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Spectral domain boundaries in cellular automata

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Abstract. Let $\mathcal{A}^{\mathbb{Z}^D}$ be the Cantor space of $\mathbb{Z}^D$-indexed configurations in a finite alphabet $\mathcal{A}$, and let σ be the $\mathbb{Z}^D$-action of shifts on $\mathcal{A}^{\mathbb{Z}^D}$. A ***cellular automaton*** is a continuous, σ-commuting self-map Φ of $\mathcal{A}^{\mathbb{Z}^D}$, and a Φ-***invariant subshift*** is a closed, (Φ,σ)-invariant subset $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$. Suppose $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ is $\mathfrak{A}$-admissible everywhere except for some small region we call a ***defect***. It has been empirically observed that such defects persist under iteration of Φ, and often propagate like 'particles' which coalesce or annihilate on contact. We use spectral theory to explain the persistence of some defects under Φ, and partly explain the outcomes of their collisions.

An often-observed phenomenon in cellular automata is the emergence and persistence of homogeneous 'domains' (each characterized by a particular spatial pattern), separated by *defects* (analogous to 'domain boundaries' or 'kinks' in a crystalline solid) which evolve over time, propagating and occasionally colliding. Such defects were first empirically observed by Grassberger in the 'elementary' cellular automata or 'ECA' (radius-one CA on $\{0,1\}^{\mathbb{Z}}$) with numbers #18, #122, #126, #146, and #182 [18, 19] and also noted in ECA #184, considered as a simple model of surface growth [33, §III.B]. Later, Boccara *et al.* empirically investigated the motion and interactions of defects in ECA #18, #54, #62, and #184 (see Figure 1), and longer range totalistic CA [1, 2]; see also [27, §3.1.2.2 & §3.1.4.4]. Eloranta made the first rigorous mathematical study of defects in [11, 12, 13, 14, 15], while Crutchfield and Hanson developed an empirical methodology called *Computational Mechanics* [4, 5, 6, 7, 8, 24]. In another paper [48], we characterized the propagation of defects in one-dimensional subshifts of finite type.

A mathematical theory of cellular automaton defect dynamics is starting to emerge, but some basic questions remain unanswered:

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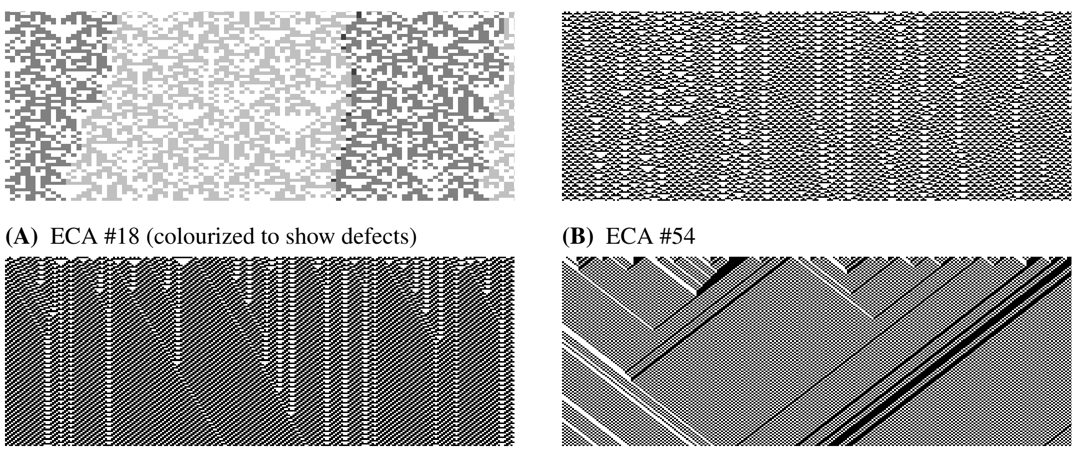
1. What is the right definition of ‘defect?’ What constitutes a ‘regular domain’?
2. Why should a defect persist under the action of cellular automata, rather than disappearing? Are there ‘topological’ constraints imposed by the structure of the underlying domain, which make defects indestructible?
3. When defects collide, they often coalesce into a new type of defect, or mutually annihilate. Is there a ‘chemistry’ governing these defect collisions?
4. Can we assign algebraic invariants to defects, which reflect **(a)** the ‘topological constraints’ of question #2 or **(b)** the ‘defect chemistry’ of question #3?

This paper is organized as follows: in §1 we propose an answer to question #1, which is a synthesis of several previous approaches (see below). We also introduce several examples which recur throughout the paper. In §2 and §3, we use spectral theory to address question #4 for two types of domain boundaries: *interfaces* and *dislocations*. We develop spectral invariants which answer question #2, and partially answer question #3.



(A) ECA #18 (colourized to show defects) **(B)** ECA #54

(C) ECA #62 **(D)** ECA #184

Figure 1. Spacetime diagrams showing defect dynamics in one-dimensional cellular automata. Each picture show 120 timesteps on a 250 pixel array (time increases downwards).

Background: Roughly speaking, there are three approaches to question #1:

(a) ‘Domains’ are identified with ‘invariant subalphabets’ of the CA. ‘Defects’ are transitions from one subalphabet to another.

(b) ‘Defects’ are identified by evaluating some numerical ‘weight function’ on the subwords of a sequence. ‘Domains’ are regions where this weight function takes the value 0.

(c) ‘Domains’ are identified with some subshift (finite type or sofic). ‘Defects’ are the (minimal-length) ‘forbidden words’ of this subshift.

Approach **(a)** was first developed in [15] to prove Lind's conjecture [41, §5] that the defects of ECA#18 perform random walks. It was then applied to ECA's #22, #54, #184 and other one-dimensional CA [11, 12], and extended to defect ensembles [13] and two-dimensional domain boundaries [14]. Approach **(b)** was proposed by Kůrka [34], and used to verify another conjecture of Lind [41, §5], that some cellular automata (such as ECA#18) converge 'in measure' to certain limit sets through a process of defect coalescence/annihilation; see also [36, 37, 38] for related ideas. Approach **(c)** was first suggested in [41, §5], and later elaborated in [4, 5, 6, 7, 8, 24], where each 'domain' was identified with a regular language (or equivalently, with a sofic shift), which could be digitally filtered out of the spacetime diagram using a finite automaton, thereby revealing the defect trajectories. Approach **(c)** was also used in [48].

Each approach has advantages and disadvantages. Approach **(b)** is the most flexible, but also seems the most artificial, since we must explicitly define the defects. In contrast, approach **(c)** allows defects to arise out of a simple and naturally occuring background. However, both **(b)** and **(c)** suffer from a problem of defect 'delocalization'. A sofic shift cannot be characterized by any finite set of forbidden words. Thus, there must exist huge yet indecomposable 'defect particles'. If we arbitrarily assign these unwieldy defects a 'location' at some point in $\mathbb{Z}^D$, then we are confronted with instantaneous long-range interactions between defects, or the apparent 'teleportation' of defects through space. To avoid this problem, we could only admit subshifts of finite type as regular domains; then the minimal-length forbidden words are bounded in length, so that defects can be localized. However, for certain CA (e.g. ECA#18), the regular domain is *not* of finite type —it is strictly sofic —hence these CA would escape our analysis.

Approach **(a)** avoids the 'delocalization' problem. For example, the 'defect-free' subshift of ECA#18 is a sofic subshift [see Example 1.1(c)], but when recoded using the alphabet $\{00, 01, 10, 11\}$, it is simply the disjoint union of two full shifts: $\mathfrak{E} = \mathcal{E}^{\mathbb{Z}}$ and $\mathfrak{O} = \mathcal{O}^{\mathbb{Z}}$, where $\mathcal{E} := \{00, 01\}$ and $\mathcal{O} := \{00, 10\}$ (the symbol 11 is forbidden). A defect is then a transition from an $\mathcal{E}$-valued sequence to a $\mathcal{O}$-valued sequence. However, **(a)** is only appropriate for codimension-one defects (i.e. 'domain boundaries') and not for defects of codimension two (e.g. 'holes' in $\mathbb{Z}^2$, 'strings' in $\mathbb{Z}^3$, etc.) or higher. Even in codimension one, not every domain/defect problem can be recoded in terms of invariant subalphabets, without obscuring important information.

In §1, we will propose a definition which combines features of **(a)**, **(b)**, and **(c)**, and which is applicable defects of any codimension, in any kind of subshift (sofic or otherwise).

Preliminaries & Notation

For any $L \leq R \in \mathbb{Z}$, we define

$$[L...R] := \{L, L{+}1, \ldots, R\}, \qquad [L...R) := [L \ldots R{-}1], \qquad (L...R] := [L{+}1 \ldots R], \quad \text{etc.}$$

We likewise define $(-\infty....R]$, $[L...\infty)$, etc. Let $\mathcal{A}$ be a finite alphabet. Let $D \geq 1$, let $\mathbb{Z}^D$ be the D-dimensional lattice, and let $\mathcal{A}^{\mathbb{Z}^D}$ be the set of all $\mathbb{Z}^D$-indexed ***configurations*** of the form $\mathbf{a} = [a_{\mathsf{z}}]_{\mathsf{z}\in\mathbb{Z}^D}$, where $a_{\mathsf{z}} \in \mathcal{A}$ for all $\mathsf{z} \in \mathbb{Z}^D$. The ***Cantor metric*** on $\mathcal{A}^{\mathbb{Z}^D}$ is defined by $d(\mathbf{a}, \mathbf{b}) = 2^{-\Delta(\mathbf{a},\mathbf{b})}$, where $\Delta(\mathbf{a}, \mathbf{b}) := \min\{|\mathsf{z}| \,;\, a_{\mathsf{z}} \neq b_{\mathsf{z}}\}$. It follows that $(\mathcal{A}^{\mathbb{Z}^D}, d)$ is a Cantor space (i.e. a compact, totally disconnected, perfect metric space). If $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$, and $\mathbb{U} \subset \mathbb{Z}^D$, then we define $\mathbf{a}_{\mathbb{U}} \in \mathcal{A}^{\mathbb{U}}$ by $\mathbf{a}_{\mathbb{U}} := [a_{\mathsf{u}}]_{\mathsf{u}\in\mathbb{U}}$. If $\mathsf{z} \in \mathbb{Z}^D$, then strictly speaking, $\mathbf{a}_{\mathsf{z}+\mathbb{U}} \in \mathcal{A}^{\mathsf{z}+\mathbb{U}}$; however, it is sometimes convenient to 'abuse notation' and treat $\mathbf{a}_{\mathsf{z}+\mathbb{U}}$ as an element of $\mathcal{A}^{\mathbb{U}}$ in the obvious way. If $\mathbb{X} \subset \mathbb{Y} \subseteq \mathbb{Z}^D$, and $\mathbf{x} \in \mathcal{A}^{\mathbb{X}}$ and $\mathbf{y} \in \mathcal{A}^{\mathbb{Y}}$, we write "$\mathbf{x} \sqsubset \mathbf{y}$" if $\mathbf{x} = \mathbf{y}_{\mathbb{X}}$.

