a}) = C'(\mathsf{z}, \Phi(\mathbf{a})) = C'(\mathsf{z}, \mathbf{a}')$. Thus, $\mathbf{a}$ has a C_1-gap iff $\mathbf{a}'$ has a C'-gap. □

3. *Homotopy Defects for Subshifts of Finite Type*

We will introduce homotopy/homology groups for Wang tile systems and subshifts of finite type, generalizing the constructions of [**CL90, GP95**]. Nontrivial elements of these groups represent codimension-$(d+1)$ 'obstructions' to the hole-filling problem, and can be used to characterize codimension-$(d+1)$ defects. This section's main results are Theorem 3.7 and Corollary 3.8.

3.1. *Canonical cell complex of $\mathbb{R}^D$:* For any $\mathsf{z} \in \mathbb{Z}^D$, let $\ulcorner\mathsf{z}\urcorner := \mathsf{z} + [0,1]^D$ be the unit cube in $\mathbb{R}^D$ with its minimal-coordinate corner at z. For all $d \in [1...D]$, let $\mathsf{e}_d := (0, ..., 0, 1, 0, ..., 0)$ be the dth unit vector. Suppose $\mathsf{z} = (z_1, \ldots, z_D)$; for all $d \in [1...D]$, let $\partial^-_d \ulcorner\mathsf{z}\urcorner := \{\mathbf{x} \in \ulcorner\mathsf{z}\urcorner\,;\ x_d = z_d\}$ and $\partial^+_d \ulcorner\mathsf{z}\urcorner := \{\mathbf{x} \in \ulcorner\mathsf{z}\urcorner\,;\ x_d = z_d + 1\}$ be the two 'faces' of $\ulcorner\mathsf{z}\urcorner$ of codimension one which are orthogonal to e_d. For any $k \in [2...D]$, we define codimension-k faces by intersecting k of these codimension-one faces (e.g. if $\mathbf{d} := \{d_1, \ldots, d_k\} \subset [1...D]$ and $\mathbf{s} := (s_1, \ldots, s_d) \in \{\pm 1\}^k$, then $\partial^{\mathbf{s}}_{\mathbf{d}} \ulcorner\mathsf{z}\urcorner := \partial^{s_1}_{d_1} \ulcorner\mathsf{z}\urcorner \cap \cdots \cap \partial^{s_k}_{d_k} \ulcorner\mathsf{z}\urcorner$). This yields a natural D-cell decomposition $\mathbf{Y}$ of $\mathbb{R}^D$, whose zero-skeleton $\mathbf{Y}^0$ is $\mathbb{Z}^D$, and whose D-skeleton $\mathbf{Y}^D$ is the set of D-cubes $\{\ulcorner\mathsf{z}\urcorner\,;\ \mathsf{z} \in \mathbb{Z}^D\}$. For any $d \in [1...D)$, the d-skeleton $\mathbf{Y}^d$ is the set of all d-dimensional edges/faces/etc. of the cubes in $\{\ulcorner\mathsf{z}\urcorner\,;\ \mathsf{z} \in \mathbb{Z}^D\}$. We will call this the *canonical cell complex* for $\mathbb{R}^D$.

3.2. *Review of Cubic (co)homology:* For any $d \in [0...D]$, let $\mathbb{Z}[\mathbf{Y}^d]$ be the free abelian group generated by $\mathbf{Y}^d$, i.e. the group of formal $\mathbb{Z}$-linear combinations of d-cells in $\mathbf{Y}$. Elements of $\mathbb{Z}[\mathbf{Y}^d]$ are called *d-chains*. Since $\mathbf{Y}^{-1} = \emptyset = \mathbf{Y}^{D+1}$, we also formally define $\mathbb{Z}[\mathbf{Y}^{-1}] := \{0\} =: \mathbb{Z}[\mathbf{Y}^{D+1}]$. Let $\partial_d : \mathbb{Z}[\mathbf{Y}^d] \longrightarrow \mathbb{Z}[\mathbf{Y}^{d-1}]$ be the *boundary homomorphism* from cubic homology theory. For example:

- For any $\mathsf{z} \in \mathbb{Z}^D$, let $\dot{\mathsf{z}} \in \mathbf{Y}^0$ be the corresponding zero-cell (i.e. vertex). Then $\partial_0(\dot{\mathsf{z}}) := 0$.
- If $\mathsf{y}, \mathsf{z} \in \mathbb{Z}^D$ are adjacent, and $(\mathsf{y} \to \mathsf{z}) \in \mathbf{Y}^1$ is the one-cell (i.e. oriented edge) from $\dot{\mathsf{y}}$ to $\dot{\mathsf{z}}$, then $\partial_1(\mathsf{y} \to \mathsf{z}) := \dot{\mathsf{z}} - \dot{\mathsf{y}}$.

- If $\begin{smallmatrix}\mathsf{z} & \leftarrow & \mathsf{y}\\ \downarrow & & \uparrow\\ \mathsf{w} & \rightarrow & \mathsf{x}\end{smallmatrix} \in \mathbf{Y}^2$ is the two-cell (i.e. oriented square) whose four corner vertices are $\dot{\mathsf{w}}, \dot{\mathsf{x}}, \dot{\mathsf{y}}, \dot{\mathsf{z}} \in \mathbf{Y}^0$, then $\partial_2\left(\begin{smallmatrix}\mathsf{z} & \leftarrow & \mathsf{y}\\ \downarrow & & \uparrow\\ \mathsf{w} & \rightarrow & \mathsf{x}\end{smallmatrix}\right) := (\mathsf{w}\to\mathsf{x}) + (\mathsf{x}\to\mathsf{y}) + (\mathsf{y}\to\mathsf{z}) + (\mathsf{z}\to\mathsf{w})$.
- If $\begin{smallmatrix} & \mathsf{z} & - & \mathsf{y}\\ \mathsf{v} & - & \mathsf{u} & \\ & \mathsf{w} & - & \mathsf{x}\\ \mathsf{s} & - & \mathsf{t} & \end{smallmatrix}$ is a three-cell (i.e. oriented cube), then $\partial_3\left(\begin{smallmatrix} & \mathsf{z} & - & \mathsf{y}\\ \mathsf{v} & - & \mathsf{u} & \\ & \mathsf{w} & - & \mathsf{x}\\ \mathsf{s} & - & \mathsf{t} & \end{smallmatrix}\right) = \begin{smallmatrix}\mathsf{v} & \leftarrow & \mathsf{u}\\ \downarrow & & \uparrow\\ \mathsf{s} & \rightarrow & \mathsf{t}\end{smallmatrix} + \begin{smallmatrix}\mathsf{u} & \leftarrow & \mathsf{y}\\ \downarrow & & \uparrow\\ \mathsf{t} & \rightarrow & \mathsf{x}\end{smallmatrix} + \begin{smallmatrix}\mathsf{y} & \leftarrow & \mathsf{z}\\ \downarrow & & \uparrow\\ \mathsf{x} & \rightarrow & \mathsf{w}\end{smallmatrix} + \begin{smallmatrix}\mathsf{z} & \leftarrow & \mathsf{v}\\ \downarrow & & \uparrow\\ \mathsf{w} & \rightarrow & \mathsf{s}\end{smallmatrix} + \begin{smallmatrix}\mathsf{z} & \leftarrow & \mathsf{y}\\ \downarrow & & \uparrow\\ \mathsf{v} & \rightarrow & \mathsf{u}\end{smallmatrix} + \begin{smallmatrix}\mathsf{t} & \leftarrow & \mathsf{s}\\ \downarrow & & \uparrow\\ \mathsf{x} & \rightarrow & \mathsf{w}\end{smallmatrix}$.

In general, ∂_d can be computed by decomposing a cubic d-cell as a formal sum of d-simplices, computing the boundary of the resulting simplicial d-chain as in standard simplicial homology [**Hat02**, §2.1], and then expressing the result as a sum of $(d-1)$-cubes.

Let $(\mathcal{G}, +)$ be an abelian group, and let $\mathcal{C}^d = \mathcal{C}^d(\mathbf{Y}, \mathcal{G})$ be the set of (d-dimensional, $\mathcal{G}$-valued) *cochains*: i.e. homomorphisms $C : \mathbb{Z}[\mathbf{Y}^d] \longrightarrow \mathcal{G}$ (or equivalently, arbitrary functions $C : \mathbf{Y}^d \longrightarrow \mathcal{G}$). We define the *coboundary* homomorphism $\delta_d : \mathcal{C}^d \longrightarrow \mathcal{C}^{d+1}$ as follows: If $C \in \mathcal{C}^d$, then $\delta_d C \in \mathcal{C}^{d+1}$ is defined by $\delta_d C(\zeta) := C(\partial_{d+1}\zeta)$ for any $\zeta \in \mathbb{Z}[\mathbf{Y}^{d+1}]$. We say C is a *cocycle* if $\delta_d C \equiv 0$ (i.e. $C(\partial_{d+1}\zeta) = 0$ for any $\zeta \in \mathbb{Z}[\mathbf{Y}_{d+1}]$). Let $\mathcal{Z}^d := \ker(\delta_d)$ be the group of cocycles. We say C is a *coboundary* if there is some *cobounding function* $b \in \mathcal{C}^{d-1}$ such that $C = \delta_{d-1} b$. Let $\mathcal{B}^d := \delta_{d-1}(\mathcal{C}^{d-1})$ be the group of coboundaries. Two cocycles $C, C' \in \mathcal{Z}^d$ are *cohomologous* (notation: $C \approx C'$) if there is some coboundary $B := \delta_{d-1} b$ so that $C' = C + B$. Let $\underline{C}$ denote the cohomology class of C.

Example 3.1: If $d = 1$, then $C : \mathbf{Y}^1 \longrightarrow \mathcal{G}$ is a cocycle iff, for any two-cell $\begin{smallmatrix}\mathsf{z} & \leftarrow & \mathsf{y}\\ \downarrow & & \uparrow\\ \mathsf{w} & \rightarrow & \mathsf{x}\end{smallmatrix}$, we have $C(\mathsf{w}\to\mathsf{x}) + C(\mathsf{x}\to\mathsf{y}) + C(\mathsf{y}\to\mathsf{z}) + C(\mathsf{z}\to\mathsf{w}) = 0$. Equivalently, $C(\mathsf{w}\to\mathsf{x}) + C(\mathsf{x}\to\mathsf{y}) = C(\mathsf{w}\to\mathsf{z}) + C(\mathsf{z}\to\mathsf{y})$. By induction, this is equivalent to saying: for any $\mathsf{w}, \mathsf{z} \in \mathbb{Z}^D$ and any two chains $\zeta, \zeta' \in \mathbb{Z}[\mathbf{Y}^1]$ with $\partial_1(\zeta) = (\dot{\mathsf{y}} - \dot{\mathsf{w}}) = \partial_1(\zeta')$ (i.e. any two 'paths' from $\dot{\mathsf{w}}$ to $\dot{\mathsf{y}}$), we have $C(\zeta) = C(\zeta')$. Thus, C defines a function $\ddot{C} : \mathbb{Z}^D \times \mathbb{Z}^D \longrightarrow \mathcal{G}$, by $\ddot{C}(\mathsf{w}, \mathsf{y}) := C(\zeta)$, where $\zeta \in \mathbb{Z}[\mathbf{Y}^1]$ is any 1-chain with $\partial_1(\zeta) = (\dot{\mathsf{y}} - \dot{\mathsf{w}})$. The function $\ddot{C}$ is a *two-point cocycle*, by which we mean:

$$\forall\ \mathsf{w}, \mathsf{x}, \mathsf{y} \in \mathbb{Z}^D, \quad (\mathbf{a})\ \ddot{C}(\mathsf{y}, \mathsf{w}) = -\ddot{C}(\mathsf{w}, \mathsf{y}) \quad \text{and} \quad (\mathbf{b})\ \ddot{C}(\mathsf{y}, \mathsf{w}) = \ddot{C}(\mathsf{y}, \mathsf{x}) + \ddot{C}(\mathsf{x}, \mathsf{w}) \tag{11}$$

Conversely, any two-point cocycle $\ddot{C} : \mathbb{Z}^D \times \mathbb{Z}^D \longrightarrow \mathcal{G}$ defines a cocycle $C : \mathbf{Y}^1 \longrightarrow \mathcal{G}$ by $C(\mathsf{x}\to\mathsf{y}) := \ddot{C}(\mathsf{x}, \mathsf{y})$.

Also, $C \in \mathcal{B}^1$ iff there exists $b : \mathbb{Z}^D \longrightarrow \mathcal{G}$ such that, for any one-cell $(\mathsf{y}\to\mathsf{z})$, we have $C(\mathsf{y}\to\mathsf{z}) = b(\mathsf{z}) - b(\mathsf{y})$. Thus, $C \approx C'$ iff there exists $b : \mathbb{Z}^D \longrightarrow \mathcal{G}$ with $C'(\mathsf{y}\to\mathsf{z}) = b(\mathsf{z}) + C(\mathsf{y}\to\mathsf{z}) - b(\mathsf{y})$. $\diamondsuit$

3.3. *Homotopy/Homology groups for Wang Tiles:* Let $\mathcal{W}$ be a set of Wang tiles, and let $\mathfrak{W} \subset \mathcal{W}^{\mathbb{Z}^D}$ be the corresponding Wang subshift. If $w_1, w_2 \in \mathcal{W}$ and $d \in [1...D]$, then we will write "$w_1 \overset{d}{\rightsquigarrow} w_2$" to mean that the face $\partial_d^+ w_1$ is compatible with the face $\partial_d^- w_2$. The *tile complex* of $\mathfrak{W}$ is defined as follows: for each $\mathsf{z} \in \mathbb{Z}^D$ and $w \in \mathcal{W}$, let ${}^{\ulcorner}_{\mathsf{z}}w^{\urcorner}_{\lrcorner}$ be a D-cell, which we imagine as a D-dimensional unit cube 'labelled' by w, with its minimal-coordinate corner 'over' z. Let

$$\widetilde{\mathbf{X}} \quad := \quad \bigsqcup_{\mathsf{z} \in \mathbb{Z}^D} \bigsqcup_{w \in \mathcal{W}} {}^{\ulcorner}_{\mathsf{z}}w^{\urcorner}_{\lrcorner}.$$

For all $d \in [1...D]$, let $\partial_d^{\pm}\ {}^{\ulcorner}_{\mathsf{z}}w^{\urcorner}_{\lrcorner}$ be the two $(D-1)$-dimensional faces of ${}^{\ulcorner}_{\mathsf{z}}w^{\urcorner}_{\lrcorner}$ which are orthogonal to e_d. For any $\mathsf{z} \in \mathbb{Z}^D$ and $\mathsf{z}' := \mathsf{z} + \mathsf{e}_d$, and any $w, w' \in \mathcal{W}$, we will 'glue together' the faces $\partial_d^+\ {}^{\ulcorner}_{\mathsf{z}}w^{\urcorner}_{\lrcorner}$ and $\partial_d^-\ {}^{\ulcorner}_{\mathsf{z}'}w'^{\urcorner}_{\lrcorner}$ if and only if $w \overset{d}{\rightsquigarrow} w'$. Let $\sim$ be the equivalence relation on $\widetilde{\mathbf{X}}$ which instantiates all these gluing operations. Then $\mathbf{X} := \widetilde{\mathbf{X}}/\sim$ is a D-dimensional cell complex with the following properties:

(a) The D-cells of $\mathbf{X}$ are in bijective correspondence with the cubes comprising $\widetilde{\mathbf{X}}$.

(b) There is a natural continuous surjection $\Pi : \mathbf{X} \longrightarrow \mathbb{R}^D$, such that $\Pi_| : {}^{\ulcorner}_{\mathsf{z}} w^{\urcorner}_{\lrcorner} \longrightarrow {}^{\ulcorner}_{\mathsf{z}} \;{}^{\urcorner}_{\lrcorner}$, is a homeomorphism for each $\mathsf{z} \in \mathbb{Z}^D$ and $w \in \mathcal{W}$. (Here, ${}^{\ulcorner}_{\mathsf{z}} \;{}^{\urcorner}_{\lrcorner}$ is as in §3.1.)

(c) Π is a *cellular map*. For each $d \in [0...D]$, let $\mathbf{Y}^d$ be the d-skeleton of $\mathbb{R}^D$ from §3.1, let $\mathbf{X}^d$ be the d-skeleton of $\mathbf{X}$, and let $\Pi^d := \Pi_{|_{\mathbf{X}^d}}$. Then $\Pi^d : \mathbf{X}^d \twoheadrightarrow \mathbf{Y}^d$ surjectively. In particular, $\Pi^0 : \mathbf{X}^0 \twoheadrightarrow \mathbf{Y}^0 = \mathbb{Z}^D$ is a surjection (and in many cases, a bijection).

(d) Any continuous *section* of Π (i.e. a function $\varsigma : \mathbb{R}^D \longrightarrow \mathbf{X}$ such that $\Pi \circ \varsigma = \mathbf{Id}_{\mathbb{R}^D}$) assigns a unique $w \in \mathcal{W}$ to each D-cube in $\mathbb{R}^D$, and thereby defines a tiling $\mathbf{w}_\varsigma \in \mathfrak{W}$. Conversely, any admissible tiling $\mathbf{w} \in \mathfrak{W}$ determines a continuous section $\varsigma_{\mathbf{w}}$ of Π.

(e) For each $\mathsf{z} \in \mathbb{Z}^D$, let $\Xi^{\mathsf{z}} : \mathbf{X} \longrightarrow \mathbf{X}$ be the self-homeomorphism of $\mathbf{X}$ induced by translating all cells by z in the obvious way [i.e. for any $\mathsf{x} \in \mathbb{Z}^D$ and $w \in \mathcal{W}$, let $\Xi^{\mathsf{z}}\left({}^{\ulcorner}_{\mathsf{x}} w^{\urcorner}_{\lrcorner}\right) := {}^{\ulcorner}_{\mathsf{y}} w^{\urcorner}_{\lrcorner}$, where $\mathsf{y} := \mathsf{x} + \mathsf{z}$]. This defines a homeomorphic $\mathbb{Z}^D$-action on $\mathbf{X}$. Then $\Pi \circ \Xi^{\mathsf{z}} = \sigma^{\mathsf{z}} \circ \Pi$, and for any $\mathbf{w} \in \mathfrak{W}$, and $\mathsf{y} \in \mathbb{R}^D$, $\varsigma_{\sigma^{\mathsf{z}}(\mathbf{w})}(\mathsf{y}) = \Xi^{-\mathsf{z}} \circ \varsigma_{\mathbf{w}}(\mathsf{y} + \mathsf{z})$.