Cellular automata: For any $\mathsf{v} \in \mathbb{Z}^D$, we define the *shift* $\sigma^{\mathsf{v}} : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ by $\sigma^{\mathsf{v}}(\mathbf{a})_{\mathsf{z}} = a_{\mathsf{z}+\mathsf{v}}$ for all $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ and $\mathsf{z} \in \mathbb{Z}^D$. A *cellular automaton* is a transformation $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ that is continuous and commutes with all shifts. Equivalently, Φ is determined by a *local rule* $\phi : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$ so that $\Phi(\mathbf{a})_{\mathsf{z}} = \phi(\mathbf{a}_{\mathsf{z}+\mathbb{H}})$ for all $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ and $\mathsf{z} \in \mathbb{Z}^D$ [25]. Here, $\mathbb{H} \subset \mathbb{Z}^D$ is a finite set which we normally imagine as a 'neighbourhood of the origin'. If $\mathbb{H} \subseteq \mathbb{B}(r) := [-r...r]^D$, we say that Φ has *radius* r.

Subshifts: A subset $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ is a *subshift* [40, 32] if $\mathfrak{A}$ is closed in the Cantor topology, and if $\sigma^{\mathsf{z}}(\mathfrak{A}) = \mathfrak{A}$ for all $\mathsf{z} \in \mathbb{Z}^D$. For any $\mathbb{U} \subset \mathbb{Z}^D$, we define $\mathfrak{A}_{\mathbb{U}} := \{\mathbf{a}_{\mathbb{U}} \; ; \; \mathbf{a} \in \mathfrak{A}\}$. In particular, for any $r > 0$, let $\mathfrak{A}_{(r)} := \mathfrak{A}_{\mathbb{B}(r)}$ be the set of *admissible r-blocks* for $\mathfrak{A}$. We say $\mathfrak{A}$ is *subshift of finite type* (SFT) if there is some $r > 0$ (the *radius* of $\mathfrak{A}$) such that $\mathfrak{A}$ is entirely described by $\mathfrak{A}_{(r)}$, in the sense that $\mathfrak{A} = \left\{\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D} \; ; \; \mathbf{a}_{\mathbb{B}(\mathsf{z},r)} \in \mathfrak{A}_{(r)}, \; \forall \mathsf{z} \in \mathbb{Z}^D\right\}$ (here, $\mathbb{B}(\mathsf{z}, r) := \mathsf{z} + [-r...r]^D$). If $D = 1$, then a *Markov subshift* is an SFT $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ determined by a set $\mathfrak{A}_{\{0,1\}} \subset \mathcal{A}^{\{0,1\}}$ of *admissible transitions*; equivalently, $\mathfrak{A}$ is the set of all bi-infinite directed paths in a digraph whose vertices are the elements of $\mathfrak{A}$, with an edge $a \rightsquigarrow b$ iff $(a, b) \in \mathfrak{A}_{\{0,1\}}$. If $D = 2$, then let $\mathbb{E}_1 := \{(0, 0), (1, 0)\}$ and $\mathbb{E}_2 := \{(0, 0), (0, 1)\}$. A *Wang subshift* is an SFT $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^2}$ determined by sets $\mathfrak{A}_{\mathbb{E}_1} \subset \mathcal{A}^{\mathbb{E}_1}$ and $\mathfrak{A}_{\mathbb{E}_2} \subset \mathcal{A}^{\mathbb{E}_2}$ of *edge-matching conditions*. Equivalently, $\mathfrak{A}$ is the set of all *tilings* of the plane $\mathbb{R}^2$ by unit square tiles (corresponding to the elements of $\mathcal{A}$) with notched edges representing the edge-matching conditions [23, Ch.11].

If $\mathcal{X}$ is any set and $F : \mathfrak{A} \longrightarrow \mathcal{X}$ is a function, then F is *locally determined* if there is some *radius* $r \in \mathbb{N}$ and some *local rule* $f : \mathfrak{A}_{(r)} \longrightarrow \mathcal{X}$ such that $F(\mathbf{a}) = f(\mathbf{a}_{\mathbb{B}(r)})$ for any $\mathbf{a} \in \mathfrak{A}$. If $\mathcal{X}$ is any discrete space, then $F : \mathfrak{A} \longrightarrow \mathcal{X}$ is continuous iff F is locally determined. For example, if $\mathcal{A}$ and $\mathcal{B}$ are finite sets, then a (subshift) *homomorphism* is a continuous, σ-commuting function $\Phi : \mathcal{B}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ (e.g. a CA is a homomorphism with $\mathcal{A} = \mathcal{B}$); it follows that $F(\mathbf{a}) = \Phi(\mathbf{a})_0$ is locally determined. If $\mathfrak{B} \subset \mathcal{B}^{\mathbb{Z}^D}$ is a subshift of finite type, and $\Psi : \mathcal{B}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ is a homomorphism, then $\mathfrak{A} := \Psi(\mathfrak{B}) \subset \mathcal{A}^{\mathbb{Z}^D}$ is called a *sofic shift*. If $D = 1$, we define the *language* of a subshift $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ by $\mathcal{L}(\mathfrak{A}) := \bigcup_{n=0}^{\infty} \mathfrak{A}_{[0...n]}$; then $\mathfrak{A}$ is sofic if and only if $\mathcal{L}(\mathfrak{A})$ is a *regular language*; i.e. a language recognized by a finite automaton [26, §2.5 & §9.1]. Equivalently, $\mathfrak{A}$ is the set of all bi-infinite directed paths in a digraph whose vertices are (nonbijectively) $\mathcal{A}$-labelled.

If $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ is a cellular automaton, then we say $\mathfrak{A}$ is *Φ-invariant* if $\Phi(\mathfrak{A}) = \mathfrak{A}$, and $\mathfrak{A}$ is *weakly Φ-invariant* if $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$ (i.e. Φ is an *endomorphism* of $\mathfrak{A}$). For example:

(a) The set $\mathsf{Fix}[\Phi] := \left\{\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D} \; ; \; \Phi(\mathbf{a}) = \mathbf{a}\right\}$ of Φ-*fixed points* is a Φ-invariant SFT. Likewise, if $p \in \mathbb{N}$ and $\mathsf{v} \in \mathbb{Z}^D$, then the set $\mathsf{Fix}[\Phi^p]$ of (Φ, p)*-periodic points* and the set $\mathsf{Fix}[\Phi^p \circ \sigma^{-p\mathsf{v}}]$ of (Φ, p, v)*-travelling waves* are Φ-invariant SFTs.

(b) If $\mathbb{P} \subset \mathbb{Z}^D$ is a finite-index subgroup, then the set $\mathcal{A}^{\mathbb{Z}^D/\mathbb{P}} := \bigcap_{\mathsf{p} \in \mathbb{P}} \mathsf{Fix}[\sigma^{\mathsf{p}}]$ of $\mathbb{P}$*-periodic configurations* is a weakly Φ-invariant SFT. (Furthermore, all Φ-orbits in $\mathcal{A}^{\mathbb{Z}^D/\mathbb{P}}$ are eventually periodic, because $\mathcal{A}^{\mathbb{Z}^D/\mathbb{P}}$ is finite.)

(c) If Φ has radius R, then for any $r > 0$, Φ induces a function $\Phi : \mathcal{A}^{\mathbb{B}(R+r)} \longrightarrow \mathcal{A}^{\mathbb{B}(r)}$. If $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ is an SFT determined by a set $\mathfrak{A}_{(r)} \subset \mathcal{A}^{\mathbb{B}(r)}$ of admissible r-blocks, then $\Big(\Phi(\mathfrak{A}) \subseteq \mathfrak{A}\Big) \iff \Big(\Phi(\mathfrak{A}_{(r+R)}) \subseteq \mathfrak{A}_{(r)}\Big)$. Furthermore, if $D = 1$ and $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ is irreducible, then $\Phi(\mathfrak{A}) = \mathfrak{A}$ (surjectively) iff Φ is finite-to-one on $\mathfrak{A}$ iff Φ has no 'diamonds' i.e. words $\mathbf{a}, \mathbf{c} \in \mathfrak{A}_{(r)}$ and $\mathbf{b} \neq \mathbf{b}' \in \mathfrak{A}_L$ ($L \geq R$) with

$\Phi(\mathbf{abc}) = \Phi(\mathbf{ab'c})$ [40, Thm.8.1.16]. The endomorphism set of an SFT is quite huge; see [32, Ch.3] or [40, §13.2].

(d) Let $\Phi^\infty(\mathcal{A}^{\mathbb{Z}^D}) := \bigcap_{t=1}^\infty \Phi^t(\mathcal{A}^{\mathbb{Z}^D})$ be the *eventual image* of Φ. Then $\Phi^\infty(\mathcal{A}^{\mathbb{Z}^D})$ is a Φ-invariant subshift (possibly non-sofic), which contains $\mathsf{Fix}\,[\Phi^p \circ \sigma^{-p\mathsf{v}}]$ for any $p \in \mathbb{N}$ and $\mathsf{v} \in \mathbb{Z}^D$.

Trails: If $\mathsf{y}, \mathsf{z} \in \mathbb{Z}^D$, then we write "$\mathsf{y} \rightsquigarrow \mathsf{z}$" if $|\mathsf{z} - \mathsf{y}| = 1$. A *trail* is a sequence $\zeta = (\mathsf{z}_1 \rightsquigarrow \mathsf{z}_2 \rightsquigarrow \cdots \rightsquigarrow \mathsf{z}_n)$. A subset $\mathbb{Y} \subset \mathbb{Z}^D$ is *trail-connected* if, for any $\mathsf{x}, \mathsf{y} \in \mathbb{Y}$, there is a trail $\mathsf{x} = \mathsf{z}_0 \rightsquigarrow \mathsf{z}_1 \rightsquigarrow \cdots \rightsquigarrow \mathsf{z}_n = \mathsf{y}$ in $\mathbb{Y}$.

Font conventions: Upper case calligraphic letters $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots)$ denote finite alphabets. Upper-case Gothic letters $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots)$ are subsets of $\mathcal{A}^{\mathbb{Z}^D}$ (e.g. subshifts), lowercase bold-faced letters $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots)$ are elements of $\mathcal{A}^{\mathbb{Z}^D}$, and Roman letters $(a, b, c, \ldots)$ are elements of $\mathcal{A}$ or ordinary numbers. Lower-case sans-serif $(\ldots, \mathsf{x}, \mathsf{y}, \mathsf{z})$ are elements of $\mathbb{Z}^D$, upper-case hollow font $(\mathbb{U}, \mathbb{V}, \mathbb{W}, \ldots)$ are subsets of $\mathbb{Z}^D$, upper-case Greek letters $(\Phi, \Psi, \ldots)$ denote functions on $\mathcal{A}^{\mathbb{Z}^D}$ (e.g. CA), and lower-case Greek letters $(\phi, \psi, \ldots)$ denote other functions (e.g. local rules).

1. Domains and Defects

Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be any subshift. If $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$, then the *defect field* $\mathcal{F}_\mathbf{a} : \mathbb{Z}^D \longrightarrow \mathbb{N} \cup \{\infty\}$ is defined

$$\forall\, \mathsf{z} \in \mathbb{Z}^D, \quad \mathcal{F}_\mathbf{a}(\mathsf{z}) \quad := \quad \max\left\{r \in \mathbb{N}\ ;\ \mathbf{a}_{\mathbb{B}(\mathsf{z},r)} \in \mathfrak{A}_{(r)}\right\}.$$

It is easy to see that $\mathcal{F}_\mathbf{a}$ is 'Lipschitz' in the sense that $|\mathcal{F}_\mathbf{a}(\mathsf{y}) - \mathcal{F}_\mathbf{a}(\mathsf{z})| \le |\mathsf{y} - \mathsf{z}|$. The *defect set* of $\mathbf{a}$ is the set $\mathbb{D}(\mathbf{a}) \subset \mathbb{Z}^D$ of local minima of $\mathcal{F}_\mathbf{a}$.