Fix $x \in \mathbf{X}^0$. For any $d \in [1...D]$, let $\overline{\pi}_d(\mathcal{W}) := \pi_d(\mathbf{X}, x)$ be the *dth homotopy group* of $\mathcal{W}$. Let $(\mathcal{G}, +)$ be an abelian group, and let $\overline{\mathcal{H}}_d(\mathcal{W}, \mathcal{G}) := \mathcal{H}_d(\mathbf{X}, \mathcal{G})$ and $\overline{\mathcal{H}}^d(\mathcal{W}, \mathcal{G}) := \mathcal{H}^d(\mathbf{X}, \mathcal{G})$ be the *dth homology group* and *cohomology group* respectively (with coefficients in $\mathcal{G}$). We will briefly review how to construct these groups in terms of the cellular structure of $\mathbf{X}$. Any element of $\overline{\pi}_d(\mathcal{W})$ can be represented as a continuous function $\xi : (\mathbb{S}^d, s) \longrightarrow (\mathbf{X}^d, x)$ (where $(\mathbb{S}^d, s)$ is as in §1.2), and two such functions represent the same element of $\overline{\pi}_d(\mathcal{W}, x)$ if and only if they are homotopic in $(\mathbf{X}^{d+1}, x)$ (in a basepoint-fixing way); see [**Hat02**, Corollary 4.12]. Let $\mathcal{C}_d := \bigoplus_{x \in \mathbf{X}_d} \mathcal{G}$ be the group of *d-dimensional $\mathcal{G}$-chains* (i.e. functions $\mathbf{X}^d \longrightarrow \mathcal{G}$ with only finitely many nontrivial entries). There is a natural 'boundary' homomorphism $\partial_d : \mathcal{C}_d \longrightarrow \mathcal{C}_{d-1}$ (see §3.2), and $\overline{\mathcal{H}}_d(\mathcal{W}, \mathcal{G}) := \mathcal{Z}_d / \mathcal{B}_d$, where $\mathcal{Z}_d := \ker(\partial_d)$ and $\mathcal{B}_d := \mathrm{img}\,(\partial_{d+1})$. Let $\mathcal{C}^d := \mathcal{G}^{\mathbf{X}_d}$ be the group of *d-dimensional $\mathcal{G}$-cochains* i.e. all functions $\mathbf{X}^d \longrightarrow \mathcal{G}$, or equivalently, all homomorphisms $\mathbb{Z}[\mathbf{X}^d] \longrightarrow \mathcal{G}$. There is a natural 'coboundary' homomorphism $\delta_d : \mathcal{C}^d \longrightarrow \mathcal{C}^{d+1}$ defined by $\delta_d(\eta) = \eta \circ \partial_d$. Then $\overline{\mathcal{H}}^d(\mathcal{W}, \mathcal{G}) := \mathcal{Z}^d / \mathcal{B}^d$, where $\mathcal{Z}^d := \ker(\delta_d)$ and $\mathcal{B}^d := \mathrm{img}\,(\delta_{d-1})$.

Example 3.2: (a) Let $D = 2$, so that $\mathcal{W}$ is a set of two-dimensional square tiles with edge-matching conditions (e.g. edge 'colours'). Then $\overline{\pi}_1(\mathcal{W})$ is the Conway-Lagarias *tile homotopy group* of [**CL90**, §3]; see also [**Thu90**], [**Pro97**] or [**Rei03**, §4].

To see this, note that $\mathbf{X}$ is a two-dimensional cell-complex obtained by taking a collection $\widetilde{\mathbf{X}}$ of $\mathcal{W}$-labelled unit squares and gluing them along their edges in accordance with the edge-matching conditions of $\mathcal{W}$. For example, Figure 3 shows a fragment of the tile complex for the domino tiles $\mathfrak{Dom}$ from Example 1.5(b). Let $\mathcal{H}$ (resp. $\mathcal{V}$) be the set of 'colours' of horizontal (resp. vertical) tile edges in $\mathcal{W}$. Assume that $\mathcal{H}$ and $\mathcal{V}$ are disjoint, and let $\mathcal{C}$ be the free group generated by $\mathcal{H} \sqcup \mathcal{V}$. Any element of $\overline{\pi}_1(\mathcal{W})$ corresponds to a continuous function $\xi : \mathbb{S}^1 \longrightarrow \mathbf{X}^1$ —i.e. a closed continuous path along the edges of the tile complex, beginning and ending at zero. The function ξ defines an element $c_1^{\pm 1} c_2^{\pm 1} \cdots c_k^{\pm 1} \in \mathcal{C}$, where

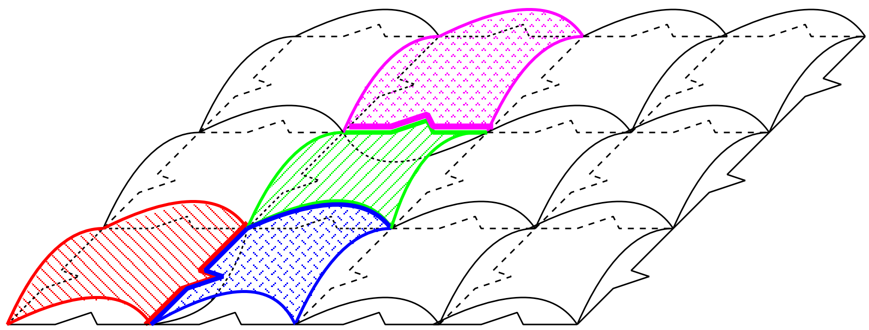


FIGURE 3. A fragment of the tile complex $\mathbf{X}$ for $\mathfrak{Dom}$. The elements of $\mathbf{X}^0$ are in bijective correspondence with $\mathbb{Z}^2$. Between each pair of adjacent vertices in $\mathbf{X}_0$, we adjoin two edges: one 'straight' edge, and one 'notched' edge; we define $\mathbf{X}^1$ to be the union of all these edges. In every square bounded by four vertices of $\mathbf{X}_0$, there are four distinct 2-cells, each of which has exactly three 'straight' edges and one 'notched' edge (four such 2-cells are depicted in the figure). We define $\mathbf{X}^2$ to be the union of all such 2-cells.

c_1 is the colour of the first edge traversed by ξ and we put c^{+1} if ξ heads east or north along this edge, and c^{-1} if ξ heads west or south. Likewise c_2 the colour of the second edge, with the same sign convention, and so on. The word $c_1^{\pm 1} c_2^{\pm 1} \cdots c_k^{\pm 1}$ satisfies two constraints:

[i] If N, E, S, and W are the total # of northward, eastward, southward, and westward edges (as indicated by the colours and sign conventions), then $N = S$ and $E = W$.

[ii] c_1 and c_k must be the colours of edges coming into or out of the vertex x.

(If $\Pi^0 : \mathbf{X}^0 \longrightarrow \mathbb{Z}^D$ is bijective, as is the case in [**CL90, Rei03, Thu90, Pro97**], then condition [ii] is trivial.) Let $\mathcal{D} \subset \mathcal{C}$ be the subgroup of elements satisfying [i] and [ii]. Let $\mathcal{N}$ be the normal subgroup of $\mathcal{C}$ generated by all words of the form $sen^{-1}w^{-1}$, where w, n, e, and s are the four edge colours of any tile in $\mathcal{W}$. Let $\mathcal{G} := \mathcal{D}/\mathcal{N}$. Then $\mathcal{G}$ is the Conway-Lagarias group, and $\mathcal{G} \cong \overline{\pi}_1(\mathcal{W})$ (because a nullhomotopy of ξ is equivalent to an algebraic reduction of $c_1^{\pm 1} c_2^{\pm 1} \cdots c_k^{\pm 1}$ to an element of $\mathcal{N}$).

Similarly, $\overline{\mathcal{H}}_1(\mathcal{W}, \mathbb{Z})$ is the Conway-Lagarias *tile homology group* [**CL90**, §5], [**Rei03**, §2].

(b) Let $\mathcal{I}$ be as in Example 1.5(a). We apply (a) to show that $\overline{\pi}_1(\mathcal{I}) \cong \mathbb{Z}$. In this case, $\mathcal{H} := \{A, V\}$ and $\mathcal{V} := \{\prec, \succ\}$, and $\mathcal{N}$ is the normal subgroup generated by

$$\{A{\prec}A^{-1}{\prec}^{-1},\ A{\succ}A^{-1}{\succ}^{-1},\ V{\prec}V^{-1}{\prec}^{-1},\ V{\succ}V^{-1}{\succ}^{-1},\ V{\prec}A^{-1}{\succ}^{-1},\ A{\succ}V^{-1}{\prec}^{-1}\}. \quad (12)$$

Let $\overline{A}, \overline{V}, \overline{\succ}$ and $\overline{\prec}$ be the corresponding generators of $\mathcal{C}/\mathcal{N}$. The first four generators in eqn.(12) make $\mathcal{C}/\mathcal{N}$ abelian, so we will switch to additive notation. Thus, any element of $\mathcal{C}/\mathcal{N}$ has the form $a\overline{A} + v\overline{V} + p\overline{\succ} + q\overline{\prec}$ for some $a, v, p, q \in \mathbb{Z}$, and $\mathcal{D}/\mathcal{N}$ is the subgroup of elements satisfying $a = -v$ and $p = -q$ (from condition [i]). Now the last two generators in eqn.(12) both say that $\overline{\prec} = \overline{A} - \overline{V} + \overline{\succ}$. Thus, $a\overline{A} + v\overline{V} + p\overline{\succ} + q\overline{\prec} = (a+q)\overline{A} + (v-q)\overline{V} + (p+q)\overline{\succ} = (a+q)\overline{A} + (-a-q)\overline{V} = (a+q)(\overline{A} - \overline{V})$. Thus, if we define $G := \overline{A} - \overline{V}$, then $\overline{\pi}_1(\mathcal{I}) \cong \mathcal{D}/\mathcal{N}$ is the cyclic group generated by G; hence $\overline{\pi}_1(\mathcal{I}) \cong \mathbb{Z}$. $\diamondsuit$

3.4. *Wang representations:* We can represent any subshift of finite type using Wang tiles. To do this requires some notation. For any $r \in \mathbb{N}$, recall that $\mathbb{B}(r) := [-r...r]^D \subset \mathbb{Z}^D$ is the cube of sidelength $(2r+1)$. For any $d \in [1...D]$, we define the 'top' and 'bottom' faces of this cube in the dth dimension by:

$$\partial_d^+ \mathbb{B}(r) := [-r...r]^{d-1} \times (-r...r] \times [-r...r]^{D-d-1} \quad \text{and} \quad \partial_d^- \mathbb{B}(r) := [-r...r]^{d-1} \times [-r...r) \times [-r...r]^{D-d-1}$$

Now, let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift of finite type defined by a set $\mathfrak{A}_{(R)} \subset \mathcal{A}^{\mathbb{B}(R)}$ of admissible R-blocks. For any $r \geq R$, the *radius r Wang representation* of $\mathfrak{A}$ is defined as follows: let $\mathcal{W}_r := \mathfrak{A}_{(r)}$, and for any $\mathbf{a}, \mathbf{b} \in \mathcal{W}_r$, and any $d \in [1...D]$, we allow $\mathbf{a} \overset{d}{\leadsto} \mathbf{b}$ if and only if $\mathbf{a}_{\partial_d^+ \mathbb{B}(r)} = \mathbf{b}_{\partial_d^- \mathbb{B}(r)}$. For example, suppose $D = 2$ and $r = 1$; and suppose

$$\mathbf{a} = \begin{bmatrix} a_{-1,1} & a_{0,1} & a_{1,1} \\ a_{-1,0} & a_{0,0} & a_{1,0} \\ a_{-1,-1} & a_{0,-1} & a_{1,-1} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_{-1,1} & b_{0,1} & b_{1,1} \\ b_{-1,0} & b_{0,0} & b_{1,0} \\ b_{-1,-1} & b_{0,-1} & b_{1,-1} \end{bmatrix}$$

Then $\mathbf{a} \overset{1}{\leadsto} \mathbf{b}$ iff $\begin{bmatrix} a_{0,1} & a_{1,1} \\ a_{0,0} & a_{1,0} \\ a_{0,-1} & a_{1,-1} \end{bmatrix} = \begin{bmatrix} b_{-1,1} & b_{0,1} \\ b_{-1,0} & b_{0,0} \\ b_{-1,-1} & b_{0,-1} \end{bmatrix}$, and $\mathbf{a} \overset{2}{\leadsto} \mathbf{b}$ iff $\begin{bmatrix} a_{-1,1} & a_{0,1} & a_{1,1} \\ a_{-1,0} & a_{0,0} & a_{1,0} \end{bmatrix} = \begin{bmatrix} b_{-1,0} & b_{0,0} & b_{1,0} \\ b_{-1,-1} & b_{0,-1} & b_{1,-1} \end{bmatrix}$.

3.5. *Projective Homotopy/Homology groups for SFTs* Using the Wang representation of §3.4, we can define homotopy/homology groups for any subshift of finite type as in §3.3. There are two problems, however:

[i] There are many different Wang tile representations for any SFT, and none of them is 'canonical'. Different Wang representations may yield non-isomorphic groups.

[ii] Wang tile representations (and hence, the corresponding homotopy/homology groups) do not behave well under subshift homomorphisms (e.g. cellular automata).

To obviate these problems, we use an inverse limit which encompasses 'all possible' Wang tile representations within a single algebraic structure.

Throughout this section, let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift of finite type of radius R. For any $r \geq R$, let $\mathcal{W}_r := \mathfrak{A}_{(r)}$, and let $\mathfrak{W}_r \subset \mathcal{W}_r^{\mathbb{Z}^D}$ be the Wang representation of $\mathfrak{A}$ from §3.4. Let $\mathbf{X}_r = (\mathbf{X}_r^0, \ldots, \mathbf{X}_r^D)$ be the corresponding tile complex; hence the D-cells of $\mathbf{X}_r$ have the form $\ulcorner_{\mathsf{z}} \mathbf{b} \urcorner$, where $\mathsf{z} \in \mathbb{Z}^D$ and $\mathbf{b} \in \mathfrak{A}_{(r)}$ is an $\mathfrak{A}$-admissible $\mathbb{B}(r)$-block. Let $\Pi_r : \mathbf{X}_r \longrightarrow \mathbb{R}^D$ be the natural projection map; then there is a natural bijective correspondence between:

- $\mathfrak{A}$-admissible configurations in $\mathcal{A}^{\mathbb{Z}^D}$.
- $\mathcal{W}_r$-tilings of $\mathbb{R}^D$ (satisfying the relevant edge-matching constraints).
- Continuous sections $\varsigma : \mathbb{R}^D \longrightarrow \mathbf{X}_r$ of Π_r.

Fix $\mathbf{a} \in \mathfrak{A}$, and let $\mathbf{a}_r := \mathbf{a}_{\mathbb{B}(r)}$, so that $\ulcorner_0 \mathbf{a}_r \urcorner$ is a D-cell in $\mathbf{X}_r$. Let $x_r = x_r(\mathbf{a})$ be the unique element of the singleton set $\Pi_r^{-1}\{0\} \cap \ulcorner_0 \mathbf{a}_r \urcorner$; then x_r is a corner vertex of $\ulcorner_0 \mathbf{a}_r \urcorner$. Define $\pi_d^r(\mathfrak{A}, \mathbf{a}) := \pi_d(\mathbf{X}_r, x_r)$. For example, if $\mathcal{W}$ is a set of Wang tiles and $\mathfrak{W} \subset \mathcal{W}^{\mathbb{Z}^D}$ is the corresponding Wang subshift, then $\pi_d^0(\mathfrak{W}, \mathbf{w}) = \overline{\pi}_d(\mathcal{W})$.

There is a natural continuous surjection $\zeta_r : \mathbf{X}_{r+1} \longrightarrow \mathbf{X}_r$ where, for any $\mathbf{a} \in \mathfrak{A}_{(r+1)}$, if $\mathbf{a}' := \mathbf{a}_{\mathbb{B}(r)}$, then for any $\mathsf{z} \in \mathbb{Z}^D$, $\zeta_| : \ulcorner_{\mathsf{z}} \mathbf{a} \urcorner \longrightarrow \ulcorner_{\mathsf{z}} \mathbf{a}' \urcorner$ is a homeomorphism. Furthermore, ζ_r is a *cellular map*: i.e. $\zeta(\mathbf{X}^d_{r+1}) \subseteq \mathbf{X}^d_r$ for all $d \in [0...D]$. Also, $\zeta_r(x_{r+1}) = x_r$. This induces homomorphisms $\pi_d\zeta_r : \pi^{r+1}_d(\mathfrak{A}, \mathbf{a}) \longrightarrow \pi^r_d(\mathfrak{A}, \mathbf{a})$ for all $d \in [1...D]$. We define

$$\pi_d(\mathfrak{A}, \mathbf{a}) \quad := \quad \varprojlim \left(\pi^R_d(\mathfrak{A}, \mathbf{a}) \xleftarrow{\pi_d\zeta_R} \pi^{R+1}_d(\mathfrak{A}, \mathbf{a}) \xleftarrow{\pi_d\zeta_{R+1}} \pi^{R+2}_d(\mathfrak{A}, \mathbf{a}) \xleftarrow{\pi_d\zeta_{R+2}} \cdots \right)$$

[Note: $\pi_d(\mathfrak{A}, \mathbf{a})$ is *not* the homotopy group of $\mathfrak{A}$ as a (zero-dimensional) topological space.] For example, if $d = 1$, then $\pi_1(\mathfrak{A})$ is the *projective fundamental group* of [**GP95**] (the cell-complex $\mathbf{X}_r$ is a dual version of the cellular realization of 'scenery space' on [**GP95**, p.1101]). We would like this definition to be independent of the choice of 'basepoint' $\mathbf{a}$. This will be true if $\mathfrak{A}$ is *projectively connected* in the sense of [**GP95**, p.1098], but we also have the following criteria. Recall that a subshift $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ is *topologically weakly mixing* if the Cartesian product system $(\mathfrak{A} \times \mathfrak{A}, \sigma \times \sigma)$ is topologically transitive.

PROPOSITION 3.3. *Fix* $\mathbf{a}, \mathbf{b} \in \mathfrak{A}$.

(a) *Suppose* $\Pi^0_r : \mathbf{X}^0_r \longrightarrow \mathbb{Z}^D$ *is injective for all large enough* $r \in \mathbb{N}$. *Then for any* $d \geq 1$, *there is a canonical isomorphism* $\pi_d(\mathfrak{A}, \mathbf{a}) \cong \pi_d(\mathfrak{A}, \mathbf{b})$.

Suppose $(\mathfrak{A}, \sigma)$ *is topologically weakly mixing. Then:*

(b) *If* $\pi_1(\mathfrak{A}, \mathbf{a})$ *is abelian, then there is a canonical isomorphism* $\pi_1(\mathfrak{A}, \mathbf{a}) \cong \pi_1(\mathfrak{A}, \mathbf{b})$.

(c) *If* $\pi_1(\mathfrak{A}, \mathbf{a})$ *is trivial, then there are canonical isomorphisms* $\pi_d(\mathfrak{A}, \mathbf{a}) \cong \pi_d(\mathfrak{A}, \mathbf{b})$ *for all* $d \in \mathbb{N}$.

Proof: **(a)** If $\Pi^0_r : \mathbf{X}^0_r \longrightarrow \mathbb{Z}^D$ is injective, then $(\Pi^0_r)^{-1}\{0\}$ is a singleton, which means $x_r(\mathbf{a}) = x_r(\mathbf{b})$. Hence, $\pi^r_d(\mathfrak{A}, \mathbf{a}) := \pi_d[\mathbf{X}_r, x_r(\mathbf{a})] = \pi_d[\mathbf{X}_r, x_r(\mathbf{b})] =: \pi^r_d(\mathfrak{A}, \mathbf{b})$. If this holds for all large $r \in \mathbb{N}$ then clearly $\pi_d(\mathfrak{A}, \mathbf{a}) = \pi_d(\mathfrak{A}, \mathbf{b})$. This proves **(a)**. For **(b,c)** we need:

CLAIM 1: *If* $(\mathfrak{A}, \sigma)$ *is topologically weakly mixing, then for all* $r \geq R$, *the space* $\mathbf{X}_r$ *is path connected.*

Proof: Fix $\mathbf{c} \in \mathfrak{A}_{(r)}$, and let $[\mathbf{c}] := \left\{\mathbf{d} \in \mathfrak{A} \; ; \; \mathbf{d}_{\mathbb{B}(r)} = \mathbf{c}\right\}$ be the corresponding cylinder set. Likewise, let $[\mathbf{a}_{\mathbb{B}(r)}]$ and $[\mathbf{b}_{\mathbb{B}(r)}]$ be the cylinder sets defined by $\mathbf{a}_{\mathbb{B}(r)}$ and $\mathbf{b}_{\mathbb{B}(r)}$. The system $(\mathfrak{A} \times \mathfrak{A}, \sigma \times \sigma)$ is transitive (because $(\mathfrak{A}, \sigma)$ is weakly mixing), so there is some z such that $(\sigma^{\mathsf{z}}[\mathbf{a}_{\mathbb{B}(r)}] \times \sigma^{\mathsf{z}}[\mathbf{b}_{\mathbb{B}(r)}]) \cap ([\mathbf{c}] \times [\mathbf{c}]) \neq \emptyset$. Hence, there exist $\mathbf{a}', \mathbf{b}' \in \mathfrak{A}$ such that $\mathbf{a}'_{\mathbb{B}(r)} = \mathbf{a}_{\mathbb{B}(r)}$ and $\mathbf{b}'_{\mathbb{B}(r)} = \mathbf{b}_{\mathbb{B}(r)}$, but also $\mathbf{a}'_{\mathbb{B}(\mathsf{z},r)} = \mathbf{c} = \mathbf{b}'_{\mathbb{B}(\mathsf{z},r)}$. Clearly, $x_r(\mathbf{a}') = x_r(\mathbf{a})$ and $x_r(\mathbf{b}') = x_r(\mathbf{b})$.