Example 1.1: (a) $\mathbf{a} \in \mathfrak{A}$ if and only if $\mathcal{F}_\mathbf{a}(\mathsf{z}) = \infty$ for some (and thus all) $\mathsf{z} \in \mathbb{Z}^D$. In this case, $\mathbb{D}(\mathbf{a}) = \emptyset$.

(b) Suppose $\mathfrak{A}$ is an SFT determined by a set $\mathfrak{A}_{(r)} \subset \mathcal{A}^{\mathbb{B}(r)}$ of admissible r-blocks, and let $\mathbb{X} := \left\{\mathsf{z} \in \mathbb{Z}^D\ ;\ \mathbf{a}_{\mathbb{B}(\mathsf{z},r)} \not\in \mathfrak{A}_{(r)}\right\}$. Assume for simplicity that $\mathfrak{A}_{(r-1)} = \mathcal{A}^{\mathbb{B}(r-1)}$. Then $\mathcal{F}_\mathbf{a}(\mathsf{z}) = r + d(\mathsf{z}, \mathbb{X})$, where $d(\mathsf{z}, \mathbb{X}) := \min_{\mathsf{x} \in \mathbb{X}} |\mathsf{z} - \mathsf{x}|$. In particular, $\mathcal{F}_\mathbf{a}(\mathsf{z}) = r$ if and only if $\mathsf{z} \in \mathbb{X}$, and this is the smallest possible value for $\mathcal{F}_\mathbf{a}(\mathsf{z})$. Thus, $\mathbb{D}(\mathbf{a}) = \mathbb{X}$.

(c) Let $\mathcal{A} := \{0, 1\}$. Let $\mathfrak{S}$ be the sofic shift defined by the $\mathcal{A}$-labelled digraph ① $\leftrightarrows$ ⓪ $\leftrightarrows$ ⓪ (this is the invariant sofic shift of ECA#18 mentioned in the introduction). Let

$$\mathbf{s} \quad := \quad [\ldots 00\ 01\ 00\ 01\ 01\ \underbrace{00\ 00\ 00\ 00\ \overbrace{00}^{\mathbb{X}}\ 00\ 00\ 00\ 00}_{\mathbb{Y}}\ 10\ 00\ 10\ 00\ 00\ 10 \ldots].$$

Then for any $\mathsf{z} \in \mathbb{Z}$, $\mathcal{F}_\mathbf{s}(\mathsf{z})$ is the distance from z to the *farthest* endpoint of $\mathbb{Y}$; this is the maximum radius r such that $\mathbf{s}_{[z-r\ldots z+r]}$ is $\mathfrak{S}$-inadmissible. Thus, $\mathcal{F}_\mathbf{s}(\mathsf{z})$ takes a minimal value of 8 (inside $\mathbb{X}$) and increases linearly in either direction. Thus, $\mathbb{D}(\mathbf{s}) = \mathbb{X}$.

(d) Let $\mathcal{A} = \mathcal{B} \cup \mathcal{D}$ (think of $\mathcal{B}$ and $\mathcal{D}$ as 'invariant subalphabets' in the sense of [11, 12]), and let $\mathfrak{A} := \mathcal{B}^{\mathbb{Z}^D} \cup \mathcal{D}^{\mathbb{Z}^D}$. Let $\mathcal{C} := \mathcal{B} \cap \mathcal{D}$ be the (possibly empty) set of 'ambiguous symbols', and let $\mathcal{B}^* := \mathcal{B} \setminus \mathcal{C}$

and $\mathcal{D}^* := \mathcal{D} \setminus \mathcal{C}$. Any $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ is a mixture of $\mathcal{B}^*$-symbols, $\mathcal{C}$-symbols, and $\mathcal{D}^*$-symbols. If $\mathsf{z} \in \mathbb{Z}^D$ and $a_\mathsf{z} \in \mathcal{B}^*$, then $\mathcal{F}_\mathbf{a}(\mathsf{z}) = \min\{|\mathsf{y} - \mathsf{z}| \ ; \ a_\mathsf{y} \in \mathcal{D}^*\}$. If $a_\mathsf{z} \in \mathcal{D}^*$, then $\mathcal{F}_\mathbf{a}(\mathsf{z}) = \min\{|\mathsf{y} - \mathsf{z}| \ ; \ a_\mathsf{y} \in \mathcal{B}^*\}$. If $a_\mathsf{z} \in \mathcal{C}$, then $\mathcal{F}_\mathbf{a}(\mathsf{z}) = \min\{r \ ; \ a_\mathsf{x} \in \mathcal{B}^* \text{ and } a_\mathsf{y} \in \mathcal{B}^* \text{ for some } \mathsf{x}, \mathsf{y} \in \mathbb{B}(\mathsf{z}, r)\}$. Thus, $\mathbb{D}(\mathbf{a})$ is the set of all points which are either on a 'boundary' between a $\mathcal{B}^*$-domain and a $\mathcal{D}^*$-domain, or which lie roughly in the middle of a $\mathcal{C}$-domain. ♢

Remarks: If $\mathfrak{A}$ is an SFT [Example 1.1(b)], then the set $\mathbb{D}(\mathbf{a})$ encodes all information about the defects of $\mathbf{a}$. However, if $\mathfrak{A}$ is *not* of finite type, then $\mathbb{D}(\mathbf{a})$ is an inadequate description of the larger-scale 'defect structures' of $\mathbf{a}$. Thus, instead of treating the defect as a precisely defined subset of $\mathbb{Z}^D$, it is better to think of it as a 'fuzzy' object residing in the low areas in the defect field $\mathcal{F}_\mathbf{a}$. This is particularly appropriate when we want to evaluate a locally determined function (such as a cellular automaton or eigenfunction) on $\mathbf{a}$ at a point 'close' (but not *too* close) to the defect. The advantage of this definition is its applicability to any kind of subshift (finite type, sofic, or otherwise). Nevertheless, most of our examples will be SFTs, and we may then refer to the specific region $\mathbb{D} \subset \mathbb{Z}^D$ as 'the defect'. This is the approach taken in [48], for example. ♢

Let

$$\widetilde{\mathfrak{A}} \quad := \quad \left\{ \mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D} \ ; \ \sup_{\mathsf{z} \in \mathbb{Z}^D} \ \mathcal{F}_\mathbf{a}(\mathsf{z}) \ = \ \infty \right\}$$

be the set of 'slightly defective' configurations. If $\mathbf{a} \in \widetilde{\mathfrak{A}} \setminus \mathfrak{A}$, then we say $\mathbf{a}$ is ***defective***. Elements of $\widetilde{\mathfrak{A}}$ may have infinitely large defects, but also have arbitrarily large non-defective regions. Clearly $\mathfrak{A} \subset \widetilde{\mathfrak{A}}$, and $\widetilde{\mathfrak{A}}$ is a σ-invariant, dense subset of $\mathcal{A}^{\mathbb{Z}^D}$ (but not a subshift).

For any $R > 0$, let $\mathbb{G}_R(\mathbf{a}) := \{\mathsf{z} \in \mathbb{Z}^D \ ; \ \mathcal{F}_\mathbf{a}(\mathsf{z}) \geq R\}$. Thus, $\mathbf{a} \in \widetilde{\mathfrak{A}}$ iff $\mathbb{G}_R(\mathbf{a}) \neq \emptyset$ for all $R > 0$. For example, if $\mathfrak{A}$ is an SFT determined by a set $\mathfrak{A}_{(r)}$ of admissible r-blocks, and $\mathbb{D} = \{\mathsf{z} \in \mathbb{Z}^D \ ; \ \mathbf{a}_{\mathbb{B}(\mathsf{z},r)} \notin \mathfrak{A}_{(r)}\}$ as in Example 1.1(b), then

$$\mathbb{G}_R(\mathbf{a}) \quad = \quad \{\mathsf{z} \in \mathbb{Z}^D \ ; \ d(\mathsf{z}, \mathbb{D}) \geq R - r\} \quad = \quad \mathbb{Z}^D \setminus \mathbb{B}(\mathbb{D}, R - r).$$

However, if $\mathfrak{A}$ is *not* an SFT, then in general $\mathbb{G}_R(\mathbf{a}) \neq \mathbb{Z}^D \setminus \mathbb{B}(\mathbb{D}, R')$ for any $R' > 0$.

Proposition 1.2. Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA with radius $r > 0$.
(a) Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a weakly Φ-invariant subshift. Then $\Phi(\widetilde{\mathfrak{A}}) \subseteq \widetilde{\mathfrak{A}}$.
(b) If $\mathbf{a} \in \widetilde{\mathfrak{A}}$, and $\mathbf{a}' = \Phi(\mathbf{a})$, then $\mathcal{F}_{\mathbf{a}'} \geq \mathcal{F}_\mathbf{a} - r$. Thus, for all $R \in \mathbb{N}$, $\mathbb{G}_{R+r}(\mathbf{a}) \subseteq \mathbb{G}_R(\mathbf{a}')$.

Proof: **(b)** Let $\mathsf{z} \in \mathbb{Z}^D$ and suppose $\mathcal{F}_\mathbf{a}(\mathsf{z}) = R$. Thus, $\mathbf{a}_{\mathbb{B}(\mathsf{z},R)} \in \mathfrak{A}_R$. But $\mathfrak{A}$ is Φ-invariant; hence $\mathbf{a}'_{\mathbb{B}(\mathsf{z},R-r)} \in \mathfrak{A}_{(R-r)}$. Hence $\mathcal{F}_{\mathbf{a}'}(\mathsf{z}) \geq R - r$. Then **(a)** follows from **(b)**. □

If $\mathbf{a} \in \widetilde{\mathfrak{A}}$, we say $\mathbf{a}$ has a ***range*** r ***domain boundary*** if $\mathbb{G}_r(\mathbf{a})$ is trail-disconnected. Domain boundaries divide $\mathbb{Z}^D$ into different 'domains', which may correspond to different transitive components of $\mathfrak{A}$ (see §2), different eigenfunction phases (see §3), or different cocycle asymptotics [47, §2.3].

Example 1.3: (a) Let $\mathcal{A} = \{\blacksquare, \square\}$, and let $\mathfrak{M}\mathfrak{o} \subset \mathcal{A}^{\mathbb{Z}^2}$ be the ***monochromatic*** SFT defined by the condition that no $\blacksquare$ can be adjacent to a $\square$. Figure 2(A) shows a domain boundary in $\mathfrak{M}\mathfrak{o}$. See also Example 2.2(b).

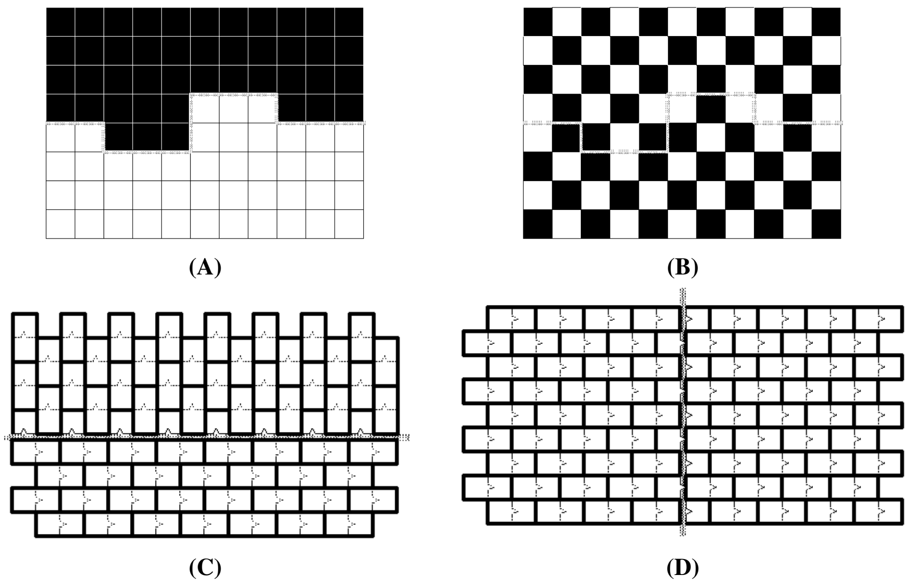


Figure 2. Domain boundaries. **(A)** An interface in $\mathfrak{Mo}$; see Examples 1.3(a) and 2.2(b). **(B)** A dislocation in $\mathfrak{Ch}$; see Examples 1.3(b) and 3.4(e). **(C,D)** Boundaries in $\mathfrak{Dom}$; see Examples 1.3(c) and 2.2(c).

(b) Let $\mathcal{A} = \{\blacksquare, \square\}$, and let $\mathfrak{Ch} \subset \mathcal{A}^{\mathbb{Z}^2}$ be the *checkerboard* SFT defined by the condition that no $\blacksquare$ can be adjacent to a $\blacksquare$, and no $\square$ can be adjacent to a $\square$. Figure 2(B) shows a domain boundary in $\mathfrak{Ch}$. See also Example 3.4(e).

(c) Let $\mathcal{D} := \left\{ \quad , \quad , \quad , \quad \right\}$, and let $\mathfrak{Dom} \subset \mathcal{D}^{\mathbb{Z}^2}$ be the *domino* SFT defined by the obvious edge-matching conditions. Figure 2(D,E) shows two domain boundaries in $\mathfrak{Dom}$; see also Example 2.2(c).

(d) Let $\mathcal{A} = \{0, 1\}$, and let $\mathfrak{S} \subset \mathcal{A}^{\mathbb{Z}^D}$ and $\mathbf{s} \in \widetilde{\mathfrak{S}}$ be as in Example 1.1(c). Then $\mathbf{s}$ has a domain boundary in the region $\mathbb{Y}$. Figure 1(A) shows defects of this type evolving under the iteration of ECA#18. $\diamondsuit$

Domain boundaries are defects of *codimension one* (they disconnect the space), and are the only kind which exist in one-dimensional cellular automata. In two-dimensional cellular automata, there are also defects of codimension *two*, which do *not* disconnect the space, but instead resemble localized 'holes'. In three-dimensional CA, a codimension-two defect has the topology of an extended 'string', while a 'hole'-shaped defect has codimension *three*. The precise definition of defect codimension involves homotopy groups; see [47]. The present paper is concerned only with defects of codimension one —i.e. domain boundaries.

Projective domain boundaries: The action of a cellular automaton may locally change the geometry of a defect, and we are mainly interested in properties that are invariant under such local modifications. If $\mathbf{a} \in \widetilde{\mathfrak{A}}$, then a connected component $\mathbb{Y}$ of $\mathbb{G}_r(\mathbf{a})$ is called *projective* if, for all $R \geq r$, $\mathbb{Y} \cap \mathbb{G}_R(\mathbf{a}) \neq \emptyset$. We say that $\mathbf{a} \in \widetilde{\mathfrak{A}}$ has a *projective domain boundary* if there is some $R \geq 0$ such that $\mathbb{G}_R(\mathbf{a})$ has at least two projective components. (Hence $\mathbb{G}_r(\mathbf{a})$ is disconnected for all $r \geq R$.)

A trail-connected subset $\mathbb{Y} \subset \mathbb{Z}^D$ is *spacious* if, for any $R > 0$, there exists $\mathsf{y} \in \mathbb{Y}$ with $\mathbb{B}(\mathsf{y}, R) \subset$

$\mathbb{Y}$. If $\mathbb{Y}$ is a connected component of $\mathbb{G}_r(\mathbf{a})$, then ($\mathbb{Y}$ is projective) $\Longrightarrow$ ($\mathbb{Y}$ is spacious). If $\mathfrak{A}$ is a subshift of finite type, then the converse is also true.

Example 1.4: (a) A proper subset $\mathbb{Y} \subset \mathbb{Z}$ is spacious iff either $\mathbb{Y} = (-\infty...Z]$ or $\mathbb{Y} = [Z...\infty)$ for some $Z \in \mathbb{Z}$.

(b) We say the defect in $\mathbf{a}$ is *finite* if $\lim_{|\mathsf{z}|\to\infty} \mathcal{F}_{\mathbf{a}}(\mathsf{z}) = \infty$. (This implies that $\mathbb{D}(\mathbf{a})$ is finite. If $\mathfrak{A}$ is a subshift of finite type, then the converse is also true.) If $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$, then $\mathbf{a} \in \widetilde{\mathfrak{A}}$ has a projective domain boundary iff the defect of $\mathbf{a}$ is finite. If R is large enough, then $\mathbb{G}_R(\mathbf{a}) = (-\infty...Z_1] \sqcup [Z_2...\infty)$ for some $Z_1, Z_2 \in \mathbb{Z}$. ♢

Essential vs. Persistent Defects: Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a cellular automaton, and suppose that $\phi(\mathfrak{A}) = \mathfrak{A}$. If $\mathbf{a} \in \widetilde{\mathfrak{A}}$ then $\mathbf{a}$ has a Φ-*persistent* defect if, for all $t \in \mathbb{N}$, $\mathbf{a}' = \Phi^t(\mathbf{a})$ is also defective. Otherwise $\mathbf{a}$ has a *transient* defect —i.e. one which eventually disappears. Our main goal is to determine when defects are persistent. We say $\mathbf{a}$ has a *removable* defect if there is some $r > 0$ and some $\mathbf{a}' \in \mathfrak{A}$ so that $a'_{\mathsf{z}} = a_{\mathsf{z}}$ for all $\mathsf{z} \in \mathbb{G}_r(\mathbf{a})$ (i.e. the defect can be erased by modifying $\mathbf{a}$ in a finite radius of the defective region). Otherwise $\mathbf{a}$ has an *essential* defect.

Example 1.5: Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift of finite type. Then $(\mathfrak{A}, \sigma)$ is topologically mixing if and only if no finite defect is essential. ♢

Proposition 1.6. Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA and let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a Φ-invariant subshift. If $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}$ is bijective, then any essential defect is Φ-persistent.

Proof: Suppose $\mathbf{a} \in \widetilde{\mathfrak{A}}$ has an essential defect and let $\mathbf{a}' := \Phi(\mathbf{a})$. We must show that $\mathbf{a}'$ is also defective. We will suppose not and derive a contradiction. Suppose that Φ has radius $H > 0$. Then for any $r > 0$, we have a map $\Phi_r : \mathfrak{A}_{(r+H)} \longrightarrow \mathfrak{A}_{(r)}$.

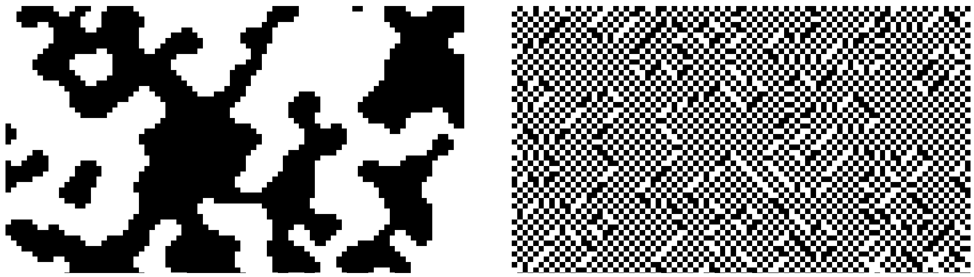
CLAIM 1: *There exists $R > 0$ such that, for all $r \geq R$, the function $\Phi_r : \mathfrak{A}_{(r+H)} \longrightarrow \mathfrak{A}_{(r)}$ is bijective.*

Proof: Suppose not. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence such that, for each $n \in \mathbb{N}$, $\Phi_{r_n} : \mathfrak{A}_{(r_n+H)} \longrightarrow \mathfrak{A}_{(r_n)}$ is not bijective. Let $\mathbf{c}_n \in \mathfrak{A}_{(r_n)}$ be a point with two Φ_{r_n}-preimages in $\mathfrak{A}_{(r_n+H)}$, say $\mathbf{b}_n$ and $\mathbf{b}'_n$. By dropping to a subsequence if necessary, we can assume $\mathbf{c}_1 \sqsubset \mathbf{c}_2 \sqsubset \cdots$ and $\mathbf{b}_1 \sqsubset \mathbf{b}_2 \sqsubset \cdots$ and $\mathbf{b}'_1 \sqsubset \mathbf{b}'_2 \sqsubset \cdots$. Thus, there are limit points $\mathbf{c}, \mathbf{b}, \mathbf{b}' \in \mathfrak{A}$ such that $\mathbf{c}_n = \mathbf{c}_{\mathbb{B}(r_n)}$, $\mathbf{b}_n = \mathbf{b}_{\mathbb{B}(r_n)}$, and $\mathbf{b}'_n = \mathbf{b}'_{\mathbb{B}(r_n)}$ for all $n \in \mathbb{N}$ (because $\mathfrak{A}$ is compact). Also, $\Phi(\mathbf{b}) = \mathbf{c} = \Phi(\mathbf{b}')$ (because Φ is continuous). But $\mathbf{b} \neq \mathbf{b}'$ (by construction) so this contradicts the supposed bijectivity of Φ. ♢ `Claim 1`

If $\mathbf{a}'$ is not defective, then $\mathbf{a}' \in \mathfrak{A}$. Let $\mathbf{b} \in \mathfrak{A}$ be the unique Φ-preimage of $\mathbf{a}'$ in $\mathfrak{A}$ (recall that $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}$ is bijective). Let $R > 0$ be as in Claim 1, and let $\mathbb{Y} := \mathbb{G}_{R+H}(\mathbf{a})$.

CLAIM 2: $\mathbf{a}_{\mathbb{Y}} = \mathbf{b}_{\mathbb{Y}}$.

Proof: If $\mathsf{y} \in \mathbb{Y}$, then $\mathbf{a}_{\mathbb{B}(\mathsf{y},R+H)} \in \mathfrak{A}_{(R+H)}$, so $\Phi(\mathbf{a}_{\mathbb{B}(\mathsf{y},R+H)}) = (\mathbf{a}')_{\mathbb{B}(\mathsf{y},R)}$ is an element of $\mathfrak{A}_{(R)}$. But by definition, $\Phi(\mathbf{b}_{\mathbb{B}(\mathsf{y},R+H)}) = \mathbf{a}'_{\mathbb{B}(\mathsf{y},R)}$ also. Thus, $\mathbf{a}_{\mathbb{B}(\mathsf{y},R+H)} = \mathbf{b}_{\mathbb{B}(\mathsf{y},R+H)}$ (because Claim 1 says $\Phi_r : \mathfrak{A}_{(r+H)} \longrightarrow \mathfrak{A}_{(r)}$ is bijective.) This is true for every $\mathsf{y} \in \mathbb{Y}$. ♢ `Claim 2`



(A) **(B)**

Figure 3. Configurations with many (nonprojective) domain boundaries. **(A)** An equilibrium of a voter CA [Example 1.8(a)], with $\mathfrak{Mo}$-domain boundaries. **(B)** An equilibrium of the zero-temperature antiferromagnet [Example 1.8(b)], with $\mathfrak{Ch}$-domain boundaries.