Let $\varsigma^r_{\mathbf{a}'}, \varsigma^r_{\mathbf{b}'} : \mathbb{R}^D \longrightarrow \mathbf{X}_r$ be the continuous sections of Π_r defined by $\mathbf{a}'$ and $\mathbf{b}'$. Let $\gamma : [0,1] \longrightarrow \mathbb{R}$ be a continuous path, with $\gamma(0) = 0$ and $\gamma(1) = \mathsf{z}$. If $\alpha := \varsigma^r_{\mathbf{a}'} \circ \gamma$, then $\alpha(0) = x_r(\mathbf{a})$ and $\alpha(1) = \Xi^{\mathsf{z}}(x_r(\mathbf{c}))$. Likewise, if $\beta := \varsigma^r_{\mathbf{b}'} \circ \gamma$. then $\beta(0) = x_r(\mathbf{b})$ and $\beta(1) = \Xi^{\mathsf{z}}(x_r(\mathbf{c}))$. If $\gamma_r := \overleftarrow{\beta} \star \alpha : [0,1] \longrightarrow \mathbf{X}_r$, then $\gamma_r(0) = x_r(\mathbf{a})$ and $\gamma_r(1) = x_r(\mathbf{b})$, as desired. $\diamond$ `Claim 1`

Let $r \geq R$. Any path $\gamma : [0,1] \longrightarrow \mathbf{X}_r$ from $x_r(\mathbf{a})$ to $x_r(\mathbf{b})$ (such as in Claim 1) yields an isomorphism $\gamma_* : \pi_d(\mathfrak{A}, \mathbf{a}) \longrightarrow \pi_d(\mathfrak{A}, \mathbf{b})$ (see §1.2). If $\eta : [0,1] \longrightarrow \mathbf{X}_r$ is another path from $x_r(\mathbf{a})$ to $x_r(\mathbf{b})$, and $\gamma \approx \eta$, then $\gamma_* = \eta_*$.

(c) If $\pi^r_1(\mathfrak{A}, \mathbf{a}) = \pi_1(\mathbf{X}_r, x_r(\mathbf{a}))$ is trivial, then any two such such paths γ and η are homotopic. Hence in this case there is a canonical isomorphism $I^r_d : \pi^r_d(\mathfrak{A}, \mathbf{a}) \longrightarrow \pi^r_d(\mathfrak{A}, \mathbf{b})$, which is independent of the choice of path. This yields a commuting ladder with canonical isomorphisms for rungs:

$$\begin{array}{cccccccc}
\pi_d^R(\mathfrak{A},\mathbf{a}) & \xleftarrow{\pi_d\zeta_R} & \pi_d^{R+1}(\mathfrak{A},\mathbf{a}) & \xleftarrow{\pi_d\zeta_{R+1}} & \pi_d^{R+2}(\mathfrak{A},\mathbf{a}) & \xleftarrow{\pi_d\zeta_{R+2}} \cdots & \text{which yields} & \pi_d(\mathfrak{A},\mathbf{a}) \\
I_d^R \Big\Downarrow & & I_d^{R+1} \Big\Downarrow & & I_d^{R+2} \Big\Downarrow & & \begin{array}{c}\text{a canonical}\\ \text{isomorphism}\end{array} & \Big\Downarrow \\
\pi_d^R(\mathfrak{A},\mathbf{b}) & \xleftarrow[\pi_d\zeta_R]{} & \pi_d^{R+1}(\mathfrak{A},\mathbf{b}) & \xleftarrow[\pi_d\zeta_{R+1}]{} & \pi_d^{R+2}(\mathfrak{A},\mathbf{b}) & \xleftarrow[\pi_d\zeta_{R+2}]{} \cdots & \text{of colimits:} & \pi_d(\mathfrak{A},\mathbf{b}).
\end{array}$$

(b) If $\pi_1(\mathbf{X}_r, x_r(\mathbf{a}))$ is not trivial, and γ and η are non-homotopic paths from $x_r(\mathbf{a})$ to $x_r(\mathbf{b})$, then in general $\gamma_* \neq \eta_*$. Hence, $\eta_*^{-1} \circ \gamma_*$ will be a nontrivial automorphism of $\pi_1(\mathbf{X}_r, x_r(\mathbf{a}))$. Indeed, if $\alpha := \overleftarrow{\eta} \star \gamma$, then α is a closed loop based at $x_r(\mathbf{a})$, hence $\underline{\alpha} \in \pi_1(\mathbf{X}_r, x_r(\mathbf{a}))$, and the automorphism $\eta_*^{-1} \circ \gamma_* = (\overleftarrow{\eta} \star \gamma)_* = \alpha_*$: $\pi_1(\mathbf{X}_r, x_r(\mathbf{a})) \longrightarrow \pi_1(\mathbf{X}_r, x_r(\mathbf{a}))$ is simply the inner automorphism $\alpha_*(\underline{\beta}) = \underline{\alpha}^{-1} \cdot \underline{\beta} \cdot \underline{\alpha}$. But if $\pi_1(\mathbf{X}_r, x_r(\mathbf{a}))$ is abelian, then all inner automorphisms are trivial; hence $\alpha_* = \mathbf{Id}$, hence $\gamma_* = \eta_*$ after all. Thus, the isomorphism $I_1^r : \pi_1^r(\mathfrak{A},\mathbf{a}) \longrightarrow \pi_1^r(\mathfrak{A},\mathbf{b})$ once again well-defined independent of the choice of path from $x_r(\mathbf{a})$ to $x_r(\mathbf{b})$. Now proceed as in **(b)**. □

If any of the conditions of Proposition 3.3 is satisfied, then we say that $\mathfrak{A}$ is *basepoint-free* in codimension $d+1$. We will then write "$\pi_d(\mathfrak{A})$" to mean "$\pi_d(\mathfrak{A},\mathbf{a})$", where $\mathbf{a} \in \mathfrak{A}$ is arbitrary.

EXAMPLE 3.4: Let $\mathfrak{Ice}$ be as in Example 1.5(c). Then $\pi_1(\mathfrak{Ice}) = \mathbb{Z}$ [**GP95**, Theorem 3]. ◇

The (co)homology groups do not require a basepoint. For any $d \in [0...D]$, any $r \geq R$, and any abelian group $(\mathcal{G},+)$, we define $\mathcal{H}_d^r(\mathfrak{A},\mathcal{G}) := \mathcal{H}_d(\mathbf{X}_r,\mathcal{G})$ and $\mathcal{H}_r^d(\mathfrak{A},\mathcal{G}) := \mathcal{H}^d(\mathbf{X}_r,\mathcal{G})$ (see §3.3). For example, if $\mathcal{W}$ is a set of Wang tiles and $\mathfrak{W} \subset \mathcal{W}^{\mathbb{Z}^D}$ is the corresponding Wang subshift, then $\mathcal{H}_d^0(\mathfrak{W},\mathcal{G}) = \overline{\mathcal{H}}_d(\mathcal{W},\mathcal{G})$ and $\mathcal{H}_0^d(\mathfrak{W},\mathcal{G}) = \overline{\mathcal{H}}^d(\mathcal{W},\mathcal{G})$. The cellular maps ζ_r induce homomorphisms $\mathcal{H}_d\zeta_r \colon \mathcal{H}_d^{r+1}(\mathfrak{A}) \to \mathcal{H}_d^r(\mathfrak{A})$ for all $d \in [0...D]$. We define

$$\mathcal{H}_d(\mathfrak{A},\mathcal{G}) \quad := \quad \varprojlim \left(\mathcal{H}_d^R(\mathfrak{A},\mathcal{G}) \xleftarrow{\mathcal{H}_d\zeta_R} \mathcal{H}_d^{R+1}(\mathfrak{A},\mathcal{G}) \xleftarrow{\mathcal{H}_d\zeta_{R+1}} \mathcal{H}_d^{R+2}(\mathfrak{A},\mathcal{G}) \xleftarrow{\mathcal{H}_d\zeta_{R+2}} \cdots \right)$$

The functions ζ_r also induce (contravariant) homomorphisms $\mathcal{H}^d\zeta_r \colon \mathcal{H}_r^d(\mathfrak{A},\mathcal{G}) \to \mathcal{H}_{r+1}^d(\mathfrak{A},\mathcal{G})$ for all $d \in [0...D]$. We define

$$\mathcal{H}^d(\mathfrak{A},\mathcal{G}) \;:=\; \varinjlim \left(\mathcal{H}_R^d(\mathfrak{A},\mathcal{G}) \xrightarrow{\mathcal{H}^d\zeta_R} \mathcal{H}_{R+1}^d(\mathfrak{A},\mathcal{G}) \xrightarrow{\mathcal{H}^d\zeta_{R+1}} \mathcal{H}_{R+2}^d(\mathfrak{A},\mathcal{G}) \xrightarrow{\mathcal{H}^d\zeta_{R+2}} \cdots \right) \quad (13)$$

(See [**Hat02**, §3.F] or [**Lan84**, §III.9] for background on direct limits.) We then have the following generalizations of [**GP95**, Theorem 1, §4]:

PROPOSITION 3.5. **(a)** *Let* $\mathfrak{A} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ *and* $\mathfrak{B} \subseteq \mathcal{B}^{\mathbb{Z}^D}$ *be* D*-dimensional SFTs, and let* $\Phi \colon \mathfrak{A} \to \mathfrak{B}$ *be a subshift homomorphism. Let* $\mathbf{a} \in \mathfrak{A}$ *and let* $\mathbf{b} := \Phi(\mathbf{a})$*. Then* Φ *induces group homomorphisms* $\pi_d\Phi \colon \pi_d(\mathfrak{A},\mathbf{a}) \longrightarrow \pi_d(\mathfrak{B},\mathbf{b})$ *for all* $d \in \mathbb{N}$*, and homomorphisms* $\mathcal{H}_d\Phi \colon \mathcal{H}_d(\mathfrak{A},\mathcal{G}) \longrightarrow \mathcal{H}_d(\mathfrak{B},\mathcal{G})$ *and* $\mathcal{H}^d\Phi \colon \mathcal{H}^d(\mathfrak{B},\mathcal{G}) \longrightarrow \mathcal{H}^d(\mathfrak{A},\mathcal{G})$*, for all* $d \in [0...D]$*.*

(b) *In particular, if* $\Phi \colon \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ *is a CA and* $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$*, then* Φ *induces a group homomorphism* $\pi_d\Phi \colon \pi_d(\mathfrak{A},\mathbf{a}) \longrightarrow \pi_d(\mathfrak{A},\mathbf{b})$ *and group endomorphisms* $\mathcal{H}_d\Phi \colon \mathcal{H}_d(\mathfrak{A},\mathcal{G}) \longrightarrow \mathcal{H}_d(\mathfrak{A},\mathcal{G})$ *and* $\mathcal{H}^d\Phi \colon \mathcal{H}^d(\mathfrak{A},\mathcal{G}) \longrightarrow \mathcal{H}^d(\mathfrak{A},\mathcal{G})$*.*

Proof: **(a)** Suppose $\mathfrak{A}$ has radius R_A, and $\mathfrak{B}$ has radius R_B, and let $R := \max\{R_A, R_B\}$. If Φ has radius q, then for any $r > R$, Φ induces a natural cellular map $\Phi_* : \mathbf{X}_{r+q}(\mathfrak{A}) \longrightarrow \mathbf{X}_r(\mathfrak{B})$, such that, for any $\mathbf{a} \in \mathfrak{A}_{(r+q)}$ and $\mathbf{b} := \Phi(\mathbf{a}) \in \mathfrak{B}_{(r)}$, and any $\mathsf{z} \in \mathbb{Z}^D$, the restriction $(\Phi_*)_| : {}_{\mathsf{z}}\ulcorner\mathbf{a}\lrcorner \longrightarrow {}_{\mathsf{z}}\ulcorner\mathbf{b}\lrcorner$ is a homeomorphism. Thus, $\Phi_*(x_{r+q}(\mathbf{a})) = x_r(\mathbf{b})$. Thus we get a homomorphism $\pi_k^r\Phi : \pi_d[\mathbf{X}_{r+q}(\mathfrak{A}), x_{r+q}(\mathbf{a})] \longrightarrow \pi_d[\mathbf{X}_r(\mathfrak{B}), x_r(\mathbf{b})]$ —i.e. a function $\pi_k^r\Phi : \pi_d^{r+q}(\mathfrak{A}, \mathbf{a}) \longrightarrow \pi_d^r(\mathfrak{B}, \mathbf{b})$. This yields a commuting ladder of homomorphisms:

$$\begin{array}{ccccccccc}
\pi_d^{r+q}(\mathfrak{A},\mathbf{a}) & \xleftarrow{\pi_d\zeta_{r+q}} & \pi_d^{r+q+1}(\mathfrak{A},\mathbf{a}) & \xleftarrow{\pi_d\zeta_{r+q+1}} & \pi_d^{r+q+2}(\mathfrak{A},\mathbf{a}) & \xleftarrow{\pi_d\zeta_{r+q+2}} & \cdots & \text{which yields a} & \pi_d(\mathfrak{A},\mathbf{a}) \\
{\scriptstyle \pi_k^r\Phi}\downarrow & & {\scriptstyle \pi_k^{r+1}\Phi}\downarrow & & {\scriptstyle \pi_k^{r+2}\Phi}\downarrow & & & \text{homomorphism} & {\scriptstyle \pi_k\Phi}\downarrow \\
\pi_d^r(\mathfrak{B},\mathbf{b}) & \xleftarrow[\pi_d\zeta_r]{} & \pi_d^{r+1}(\mathfrak{B},\mathbf{b}) & \xleftarrow[\pi_d\zeta_{r+1}]{} & \pi_d^{r+2}(\mathfrak{B},\mathbf{b}) & \xleftarrow[\pi_d\zeta_{r+2}]{} & \cdots & \text{of colimits:} & \pi_d(\mathfrak{B},\mathbf{b}).
\end{array}$$

The (co)homology group proof is analogous. **(b)** follows from **(a)**. □

For any $\mathbf{a} \in \widetilde{\mathfrak{A}}$ and $r \geq R$, there is a continuous section $\varsigma_{\mathbf{a}}^r : \mathbb{G}_r(\mathbf{a}) \longrightarrow \mathbf{X}_r$ such that $\Pi_r \circ \varsigma_{\mathbf{a}}^r = \mathbf{Id}_{\mathbb{G}_r(\mathbf{a})}$. If $k \in \mathbb{N}$ and $\mathbf{a}$ has a range-r, codimension-$(k+1)$ defect, and $0 \in \mathbb{G}_r(\mathbf{a})$, then $\pi_k[\mathbb{G}_r(\mathbf{a}), 0]$ is nontrivial. Then $\varsigma_{\mathbf{a}}^r$ induces a group homomorphism $\pi_k^r\mathbf{a} : \pi_k[\mathbb{G}_r(\mathbf{a}), 0] \longrightarrow \pi_k(\mathbf{X}_r, x_r(\mathbf{a})) = \pi_k^r(\mathfrak{A}, \mathbf{a})$, defined by $\pi_k^r\mathbf{a}(\underline{\gamma}) := \underline{\varsigma_{\mathbf{a}}^r \circ \gamma}$ for all $\underline{\gamma} \in \pi_k[\mathbb{G}_r(\mathbf{a}), 0]$.

EXAMPLE 3.6: Recall from Example 3.2(b) that $\pi_1^1(\mathfrak{I}_{\mathfrak{ce}}) \cong \overline{\pi}_1(\mathcal{I}) \cong \mathbb{Z}$, and is generated by the element $G = \overline{V} - \overline{A}$. Let $\mathbf{i} \in \widetilde{\mathfrak{I}_{\mathfrak{ce}}}$ be as in Figure 1(D). Then $\mathbb{G}_1(\mathbf{i})$ is a punctured plane, so $\pi_1[\mathbb{G}_1(\mathbf{i})] \cong \mathbb{Z}$. If ζ is a path in $\mathbb{G}_1(\mathbf{i})$ that goes once counterclockwise around the defect, then ζ generates $\pi_1[\mathbb{G}_1(\mathbf{i})]$, and $\pi_1^1\mathbf{i}(\zeta) = 2\overline{V} + 2\overline{>} - 2\overline{A} - 2\overline{<} = 4G$. Hence, for all $n \in \mathbb{Z}$, $\pi_1^1\mathbf{i}(\zeta^n) = 4nG$, so $\pi_1^1\mathbf{i}$ is equivalent to the function $\mathbb{Z} \ni n \mapsto 4n \in \mathbb{Z}$. ♢

If $(\mathcal{G}, +)$ is an abelian group, then for any $k \in [0...D]$, we likewise get homomorphisms $\mathcal{H}_k^r\mathbf{a} : \mathcal{H}_k^r[\mathbb{G}_r(\mathbf{a}), \mathcal{G}] \longrightarrow \mathcal{H}_k^r(\mathfrak{A}, \mathcal{G})$ and $\mathcal{H}_r^k\mathbf{a} : \mathcal{H}_r^k(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}_r^k[\mathbb{G}_r(\mathbf{a}), \mathcal{G}]$.