Thus, $\mathbf{a}$ and $\mathbf{b}$ are equal on $\mathbb{G}_{R+H}(\mathbf{a})$, so the defect in $\mathbf{a}$ is removable. Contradiction. □

Corollary 1.7. Suppose $\mathfrak{A} \subseteq \mathsf{Fix}[\Phi]$ or $\mathfrak{A} \subseteq \mathsf{Fix}[\Phi^p \circ \sigma^{p\mathsf{v}}]$ (for some $p \in \mathbb{N}$ and $\mathsf{v} \in \mathbb{Z}^D$). If $\mathbf{a} \in \widetilde{\mathfrak{A}}$, then any essential defect in $\mathbf{a}$ is Φ-persistent. □

Example 1.8: (a) Let $\mathfrak{Mo}$ be as in Example 1.3(a). The domain boundary in Figure 2(A) is essential because it separates two infinite domains of opposite colour (and one infinite domain would have to be completely erased to eliminate the defect). Let $\Phi : \mathcal{A}^{\mathbb{Z}^2} \longrightarrow \mathcal{A}^{\mathbb{Z}^2}$ be a *Voter CA* [52] with local rule $\phi : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$ defined

$$\phi(\mathbf{a}) \quad := \quad \begin{cases} \blacksquare & \text{if} \quad N(\mathbf{a}) < \theta; \\ \square & \text{if} \quad N(\mathbf{a}) \geq \theta; \end{cases} \qquad \text{where} \quad N(\mathbf{a}) \ := \ \frac{\#\{\mathsf{h} \in \mathbb{H} \ ; \ a_{\mathsf{h}} = \square\}}{\#(\mathbb{H})},$$

and where $\theta \in [0,1]$ is some *threshold*. Then $\mathfrak{Mo} \subset \mathsf{Fix}[\Phi]$; hence the domain boundary in Figure 2(A) is Φ-persistent. If θ is close to 1/2 (for example, in the *zero-temperature ferromagnet* CA, $\theta := 1/2$), then domain boundaries like Figure 2(A) are roughly stationary. The CA rapidly evolves from random initial conditions to a mottled equilibrium configuration with infinitely many such boundaries, as in Figure 3(A) (these boundaries are not essential, because the domains are finite). If θ is close to 0 (respectively to 1), then the boundary in Figure 2(A) will rapidly propagate north (respectively south). Likewise, a small 'seed' of one colour will grow monotonically like a crystal. (The asymptotic shape of these 'crystals' has been studied in [54, 20, 21, 22].)

(b) Let $\mathfrak{Ch}$ be as in Example 1.3(b). The domain boundary in Figure 2(B) is essential because it separates two infinite domains of opposite 'phase'. Let $\Phi\colon \mathcal{A}^{\mathbb{Z}^2} \to \mathcal{A}^{\mathbb{Z}^2}$ be a *zero-temperature antiferromagnet* CA, with local rule $\phi : \mathcal{A}^{\mathbb{H}} \longrightarrow \mathcal{A}$ defined

$$\phi(\mathbf{a}) \ := \ \begin{cases} \blacksquare & \text{if} \quad N_0(\mathbf{a}) - N_1(\mathbf{a}) < 0; \\ \square & \text{if} \quad N_0(\mathbf{a}) - N_1(\mathbf{a}) \geq 0; \end{cases}$$

$$\begin{aligned} \text{where} \quad N_0(\mathbf{a}) \ &:= \ \#\{\mathsf{h} = (h_0, h_1) \in \mathbb{H} \ ; \ a_{\mathsf{h}} = \square \text{ and } \ h_0 + h_1 = 0 \bmod 2\}, \\ \text{and} \ N_1(\mathbf{a}) \ &:= \ \#\{\mathsf{h} = (h_0, h_1) \in \mathbb{H} \ ; \ a_{\mathsf{h}} = \square \text{ and } \ h_0 + h_1 = 1 \bmod 2\}. \end{aligned}$$

Then $\mathfrak{Ch} \subset \mathsf{Fix}[\Phi]$; hence the domain boundary in Figure 2(B) is Φ-persistent. Figure 3(B) shows a Φ-fixed configuration with many $\mathfrak{Ch}$-domain boundaries. $\diamondsuit$

2. Interfaces

A Φ-invariant subshift $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ is (Φ, σ)*-transitive* if, for any nonempty open subset $\mathfrak{O} \subset \mathfrak{A}$, the union

$$\bigcup_{t \in \mathbb{N}} \bigcup_{\mathsf{z} \in \mathbb{Z}^D} \Phi^{-t} \sigma^{-\mathsf{z}}(\mathfrak{O})$$

is dense in $\mathfrak{A}$. Equivalently, there exists $\mathbf{a} \in \mathfrak{A}$ such that the orbit $\{\Phi^t \circ \sigma^{\mathsf{z}}(\mathbf{a})\}_{t \in \mathbb{N}, \mathsf{z} \in \mathbb{Z}^D}$ is dense in $\mathfrak{A}$.

Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a (Φ, σ)-nontransitive subshift, and suppose $\mathfrak{A} = \mathfrak{A}_1 \sqcup \cdots \sqcup \mathfrak{A}_K$, where $\mathfrak{A}_1, \ldots, \mathfrak{A}_K$ are clopen (Φ, σ)-transitive components. As $\mathfrak{A}_1, \ldots, \mathfrak{A}_K$ are clopen, their indicator functions are locally determined; hence there is some $r > 0$, and a function $\kappa : \mathfrak{A}_{(r)} \longrightarrow [1...K]$ such that, for any $\mathbf{a} \in \mathfrak{A}$, we have $(\mathbf{a} \in \mathfrak{A}_k) \Leftrightarrow \left(\kappa(\mathbf{a}_{\mathbb{B}(r)}) = k\right)$. For any $\mathsf{z} \in \mathbb{Z}^D$, let $\kappa_{\mathsf{z}}(\mathbf{a}) := \kappa(\mathbf{a}_{\mathbb{B}(\mathsf{z},r)})$. Hence, if $\mathbf{a} \in \mathfrak{A}_k$, then $\kappa_{\mathsf{z}}(\mathbf{a}) = k$ for all $\mathsf{z} \in \mathbb{Z}^D$. However, $\kappa_{\mathsf{z}}(\mathbf{a})$ is also well-defined for any $\mathbf{a} \in \widetilde{\mathfrak{A}}$ such that $\mathbf{a}_{\mathbb{B}(\mathsf{z},r)}$ is $\mathfrak{A}$-admissible. Hence, if $\mathbf{a} \in \widetilde{\mathfrak{A}}$, then $\kappa_{\mathsf{z}}(\mathbf{a})$ is well-defined for all $\mathsf{z} \in \mathbb{G}_r(\mathbf{a})$. If $\mathsf{y}, \mathsf{z} \in \mathbb{G}_r(\mathbf{a})$, then we say that $\mathbf{a}$ has an $(\mathfrak{A}, \Phi)$*-interface* between y and z if $\kappa_{\mathsf{y}}(\mathbf{a}) \neq \kappa_{\mathsf{z}}(\mathbf{a})$. (We will simply call this an *interface* when $\mathfrak{A}$ and Φ are clear from context).

Proposition 2.1. Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA and let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$. Suppose $\mathbf{a} \in \widetilde{\mathfrak{A}}$ has an $(\mathfrak{A}, \Phi)$-interface. Let κ have radius r' and let $r \geq r' + 1$. Then

(a) The interface in $\mathbf{a}$ is a range r domain boundary.

(b) $\kappa(\mathbf{a})$ is constant on each connected component of $\mathbb{G}_r(\mathbf{a})$. Suppose $\mathbb{G}_r(\mathbf{a})$ has connected components $\{\mathbb{Y}_n\}_{n=1}^N$ (where $N \in \mathbb{N} \cup \{\infty\}$). There is a function $\mathcal{K} : [1...N] \longrightarrow [1...K]$ such that, for any $n \in [1...N]$ and any $\mathsf{y} \in \mathbb{Y}_n$, $\kappa_{\mathsf{y}}(\mathbf{a}) = \mathcal{K}(n)$.

(c) If two of the components $\mathbb{Y}_n$ and $\mathbb{Y}_m$ are projective, and $\mathcal{K}(n) \neq \mathcal{K}(m)$, then there is an essential domain boundary between $\mathbb{Y}_n$ and $\mathbb{Y}_m$.

Proof: **(b)** Suppose x and z are in the same connected component of $\mathbb{G}_r(\mathbf{a})$. Let $\mathsf{x} = \mathsf{y}_0 \rightsquigarrow \mathsf{y}_1 \rightsquigarrow \ldots \rightsquigarrow \mathsf{y}_N = \mathsf{z}$ be a trail from x to z in $\mathbb{G}_r(\mathbf{a})$.

CLAIM 1: *For all* $n \in [1...N]$, $\kappa_{\mathsf{y}_n}(\mathbf{a}) = \kappa_{\mathsf{y}_{n-1}}(\mathbf{a})$.

Proof: If $\mathbb{U} := \mathbb{B}(\mathsf{y}_n, r)$ then $\mathbf{a}_{\mathbb{U}} \in \mathfrak{A}_{\mathbb{U}}$ (because $\mathcal{F}_{\mathbf{a}}(\mathsf{y}_n) \geq r$ by definition). Thus, there exists $\mathbf{b} \in \mathfrak{A}$ such that $\mathbf{b}_{\mathbb{U}} = \mathbf{a}_{\mathbb{U}}$. There is a (unique) $k \in [1...K]$ such that $\mathbf{b} \in \mathfrak{A}_k$. Thus, $\kappa_{\mathsf{y}_{n-1}}(\mathbf{a}) \underset{(*)}{=} \kappa_{\mathsf{y}_{n-1}}(\mathbf{b}) = k = \kappa_{\mathsf{y}_n}(\mathbf{b}) \underset{(\dagger)}{=} \kappa_{\mathsf{y}_n}(\mathbf{a})$, where $(*)$ is because $\mathbb{B}(\mathsf{y}_{n-1}, r') \subset \mathbb{U}$ and $(\dagger)$ is because $\mathbb{B}(\mathsf{y}_n, r') \subset \mathbb{U}$. $\diamondsuit$ `Claim 1`

Apply Claim 1 inductively to conclude that $\kappa_{\mathsf{x}}(\mathbf{a}) = \kappa_{\mathsf{z}}(\mathbf{a})$.

(a) If there exist $\mathsf{y}, \mathsf{z} \in \mathbb{G}_r(\mathbf{a})$ with $\kappa_{\mathsf{y}}(\mathbf{a}) \neq \kappa_{\mathsf{z}}(\mathbf{a})$, then **(b)** says y and z must be in different connected components of $\mathbb{G}_r(\mathbf{a})$. **(c)** follows immediately. $\square$

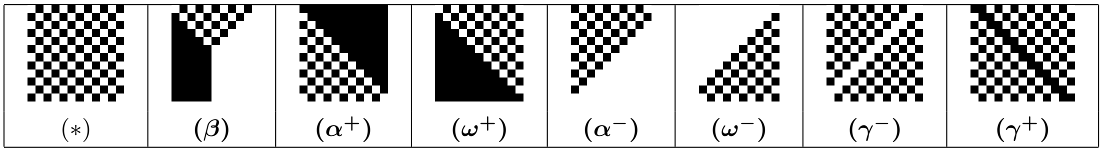


Figure 4. $(*)$ The periodic background generated by ${}_{\varepsilon}\Phi_{184}$ acting on $\mathfrak{G}$. $(\boldsymbol{\alpha}^{\pm}, \boldsymbol{\beta}, \boldsymbol{\omega}^{\pm})$: Interfaces in $\mathfrak{G}$; see Example 2.2(a). $(\boldsymbol{\gamma}^{\pm})$: Dislocations in $\mathfrak{G}$; see Examples 3.4(c) and 3.7(c). (In the nomenclature of [1, §III(A)], $\gamma^+ = \vec{0}_2$, $\gamma^- = \overleftarrow{1}_2$, while both α^+ and ω^+ are "$\vec{0}_\infty$", and both α^- and ω^- are "$\overleftarrow{1}_\infty$"). See also [48, Example 1.2(a)]

.

By reordering if necessary, assume that the projective components are $\mathbb{Y}_1, \ldots, \mathbb{Y}_M$. We call the restricted function $\mathcal{K}_* : [1...M] \longrightarrow [1...K]$ the ***signature*** of the interface. Proposition 2.1(c) says the interface is essential if $\mathcal{K}_*$ is not constant.

Example 2.2: (a) (ECA#184) Let $\mathcal{A} = \{\blacksquare, \square\}$, and let $\mathfrak{G} = \mathfrak{G}_0 \sqcup \mathfrak{G}_1 \sqcup \mathfrak{G}_*$, where $\mathfrak{G}_0 := \{\overline{\blacksquare}\}$, $\mathfrak{G}_1 := \{\overline{\square}\}$, and $\mathfrak{G}_* := \{\overline{\blacksquare\square}, \overline{\square\blacksquare}\}$. (Here, $\overline{\blacksquare} := [\ldots\blacksquare\blacksquare.\blacksquare\blacksquare\ldots]$ and $\overline{\blacksquare\square} := [\ldots\blacksquare\square.\blacksquare\square\blacksquare\square\ldots]$, etc., where the period is before the zeroth coordinate). If ${}_{\varepsilon}\Phi_{184}$ is ECA#184, then $\mathfrak{G}_0 \cup \mathfrak{G}_1 \subset \mathsf{Fix}\left[{}_{\varepsilon}\Phi_{184}\right]$, while ${}_{\varepsilon}\Phi_{184}|_{\mathfrak{G}_*} = \sigma$, as shown in Figure 4$(*)$. Thus $\mathfrak{G}$ has three $({}_{\varepsilon}\Phi_{184}, \sigma)$-transitive components, so there are six possible interfaces:

α^+ :	$\mathfrak{G}_*$	…■□■□■□	■■■■■■…	$\mathfrak{G}_0$	α^- :	$\mathfrak{G}_*$	…■□■□■□	□□□□□□…	$\mathfrak{G}_1$
ω^+ :	$\mathfrak{G}_0$	…■■■■■■	■□■□■□…	$\mathfrak{G}_*$	ω^- :	$\mathfrak{G}_1$	…□□□□□□	■□■□■□…	$\mathfrak{G}_*$
β :	$\mathfrak{G}_0$	…■■■■■■	□□□□□□…	$\mathfrak{G}_1$	ϵ :	$\mathfrak{G}_0$	…□□□□□□	■■■■■■…	$\mathfrak{G}_1$

Figure 4$(\alpha^{\pm}, \beta, \omega^{\pm})$ shows the ${}_{\varepsilon}\Phi_{184}$-evolution of these defects. (The ϵ defect is unstable, and immedately 'decays' into ω^- and α^+ defects travelling in opposite directions.) Figure 1(D) showed the long-term ${}_{\varepsilon}\Phi_{184}$-evolution of these defects. If $\mathbf{g} \in \widetilde{\mathfrak{G}}$ has a finite defect, then $\mathbf{g}$ can be written as an ensemble of range-r defects $\mathbf{d}_1, \ldots, \mathbf{d}_N$ arranged along a line, with $\mathbb{Y}_0, \ldots, \mathbb{Y}_N$ being the $\mathfrak{G}$-admissible intervals between these defects:

$$\cdots -\!\!-\mathbb{Y}_0 \longrightarrow \mathbf{d}_1 \longleftarrow \mathbb{Y}_1 \longrightarrow \mathbf{d}_2 \longleftarrow \mathbb{Y}_2 \longrightarrow \cdots \longleftarrow \mathbb{Y}_{N-1} \longrightarrow \mathbf{d}_N \longleftarrow \mathbb{Y}_N -\!\!- \cdots$$

The projective components are $\mathbb{Y}_0$ and $\mathbb{Y}_N$. Hence the interface is essential if $\mathcal{K}(0) \neq \mathcal{K}(N)$.

(b) Let $\mathcal{A} = \{\blacksquare, \square\}$ and let $\mathfrak{Mo} \subset \mathcal{A}^{\mathbb{Z}^2}$ be as in Example 1.3(a). Then $\mathfrak{Mo} = \{\blacksquare^\infty, \square^\infty\}$, where $\blacksquare^\infty$ is the solid black configuration, and $\square^\infty$ is the solid black configuration. Let $\Phi : \mathcal{A}^{\mathbb{Z}^2} \longrightarrow \mathcal{A}^{\mathbb{Z}^2}$ be an CA such that $\mathfrak{Mo} \subset \mathsf{Fix}[\Phi]$ [e.g. a voter CA from Example 1.8(a)] Then $\mathfrak{Mo}$ has two (Φ, σ)-transitive components, $\mathfrak{M}_0 := \{\blacksquare^\infty\}$ and $\mathfrak{M}_1 := \{\square^\infty\}$. Figure 2(A) shows a domain boundary in $\mathfrak{Mo}$. If $\mathbb{Y}_0$ is the northern connected component and $\mathbb{Y}_1$ is the southern component, then we have $\mathcal{K}(0) = 0$ and $\mathcal{K}(1) = 1$. Both components are projective, so this is an essential interface.

(c) Let $\mathfrak{Dom}$ be as in Example 1.3(c). Despite appearances, the domain boundary in Figure 2(C) is *not* an interface, because $(\mathfrak{Dom}, \sigma)$ is topologically transitive [10, Lemma 2.1]. Instead, this is a 'gap' defect; see [47, Example 2.14(b)]. ◇

The next result implies that the defects in Examples 2.2(a,b) must be persistent.

Proposition 2.3. Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA. If $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ is a subshift with $\Phi(\mathfrak{A}) = \mathfrak{A}$, then any essential $(\mathfrak{A}, \Phi)$-interface is Φ-persistent. If $\mathbf{a} \in \widetilde{\mathfrak{A}}$ has an essential interface, then $\Phi(\mathbf{a})$ also has an essential interface, with the same signature as $\mathbf{a}$.

Proof: Let $\mathbf{a}' := \Phi(\mathbf{a})$ and suppose Φ has radius $R > 0$. Then each projective component of $\mathbb{G}_{r+R}(\mathbf{a})$ is contained in a projective component of $\mathbb{G}_r(\mathbf{a}')$, because $\mathbb{G}_{r+R}(\mathbf{a}) \subseteq \mathbb{G}_r(\mathbf{a}')$ by Lemma 1.2(b). Let $\mathbb{G}_{R+r}(\mathbf{a})$ have projective components $\mathbb{Y}_1, \ldots, \mathbb{Y}_M$, and let $\mathbb{G}_r(\mathbf{a}')$ have projective components $\mathbb{Y}'_1, \ldots, \mathbb{Y}'_M$, where $\mathbb{Y}_m \subset \mathbb{Y}'_m$ for all $m \in [1...M]$. If $\mathsf{y} \in \mathbb{Y}_m \subset \mathbb{Y}'_m$, then $\kappa_{\mathsf{y}}(\mathbf{a})$ and $\kappa_{\mathsf{y}}(\mathbf{a}')$ are well-defined, and, in the notation of Proposition 2.1(b), we must have $\kappa_{\mathsf{y}}(\mathbf{a}') = \mathcal{K}(m) = \kappa_{\mathsf{y}}(\mathbf{a})$. □

Remark 2.4: If $\mathfrak{A}$ is not σ-transitive, then a ***σ-transitive decomposition*** of $\mathfrak{A}$ is a collection of disjoint clopen σ-transitive components $\mathfrak{A}_1, \ldots, \mathfrak{A}_R$ such that $\mathfrak{A} = \mathfrak{A}_1 \sqcup \cdots \sqcup \mathfrak{A}_R$. (Not all non-transitive subshifts admit such a decomposition.) If $\mathfrak{A}$ has a σ-transitive decomposition, and $\Phi(\mathfrak{A}) = \mathfrak{A}$, then Φ induces a permutation $\varphi : \{\mathfrak{A}_1, ..., \mathfrak{A}_R\} \longrightarrow \{\mathfrak{A}_1, ..., \mathfrak{A}_R\}$, and each (Φ, σ)-transitive component of $\mathfrak{A}$ is a union of all elements of $\{\mathfrak{A}_1, ..., \mathfrak{A}_R\}$ in some ς-orbit. (In particular, if $\varphi = \mathbf{Id}$, then the (Φ, σ)-transitive components of $\mathfrak{A}$ are also $\mathfrak{A}_1, \ldots, \mathfrak{A}_R$.)

3. Dislocations

In a periodic crystalline solid, a *dislocation* (or *fault line*) is an internal surface separating two domains whose crystal structures are spatially out of phase. We will use the word ***dislocation*** to describe an analogous domain boundary in a configuration in $\mathcal{A}^{\mathbb{Z}^D}$. The main results of this section are Theorems 3.3, 3.6 3.12, and 3.14.

Example 3.1: If $D = 1$ and $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ is a σ-transitive SFT, then any dislocation in $\mathfrak{A}$ takes a simple form. For simplicity, suppose $\mathfrak{A}$ is a Markov subshift. For any $a, c \in \mathcal{A}$, say that c is ***reachable from a in time t*** if there is a word $\mathbf{b} \in \mathcal{A}^{t-1}$ such that $a\mathbf{b}c$ is $\mathfrak{A}$-admissible. There is some $P = P(\mathfrak{A}) \in \mathbb{N}$ (the ***period*** of $\mathfrak{A}$) and a ***phase partition*** $\mathcal{A} = \mathcal{A}_0 \sqcup \cdots \sqcup \mathcal{A}_{P-1}$ such that, if $a \in \mathcal{A}_n$ and $c \in \mathcal{A}_m$, then c is reachable from a in time t only if $t \equiv m - n \pmod{P}$ [40, Prop.4.5.6]. Hence, $\mathfrak{A}$ is mixing iff $P = 1$. A sequence $\mathbf{a} = [\ldots a_{-1}a_0a_1 \ldots] \in \mathcal{A}^{\mathbb{Z}}$ thus has a ***dislocation*** at 0 if $a_0 \in \mathcal{A}_n$ and $a_1 \in \mathcal{A}_m$, but $m \neq n + 1 \pmod{P}$. The ***phase gap*** of the dislocation is the value $m - (n + 1)$ (as an element of $\mathbb{Z}_{/P}$). Two such defects can 'cancel out' if and only if their phase gaps together sum to zero, mod P.