THEOREM 3.7. *Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$ have a defect of projective codimension $(k+1)$.*

(a) *For any abelian group $(\mathcal{G}, +)$ there are homomorphisms $\mathcal{H}_k\mathbf{a}: \mathcal{H}_k[\mathbb{G}_\infty(\mathbf{a}), \mathcal{G}] \longrightarrow \mathcal{H}_k(\mathfrak{A}, \mathcal{G})$ and $\mathcal{H}^k\mathbf{a}: \mathcal{H}^k(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^k[\mathbb{G}_\infty(\mathbf{a}), \mathcal{G}]$.*

(b) *Suppose $\mathfrak{A}$ is basepoint-free in codimension $(k+1)$, and let $\omega : [0, \infty) \longrightarrow \mathbb{R}^D$ be a proper base ray. Then $\mathbf{a}$ induces a homomorphism $\pi_k\mathbf{a}: \pi_k[\mathbb{G}_\infty(\mathbf{a}), \omega] \longrightarrow \pi_k(\mathfrak{A})$.*

(c) *Let $\Phi: \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$, and let $\Phi(\mathbf{a}) = \mathbf{b}$. Then we have homomorphisms $\mathcal{H}_k\iota$ and $\mathcal{H}^k\iota$ and commuting diagrams:*

$$\begin{array}{ccc}
\mathcal{H}_k[\mathbb{G}_\infty(\mathbf{a}),\mathcal{G}] & \xrightarrow{\mathcal{H}_k\iota} & \mathcal{H}_k[\mathbb{G}_\infty(\mathbf{b}),\mathcal{G}] \\
{\scriptstyle \mathcal{H}_k\mathbf{a}}\downarrow & & \downarrow{\scriptstyle \mathcal{H}_k\mathbf{b}} \\
\mathcal{H}_k(\mathfrak{A},\mathcal{G}) & \xrightarrow{\mathcal{H}_k\Phi} & \mathcal{H}_k(\mathfrak{A},\mathcal{G})
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathcal{H}^k[\mathbb{G}_\infty(\mathbf{a}),\mathcal{G}] & \xleftarrow{\mathcal{H}^k\iota} & \mathcal{H}^k[\mathbb{G}_\infty(\mathbf{b}),\mathcal{G}] \\
{\scriptstyle \mathcal{H}^k\mathbf{a}}\uparrow & & \uparrow{\scriptstyle \mathcal{H}^k\mathbf{b}} \\
\mathcal{H}^k(\mathfrak{A},\mathcal{G}) & \xleftarrow{\mathcal{H}^k\Phi} & \mathcal{H}^k(\mathfrak{A},\mathcal{G})
\end{array}$$

*Assuming the hypothesis of **(b)**, we have a homomorphism $\pi_k\iota$ and a commuting diagram:*

$$\begin{array}{ccc}
\pi_k[\mathbb{G}_\infty(\mathbf{a}),\omega] & \xrightarrow{\pi_k\iota} & \pi_k[\mathbb{G}_\infty(\mathbf{b}),\omega] \\
{\scriptstyle \pi_k\mathbf{a}}\downarrow & & \downarrow{\scriptstyle \pi_k\mathbf{b}} \\
\pi_k(\mathfrak{A}) & \xrightarrow{\pi_k\Phi} & \pi_k(\mathfrak{A})
\end{array} \qquad (14)$$

Proof: We will prove the statements for π_k. The (co)homological versions are analogous.

(b) Let R be the radius of $\mathfrak{A}$, and recall that $\mathbb{G}_R(\mathbf{a}) \supseteq \mathbb{G}_{R+1}(\mathbf{a}) \supseteq \mathbb{G}_{R+2}(\mathbf{a}) \supseteq \cdots$. For each $r \geq R$, the inclusion map $\alpha_r : \mathbb{G}_{r+1}(\mathbf{a}) \hookrightarrow \mathbb{G}_r(\mathbf{a})$ induces a canonical homomorphism $\alpha_r^* : \pi_k[\mathbb{G}_{r+1}(\mathbf{a}), \omega] \longrightarrow \pi_k[\mathbb{G}_r(\mathbf{a}), \omega]$. Let $w_r \in \omega[0,\infty) \cap \mathbb{G}_r(\mathbf{a})$ (so $\pi_k[\mathbb{G}_r(\mathbf{a}), \omega] \cong \pi_k[\mathbb{G}_r(\mathbf{a}), w_r]$ canonically), let $a_r := \varsigma^r_{\mathbf{a}}(w_r) \in \mathbf{X}_r$, and define $\pi_k^r(\mathfrak{A}, a_r) := \pi_k(\mathbf{X}_r, a_r)$. There is a canonical homomorphism $\zeta_r^* : \pi_k^{r+1}(\mathfrak{A}, a_{r+1}) \longrightarrow \pi_k^r(\mathfrak{A}, a_r)$, because $\mathfrak{A}$ is basepoint-free. This yields a commuting ladder of homomorphisms, which defines homomorphism of inverse limits:

$$\begin{array}{ccccccccc}
\pi_d[\mathbb{G}_R(\mathbf{a}), \omega] & \xleftarrow{\alpha_R^*} & \pi_d[\mathbb{G}_{R+1}(\mathbf{a}), \omega] & \xleftarrow{\alpha_{R+1}^*} & \pi_d[\mathbb{G}_{R+2}(\mathbf{a}), \omega] & \xleftarrow{\alpha_{R+2}^*} & \cdots & & \pi_d[\mathbb{G}_\infty(\mathbf{a}), \omega] \\
{\scriptstyle \pi_k^R \mathbf{a}} \big\downarrow & & {\scriptstyle \pi_k^{R+1} \mathbf{a}} \big\downarrow & & {\scriptstyle \pi_k^{R+2} \mathbf{a}} \big\downarrow & & & & {\scriptstyle \pi_k \mathbf{a}} \big\downarrow \\
\pi_d^R(\mathfrak{A}, a_R) & \underset{\zeta_R^*}{\longleftarrow} & \pi_d^{R+1}(\mathfrak{A}, a_{R+1}) & \underset{\zeta_{R+1}^*}{\longleftarrow} & \pi_d^{R+2}(\mathfrak{A}, a_{R+2}) & \underset{\zeta_{R+2}^*}{\longleftarrow} & \cdots & & \pi_d(\mathfrak{A}).
\end{array} \tag{15}$$

(c) For any $r \geq R$, the inclusion map $\beta_r : \mathbb{G}_{r+1}(\mathbf{b}) \hookrightarrow \mathbb{G}_r(\mathbf{b})$ induces a canonical homomorphism $\beta_r^* : \pi_k[\mathbb{G}_{r+1}(\mathbf{b}), \omega] \longrightarrow \pi_k[\mathbb{G}_r(\mathbf{b}), \omega]$. Assume that $w_r \in \mathbb{G}_r(\mathbf{b})$, let $b_r := \varsigma^r_{\mathbf{a}}(w_r) \in \mathbf{X}_r$, and define $\pi_k^r(\mathfrak{B}, b_r) := \pi_k(\mathbf{X}_r, b_r)$. This yields a commuting ladder like eqn.(15), only with $\mathbf{b}$ instead of $\mathbf{a}$, β_r^* instead of α_r^*, and b_r instead of a_r. Suppose Φ has radius $q > 0$. If $\Phi_* : \mathbf{X}_{r+q} \longrightarrow \mathbf{X}_r$ is the cellular map induced by Φ, then $\Phi_*(a_{q+r}) = b_r$ [because $\Phi(\mathbf{a}_{\mathbb{B}(q+r)}) = \mathbf{b}_{\mathbb{B}(r)}$]. This yields homomorphisms $\pi_d^r \Phi : \pi_k^{r+q}(\mathfrak{A}, a_{r+q}) \longrightarrow \pi_k^r(\mathfrak{A}, b_r)$, for all $r \geq R$, as in the proof of Proposition 3.5.

For any $r \geq R$, Proposition 1.2(b) says $\mathbb{G}_{r+q}(\mathbf{a}) \subseteq \mathbb{G}_r(\mathbf{b})$. The inclusion map $\iota_r : \mathbb{G}_{r+q}(\mathbf{a}) \hookrightarrow \mathbb{G}_r(\mathbf{b})$ induces a (canonical) homomorphism $\iota_r^* : \pi_k[\mathbb{G}_{r+q}(\mathbf{a}), \omega] \longrightarrow \pi_k[\mathbb{G}_r(\mathbf{b}), \omega]$, and we have a commuting diagram

$$\begin{array}{ccc}
\pi_k[\mathbb{G}_{r+q}(\mathbf{a}), \omega] & \xrightarrow{\iota_r^*} & \pi_k[\mathbb{G}_r(\mathbf{b}), \omega] \\
{\scriptstyle \pi_k^{r+q} \mathbf{a}} \big\downarrow & & \big\downarrow {\scriptstyle \pi_k^r \mathbf{b}} \\
\pi_k^{r+q}(\mathfrak{A}, a_{r+q}) & \xrightarrow{\pi_d^r \Phi} & \pi_k^r(\mathfrak{A}, b_r)
\end{array} \tag{16}$$

Combining the commuting ladders (15) for $\mathbf{a}$ and $\mathbf{b}$, along with copies of the square (16) for each $r \in \mathbb{N}$, we obtain a 'commuting girder' of homomorphisms:

$$\begin{tikzcd}[column sep=small]
& \pi_d[\mathbb{G}_{q+R}(\mathbf{a})] \arrow[dl, "\iota_R^*"'] \arrow[dd, "\pi_d^{q+R}\mathbf{a}"] & & \pi_d[\mathbb{G}_{q+R+1}(\mathbf{a})] \arrow[ll, "\alpha_{q+R}^*"'] \arrow[dl, "\iota_{R+1}^*"'] \arrow[dd, "\pi_d^{q+R+1}\mathbf{a}"] & & \pi_d[\mathbb{G}_{q+R+2}(\mathbf{a})] \arrow[ll, "\alpha_{q+R+1}^*"'] \arrow[dl, "\iota_{R+2}^*"'] \arrow[dd, "\pi_d^{q+R+2}\mathbf{a}"] & & \cdots \arrow[ll, "\alpha_{q+R+2}^*"'] & & \pi_d[\mathbb{G}_\infty(\mathbf{a})] \arrow[dl, "\iota^*"'] \arrow[dd, "\pi_d\mathbf{a}"] \\
\pi_d[\mathbb{G}_R(\mathbf{b})] \arrow[dd, "\pi_d^R\mathbf{b}"'] & & \pi_d[\mathbb{G}_{R+1}(\mathbf{b})] \arrow[ll, "\beta_R^*"'] \arrow[dd, "\pi_d^{R+1}\mathbf{b}"'] & & \pi_d[\mathbb{G}_{R+2}(\mathbf{b})] \arrow[ll, "\beta_{R+1}^*"'] \arrow[dd, "\pi_d^{R+2}\mathbf{b}"'] & & \cdots \arrow[ll, "\beta_{R+2}^*"'] & & \pi_d[\mathbb{G}_\infty(\mathbf{b})] \arrow[dd, "\pi_d\mathbf{b}"'] & \\
& \pi_d^{q+R}(\mathfrak{A}) \arrow[dl, "\pi_d^R\Phi"] & & \pi_d^{q+R+1}(\mathfrak{A}) \arrow[ll, "\zeta_{q+R}^*"'] \arrow[dl, "\pi_d^{R+1}\Phi"] & & \pi_d^{q+R+2}(\mathfrak{A}) \arrow[ll, "\zeta_{q+R+1}^*"'] \arrow[dl, "\pi_d^{R+2}\Phi"] & & \cdots \arrow[ll, "\zeta_{q+R+2}^*"'] & & \pi_d(\mathfrak{A}) \arrow[dl, "\pi_d\Phi"] \\
\pi_d^R(\mathfrak{A}) & & \pi_d^{R+1}(\mathfrak{A}) \arrow[ll, "\zeta_R^*"] & & \pi_d^{R+2}(\mathfrak{A}) \arrow[ll, "\zeta_{R+1}^*"] & & \cdots \arrow[ll, "\zeta_{R+2}^*"] & & \pi_d(\mathfrak{A}) &
\end{tikzcd}$$

which yields the commuting square (14) of colimit homomorphisms. □

We call $\pi_k\mathbf{a}$ (resp. $\mathcal{H}_k\mathbf{a}$ or $\mathcal{H}^k\mathbf{a}$) the k*th homotopy* (resp. *(co)homology*) *signature* of $\mathbf{a}$; if it is nontrivial, we say $\mathbf{a}$ has a *homotopy* (resp. (*co*)*homology*) *defect* of codimension $(k+1)$. The next result is analogous to Proposition 2.11(b):

Corollary 3.8. *Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA and with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$.*

(a) *Suppose $\mathfrak{A}$ is basepoint-free in codimension $(k+1)$. If $\pi_k\Phi : \pi_k(\mathfrak{A}) \longrightarrow \pi_k(\mathfrak{A})$ is injective, then every homotopy defect of codimension $(k+1)$ is Φ-persistent.*

(b) *If $\mathcal{H}_k\Phi : \mathcal{H}_k(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}_k(\mathfrak{A}, \mathcal{G})$ is injective, then every homology defect of codimension $(k+1)$ is Φ-persistent.*

(c) *If $\mathcal{H}^k\Phi : \mathcal{H}^k(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^k(\mathfrak{A}, \mathcal{G})$ is surjective, then every cohomology defect of codimension $(k+1)$ is Φ-persistent.*

Proof: **(a)** If $\pi_k\mathbf{a}$ is nontrivial, and $\pi_k\Phi$ is a monomorphism, then $\pi_k\Phi \circ \pi_k\mathbf{a}$ must also be nontrivial. Then diagram (14) says that $\pi_k\mathbf{b} \circ \pi_k\iota$ must be nontrivial, hence $\pi_k\mathbf{b}$ is nontrivial, hence $\mathbf{b}$ has a homotopy defect in codimension $(k+1)$. The proofs of **(b,c)** are similar. □

Remark: Using the machinery of Appendix 4.3 (below), we could compute $\mathcal{H}^k(\mathfrak{A}, \mathcal{G})$ as follows. For each $r \geq R$, the function $\zeta_r : \mathbf{X}_{r+1} \longrightarrow \mathbf{X}_r$ is surjective, so the contravariant homomorphism $\mathcal{C}^d\zeta_r : \mathcal{C}^d(\mathbf{X}_r, \mathcal{G}) \to \mathcal{C}^d(\mathbf{X}_{r+1}, \mathcal{G})$ (defined $C \mapsto C \circ \zeta_r$) is injective. Thus, we can identify $\mathcal{C}^d(\mathbf{X}_r, \mathcal{G})$ as a subgroup of $\mathcal{C}^d(\mathbf{X}_{r+1}, \mathcal{G})$ in a natural way. This yields an ascending chain $\mathcal{C}^d(\mathbf{X}_R, \mathcal{G}) \subseteq \mathcal{C}^d(\mathbf{X}_{R+1}, \mathcal{G}) \subseteq \mathcal{C}^d(\mathbf{X}_{R+2}, \mathcal{G}) \subseteq \cdots$. Let $\mathcal{C}^d_\infty := \bigcup_{r=R}^\infty \mathcal{C}^d(\mathbf{X}_r, \mathcal{G})$. The system of coboundary maps $\{\partial_d : \mathcal{C}^d(\mathbf{X}_r, \mathcal{G}) \longrightarrow \mathcal{C}^{d+1}(\mathbf{X}_r, \mathcal{G})\}_{r=R}^\infty$ then defines a coboundary map $\partial_d : \mathcal{C}^d_\infty \longrightarrow \mathcal{C}^{d+1}_\infty$. This yields a chain complex $\mathbf{C}_\infty := \{\mathcal{C}^d_\infty, \partial_d\}_{d=0}^\infty$, which is the colimit of the system of chain complexes ($\mathbf{C}_R \xrightarrow{\zeta_R} \mathbf{C}_{R+1} \xrightarrow{\zeta_{R+1}} \cdots$) (where $\mathbf{C}_r := \{\mathcal{C}^d(\mathbf{X}_r, \mathcal{G}), \partial_d\}_{d=0}^\infty$ for each $r \geq R$). Lemma 4.15 (in Appendix 4.3) then implies that $\mathcal{H}^k(\mathfrak{A}, \mathcal{G}) = \mathcal{H}^k(\mathbf{C}_\infty)$.

Unfortunately, this argument doesn't dualize to homology or homotopy groups. ◊

4. *Equivariant vs. Invariant Cohomology*

By relating the dynamical cohomology of §2.1 to the tiling cohomology of §3.5, we can generalize the results of §2.2 to defects of higher codimensions. The main results of this section are Theorems 4.4, 4.10, and 4.11, and Proposition 4.9.

4.1. *Equivariant Cohomology of Subshifts:*

Let $\mathbf{Y} = (\mathbf{Y}^0, \ldots, \mathbf{Y}^D)$ be as in §3.1. Let $(\mathcal{G}, +)$ be an abelian topological group, and for all $d \in [1...D]$, give $\mathcal{C}^d(\mathbf{Y}, \mathcal{G}) \cong \mathcal{G}^{\mathbf{Y}^d}$ the Tychonoff product topology. For any $\mathsf{z} \in \mathbb{Z}^D$, let $\Upsilon^{\mathsf{z}}_d : \mathbf{Y}^d \longrightarrow \mathbf{Y}^d$ be the obvious translation function. For example, we define $\Upsilon^{\mathsf{z}}_0 : \mathbf{Y}^0 \longrightarrow \mathbf{Y}^0$ by $\Upsilon^{\mathsf{z}}_0(\dot{\mathsf{x}}) := \dot{\mathsf{y}}$ where $\mathsf{y} := \mathsf{x} + \mathsf{z}$, and we define $\Upsilon^{\mathsf{z}}_1 : \mathbf{Y}^1 \longrightarrow \mathbf{Y}^1$ by $\Upsilon^{\mathsf{z}}_1(\mathsf{x}{\rightarrow}\mathsf{y}) := \mathsf{x}'{\rightarrow}\mathsf{y}'$ where $\mathsf{x}' := \mathsf{x} + \mathsf{z}$ and $\mathsf{y}' := \mathsf{y} + \mathsf{z}$. We extend this to a function $\Upsilon^{\mathsf{z}}_d : \mathbb{Z}[\mathbf{Y}^d] \longrightarrow \mathbb{Z}[\mathbf{Y}^d]$ by linearity. Clearly, $\partial_d \circ \Upsilon^{\mathsf{z}}_d = \Upsilon^{\mathsf{z}}_{d-1} \circ \partial_d$. We define $\Upsilon^d_{\mathsf{z}} : \mathcal{C}^d \longrightarrow \mathcal{C}^d$ by $\Upsilon^d_{\mathsf{z}}(C)(\zeta) = C \circ \Upsilon^{\mathsf{z}}_d(\zeta)$ for any $C \in \mathcal{C}^d$ and $\zeta \in \mathbb{Z}[\mathbf{Y}^d]$. Thus, $\delta_d \circ \Upsilon^d_{\mathsf{z}} = \Upsilon^{d+1}_{\mathsf{z}} \circ \delta_d$.

Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift. A (d-dimensional, $\mathcal{G}$-valued, continuous) ***equivariant cochain*** is a continuous function $C : \mathfrak{A} \longrightarrow \mathcal{C}^d$ so that, for any $\mathbf{a} \in \mathfrak{A}$ and $\mathsf{z} \in \mathbb{Z}^D$, $C(\sigma^{\mathsf{z}}(\mathbf{a})) = \Upsilon^d_{\mathsf{z}} \circ C(\mathbf{a})$. Equivalently, an equivariant cochain is a continuous function $C : \mathbb{Z}[\mathbf{Y}^d] \times \mathfrak{A} \longrightarrow \mathcal{G}$ such that:

$$\text{For all } \zeta \in \mathbb{Z}[\mathbf{Y}^d],\ \mathbf{a} \in \mathfrak{A},\ \text{and } \mathsf{z} \in \mathbb{Z}^D, \qquad C(\zeta, \sigma^{\mathsf{z}}(\mathbf{a})) = C(\Upsilon^{\mathsf{z}}_d(\zeta), \mathbf{a}). \tag{17}$$

EXAMPLE 4.1: (a) ($d = 0$) An equivariant zero-cochain is equivalent to a continuous function $B : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ such that $B(\mathsf{x}, \sigma^{\mathsf{y}}(\mathbf{a})) = B(\mathsf{x} + \mathsf{y}, \mathbf{a})$ for any $\mathsf{x}, \mathsf{y} \in \mathbb{Z}^D$ and $\mathbf{a} \in \mathfrak{A}$.

(b) ($d = 1$) An equivariant 1-cochain is equivalent to a continuous function $C : \mathbf{Y}^1 \times \mathfrak{A} \longrightarrow \mathcal{G}$ such that, for any $\mathbf{a} \in \mathfrak{A}$ and $\mathsf{x}, \mathsf{y}, \mathsf{z} \in \mathbb{Z}^D$, if $\mathsf{x}' := \mathsf{x} + \mathsf{z}$ and $\mathsf{y}' := \mathsf{y} + \mathsf{z}$, then $C(\mathsf{x}{\to}\mathsf{y}; \sigma^{\mathsf{z}}(\mathbf{a})) = C(\mathsf{x}'{\to}\mathsf{y}'; \mathbf{a})$. $\diamondsuit$

Let $\mathcal{C}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$ be the abelian group of all equivariant cochains (with pointwise addition). We define the *coboundary* homomorphism $\delta_d : \mathcal{C}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{C}^{d+1}_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$ by applying the cubic coboundary map (from §3.2) pointwise. That is, for any $\zeta \in \mathbb{Z}[\mathbf{Y}^{d+1}]$ and $\mathbf{a} \in \mathfrak{A}$, let $\delta_d C(\zeta, \mathbf{a}) := C(\partial_{d+1}\zeta, \mathbf{a})$. We say that C is an *equivariant cocycle* if $\delta_d C = 0$, and we say that C is an *equivariant coboundary* if $C = \delta_{d-1} b$ for some $b \in \mathcal{C}^{d-1}_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$.

EXAMPLE 4.2: Let $\mathfrak{Q} \subset \mathcal{Q}^{\mathbb{Z}^3}$ be the 'ice-cube' shift from Example 1.5(c). Define $C : \mathbf{Y}^2 \times \mathfrak{Q} \longrightarrow \mathbb{Z}$ as follows. Any two-cell $\begin{smallmatrix} \mathsf{z} & \leftarrow & \mathsf{y} \\ \downarrow & & \uparrow \\ \mathsf{w} & \rightarrow & \mathsf{x} \end{smallmatrix}$ is an oriented square frame, with a unique normal vector $\vec{V}$ defined by the right-hand rule. For any $\mathbf{q} \in \mathfrak{Q}$ (seen as a 'ball-and-pin' assembly), there is a unique 'pin' passing through $\begin{smallmatrix} \mathsf{z} & \leftarrow & \mathsf{y} \\ \downarrow & & \uparrow \\ \mathsf{w} & \rightarrow & \mathsf{x} \end{smallmatrix}$, from one the two adjoining cubes into the other. Define $C(\begin{smallmatrix} \mathsf{z} & \leftarrow & \mathsf{y} \\ \downarrow & & \uparrow \\ \mathsf{w} & \rightarrow & \mathsf{x} \end{smallmatrix}, \mathbf{q}) = +1$ if this pin is parallel to $\vec{V}$, and -1 if it is antiparallel. Then C is a 2-dimensional equivariant cocycle. $\diamondsuit$

Let $\mathcal{Z}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G}) := \ker(\delta_d)$ and $\mathcal{B}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G}) := \mathrm{img}\,(\delta_{d-1})$ be the subgroups of cocycles and coboundaries respectively; then $\mathcal{B}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G}) \subseteq \mathcal{Z}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$, and the quotient $\mathcal{H}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G}) := \mathcal{Z}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G}) / \mathcal{B}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$ is the *dth equivariant cohomology group* of $\mathfrak{A}$ (with coefficients in $\mathcal{G}$). Two cocycles C_1 and C_2 are *cohomologous* ($C_1 \approx C_2$) if they project to the same coset in $\mathcal{H}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$ —i.e. if $C_1 = C_2 + \delta_{d-1} B$ for some $B \in \mathcal{C}^{d-1}_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$.

EXAMPLE 4.3: If $C_1, C_2 \in \mathcal{Z}^1_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$, then $C_1 \approx C_2$ iff there is an equivariant zero-cochain $B : \mathbf{Y}^0 \times \mathfrak{A} \longrightarrow \mathcal{G}$ such that, for any $(\mathsf{x}{\to}\mathsf{y}) \in \mathbf{Y}^1$ and $\mathbf{a} \in \mathfrak{A}$, $C_2[(\mathsf{x}{\to}\mathsf{y}), \mathbf{a}] = B(\dot{\mathsf{y}}, \mathbf{a}) + C_1[(\mathsf{x}{\to}\mathsf{y}), \mathbf{a}] - B(\dot{\mathsf{x}}, \mathbf{a})$. $\diamondsuit$

Equivariant cocycles are the natural generalization of the 'dynamical' cocycles from §2.1. To see this, recall that a (continuous, $\mathcal{G}$-valued) *dynamical cocycle* on $\mathfrak{A}$ is a continuous function $\widetilde{C} : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ satisfying the cocycle equation (2), which, in additive notation, reads: $\widetilde{C}(\mathsf{y} + \mathsf{z}, \mathbf{a}) = \widetilde{C}[\mathsf{y}, \sigma^{\mathsf{z}}(\mathbf{a})] + \widetilde{C}(\mathsf{z}, \mathbf{a})$. Two dynamical cocycles $\widetilde{C}_1, \widetilde{C}_2$ are *cohomologous* if there is some $\widetilde{B} : \mathfrak{A} \longrightarrow \mathcal{G}$ such that $\widetilde{C}_2(\mathsf{z}, \mathbf{a}) = \widetilde{B}(\sigma^{\mathsf{z}}(\mathbf{a})) + \widetilde{C}_1(\mathsf{z}, \mathbf{a}) - \widetilde{B}(\mathbf{a})$ for any $\mathsf{z} \in \mathbb{Z}^D$ and $\mathbf{a} \in \mathfrak{A}$. Let $\mathcal{Z}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$ be the additive group of dynamical cocycles, and let $\mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$ be the dynamical cohomology group. The main result of §4.1 is:

THEOREM 4.4. *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift and let $(\mathcal{G}, +)$ be an abelian group. There are canonical isomorphisms $\mathcal{Z}^1_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G}) \cong \mathcal{Z}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$, and $\mathcal{H}^1_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G}) \cong \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G})$.*

To prove Theorem 4.4, we will introduce another intermediate notion of one-dimensional cocycle. A (continuous, $\mathcal{G}$-valued) *two-point cocycle* on $\mathfrak{A}$ is a continuous function $\ddot{C} : \mathbb{Z}^D \times \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ such that, for any fixed $\mathbf{a} \in \mathfrak{A}$,

($\ddot{\mathbf{C}}$1) The function $\ddot{C}(\_, \_; \mathbf{a}) : \mathbb{Z}^D \times \mathbb{Z}^D \longrightarrow \mathcal{G}$ is a two-point cocycle in the sense of eqn.(11) in Example 3.1.

($\ddot{\mathbf{C}}$2) For any $\mathsf{w}, \mathsf{y}, \mathsf{z} \in \mathbb{Z}^D$, we have $\ddot{C}[\mathsf{w}, \mathsf{y}; \sigma^{\mathsf{z}}(\mathbf{a})] = \ddot{C}(\mathsf{w} + \mathsf{z}, \mathsf{y} + \mathsf{z}; \mathbf{a})$.

Let $\ddot{\mathcal{Z}}^1(\mathfrak{A}, \mathcal{G})$ be the group of two-point cocycles. Then Theorem 4.4 follows from:

LEMMA 4.5. *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift, and let $(\mathcal{G},+)$ be an abelian group.*

(a) *Let $C \in \mathcal{Z}^1_{\mathrm{eq}}(\mathfrak{A},\mathcal{G})$. For any $\mathsf{y},\mathsf{z} \in \mathbb{Z}^D$ and $\mathbf{a} \in \mathfrak{A}$, define $\ddot{C}(\mathsf{y},\mathsf{z};\mathbf{a}) := C(\zeta,\mathbf{a})$, where $\zeta \in \mathbb{Z}[\mathbf{Y}^1]$ is any chain such that $\partial_1(\zeta) = \dot{\mathsf{z}} - \dot{\mathsf{y}}$. Then $\ddot{C}$ well-defined, and $\ddot{C} \in \ddot{\mathcal{Z}}^1(\mathfrak{A},\mathcal{G})$.*

(b) *Let $\ddot{C} \in \ddot{\mathcal{Z}}^1(\mathfrak{A},\mathcal{G})$. Define $\widetilde{C}(\mathsf{z},\mathbf{a}) := \ddot{C}(0,\mathsf{z};\mathbf{a})$ for all $\mathsf{z} \in \mathbb{Z}^D$ and $\mathbf{a} \in \mathfrak{A}$. Then $\widetilde{C} \in \mathcal{Z}^1_{\mathrm{dy}}(\mathfrak{A},\mathcal{G})$.*

(c) *Let $\widetilde{C} \in \mathcal{Z}^1_{\mathrm{dy}}(\mathfrak{A},\mathcal{G})$. Define $C(\mathsf{x}\to\mathsf{y};\mathbf{a}) := \widetilde{C}[(\mathsf{y}-\mathsf{x}),\sigma^{\mathsf{x}}(\mathbf{a})]$ for all $(\mathsf{x}\to\mathsf{y}) \in \mathbf{Y}^1$ and $\mathbf{a} \in \mathfrak{A}$, and extend to $C : \mathbb{Z}[\mathbf{Y}^1] \times \mathfrak{A} \longrightarrow \mathcal{G}$ by linearity. Then $C \in \mathcal{Z}^1_{\mathrm{eq}}(\mathfrak{A},\mathcal{G})$.*

(d) *Any two of the following statements imply the third:* [i] *$\ddot{C}$ comes from C via* (a). [ii] *$\widetilde{C}$ comes from $\ddot{C}$ via* (b). [iii] *C comes from $\widetilde{C}$ via* (c).

(e) *If $C_1, C_2 \in \mathcal{Z}^1_{\mathrm{eq}}(\mathfrak{A},\mathcal{G})$, and $\widetilde{C}_1, \widetilde{C}_2 \in \mathcal{Z}^1_{\mathrm{dy}}(\mathfrak{A},\mathcal{G})$ are related to C_1 and C_2 as in* (c), *then $(C_1 \approx C_2) \iff (\widetilde{C}_1 \approx \widetilde{C}_2)$.*

Proof: **(a)** Reason as in Example 3.1 to get **($\ddot{\mathbf{C}}$1)**. Then use eqn.(17) to get **($\ddot{\mathbf{C}}$2)**.

(b) $\widetilde{C}(\mathsf{y}+\mathsf{z},\mathbf{a}) \underset{(\ddagger)}{=\!=} \ddot{C}(0,\mathsf{y}+\mathsf{z};\mathbf{a}) \underset{(*)}{=\!=} \ddot{C}(\mathsf{z},\mathsf{y}+\mathsf{z};\mathbf{a}) + \ddot{C}(0,\mathsf{z};\mathbf{a}) \underset{(\dagger)}{=\!=} \ddot{C}[0,\mathsf{y};\sigma^{\mathsf{z}}(\mathbf{a})] + \ddot{C}(0,\mathsf{z};\mathbf{a}) \underset{(\ddagger)}{=\!=} \widetilde{C}[\mathsf{y},\sigma^{\mathsf{z}}(\mathbf{a})] + \widetilde{C}(\mathsf{z},\mathbf{a})$. Here $(\ddagger)$ is the definition of $\widetilde{C}$ in part (b). $(*)$ is by **($\ddot{\mathbf{C}}$1)** and eqn.(11b), while $(\dagger)$ is by **($\ddot{\mathbf{C}}$2)**.

(c) Let $\begin{smallmatrix} \mathsf{v} & \leftarrow & \mathsf{u} \\ \vdots & & \uparrow \\ \mathsf{s} & \to & \mathsf{t} \end{smallmatrix}$ be a two-cell and fix $\mathbf{b} \in \mathfrak{A}$. To show that $C(\mathbf{b})$ is a one-cycle, it suffices to check that $C(\mathsf{s}\to\mathsf{t};\mathbf{b}) + C(\mathsf{t}\to\mathsf{u};\mathbf{b}) = C(\mathsf{s}\to\mathsf{v};\mathbf{b}) + C(\mathsf{v}\to\mathsf{u};\mathbf{b})$ (see Example 3.1). But

$$
\begin{aligned}
&C(\mathsf{s}\to\mathsf{t};\mathbf{b}) + C(\mathsf{t}\to\mathsf{u};\mathbf{b}) \\
&\underset{(\ddagger)}{=\!=} \widetilde{C}[\mathsf{t}-\mathsf{s};\sigma^{\mathsf{s}}(\mathbf{b})] + \widetilde{C}[\mathsf{u}-\mathsf{t};\sigma^{\mathsf{t}}(\mathbf{b})] \quad = \quad \widetilde{C}[\mathsf{t}-\mathsf{s};\sigma^{\mathsf{s}}(\mathbf{b})] + \widetilde{C}[\mathsf{u}-\mathsf{t};\sigma^{\mathsf{t}-\mathsf{s}}(\sigma^{\mathsf{s}}(\mathbf{b}))] \\
&\underset{(*)}{=\!=} \widetilde{C}[\mathsf{u}-\mathsf{s};\sigma^{\mathsf{s}}(\mathbf{b})] \underset{(\dagger)}{=\!=} \widetilde{C}[\mathsf{v}-\mathsf{s};\sigma^{\mathsf{s}}(\mathbf{b})] + \widetilde{C}[\mathsf{u}-\mathsf{v};\sigma^{\mathsf{v}-\mathsf{s}}(\sigma^{\mathsf{s}}(\mathbf{b}))] \\
&= \widetilde{C}[\mathsf{v}-\mathsf{s};\sigma^{\mathsf{s}}(\mathbf{b})] + \widetilde{C}[\mathsf{u}-\mathsf{v};\sigma^{\mathsf{v}}(\mathbf{b})] \underset{(\ddagger)}{=\!=} C(\mathsf{s}\to\mathsf{v};\mathbf{b}) + C(\mathsf{v}\to\mathsf{u};\mathbf{b}), \quad \text{as desired.}
\end{aligned}
$$

Here, $(\ddagger)$ is the definition of C in part (c). $(*)$ is by eqn.(2) with $\mathbf{a} := \sigma^{\mathsf{s}}(\mathbf{b})$, $\mathsf{z} := \mathsf{t}-\mathsf{s}$ and $\mathsf{y} := \mathsf{u}-\mathsf{t}$, while $(\dagger)$ is by eqn.(2) with $\mathbf{a} := \sigma^{\mathsf{s}}(\mathbf{b})$, $\mathsf{z} := \mathsf{v}-\mathsf{s}$ and $\mathsf{y} := \mathsf{u}-\mathsf{v}$.

To see that C is equivariant, let $\mathbf{a} \in \mathfrak{A}$ and $\mathsf{x},\mathsf{y},\mathsf{z} \in \mathbb{Z}^D$. If $\mathsf{x}' := \mathsf{x}+\mathsf{z}$ and $\mathsf{y}' := \mathsf{y}+\mathsf{z}$, then $C[\mathsf{x}\to\mathsf{y};\sigma^{\mathsf{z}}(\mathbf{a})] \underset{(\dagger)}{=\!=} \widetilde{C}[\mathsf{y}-\mathsf{x};\sigma^{\mathsf{x}}(\sigma^{\mathsf{z}}(\mathbf{a}))] = \widetilde{C}[(\mathsf{y}+\mathsf{z})-(\mathsf{x}+\mathsf{z});\sigma^{\mathsf{x}+\mathsf{z}}(\mathbf{a})] = \widetilde{C}[\mathsf{y}'-\mathsf{x}';\sigma^{\mathsf{x}'}(\mathbf{a})] \underset{(\dagger)}{=\!=} C(\mathsf{x}'\to\mathsf{y}';\mathbf{a})$, as desired. Here $(\dagger)$ is the definition of C in part (c).

(d) Straightforward calculation.

(e) Given any continuous function $\widetilde{B} : \mathfrak{A} \longrightarrow \mathcal{G}$, define $B : \mathbf{Y}^0 \times \mathfrak{A} \longrightarrow \mathcal{G}$ by $B(\dot{\mathsf{z}},\mathbf{a}) := \widetilde{B}(\sigma^{\mathsf{z}}(\mathbf{a}))$ for all $\mathsf{z} \in \mathbb{Z}^D$ and $\mathbf{a} \in \mathfrak{A}$. Then B is an equivariant zero-cochain, i.e. $B[\dot{\mathsf{x}},\sigma^{\mathsf{y}}(\mathbf{a})] = B(\dot{\mathsf{z}},\mathbf{a})$ (where $\mathsf{z} = \mathsf{x}+\mathsf{y}$). Conversely, any equivariant zero-cochain arises in this manner: given $B : \mathbf{Y}^0 \times \mathfrak{A} \longrightarrow \mathcal{G}$, define $\widetilde{B}(\mathbf{a}) := B(\dot{0},\mathbf{a})$; then $B(\dot{\mathsf{z}},\mathbf{a}) = \widetilde{B}(\sigma^{\mathsf{z}}(\mathbf{a}))$ for all $\mathsf{z} \in \mathbb{Z}^D$. If B and $\widetilde{B}$ are related in this way, and C_n is related to $\widetilde{C}_n$ as in part (c) for $n = 1,2$, then it is easy to check that C_1 is cohomologous to C_2 via B iff $\widetilde{C}_1$ is cohomologous to $\widetilde{C}_2$ via $\widetilde{B}$. $\Box$

EXAMPLE 4.6: Let $(\mathcal{G},+)$ be an abelian group, and let $\widetilde{C} : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ be a dynamical cocycle, [e.g. Examples 2.1(c,d,e)], and define $C \in \mathcal{Z}^1_{\mathrm{eq}}(\mathfrak{A},\mathcal{G})$ from $\widetilde{C}$ as in Lemma 4.5(c).

Let $\widetilde{\zeta} := (\mathsf{z}_0 \rightsquigarrow \mathsf{z}_1 \rightsquigarrow \cdots \rightsquigarrow \mathsf{z}_N)$ be a trail in $\mathbb{Z}^D$, and let $\zeta := \sum_{n=0}^{N-1}(\mathsf{z}_n \to \mathsf{z}_{n+1})$ be the 1-chain 'representing' $\widetilde{\zeta}$. For all $n \in [1...N]$, let $\mathsf{z}'_n := \mathsf{z}_n - \mathsf{z}_{n-1}$, Then

$$C(\zeta, \mathbf{a}) \quad = \quad \sum_{n=1}^{N} C(\mathsf{z}_{n-1} \to \mathsf{z}_n;\ \mathbf{a}) \quad = \quad \sum_{n=1}^{N} \widetilde{C}[\mathsf{z}'_n, \sigma^{\mathsf{z}_{n-1}}(\mathbf{a})] \underset{(*)}{=} \widetilde{C}(\widetilde{\zeta}, \mathbf{a}), \tag{18}$$

where $(*)$ is the additive version of eqn.(4). $\diamondsuit$

To generalize the notion of 'locally determined' cocycles from §2.1, we need some notation. For any $i \in [1...D]$, recall (from §3.1) that $\partial_i^- \ulcorner 0 \lrcorner$ is a $(D-1)$-dimensional face of the cube $\ulcorner 0 \lrcorner$ which is orthogonal to the ith axis. Also, for any $r \in \mathbb{N}$, recall (from §3.4) that $\partial_i^- \mathbb{B}(r) := [-r...r]^{i-1} \times [-r...r) \times [-r...r]^{D-i-1}$. For any $d \in [0...D]$, let $\mathbf{Y}_0^d$ be the set of all d-cells in $\mathbf{Y}^d$ contained in the half-open unit cube $[0,1)^D$. Let $k := D - d$. If $x \in \mathbf{Y}_0^d$, then $x = \partial_{i_1}^- \ulcorner 0 \lrcorner \cap \partial_{i_2}^- \ulcorner 0 \lrcorner \cap \cdots \cap \partial_{i_k}^- \ulcorner 0 \lrcorner$, for some $\{i_1, \ldots, i_k\} \subset [1...D]$ (see §3.1). We then define $\partial_x \mathbb{B}(r) := \partial_{i_1}^- \mathbb{B}(r) \cap \partial_{i_2}^- \mathbb{B}(r) \cap \cdots \cap \partial_{i_k}^- \mathbb{B}(r)$; this is the d-dimensional 'face' of $\mathbb{B}(r)$ corresponding to x. For any $\mathsf{z} \in \mathbb{Z}^D$, we define $\partial_x \mathbb{B}(\mathsf{z}, r) := \partial_x \mathbb{B}(r) + \mathsf{z}$.