For example, let $\mathcal{A} := \{a, b, c\}$ and let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ be the Markov subshift defined by the $\mathcal{A}$-labelled digraph $a \leftrightarrows b \leftrightarrows c$. Then $P = 2$, with $\mathfrak{A}_0 = \{a, c\}$ and $\mathfrak{A}_1 = \{b\}$. Hence, the sequence $[\ldots ababcbabc.abcbcbab\ldots]$ has a dislocation at the decimal point. ◇

If $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ is a non-finite type subshift, or if $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ is of finite type, for $D \geq 2$, then dislocations and their 'phase gaps' can take a more subtle form than in Example 3.1. Spectral theory provides the tools to characterize these dislocations.

3.1. Rational Dislocations

Let $\mathcal{C}(\mathfrak{A})$ be the $\mathbb{C}$-vector space of continuous $\mathbb{C}$-valued functions on $\mathfrak{A}$. Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle. A (Φ, σ)-***eigenfunction*** of $\mathfrak{A}$ is any $f \in \mathcal{C}(\mathfrak{A})$ admitting some ***generalized eigenvalue $\boldsymbol{\lambda}$*** $= (\lambda_0;\ \lambda_1, \ldots, \lambda_D) \in \mathbb{T}^{D+1}$ such that:

(a) $f \circ \Phi = \lambda_0 f$.

(b) For any $\mathsf{z} = (z_1, \ldots, z_D) \in \mathbb{Z}^D$, $f \circ \sigma^{\mathsf{z}} = \boldsymbol{\lambda}^{\mathsf{z}} f$, where $\boldsymbol{\lambda}^{\mathsf{z}} := \lambda_1^{z_1} \cdots \lambda_D^{z_D}$.

Let $\mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma) \subset \mathbb{T}^{D+1}$ be the set of all such eigenvalues. For any $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma)$, let

$$\mathcal{E}^{igen}_{\boldsymbol{\lambda}} \quad := \quad \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}, \Phi, \sigma) \quad = \quad \left\{ f \in \mathcal{C}(\mathfrak{A}) \; ; \; f \circ \Phi = \lambda_0 f \text{ and } f \circ \sigma^{\mathsf{z}} = \boldsymbol{\lambda}^{\mathsf{z}}, \; \forall \mathsf{z} \in \mathbb{Z}^D \right\}$$

be the ***eigenspace*** of $\boldsymbol{\lambda}$. We next relate $\mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma)$ to $\mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma)$, and review basic spectral theory.

Lemma 3.1. Let $\Phi\colon \mathcal{A}^{\mathbb{Z}^D} \to \mathcal{A}^{\mathbb{Z}^D}$ be a CA. Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift with $\Phi(\mathfrak{A}) = \mathfrak{A}$.

(a) $\mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma)$ is a multiplicative subgroup of $\mathbb{T}^{D+1}$. Let $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma)$ with $f_1 \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}_1}$ and $f_2 \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}_2}$. If $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2$ and $f = f_1 \cdot f_2$, then $f \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}$.

(b) If $(\mathfrak{A}, \Phi, \sigma)$ is transitive, then $\dim(\mathcal{E}^{igen}_{\boldsymbol{\lambda}}) = 1$ for each $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma)$.

(c) Suppose $\Psi : \mathcal{B}^{\mathbb{Z}^D} \longrightarrow \mathcal{B}^{\mathbb{Z}^D}$ is another CA, and $\mathfrak{B} \subset \mathcal{B}^{\mathbb{Z}^D}$ is a Ψ-invariant subshift. Let $\xi : (\mathfrak{A}, \Phi, \sigma) \longrightarrow (\mathfrak{B}, \Psi, \sigma)$ be an epimorphism. Then $\mathrm{S}^{\mathrm{pec}}(\mathfrak{B}) \subseteq \mathrm{S}^{\mathrm{pec}}(\mathfrak{A})$. For any $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{B})$, there is a linear monomorphism $\xi_* : \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{B}) \longrightarrow \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A})$ defined by $\xi_*(f) = f \circ \xi$.

(d) If $\mathfrak{A}$ is σ-transitive, then there is a homomorphism $\tau : \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma) \longrightarrow \mathbb{T}$ such that

$$\mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma) \quad = \quad \{(\lambda_0; \boldsymbol{\lambda}) \; ; \; \boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma) \text{ and } \lambda_0 = \tau(\boldsymbol{\lambda})\} \quad \cong \quad \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma).$$

[For example, if $\Phi_{|_{\mathfrak{A}}} \equiv \sigma^{\mathsf{z}}$, then $\tau(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^{\mathsf{z}}$, for all $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma)$.]
Furthermore, if $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma)$ and $\lambda_0 := \tau(\boldsymbol{\lambda})$, then $\mathcal{E}^{igen}_{(\lambda_0; \boldsymbol{\lambda})}(\mathfrak{A}, \Phi, \sigma) = \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}, \sigma)$.

(e) Suppose $\mathfrak{A}$ is not σ-transitive, but has σ-transitive decomposition $\mathfrak{A} = \bigsqcup_{n=1}^{N} \mathfrak{A}_n$. Then:

[i] Φ induces a permutation $\varphi : \{\mathfrak{A}_1, ..., \mathfrak{A}_K\} \longrightarrow \{\mathfrak{A}_1, ..., \mathfrak{A}_K\}$, and $(\mathfrak{A}, \Phi, \sigma)$ is transitive iff φ is transitive. In this case, $\mathrm{S}^{\mathrm{pec}}(\mathfrak{A}_1, \sigma) = \cdots = \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}_K, \sigma)$.

[ii] There is then a homomorphism $\tau : \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}_1, \sigma) \longrightarrow \mathbb{T}$ such that

$$\begin{aligned} \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma) &= \{(\rho \cdot \tau(\boldsymbol{\lambda}); \; \boldsymbol{\lambda}) \; ; \; \boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}_1, \sigma) \text{ and } \rho \in \mathbb{T} \text{ is a } K\text{th root of unity}\} \\ &\cong \mathbb{Z}_{/K} \times \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma). \end{aligned}$$

(f) Let $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma)$. The following are equivalent:

[i] The subgroup $\left\{ \boldsymbol{\lambda}^{\mathsf{z}} \; ; \; \mathsf{z} \in \mathbb{Z}^D \right\} \subset \mathbb{T}$ is finite.

[ii] $\boldsymbol{\lambda} = (\lambda_0, \ldots, \lambda_D)$, where $\lambda_0, \ldots, \lambda_D$ are complex roots of unity.

[iii] Every $F \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}$ is locally determined (hence $F(\mathfrak{A}) \subset \mathbb{C}$ is finite). □

Proof: **(a,b,c,f):** See [16, §1.5], [53, §5.5], or [35, Prop.2.53].

(d) For each $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma)$, part **(c)** yields a linear map $\Phi_*\colon \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}, \sigma) \longrightarrow \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}, \sigma)$. Part **(b)** says that $\dim[\mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}, \sigma)] = 1$, so there exists $\lambda_0 \in \mathbb{C}$ such that $f \circ \Phi = \lambda_0 f$ for all $f \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}, \sigma)$. But $|f \circ \Phi| = |f|$, hence $|\lambda_0| = 1$, so $\lambda_0 \in \mathbb{T}$. Define $\tau(\boldsymbol{\lambda}) := \lambda_0$. It follows that $f \in \mathcal{E}^{igen}_{(\lambda_0; \boldsymbol{\lambda})}$. Use part **(a)** to check that τ is a homomorphism.

(e) [i] follows from Remark 2.4 and part **(c)**.

[ii] By reordering if necessary, assume $\varphi(\mathfrak{A}_k) = \mathfrak{A}_{k-1}$ for all $k \in [1...K]$ (where $k-1$ is mod K). For any $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}_k, \sigma)$, part **(c)** yields a monomorphism $\Phi_* \colon \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}_k, \sigma) \to \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}_{k+1}, \sigma)$. Let $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}_1, \sigma)$, let $f_1 \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}_1, \sigma)$, and let $f_k := f_1 \circ \Phi^{k-1}$, for all $k \in [2...K]$. Then $f_k \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}_k, \sigma)$, and $\{f_1, \ldots, f_K\}$ is a $\mathbb{C}$-basis for $\mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}, \sigma) = \bigoplus_{k=1}^{K} \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}_k, \sigma)$. Furthermore, $\Phi_*^K : \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}_1, \sigma) \longrightarrow \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}_1, \sigma)$ is a linear map of a one-dimensional space; hence, just as in **(d)**, there is some $\tau = \tau(\boldsymbol{\lambda}) \in \mathbb{T}$ such that $f_1 \circ \Phi^K = \tau^K \cdot f_1$. Thus, $f_k \circ \Phi^K = \tau^K \cdot f_k$ for all $k \in [1...K]$, and thus, $F \circ \Phi^K = \tau^K F$ for every $F \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}, \sigma)$ (because $\mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}, \sigma)$ is spanned by $\{f_1, \ldots, f_K\}$).

"$\supseteq$" Let ρ be a Kth root of unity and let $\lambda_0 := \rho \cdot \tau$. If $F := \sum_{k=1}^{K} \lambda_0^{K-k} f_k$, then $F \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}, \sigma)$, and $F \circ \Phi = \lambda_0 F$, so $F \in \mathcal{E}^{igen}_{\lambda_0}(\mathfrak{A}, \Phi)$; hence $F \in \mathcal{E}^{igen}_{(\lambda_0; \boldsymbol{\lambda})}(\mathfrak{A}, \Phi, \sigma)$ as desired.

"$\subseteq$" If $\lambda_0 \in \mathbb{T}$ and $F \in \mathcal{E}^{igen}_{(\lambda_0; \boldsymbol{\lambda})}(\mathfrak{A}, \Phi, \sigma)$, then $\lambda_0^K F = F \circ \Phi^K = \tau^K F$ (where $\tau = \tau(\boldsymbol{\lambda})$), so $\lambda_0^K = \tau^K$, so $\lambda_0 = \rho\tau$, for some Kth root of unity ρ. □

We say that $\boldsymbol{\lambda}$ is ***rational*** if any (and thus all) of the conditions in Lemma 3.1(f) hold. If $(\mathfrak{A}, \Phi, \sigma)$ is transitive, then we define the ***radius*** of $\boldsymbol{\lambda}$ to be the radius of any nontrivial $F \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}$ [this is finite by Lemma 3.1(f)[iii], and independent of F by Lemma 3.1(b)]. Let $\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \Phi, \sigma) \subset \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma)$ be the subgroup of rational eigenvalues. Lemma 3.1(d,e) implies that $\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \Phi, \sigma)$ is nontrivial iff $\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \sigma)$ is nontrivial. Meanwhile, $\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \sigma)$ is nontrivial iff $(\mathfrak{A}, \sigma)$ has a periodic factor.