For any $y \in \mathbf{Y}^d$, there is a unique $x \in \mathbf{Y}_0^d$ and $\mathsf{z} \in \mathbb{Z}^D$ so that $y = \Upsilon_d^{\mathsf{z}}(x)$. An equivariant cocycle $C \in \mathcal{Z}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$ is *locally determined* with *radius* $r > 0$ if, for each $x \in \mathbf{Y}_0^d$, there is some *local rule* $c_x : \mathfrak{A}_{\partial_x \mathbb{B}(r)} \longrightarrow \mathcal{G}$ such that, for any $y \in \mathbf{Y}^d$ and $\mathbf{a} \in \mathfrak{A}_{(r)}$, if $y = \Upsilon_d^{\mathsf{z}}(x)$ for $\mathsf{z} \in \mathbb{Z}^D$, then $C(y, \mathbf{a}) \ := \ c_x(\mathbf{a}_{\partial_x \mathbb{B}(\mathsf{z}, r)})$.

Example 4.7: (a) Let $d = 1$. If $C \in \mathcal{Z}^1_{\mathrm{eq}}$ and $\widetilde{C} \in \mathcal{Z}^1_{\mathrm{dy}}$ are related as in Lemma 4.5(c), then C is locally determined iff $\widetilde{C}$ is locally determined in the sense of §2.1.

(b) The cocycle in Example 4.2 is locally determined, with radius 0.

(c) If $\mathcal{G}$ is discrete, then every continuous equivariant cocycle is locally determined. $\diamondsuit$

Fix $d \in [0..D]$. If $C \in \mathcal{Z}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$ is locally determined with radius r, and $\mathbf{a} \in \widetilde{\mathfrak{A}}$, then $C(y, \mathbf{a})$ is well-defined for any $y \in \mathbf{Y}^d \cap \mathbb{G}_r(\mathbf{a})$. A *$d$-cycle* is a d-chain $\zeta \in \mathbb{Z}[\mathbf{Y}^d]$ such that $\partial_d(\zeta) = 0$; let $\mathcal{Z}_d = \mathcal{Z}_d(\mathbf{Y}; \mathbb{Z})$ denote the group of *d-cycles*. We say that $\mathbf{a}$ has a *C-pole of range* r if there is some d-cycle $\zeta \in \mathcal{Z}_d[\mathbb{G}_r(\mathbf{a})]$ such that $C(\zeta, \mathbf{a}) \neq 0$. We say that $\mathbf{a}$ has a *projective C-pole* if $\mathbf{a}$ has a *C-pole of range* r for all large enough $r \in \mathbb{N}$. We say that $\mathbf{a}$ has a *projective $(\mathcal{G}, d)$-pole* if $\mathbf{a}$ has a projective C-pole for some $C \in \mathcal{Z}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$. We will call this a "$d$-pole" (resp. "$\mathcal{G}$-pole") if $\mathcal{G}$ (resp. d) is either arbitrary or clear from context.

Example 4.8: (a) If $d = 1$, then a 1-pole is a pole in the sense of §2.2 (apply Example 4.6).

(b) Let $C : \mathbf{Y}^2 \times \mathfrak{Q} \longrightarrow \mathbb{Z}$ be as in Example 4.2, and let $\mathbf{q} \in \widetilde{\mathfrak{Q}}$ be the configuration in Figure 2(D), having a codimension-three defect. Let $\beta \in \mathbb{Z}[\mathbf{Y}^3]$ be the three-chain consisting of the the twenty-seven cubes containing the twenty-seven balls in Figure 2(D), and let $\zeta := \partial_3(\beta)$. Then $C(\zeta, \mathbf{q}) = 6 \neq 0$, so this is a projective $(\mathbb{Z}, 2)$-pole. $\diamondsuit$

The next result is analogous to Theorem 2.8(a):

Proposition 4.9. *If $\mathbf{a} \in \widetilde{\mathfrak{A}}$ has a projective d-pole, then $\mathbf{a}$ has an essential defect.*

Proof: (by contradiction) Suppose $\mathbf{a}$ had a removable defect; we will show that $\mathbf{a}$ has no projective d-poles. Let $\mathbf{a}' \in \mathfrak{A}$ and suppose $\mathbf{a}'$ agrees with $\mathbf{a}$ on $\mathbb{G}_r(\mathbf{a})$ for some $r \in \mathbb{N}$.

Claim 1: *For all $R \geq r$, $\mathbf{a}$ has no d-poles of range R.*

Proof: Let $(\mathcal{G},+)$ be a group and suppose $C \in \mathcal{Z}^d_{\text{eq}}(\mathfrak{A},\mathcal{G})$ has range R. Let $\zeta \in \mathcal{Z}_d[\mathbb{G}_R(\mathbf{a})]$ be any d-cycle. Then $\zeta = \partial_{d+1}\beta$ for some $\beta \in \mathbb{Z}[\mathbf{Y}^{d+1}]$ (because $\mathbf{Y} = \mathbb{R}^D$ has trivial homology in all dimensions). Thus $C(\zeta,\mathbf{a}) \underset{(*)}{=} C(\zeta,\mathbf{a}') = C(\partial_{d+1}\beta,\mathbf{a}') = \delta_d C(\mathbf{a}')[\zeta] \underset{(\dagger)}{=} 0$. Here, $(*)$ is because $\mathbf{a}$ and $\mathbf{a}'$ agree on $\mathbb{G}_R(\mathbf{a})$, while $(\dagger)$ is because $\delta_d C(\mathbf{a}') = 0$ because $\mathbf{a}' \in \mathfrak{A}$ and C is a cocycle on $\mathfrak{A}$. ◇ `Claim 1`

Claim 1 holds for all $\mathcal{G}$ and all large enough R, so $\mathbf{a}$ has no projective d-pole. □

Remarks: (a) In §2.2, it was necessary to define 1-poles in terms of *trails* in $\pi_1[\mathbb{G}_r(\mathbf{a})]$ (rather than *one-chains* in $\mathbb{Z}[\mathbf{Y}^1]$), because §2.2 dealt with potentially *nonabelian* cocycles [e.g. Example 2.1(f)], where the (ordered) product in eqn.(4) is well-defined, but where the (unordered) sum in eqn.(18) is not.

(b) If $\mathbf{a}$ has a projective C-pole, then for every large $r \in \mathbb{N}$, there is some $\zeta_r \in \mathcal{Z}_d[\mathbb{G}_r(\mathbf{a})]$ such that $C(\zeta_r,\mathbf{a}) \neq 0$. We can further assume that for every $R \geq r$, the cycle ζ_R is homologous to ζ_r in $\mathbb{G}_r(\mathbf{a})$. We could then define 'projective C-poles' by treating C as a function on the inverse limit group $\mathcal{H}_d[\mathbb{G}_\infty(\mathbf{a})]$ in the obvious way, but we will restrain ourselves. ◇

4.2. *Invariant Cohomology for Subshifts of Finite Type:* The goal of this section is to determine when d-poles are persistent under a cellular automaton. To do this, we will introduce another cohomology group $\mathcal{H}^d_{\text{inv}}(\mathfrak{A},\mathcal{G})$, which acts as a bridge from the group $\mathcal{H}^d(\mathfrak{A},\mathcal{G})$ of §3.5 to the group $\mathcal{H}^d_{\text{eq}}(\mathfrak{A},\mathcal{G})$ of §4.1. We will then prove:

THEOREM 4.10. *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be an SFT. Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \to \mathcal{A}^{\mathbb{Z}^D}$ be a CA with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$.*

(a) *For all $d \in [0...D]$, Φ induces endomorphisms $\mathcal{H}^d_{\text{inv}}\Phi : \mathcal{H}^d_{\text{inv}}(\mathfrak{A},\mathcal{G}) \longrightarrow \mathcal{H}^d_{\text{inv}}(\mathfrak{A},\mathcal{G})$.*

(b) *Suppose $\mathcal{H}^d_{\text{inv}}\Phi$ is an epimorphism.*

[i] *If $\mathbf{a} \in \widetilde{\mathfrak{A}}$ has a projective $(\mathcal{G},d)$-pole, and $\mathbf{b} := \Phi(\mathbf{a})$, and $\operatorname{Ext}(\mathcal{H}_{d-1}[\mathbb{G}_r(\mathbf{b})],\mathcal{G}) = 0$ for all large $r \in \mathbb{N}$, then $\mathbf{b}$ also has a projective $(\mathcal{G},d)$-pole.*

[ii] *In particular, any projective 1-pole or D-pole is Φ-persistent.*

[iii] *If $\mathcal{G}$ is the additive group of a field* (e.g. $\mathcal{G} = \mathbb{Z}_{/p}$ for p prime), *then all projective $\mathcal{G}$-poles are Φ-persistent.*

(Here, $\mathcal{H}_{d-1}[\mathbb{G}_r(\mathbf{b})]$ is homology with coefficients in $\mathbb{Z}$. See Appendix §4.3 for the definition of Ext, and for other homological background for what follows.) Suppose $\mathfrak{A}$ has radius $R > 0$, and fix $r \geq R$. Let $\mathbf{X}_r = (\mathbf{X}^0_r,\ldots,\mathbf{X}^D_r)$ be the cellular complex induced by the radius-r Wang representation of $\mathfrak{A}$, as in §3.5. For any $\mathsf{z} \in \mathbb{Z}^D$, we define a self-homeomorphism $\Xi^{\mathsf{z}} : \mathbf{X}_r \longrightarrow \mathbf{X}_r$ such that, for any $\mathsf{x} \in \mathbb{Z}^D$ and $\mathbf{a} \in \mathfrak{A}_{(r)}$, if $\mathsf{y} = \mathsf{x}+\mathsf{z}$, then $\Xi^{\mathsf{z}}\big| : {}_{\mathsf{x}}^{\ulcorner}\mathbf{a}_{\lrcorner}^{\urcorner} \longrightarrow {}_{\mathsf{y}}^{\ulcorner}\mathbf{a}_{\lrcorner}^{\urcorner}$ is a homeomorphism (see also 'property (e)' in §3.3). Also, Ξ^{z} is a cellular map; i.e. for all $d \in [0...D]$, if $\Xi^{\mathsf{z}}_d := \Xi^{\mathsf{x}}\big|_{\mathbf{X}^d_r}$, then $\Xi^{\mathsf{z}}_d : \mathbf{X}^d_r \longrightarrow \mathbf{X}^d_r$. Thus, Ξ^{z} induces a (contravariant) automorphism $\Xi^d_{\mathsf{z}} : \mathcal{C}^d(\mathbf{X}_r,\mathcal{G}) \longrightarrow \mathcal{C}^d(\mathbf{X}_r,\mathcal{G})$ defined by $\Xi^d_{\mathsf{z}}(C) = C \circ \Xi^{\mathsf{z}}_d$. A cochain $C \in \mathcal{C}^d(\mathbf{X}_r,\mathcal{G})$ is Ξ-*invariant* if $C = C \circ \Xi^{\mathsf{z}}_d$ for all $\mathsf{z} \in \mathbb{Z}^D$. Let $\mathcal{C}^d_{\text{inv}} = \mathcal{C}^d_{\text{inv}}(\mathbf{X}_r,\mathcal{G})$ be the subgroup of Ξ-invariant cochains. Then $\delta_d[\mathcal{C}^d_{\text{inv}}] \subseteq \mathcal{C}^{d+1}_{\text{inv}}$, because $\delta_d \circ \Xi^d_{\mathsf{z}} = \Xi^{d+1}_{\mathsf{z}} \circ \delta_d$ for all $d \in [0...D]$. Thus we get a chain complex $\mathbf{C}_{\text{inv}} = (\mathcal{C}^0_{\text{inv}} \xrightarrow{\delta_0} \mathcal{C}^1_{\text{inv}} \xrightarrow{\delta_1} \mathcal{C}^2_{\text{inv}} \xrightarrow{\delta_2} \cdots)$, and for each $d \in [0...D]$, we define $\mathcal{H}^d_{\text{inv}}(\mathbf{X}_r,\mathcal{G}) := \mathcal{H}^d(\mathbf{C}_{\text{inv}})$ (see Appendix §4.3).

If $\zeta_r : \mathbf{X}_{r+1} \longrightarrow \mathbf{X}_r$ is the surjection defined in §3.5, then for each $d \in [0...D]$, ζ_r induces a contravariant homomorphism $\zeta_r^d : \mathcal{C}^d_{\text{inv}}(\mathbf{X}_r, \mathcal{G}) \longrightarrow \mathcal{C}^d_{\text{inv}}(\mathbf{X}_{r+1}, \mathcal{G})$ defined by $\zeta_r^d(C) := C \circ \zeta_r$. (This works because $\zeta_r \circ \Xi_d^{\mathsf{z}} = \Xi_d^{\mathsf{z}} \circ \zeta_r$ for all $\mathsf{z} \in \mathbb{Z}^D$, so if C is Ξ-invariant then so is $C \circ \zeta_r$). These homomorphisms together comprise a chain map $\boldsymbol{\zeta}_r : \mathbf{C}_{\text{inv}}(\mathbf{X}_r) \longrightarrow \mathbf{C}_{\text{inv}}(\mathbf{X}_{r+1})$, which yields homomorphisms $\mathcal{H}^d\zeta_r : \mathcal{H}^d_{\text{inv}}(\mathbf{X}_r, \mathcal{G}) \longrightarrow \mathcal{H}^d_{\text{inv}}(\mathbf{X}_{r+1}, \mathcal{G})$ for all $d \in [0...D]$. We then define the dth *invariant cohomology group* to be the direct limit:

$$\mathcal{H}^d_{\text{inv}}(\mathfrak{A}, \mathcal{G}) := \qquad (19)$$
$$\varinjlim \left(\mathcal{H}^d_{\text{inv}}(\mathbf{X}_R, \mathcal{G}) \xrightarrow{\mathcal{H}^d\zeta_R} \mathcal{H}^d_{\text{inv}}(\mathbf{X}_{R+1}, \mathcal{G}) \xrightarrow{\mathcal{H}^d\zeta_{R+1}} \mathcal{H}^d_{\text{inv}}(\mathbf{X}_{R+2}, \mathcal{G}) \xrightarrow{\mathcal{H}^d\zeta_{R+2}} \cdots \right)$$

Fix $d \in [0...D]$, and let $\mathcal{C}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$ be as in §4.1. For any $r \geq R$, let ${}_r\mathcal{C}^d_{\text{eq}} := {}_r\mathcal{C}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$ be the set of locally determined equivariant d-dimensional cochains of radius r on $\mathfrak{A}$. Then ${}_1\mathcal{C}^d_{\text{eq}} \subseteq {}_2\mathcal{C}^d_{\text{eq}} \subseteq {}_3\mathcal{C}^d_{\text{eq}} \subseteq \cdots \subseteq \mathcal{C}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$. Let ${}_\infty\mathcal{C}^d_{\text{eq}} := \bigcup_{r=1}^\infty {}_r\mathcal{C}^d_{\text{eq}}$.

Now fix $r \geq R$, and observe that $\delta_d({}_r\mathcal{C}^d_{\text{eq}}) \subseteq {}_r\mathcal{C}^{d+1}_{\text{eq}}$ for each $d \in \mathbb{N}$. Thus, we get a chain complex ${}_r\mathbf{C}_{\text{eq}} := ({}_r\mathcal{C}^0_{\text{eq}} \xrightarrow{\delta_0} {}_r\mathcal{C}^1_{\text{eq}} \xrightarrow{\delta_1} \cdots)$. For any $d \in [0...D]$, let ${}_r\mathcal{H}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G}) := \mathcal{H}^d({}_r\mathbf{C}_{\text{eq}})$ be the dth *radius-r equivariant cohomology group*. Likewise, $\delta_d({}_\infty\mathcal{C}^d_{\text{eq}}) \subseteq {}_\infty\mathcal{C}^{d+1}_{\text{eq}}$, yielding a chain complex ${}_\infty\mathbf{C}_{\text{eq}} := ({}_\infty\mathcal{C}^0_{\text{eq}} \xrightarrow{\delta_0} {}_\infty\mathcal{C}^1_{\text{eq}} \xrightarrow{\delta_1} \cdots)$. For any $d \in [0...D]$, we define ${}_\infty\mathcal{H}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G}) := \mathcal{H}^d({}_\infty\mathbf{C}_{\text{eq}})$. This section's other main result is:

THEOREM 4.11. *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be an SFT, and let $(\mathcal{G}, +)$ be an abelian group.*

(a) *There is a natural isomorphism $\mathcal{H}^d_{\text{inv}}(\mathfrak{A}, \mathcal{G}) \cong {}_\infty\mathcal{H}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$.*

(b) *If $\mathcal{G}$ is discrete, then ${}_\infty\mathcal{H}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G}) = \mathcal{H}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$, so $\mathcal{H}^d_{\text{inv}}(\mathfrak{A}, \mathcal{G}) \cong \mathcal{H}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$. In particular, $\mathcal{H}^1_{\text{inv}}(\mathfrak{A}, \mathcal{G}) \cong \mathcal{H}^1_{\text{dy}}(\mathfrak{A}, \mathcal{G})$.*

This, in turn, will follow from:

PROPOSITION 4.12. *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be an SFT of radius $R > 0$. Let $(\mathcal{G}, +)$ be an abelian group. For any $r \geq R$ and $d \in [0...D]$, there is a canonical isomorphism $\mathcal{H}^d_{\text{inv}}(\mathbf{X}_r, \mathcal{G}) \cong {}_r\mathcal{H}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$.*

To prove Proposition 4.12, let $\mathbf{Y} = (\mathbf{Y}_0, \ldots, \mathbf{Y}_D)$ be the canonical cell complex for $\mathbb{R}^D$ (see §3.1), and let $\mathcal{C}^d(\mathbf{Y}, \mathcal{G})$ be its cubic cohomology group (see §3.2). If $C \in \mathcal{C}^d(\mathbf{X}_r, \mathcal{G})$ is any cochain on the tile complex $\mathbf{X}_r$, then C induces a function $\widetilde{C} : \mathfrak{A} \longrightarrow \mathcal{C}^d(\mathbf{Y}, \mathcal{G})$ defined by $\widetilde{C}(\mathbf{a}) = C \circ \varsigma^r_{\mathbf{a}}$, for all $\mathbf{a} \in \mathfrak{A}$ (where $\varsigma^r_{\mathbf{a}} : \mathbf{Y} \longrightarrow \mathbf{X}_r$ is the continuous section of Π_r induced by $\mathbf{a}$, as in §3.5). Proposition 4.12 then follows immediately from the next result:

LEMMA 4.13. **(a)** *C is a Ξ-invariant cochain on $\mathbf{X}_r$ iff $\widetilde{C}$ is an equivariant cochain of radius r on $\mathfrak{A}$.*

(b) *This defines an isomorphism $\psi^d_r : \mathcal{C}^d_{\text{inv}}(\mathbf{X}_r, \mathcal{G}) \ni C \mapsto \widetilde{C} \in {}_r\mathcal{C}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$.*

(c) *$C \in \mathcal{Z}^d_{\text{inv}}(\mathbf{X}_r, \mathcal{G})$ iff $\widetilde{C} \in {}_r\mathcal{Z}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$, and $C \in \mathcal{B}^d_{\text{inv}}(\mathbf{X}_r, \mathcal{G})$ iff $\widetilde{C} \in {}_r\mathcal{B}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$.*

Proof: **(a)** If $\mathbf{a} \in \mathfrak{A}$ and $\mathsf{z} \in \mathbb{Z}^D$, then for any $\zeta \in \mathbb{Z}[\mathbf{Y}^d]$,

$$\begin{array}{rcl} \widetilde{C}(\Upsilon^{\mathsf{z}}_d(\zeta), \mathbf{a}) & = & C \circ \varsigma^r_{\mathbf{a}} \circ \Upsilon^{\mathsf{z}}_d(\zeta) \\ \text{and} \quad \widetilde{C}(\zeta, \sigma^{\mathsf{z}}(\mathbf{a})) & = & C \circ \varsigma^r_{\sigma^{\mathsf{z}}(\mathbf{a})}(\zeta) \underset{(\diamond)}{=\!=} C \circ \Xi^{-\mathsf{z}}_d \circ \varsigma^r_{\mathbf{a}} \circ \Upsilon^{\mathsf{z}}_d(\zeta) \end{array} \qquad (20)$$

Here, $(\diamond)$ is because $\varsigma^r_{\sigma^{\mathsf{z}}(\mathbf{a})} = \Xi^{-\mathsf{z}} \circ \varsigma^r_{\mathbf{a}} \circ \Upsilon^{\mathsf{z}}$. Thus,

$$\left(\widetilde{C} \in \mathcal{C}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})\right) \underset{(*)}{\iff} \left(\forall\, \zeta \in \mathbb{Z}[\mathbf{Y}^d],\ \mathbf{a} \in \mathfrak{A}, \text{ and } \mathsf{z} \in \mathbb{Z}^D,\ \ \widetilde{C}(\zeta, \sigma^{\mathsf{z}}(\mathbf{a})) \;=\; \widetilde{C}(\Upsilon^{\mathsf{z}}_d(\zeta), \mathbf{a})\right)$$
$$\underset{(\dagger)}{\iff} \left(\forall\ \mathsf{z} \in \mathbb{Z}^D,\ C \circ \Xi^{-\mathsf{z}}_d \;=\; C\right) \underset{(\ddagger)}{\iff} \left(C \in \mathcal{C}^d_{\mathrm{inv}}(\mathbf{X}_r, \mathcal{G})\right).$$

$(*)$ is defining eqn.(17) from §4.1. $(\ddagger)$ is the definition of $\mathcal{C}^d_{\mathrm{inv}}(\mathbf{X}_r, \mathcal{G})$. Then '$\underset{(\dagger)}{\Longleftarrow}$' is by eqn.(20), while '$\underset{(\dagger)}{\Longrightarrow}$' is by eqn.(20) and the fact that $\bigcup_{\mathbf{a}\in\mathfrak{A}} \varsigma^r_{\mathbf{a}}(\mathbf{Y}^d) = \mathbf{X}^d_r$.

$\widetilde{C}$ has radius r because C is defined on the d-cells of $\mathbf{X}^d_r$, each of which is 'labelled' by a block in $\mathfrak{A}_{\partial_y \mathbb{B}(r)}$ for some $y \in \mathbf{Y}^d_0$ (see §4.1). Hence, for any $y \in \mathbf{Y}^d_0$, $\widetilde{C}(\mathbf{a}, y) = C \circ \varsigma^r_{\mathbf{a}}(y)$ depends only on $\mathbf{a}_{\partial_y \mathbb{B}(r)}$.

(b) *ψ^d is injective:* If $\widetilde{C}_1 = \widetilde{C}_2$, then $C_1 \circ \varsigma^r_{\mathbf{a}} = C_2 \circ \varsigma^r_{\mathbf{a}}$ for all $\mathbf{a} \in \mathfrak{A}$. But $\bigcup_{\mathbf{a}\in\mathfrak{A}} \varsigma^r_{\mathbf{a}}(\mathbf{Y}^d) = \mathbf{X}^d_r$, so this means $C_1(x) = C_2(x)$ for all $x \in \mathbf{X}^d_r$, which means $C_1 = C_2$.

ψ^d is surjective: Let $C' \in {}_r\mathcal{C}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G})$ be a radius-r equivariant cochain; we seek some $C \in \mathcal{C}^d_{\mathrm{inv}}(\mathbf{X}_r, \mathcal{G})$ so that $\widetilde{C} = C'$.

CLAIM 1: *Let $x \in \mathbf{X}^d_r$, and let $y := \Pi^d_r(x) \in \mathbf{Y}^d$.*

(a) *There exists $\mathbf{a} \in \mathfrak{A}$ such that $x = \varsigma^r_{\mathbf{a}}(y)$.*

(b) *If $\mathbf{a}' \in \mathfrak{A}$ is another element with $\varsigma^r_{\mathbf{a}'}(y) = x$, then $C'(y, \mathbf{a}) = C'(y, \mathbf{a}')$.*

Proof: (a) Suppose x is a d-dimensional face of $\ulcorner_{\mathsf{z}}\mathbf{b}\lrcorner$ for some $\mathbf{b} \in \mathfrak{A}_{(r)}$ and $\mathsf{z} \in \mathbb{Z}^D$. Let $\mathbb{F} := \partial_y \mathbb{B}(\mathsf{z}, r)$ (defined in §4.1), and find $\mathbf{a} \in \mathfrak{A}$ such that $\mathbf{a}_{\mathbb{F}} = \mathbf{b}_{\mathbb{F}}$. Then $\varsigma^r_{\mathbf{a}}(y) = x$.

(b) $C'(y, \mathbf{a}) \underset{(*)}{=} c_y(\mathbf{a}_{\mathbb{F}}) \underset{(\dagger)}{=} c_y(\mathbf{a}'_{\mathbb{F}}) \underset{(*)}{=} C'(y, \mathbf{a}')$. $(*)$ is because C' is locally determined with radius r. $(\dagger)$ is because $\mathbf{a}_{\mathbb{F}} = \mathbf{b}_{\mathbb{F}} = \mathbf{a}'_{\mathbb{F}}$, because $\varsigma^r_{\mathbf{a}}(y) = x = \varsigma^r_{\mathbf{a}'}(y)$. $\diamond$ `Claim 1`

Define $C \in \mathcal{C}^d(\mathbf{X}_r, \mathcal{G})$ as follows: for any $x \in \mathbf{X}^d_r$, $C(x) := C'(y, \mathbf{a})$, where $\mathbf{a}$ and y are as in Claim 1(a). Then $C(x)$ is well-defined independent of the choice of $\mathbf{a}$, by Claim 1(b).

CLAIM 2: *$C \in \mathcal{C}^r_{\mathrm{inv}}(\mathfrak{A}, \mathcal{G})$, and $\widetilde{C} = C'$.*

Proof: Let $y \in \mathbf{Y}^d$ and let $\mathbf{a} \in \mathfrak{A}$. If $x = \varsigma^r_{\mathbf{a}}(y)$, then x, y, and $\mathbf{a}$ are related as in Claim 1, so $\widetilde{C}(y, \mathbf{a}) = C \circ \varsigma^r_{\mathbf{a}}(y) = C(x) := C'(y, \mathbf{a})$, as desired. Thus, $\widetilde{C} = C'$. But then (a) implies that $C \in \mathcal{C}^r_{\mathrm{inv}}(\mathfrak{A}, \mathcal{G})$ (because C' is equivariant). $\diamond$ `Claim 2`

(c) This follows from the fact that $\widetilde{\delta_d C} = \delta_d \widetilde{C}$ for any $C \in \mathcal{C}^d(\mathbf{X}^r, \mathcal{G})$. This, in turn, is because $\partial_{d+1} \circ \varsigma^r_{\mathbf{a}}(\zeta) = \varsigma^r_{\mathbf{a}} \circ \partial_{d+1}(\zeta)$, for any $\mathbf{a} \in \mathfrak{A}$ and $\zeta \in \mathbb{Z}[\mathbf{Y}^{d+1}]$. □

Proof of Theorem 4.11: **(a)** For each $d \in [0...D]$, recall that ${}_1\mathcal{C}^d_{\mathrm{eq}} \subseteq {}_2\mathcal{C}^d_{\mathrm{eq}} \subseteq {}_3\mathcal{C}^d_{\mathrm{eq}} \subseteq \cdots$. The inclusion maps ${}_1\mathcal{C}^d_{\mathrm{eq}} \overset{\iota_1}{\hookrightarrow} {}_2\mathcal{C}^d_{\mathrm{eq}} \overset{\iota_2}{\hookrightarrow} {}_3\mathcal{C}^d_{\mathrm{eq}} \overset{\iota_3}{\hookrightarrow} \cdots$ define a sequence of chain maps $({}_1\mathbf{C}_{\mathrm{eq}} \xrightarrow{\boldsymbol{\iota}_1} {}_2\mathbf{C}_{\mathrm{eq}} \xrightarrow{\boldsymbol{\iota}_2} {}_3\mathbf{C}_{\mathrm{eq}} \xrightarrow{\boldsymbol{\iota}_3} \cdots)$, which yields a sequence of cohomology homomorphisms $({}_1\mathcal{H}^d_{\mathrm{eq}} \xrightarrow{\mathcal{H}^d\iota_1} {}_2\mathcal{H}^d_{\mathrm{eq}} \xrightarrow{\mathcal{H}^d\iota_2} {}_3\mathcal{H}^d_{\mathrm{eq}} \xrightarrow{\mathcal{H}^d\iota_3} \cdots)$. Also, ${}_\infty\mathbf{C}_{\mathrm{eq}} = \varinjlim\, ({}_1\mathbf{C}_{\mathrm{eq}} \xrightarrow{\boldsymbol{\iota}_1} {}_2\mathbf{C}_{\mathrm{eq}} \xrightarrow{\boldsymbol{\iota}_2} \cdots)$, because for each $d \in [0...D]$, ${}_\infty\mathcal{C}^d_{\mathrm{eq}} = \bigcup_{r=1}^{\infty} {}_r\mathcal{C}^d_{\mathrm{eq}} = \varinjlim\, ({}_1\mathcal{C}^d_{\mathrm{eq}} \overset{\iota_1}{\hookrightarrow} {}_2\mathcal{C}^d_{\mathrm{eq}} \overset{\iota_2}{\hookrightarrow} \cdots)$. Thus, Lemma 4.15 (in Appendix §4.3) says

$$\text{For all } d \in [0...D], \quad {}_\infty\mathcal{H}^d_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G}) \;=\; \varinjlim\, ({}_1\mathcal{H}^d_{\mathrm{eq}} \xrightarrow{\mathcal{H}^d\iota_1} {}_2\mathcal{H}^d_{\mathrm{eq}} \xrightarrow{\mathcal{H}^d\iota_2} {}_3\mathcal{H}^d_{\mathrm{eq}} \xrightarrow{\mathcal{H}^d\iota_3} \cdots). \quad (21)$$

Suppose $\mathfrak{A}$ has radius $R > 0$. Then Proposition 4.12, eqn.(19), and eqn.(21) together yield a commuting ladder with isomorphism rungs, which yields an isomorphism of direct limits, as shown:

$$\begin{array}{ccccccccc}
\mathcal{H}^d_{\mathrm{inv}}(\mathbf{X}_R,\mathcal{G}) & \xrightarrow{\mathcal{H}^d\varsigma_R} & \mathcal{H}^d_{\mathrm{inv}}(\mathbf{X}_{R+1},\mathcal{G}) & \xrightarrow{\mathcal{H}^d\varsigma_{R+1}} & \mathcal{H}^d_{\mathrm{inv}}(\mathbf{X}_{R+2},\mathcal{G}) & \xrightarrow{\mathcal{H}^d\varsigma_{R+2}} & \cdots & & \mathcal{H}^d_{\mathrm{inv}}(\mathfrak{A},\mathcal{G}) \\
\psi^d_R \Downarrow & & \psi^d_{R+1} \Downarrow & & \psi^d_{R+2} \Downarrow & & & & \Downarrow \\
{}_R\mathcal{H}^d_{\mathrm{eq}}(\mathfrak{A},\mathcal{G}) & \xrightarrow{\mathcal{H}^d\iota_R} & {}_{R+1}\mathcal{H}^d_{\mathrm{eq}}(\mathfrak{A},\mathcal{G}) & \xrightarrow{\mathcal{H}^d\iota_{R+1}} & {}_{R+2}\mathcal{H}^d_{\mathrm{eq}}(\mathfrak{A},\mathcal{G}) & \xrightarrow{\mathcal{H}^d\iota_{R+2}} & \cdots & & {}_\infty\mathcal{H}^d_{\mathrm{eq}}(\mathfrak{A},\mathcal{G}).
\end{array}$$

(b) If $\mathcal{G}$ is discrete, then every continuous $\mathcal{G}$-valued cocycle is locally determined, hence ${}_\infty\mathcal{C}^d_{\mathrm{eq}}(\mathfrak{A},\mathcal{G}) = \mathcal{C}^d_{\mathrm{eq}}(\mathfrak{A},\mathcal{G})$ for every $d \in [0...D]$. This yields an equality ${}_\infty\mathbf{C}_{\mathrm{eq}}(\mathfrak{A},\mathcal{G}) = \mathbf{C}_{\mathrm{eq}}(\mathfrak{A},\mathcal{G})$ of chain complexes, so that ${}_\infty\mathcal{H}^d_{\mathrm{eq}}(\mathfrak{A},\mathcal{G}) = \mathcal{H}^d_{\mathrm{eq}}(\mathfrak{A},\mathcal{G})$ for every $d \in [0...D]$. This, together with part (a), implies that $\mathcal{H}^d_{\mathrm{inv}}(\mathfrak{A},\mathcal{G}) \cong \mathcal{H}^d_{\mathrm{eq}}(\mathfrak{A},\mathcal{G})$. Then Theorem 4.4 implies that $\mathcal{H}^1_{\mathrm{inv}}(\mathfrak{A},\mathcal{G}) \cong \mathcal{H}^1_{\mathrm{dy}}(\mathfrak{A},\mathcal{G})$. □

We now turn to Theorem 4.10. If $\mathbf{a} \in \widetilde{\mathfrak{A}}$, then for all $r \in \mathbb{N}$, let $\varsigma^r_{\mathbf{a}} : \mathbb{G}_r(\mathbf{a}) \longrightarrow \mathbf{X}_r$ be the continuous section of Π_r from §3.5, which induces a homomorphism ${}_r\mathcal{H}^d_{\mathrm{inv}}\mathbf{a}\colon \mathcal{H}^d_{\mathrm{inv}}(\mathbf{X}_r,\mathcal{G}) \longrightarrow \mathcal{H}^d[\mathbb{G}_r(\mathbf{a}),\mathcal{G}]$, defined by ${}_r\mathcal{H}^d_{\mathrm{inv}}\mathbf{a}(\underline{C}) := \underline{C \circ \varsigma^r_{\mathbf{a}}}$. These homomorphisms converge to a direct limit homomorphism $\mathcal{H}^d_{\mathrm{inv}}\mathbf{a}\colon \mathcal{H}^d_{\mathrm{inv}}(\mathfrak{A},\mathcal{G}) \longrightarrow \mathcal{H}^d[\mathbb{G}_\infty(\mathbf{a}),\mathcal{G}]$, by an argument analogous to Theorem 3.7(a).

PROPOSITION 4.14. *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be an SFT. Let $(\mathcal{G},+)$ be abelian. Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$.*

(a) *For any $r \in \mathbb{N}$, ($\mathbf{a}$ has a d-pole of range r) $\Longrightarrow$ (${}_r\mathcal{H}^d_{\mathrm{inv}}\mathbf{a}$ is nontrivial).*
If $\mathrm{Ext}\,(\mathcal{H}_{d-1}[\mathbb{G}_r(\mathbf{a})],\mathcal{G}) = 0$, *then '$\Longleftarrow$' is also true.*

(b) *($\mathbf{a}$ has a projective d-pole) $\Longrightarrow$ ($\mathcal{H}^d_{\mathrm{inv}}\mathbf{a}$ is nontrivial).*
If $\mathrm{Ext}\,(\mathcal{H}_{d-1}[\mathbb{G}_r(\mathbf{a})],\mathcal{G}) = 0$ *for all large $r \in \mathbb{N}$, then '$\Longleftarrow$' is also true.*

Proof: **(a)** '$\Longrightarrow$' Let $\mathbb{G}_r := \mathbb{G}_r(\mathbf{a})$. If $\mathbf{a}$ has a d-pole of range r, then there exists $C' \in {}_r\mathcal{Z}^d_{\mathrm{eq}}(\mathfrak{A},\mathcal{G})$ and a d-cycle $\zeta \in \mathbb{Z}[\mathbf{Y}^d \cap \mathbb{G}_r]$ such that $C'(\zeta,\mathbf{a}) \neq 0$. Lemma 4.13(b,c) yields $C \in \mathcal{Z}^d_{\mathrm{inv}}(\mathbf{X}_r,\mathcal{G})$ with $\widetilde{C} = C'$. Then $C \circ \varsigma^r_{\mathbf{a}}(\zeta) = \widetilde{C}(\zeta,\mathbf{a}) = C'(\zeta,\mathbf{a}) \neq 0$. Hence, the cocycle $C \circ \varsigma^r_{\mathbf{a}}$ cannot be nullhomologous on $\mathbb{G}_r$. But $\underline{C \circ \varsigma^r_{\mathbf{a}}} = {}_r\mathcal{H}^d_{\mathrm{inv}}\mathbf{a}(\underline{C})$. Hence ${}_r\mathcal{H}^d_{\mathrm{inv}}\mathbf{a}$ is nontrivial.

(a) '$\Longleftarrow$' Suppose ${}_r\mathcal{H}^d_{\mathrm{inv}}\mathbf{a}$ is nontrivial; hence there exists $\underline{C} \in \mathcal{H}^d_{\mathrm{inv}}(\mathbf{X}_r,\mathcal{G})$ such that $\underline{C'} := {}_r\mathcal{H}^d_{\mathrm{inv}}\mathbf{a}(\underline{C})$ is a nontrivial cohomology class in $\mathcal{H}^d(\mathbb{G}_r,\mathcal{G})$.

CLAIM 1: *There is some d-cycle $\zeta \in \mathcal{Z}_d(\mathbb{G}_r,\mathbb{Z})$ such that $C'(\zeta) \neq 0$.*

Proof: For any $C' \in \mathcal{Z}^d(\mathbb{G}_r,\mathcal{G})$ and $\zeta \in \mathcal{Z}_d(\mathbb{G}_r,\mathbb{Z})$, the value of $C'(\zeta)$ depends only on the cohomology class of C' and the homology class of ζ. Thus, if $\mathcal{H}_d := \mathcal{H}_d(\mathbb{G}_r,\mathbb{Z})$, then $\underline{C'}$ defines a function $\mathcal{H}_d \longrightarrow \mathcal{G}$. This construction yields a homomorphism $\mathcal{H}^d(\mathbb{G}_r,\mathcal{G}) \longrightarrow \mathsf{Hom}\,(\mathcal{H}_d,\mathcal{G})$. The Universal Coefficient Theorem [**Hat02**, Theorem 3.2] says that the kernel of this homomorphism is $\mathrm{Ext}\,(\mathcal{H}_{d-1},\mathcal{G})$. Hence, if $\mathrm{Ext}\,(\mathcal{H}_{d-1},\mathcal{G}) = 0$, then any nontrivial cohomology class $\underline{C'} \in \mathcal{H}^d(\mathbb{G}_r,\mathcal{G})$ defines a nontrivial element of $\mathsf{Hom}\,(\mathcal{H}_d,\mathcal{G})$, which means $C'(\zeta) \neq 0$ for some $\zeta \in \mathcal{Z}_d(\mathbb{G}_r,\mathbb{Z})$. ◇ `Claim 1`

If $\underline{C'} := {}_r\mathcal{H}^d_{\mathrm{inv}}\mathbf{a}(\underline{C})$, then $C' = C \circ \varsigma^r_{\mathbf{a}}$, so Claim 1 means that $\widetilde{C}(\zeta,\mathbf{a}) \neq 0$, where $\widetilde{C} \in {}_r\mathcal{H}^d_{\mathrm{eq}}(\mathfrak{A},\mathcal{G})$ is defined as prior to Lemma 4.13. Thus, $\mathbf{a}$ has a d-pole of range r.

(b) ($\mathbf{a}$ has a projective d-pole) $\iff$ ($\mathbf{a}$ has a d-pole of range r for all large $r \in \mathbb{N}$) $\underset{(*)}{\Longrightarrow}$ (${}_r\mathcal{H}^d_{\rm inv}\mathbf{a}$ is nontrivial for all large $r \in \mathbb{N}$) $\iff$ ($\mathcal{H}^d_{\rm inv}\mathbf{a}$ is nontrivial). Here "$\underset{(*)}{\Longrightarrow}$" is by part (a), and becomes a "$\underset{(*)}{\Longleftrightarrow}$" if $\mathrm{Ext}\,(\mathcal{H}_{d-1}[\mathbb{G}_r(\mathbf{a})], \mathcal{G}) = 0$ for all large $r \in \mathbb{N}$. □

Proof of Theorem 4.10: **(a)** Suppose Φ has radius q. Fix $r \in \mathbb{N}$, and recall that Φ is σ-commuting. Hence, for all $d \in [0...D]$, in the proof of Proposition 3.5(a), the induced cellular map $\Phi_* : \mathbf{X}^d_{r+q} \longrightarrow \mathbf{X}^d_r$ is Ξ_d-commuting, so the induced homomorphism $\mathcal{C}^d_r\Phi : \mathcal{C}^d(\mathbf{X}_r, \mathcal{G}) \longrightarrow \mathcal{C}^d(\mathbf{X}_{r+q}, \mathcal{G})$ is Ξ^d-commuting. Thus, $\mathcal{C}^d_r\Phi$ restricts to a map $\mathcal{C}^d_r\Phi : \mathcal{C}^d_{\rm inv}(\mathbf{X}_r, \mathcal{G}) \longrightarrow \mathcal{C}^d_{\rm inv}(\mathbf{X}_{r+q}, \mathcal{G})$. This yields a chain map $\mathcal{C}_r\Phi : \mathbf{C}_{\rm inv}(\mathbf{X}_r, \mathcal{G}) \longrightarrow \mathbf{C}_{\rm inv}(\mathbf{X}_{r+q}, \mathcal{G})$, which yields cohomology homomorphisms $\mathcal{H}^d_r\Phi : \mathcal{H}^d_{\rm inv}(\mathbf{X}_r, \mathcal{G}) \longrightarrow \mathcal{H}^d_{\rm inv}(\mathbf{X}_{r+q}, \mathcal{G})$ for all $d \in [0...D]$. Thus, taking the direct limit (as in Proposition 3.5) yields a homomorphism $\mathcal{H}^d_{\rm inv}\Phi : \mathcal{H}^d_{\rm inv}(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^d_{\rm inv}(\mathfrak{A}, \mathcal{G})$.

(b)[i] If $\mathbf{a} \in \widetilde{\mathfrak{A}}$ and $\mathbf{b} := \Phi(\mathbf{a})$, then part (a) and an argument analogous to Theorem 3.7(c) yield a commuting square

$$\begin{array}{ccc}
\mathcal{H}^k[\mathbb{G}_\infty(\mathbf{a}), \mathcal{G}] & \xleftarrow{\mathcal{H}^k\iota} & \mathcal{H}^k[\mathbb{G}_\infty(\mathbf{b}), \mathcal{G}] \\
{\scriptstyle \mathcal{H}^k_{\rm inv}\mathbf{a}}\uparrow & & \uparrow{\scriptstyle \mathcal{H}^k_{\rm inv}\mathbf{b}} \\
\mathcal{H}^k_{\rm inv}(\mathfrak{A}, \mathcal{G}) & \xleftarrow{\mathcal{H}^k_{\rm inv}\Phi} & \mathcal{H}^k_{\rm inv}(\mathfrak{A}, \mathcal{G})
\end{array}$$

Thus, if $\mathcal{H}^k_{\rm inv}\Phi$ is surjective and $\mathcal{H}^k_{\rm inv}\mathbf{a}$ is nontrivial, then $\mathcal{H}^k_{\rm inv}\mathbf{b}$ must also be nontrivial. Then Proposition 4.14(b) says: if $\mathbf{a}$ has a projective d-pole, and $\mathrm{Ext}\,(\mathcal{H}_{d-1}[\mathbb{G}_r(\mathbf{b})], \mathcal{G}) = 0$ for all large $r \in \mathbb{N}$, then $\mathbf{b}$ has a projective d-pole.

(b)[ii] follows from (b)[i] because $\mathcal{H}_0[\mathbb{G}_r(\mathbf{b})] = \mathbb{Z}^K$, where K is the number of connected components of $\mathbb{G}_r(\mathbf{b})$ [**Hat02**, Proposition 2.7], so $\mathrm{Ext}\,(\mathcal{H}_0[\mathbb{G}_r(\mathbf{b})], \mathcal{G}) = 0$. Also, for each $r \in \mathbb{N}$, $\mathbf{G}_r(\mathbf{b})$ is homotopic to an (orientable) D-dimensional submanifold of $\mathbb{R}^D$, so $\mathcal{H}_{D-1}[\mathbb{G}_r(\mathbf{b})]$ is torsion-free [**Hat02**, Corollary 3.28], so $\mathrm{Ext}\,(\mathcal{H}_{D-1}[\mathbb{G}_r(\mathbf{b})], \mathcal{G}) = 0$.

(b)[iii] follows from (b)[i] because if $\mathcal{G}$ is the additive group of a field, then $\mathrm{Ext}\,(\mathcal{H}, \mathcal{G}) = 0$ for any group $\mathcal{H}$. □

4.3. *Appendix on Homological Algebra:* If $\mathcal{H}$ is any abelian group, we can always write $\mathcal{H} \cong \mathcal{F}_1/\mathcal{F}_2$, where $\mathcal{F}_1$ is a free abelian group and $\mathcal{F}_2$ is a subgroup. Let $\mathcal{G}$ be another abelian group. Let $\mathcal{H}^* := \mathsf{Hom}\,(\mathcal{H}, \mathcal{G})$, $\mathcal{F}_1^* := \mathsf{Hom}\,(\mathcal{F}_1, \mathcal{G})$, and $\mathcal{F}_2^* := \mathsf{Hom}\,(\mathcal{F}_2, \mathcal{G})$, and observe that the short exact sequence $0 \longrightarrow \mathcal{F}_2 \overset{i}{\rightarrowtail} \mathcal{F}_1 \overset{q}{\twoheadrightarrow} \mathcal{H} \longrightarrow 0$ induces a sequence $\mathcal{F}_2^* \xleftarrow{i^*} \mathcal{F}_1^* \xleftarrow{q^*} \mathcal{H}$ where $i^*(\phi) := \phi \circ i$ and $q^*(\phi) := \phi \circ q$. Now, $i^* \circ q^* = 0$ because $q \circ i = 0$ (by definition); hence $\mathrm{img}\,(q^*) \subseteq \ker(i^*)$. We thus define $\mathrm{Ext}\,(\mathcal{H}, \mathcal{G}) := \ker(i^*)/\mathrm{img}\,(q^*)$. This definition is independent of the choice of 'free resolution' $(\mathcal{F}_2, \mathcal{F}_1)$. Furthermore, if $\mathcal{H}$ is a finitely generated abelian group, so that $\mathcal{H} \cong \mathbb{Z}^R \oplus \mathbb{Z}_{/n_1} \oplus \cdots \oplus \mathbb{Z}_{/n_K}$ (for some R and $n_1, \ldots, n_K$), then $\mathrm{Ext}\,(\mathcal{H}, \mathcal{G}) \cong \bigoplus_{k=1}^K (\mathcal{G}/n_k\mathcal{G})$. In particular, if $\mathcal{H} \cong \mathbb{Z}^R$, or if $(\mathcal{G}, +)$ is the additive group of a field, then $\mathrm{Ext}\,(\mathcal{H}, \mathcal{G}) = 0$. See [**Hat02**, p.195 of §3.1].

A *chain complex* is an infinite sequence of abelian groups and homomorphisms $\mathcal{C}^0 \xrightarrow{\delta^0} \mathcal{C}^1 \xrightarrow{\delta^1} \mathcal{C}^2 \xrightarrow{\delta^2} \cdots$ such that $\delta^{n+1} \circ \delta^n = 0$ for all $n \in \mathbb{N}$. We represent this structure as

$\mathbf{C} := \{\mathcal{C}^n, \delta^n\}_{n=0}^{\infty}$. (In fact, for our purposes, only the groups $\mathcal{C}^0, \ldots, \mathcal{C}^D$ are nontrivial; however it is both conventional and convenient to develop the theory for infinite chain complexes.) If $\mathcal{Z}^n := \ker(\delta^n)$ and $\mathcal{B}^n := \operatorname{img}\left(\delta^{n-1}\right)$, then $\mathcal{B}^n \subseteq \mathcal{Z}^n$. We define $\mathcal{H}^n(\mathbf{C}) := \mathcal{Z}^n/\mathcal{B}^n$, to be the nth *cohomology group* of the chain complex $\mathbf{C}$ (we formally define $\mathcal{B}^0 := \{0\}$, so $\mathcal{H}^0 = \mathcal{Z}^0$). See [**Hat02**, §2.1] or [**Lan84**, §IV.2].

If $\mathbf{C}_1 := \{\mathcal{C}_1^n, \delta_1^n\}_{n=0}^{\infty}$ is another chain complex, then a *chain map* from $\mathbf{C}$ to $\mathbf{C}_1$ is a sequence of homomorphisms $\boldsymbol{\phi} := \{\phi^n : \mathcal{C}^n \longrightarrow \mathcal{C}_1^n\}_{n=0}^{\infty}$ such that $\delta_1^n \circ \phi^n = \phi^{n+1} \circ \delta^n$ for all $n \in \mathbb{N}$. We indicate this by writing "$\boldsymbol{\phi} : \mathbf{C} \longrightarrow \mathbf{C}_1$". The set of all chain complexes and chain maps forms a category $\mathfrak{C}$, and cohomology yields functors $\mathcal{H}^n$ from $\mathfrak{C}$ to the category $\mathfrak{A}$ of abelian groups. To be precise if $\boldsymbol{\phi} : \mathbf{C} \longrightarrow \mathbf{C}_1$ is a chain map, then there is a homomomorphism $\mathcal{H}^n\boldsymbol{\phi} : \mathcal{H}^n(\mathbf{C}) \longrightarrow \mathcal{H}^n(\mathbf{C}_1)$ defined by $\mathcal{H}^n\boldsymbol{\phi}(z + \mathcal{B}^n) = \phi^n(z) + \mathcal{B}_1^n$ for any $z \in \mathcal{Z}^n$. (Recall that elements of $\mathcal{H}^n(\mathbf{C})$ are cosets of $\mathcal{B}^n$ in $\mathcal{Z}^n$. The function $\mathcal{H}^n\boldsymbol{\phi}$ is well-defined because $\phi^n(\mathcal{Z}^n) \subseteq \mathcal{Z}_1^n$ and $\phi^n(\mathcal{B}^n) \subseteq \mathcal{B}_1^n$). See [**Hat02**, Prop. 2.9].

Let $\mathbf{C}_1 \xrightarrow{\boldsymbol{\iota}_1} \mathbf{C}_2 \xrightarrow{\boldsymbol{\iota}_2} \mathbf{C}_3 \xrightarrow{\boldsymbol{\iota}_3} \cdots$ be an infinite sequence of chain complexes and chain maps. The *direct limit* is the chain complex $\mathbf{C} := \{\mathcal{C}^n, \delta^n\}_{n=0}^{\infty}$, where for each $n \in \mathbb{N}$, $\mathcal{C}^n := \varinjlim \left(\mathcal{C}_1^n \xrightarrow{\iota_1^n} \mathcal{C}_2^n \xrightarrow{\iota_2^n} \mathcal{C}_3^n \xrightarrow{\iota_3^n} \cdots\right)$, and where the maps $\delta^n : \mathcal{C}^n \longrightarrow \mathcal{C}^{n+1}$ arise from the commuting grid:

$$
\begin{tikzcd}
\vdots \arrow[d, "\delta_1^{n-1}"'] & \vdots \arrow[d, "\delta_2^{n-1}"'] & \vdots \arrow[d, "\delta_3^{n-1}"'] & & \vdots \arrow[d, "\delta^{n-1}"'] \\
\mathcal{C}_1^n \arrow[r, "\iota_1^n"] \arrow[d, "\delta_1^n"'] & \mathcal{C}_2^n \arrow[r, "\iota_2^n"] \arrow[d, "\delta_2^n"'] & \mathcal{C}_3^n \arrow[r, "\iota_3^n"] \arrow[d, "\delta_3^n"'] & \cdots & \mathcal{C}^n \arrow[d, "\delta^n"'] \\
\mathcal{C}_1^{n+1} \arrow[r, "\iota_1^{n+1}"] \arrow[d, "\delta_1^{n+1}"'] & \mathcal{C}_2^{n+1} \arrow[r, "\iota_2^{n+1}"] \arrow[d, "\delta_2^{n+1}"'] & \mathcal{C}_3^{n+1} \arrow[r, "\iota_3^{n+1}"] \arrow[d, "\delta_3^{n+1}"'] & \cdots & \mathcal{C}^{n+1} \arrow[d, "\delta^{n+1}"'] \\
\vdots & \vdots & \vdots & & \vdots
\end{tikzcd}
$$

Fix $n \in \mathbb{N}$. The sequence of chain maps $(\mathbf{C}_1 \xrightarrow{\boldsymbol{\iota}_1} \mathbf{C}_2 \xrightarrow{\boldsymbol{\iota}_2} \cdots)$ induces a sequence of cohomology homomorphisms $(\mathcal{H}^n\mathbf{C}_1 \xrightarrow{\mathcal{H}^n\boldsymbol{\iota}_1} \mathcal{H}^n\mathbf{C}_2 \xrightarrow{\mathcal{H}^n\boldsymbol{\iota}_2} \cdots)$.

LEMMA 4.15. *For any* $n \in \mathbb{N}$, $\mathcal{H}^n\mathbf{C} = \varinjlim \left(\mathcal{H}^n\mathbf{C}_1 \xrightarrow{\mathcal{H}^n\boldsymbol{\iota}_1} \mathcal{H}^n\mathbf{C}_2 \xrightarrow{\mathcal{H}^n\boldsymbol{\iota}_2} \mathcal{H}^n\mathbf{C}_3 \xrightarrow{\mathcal{H}^n\boldsymbol{\iota}_3} \cdots\right)$.

Proof: For every $r \in \mathbb{N}$, we have short exact sequences $0 \longrightarrow \mathcal{B}_r^n \rightarrowtail \mathcal{Z}_r^n \twoheadrightarrow \mathcal{H}^n(\mathbf{C}_r) \longrightarrow 0$, where $\mathcal{B}_r^n := \operatorname{img}\left(\delta_r^{n-1}\right) \subseteq \mathcal{Z}_r^n := \ker(\delta_r^n) \subseteq \mathcal{C}_r^n$. Also, $\iota_r^n(\mathcal{Z}_r^n) \subseteq \mathcal{Z}_{r+1}^n$ and $\iota_r^n(\mathcal{B}_r^n) \subseteq \mathcal{B}_{r+1}^n$. If $\mathcal{B}^n := \varinjlim \left(\mathcal{B}_1^n \xrightarrow{\iota_1^n} \mathcal{B}_2^n \xrightarrow{\iota_2^n} \cdots\right)$ and $\mathcal{Z}^n := \varinjlim \left(\mathcal{Z}_1^n \xrightarrow{\iota_1^n} \mathcal{Z}_2^n \xrightarrow{\iota_2^n} \cdots\right)$, then $\mathcal{B}^n = \operatorname{img}\left(\delta^{n-1}\right) \subseteq \mathcal{Z}^n = \ker(\delta^n) \subseteq \mathcal{C}^n$. If $\widetilde{\mathcal{H}}^n := \varinjlim \left(\mathcal{H}^n\mathbf{C}_1 \xrightarrow{\mathcal{H}^n\iota_1} \mathcal{H}^n\mathbf{C}_2 \xrightarrow{\mathcal{H}^n\iota_2} \cdots\right)$, then these short exact sequences converge to a short exact sequence $0 \longrightarrow \mathcal{B}^n \rightarrowtail \mathcal{Z}^n \twoheadrightarrow \widetilde{\mathcal{H}}^n \longrightarrow 0$. Thus, $\widetilde{\mathcal{H}}^n \cong \mathcal{Z}^n/\mathcal{B}^n$. But $\mathcal{Z}^n/\mathcal{B}^n = \mathcal{H}^n\mathbf{C}$ by definition. □

Conclusion: We have developed algebraic invariants which help to explain the emergence, persistence, and interaction of defects in cellular automata. However, many questions remain.

1. Example 2.9(c) shows that our set of algebraic invariants is not yet sufficient to detect all essential defects. Are there other algebraic invariants?

2. Proposition 2.11(b), Theorem 2.15(b), Corollary 3.8 and Theorem 4.10(b) all say that homotopic or (co)homological defects are Φ-persistent if the homotopy/(co)homology homomorphism induced by Φ is injective/surjective. When is this the case?

3. With the exception of a few examples of π_1 computed in in [**GP95**], there are no explicit computations of the homotopy/(co)homology groups of §3.5 and §4.2, partly because of the difficulty of taking the required inverse/direct limits. This limits the applicability of the theory. Is there an easy way to compute these groups?

4. The pole/residue theory of §2.2 suggests an appealing analogy between two-dimensional symbolic dynamics and complex analysis. Is there a deeper relationship beyond this analogy?

5. Conway has shown that the Penrose tiling has exactly 61 distinct σ-homocliny classes of essential codimension-two defects, by using the fact that any Penrose tiling can be cross-hatched by 'Ammann bars' [**GS87**, §10.5, p.566]. Can this method be extended to some two-dimensional subshifts of finite type?

6. If $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$, and there is a CA Φ with $\Phi^n(\mathcal{A}^{\mathbb{Z}^D}) \subseteq \mathfrak{A} \subseteq \mathsf{Fix}[\Phi]$, then $\mathfrak{A}$ admits no essential defects. The converse is also true, when $\mathfrak{A}$ is a one-dimensional sofic shift with a σ-fixed point [**Maa95**]. Is the converse true in higher dimensions?

7. Even when $\mathfrak{A}$ admits essential defects, Kůrka and Maass [**KM00, KM02, Kůr03, Kůr05**] have described how a one-dimensional CA can 'converge in measure' to $\mathfrak{A}$ through a gradual process of defect coalescence/annihilation. Given a subshift $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$, is it possible to build a CA which converges to $\mathfrak{A}$ in this sense?

Finally, we remark that many of the results here should generalize to random cellular automata (i.e. CA-valued stochastic processes) which almost-surely preserve a given subshift. For example, these include zero-temperature (anti)ferromagnet models acting on $\mathfrak{Mo}$ or $\mathfrak{Ch}$ with random boundary motions [**Elo94**], and random tile-rearranging processes on $\mathfrak{Dom}$ [**CEP96, CKP01**] or $\mathfrak{Ice}$ [**Elo99, Elo03, Elo05**].

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