Let $\widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \Phi, \sigma) := \{\text{continuous homomorphisms } \delta \colon \mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \Phi, \sigma) \to \mathbb{T}\}$ be the dual group of $\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \Phi, \sigma)$. For any $(t; \mathsf{z}) \in \mathbb{Z} \times \mathbb{Z}^D$, define $\delta_{\overrightarrow{(t;\mathsf{z})}} \in \widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \Phi, \sigma)$ by $\delta_{\overrightarrow{(t;\mathsf{z})}}(\boldsymbol{\lambda}) := \lambda_0^t \lambda_1^{z_1} \cdots \lambda_D^{z_D}$. We will see that $\delta_{\overrightarrow{(t;\mathsf{z})}}$ corresponds to a 'displacement' in time by t and in space by z. The group homomorphism $\boldsymbol{\delta} : \mathbb{Z} \times \mathbb{Z}^D \ni (t; \mathsf{z}) \mapsto \delta_{\overrightarrow{(t;\mathsf{z})}} \in \widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \Phi, \sigma)$ has dense image $\widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \Phi, \sigma)$ (and in most of our examples, is surjective). Hence we will regard elements of $\widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \Phi, \sigma)$ as 'generalized space-time shifts', and call them ***displacements***. If $\mathfrak{A}$ is σ-transitive, then Lemma 3.1(d) yields a natural isomorphism $\widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \Phi, \sigma) \cong \widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \sigma)$, so all displacements are 'space shifts'.

Example 3.2: (a) Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ be a one-dimensional σ-transitive SFT, with period P and phase partition $\mathcal{A}_0 \sqcup \cdots \sqcup \mathcal{A}_{P-1}$, as in Example 3.1. If $\lambda := e^{2\pi \mathbf{i}/P}$, then

$$\mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma) \quad = \quad \mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \sigma) \quad = \quad \{\lambda^q\}_{q=0}^{P-1} \quad \cong \quad \mathbb{Z}_{/P}.$$

(the group of Pth roots of unity). For example, define $f : \mathfrak{A} \longrightarrow \mathbb{T}$ by $f(\mathbf{a}) := \lambda^q$ iff $a_0 \in \mathcal{A}_q$. Then $f \in \mathcal{E}^{igen}_{\lambda}(\mathfrak{A}, \sigma)$.

If $\Phi(\mathfrak{A}) = \mathfrak{A}$, then there is ***phase rotation*** $r \in \mathbb{Z}_{/P}$ such that, for any $\mathbf{a} \in \mathfrak{A}$, if $a_0 \in \mathcal{A}_p$, then $\Phi(\mathbf{a})_0 \in \mathcal{A}_{p+r}$. The homomorphism $\tau : \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma) \longrightarrow \mathbb{T}$ in Lemma 3.1(d) is then defined $\tau(\lambda^q) := \lambda^{rq}$ for all $q \in \mathbb{Z}_{/P}$. To see this, note that $f \circ \Phi = \lambda^{rq} f$ for any $f \in \mathcal{E}^{igen}_{\lambda^q}(\mathfrak{A}, \sigma)$; hence $\mathcal{E}^{igen}_{\lambda^q}(\mathfrak{A}, \sigma) = \mathcal{E}^{igen}_{(\lambda^{rq}; \lambda^q)}(\mathfrak{A}, \Phi, \sigma)$. In this case, $\widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \Phi, \sigma) \cong \widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \sigma) \cong \mathbb{Z}_{/P}$, and the group homomorphism $\boldsymbol{\delta} : \mathbb{Z} \ni \mathsf{z} \mapsto \delta_{\overrightarrow{(0,\mathsf{z})}} \in \widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \Phi, \sigma)$ is surjective, with kernel $P\mathbb{Z}$.

(b) Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be σ-transitive, and let $\mathbb{P} \subset \mathbb{Z}^D$ be a finite-index subgroup. We say that $\mathfrak{A}$ is $\mathbb{P}$-*periodic* if $\mathfrak{A} \subset \mathsf{Fix}[\sigma^{\mathsf{p}}]$ for all $\mathsf{p} \in \mathbb{P}$. Let $\widetilde{\mathsf{Z}} := \mathbb{Z}^D/\mathbb{P}$ be the quotient group, with quotient map $\mathbb{Z}^D \ni \mathsf{z} \mapsto \widetilde{\mathsf{z}} \in \widetilde{\mathsf{Z}}$. Let $\mathsf{e}_1, \ldots, \mathsf{e}_D \in \mathbb{Z}^D$ be the unit vectors. Then $\{\widetilde{\mathsf{e}}_1, \ldots, \widetilde{\mathsf{e}}_D\}$ generates $\widetilde{\mathsf{Z}}$. For each $d \in [1...D]$, let $p_d \in \mathbb{N}$ be the (finite) order of $\widetilde{\mathsf{e}}_d$ in $\widetilde{\mathsf{Z}}$, and let $\lambda_d := e^{2\pi \mathbf{i}/p_d}$. Then

$$\mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma) \quad = \quad \mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \sigma) \quad = \quad \big\{(\lambda_1^{n_1}, ..., \lambda_D^{n_D}) \;;\; n_1, \ldots, n_D \in \mathbb{Z}\big\}.$$

[For example, fix $\mathbf{a}_0 \in \mathfrak{A}$, and $\forall\, \widetilde{\mathsf{z}} \in \widetilde{\mathsf{Z}}$, let $\mathbf{a}_{\widetilde{\mathsf{z}}} := \sigma^{\mathsf{z}}(\mathbf{a}_0)$ (well-defined because $\mathbf{a}_0$ is $\mathbb{P}$-periodic). Then $\mathfrak{A} = \{\mathbf{a}_{\widetilde{\mathsf{z}}}\}_{\widetilde{\mathsf{z}} \in \widetilde{\mathsf{Z}}}$ because $\mathfrak{A}$ is σ-transitive. Let $\boldsymbol{\lambda} := (\lambda_1^{n_1}, ..., \lambda_D^{n_D})$, and define $f\colon \mathfrak{A} \to \mathbb{T}$ by $f(\mathbf{a}_{\widetilde{\mathsf{z}}}) := \boldsymbol{\lambda}^{\mathsf{z}}$, $\forall\, \mathsf{z} \in \mathbb{Z}^D$. Then $f \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}(\mathfrak{A}, \sigma)$.] The homomorphism $\boldsymbol{\delta} : \mathbb{Z}^D \ni \mathsf{z} \mapsto \delta_{\overrightarrow{(0,\,\mathsf{z})}} \in \widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \sigma)$ is surjective with kernel $\mathbb{P}$, so $\widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \sigma) \cong \widetilde{\mathsf{Z}}$. If $\Phi(\mathfrak{A}) = \mathfrak{A}$, then there exists $\widetilde{\mathsf{z}} \in \widetilde{\mathsf{Z}}$ such that $\Phi_{|_{\mathfrak{A}}} = \sigma^{\mathsf{z}}$. The function $\tau : \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma) \longrightarrow \mathbb{T}$ in Lemma 3.1(d) is then defined by $\tau(\boldsymbol{\lambda}) := \boldsymbol{\lambda}^{\mathsf{z}}$. ◇

Remarks: The homomorphism $\boldsymbol{\delta} : \mathbb{Z} \longrightarrow \widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \sigma)$ is not always surjective. For example, if $p \in \mathbb{N}$, and $\mathfrak{A}$ is a p-adic Toeplitz shift [9] or a nonperiodic substitution shift induced by a substitution $\mathcal{A} \longrightarrow \mathcal{A}^p$ [16, Thm.7.3.1], then $\widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{A}, \Phi, \sigma) \cong \mathbb{Z}(p)$ is the p-adic integers, and $\boldsymbol{\delta}$ is the natural embedding $\mathbb{Z} \hookrightarrow \mathbb{Z}(p)$, which is *not* surjective. ◇

Suppose $(\mathfrak{A}, \Phi, \sigma)$ is transitive. Let $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \Phi, \sigma)$ have radius $r > 0$, and let $f \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}$. Then for any $\mathbf{a} \in \mathfrak{A}$ we can write $f(\mathbf{a}) = f(\mathbf{a}_{\mathbb{B}(r)})$. For any $\mathsf{z} \in \mathbb{Z}^D$, let $f_{\mathsf{z}}(\mathbf{a}) := f(\mathbf{a}_{\mathbb{B}(\mathsf{z},r)})$. Hence, if $\mathbf{a} \in \mathfrak{A}$, then $f_{\mathsf{z}}(\mathbf{a}) = f(\sigma^{\mathsf{z}}(\mathbf{a})) = \boldsymbol{\lambda}^{\mathsf{z}} f(\mathbf{a})$. However, $f_{\mathsf{z}}(\mathbf{a})$ is also well-defined on any $\mathbf{a} \in \widetilde{\mathfrak{A}}$ such that $\mathbf{a}_{\mathbb{B}(\mathsf{z},r)}$ is $\mathfrak{A}$-admissible. Hence $f_{\mathsf{z}}(\mathbf{a})$ is well-defined for all $\mathsf{z} \in \mathbb{G}_r(\mathbf{a})$. If $\mathbf{a} \in \widetilde{\mathfrak{A}}$ and $\mathsf{z}, \mathsf{y} \in \mathbb{G}_r(\mathbf{a})$, then we say that $\mathbf{a}$ has a $\boldsymbol{\lambda}$-*dislocation between* y *and* z if $f_{\mathsf{y}}(\mathbf{a}) \neq \boldsymbol{\lambda}^{\mathsf{y}-\mathsf{z}} f_{\mathsf{z}}(\mathbf{a})$. Let

$$\gamma_{\mathsf{y},\mathsf{z}}(\boldsymbol{\lambda}) \quad := \quad \boldsymbol{\lambda}^{\mathsf{y}-\mathsf{z}} \cdot f_{\mathsf{z}}(\mathbf{a})/f_{\mathsf{y}}(\mathbf{a}). \tag{1}$$

If $(\mathfrak{A}, \Phi, \sigma)$ is transitive, then $\gamma_{\mathsf{y},\mathsf{z}}(\boldsymbol{\lambda})$ is independent of the choice $f \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}$, by Lemma 3.1(b).

Theorem 3.3. Let $(\mathfrak{A}, \Phi, \sigma)$ be transitive. Let $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \Phi, \sigma)$ (with radius $r' > 0$) and let $\mathbf{a} \in \widetilde{\mathfrak{A}}$ have a $\boldsymbol{\lambda}$-dislocation. Let $r \geq r' + 1$. Then:

(a) $\mathbf{a}$ has a range r domain boundary.

(b) Suppose $\mathbb{G}_r(\mathbf{a})$ has connected components $\{\mathbb{Y}_n\}_{n=1}^N$, where $N \in \mathbb{N} \cup \{\infty\}$. There is a matrix of displacements $\Delta := [\delta_{nm}]_{n,m=1}^N$ such that

[i] $\gamma_{\mathsf{y},\mathsf{z}}(\boldsymbol{\lambda}) = \delta_{nm}(\boldsymbol{\lambda})$ for any $\mathsf{y} \in \mathbb{Y}_n$ and $\mathsf{z} \in \mathbb{Y}_m$.

[ii] (Cocycle property) $\delta_{n\ell}(\boldsymbol{\lambda}) = \delta_{nm}(\boldsymbol{\lambda})\delta_{m\ell}(\boldsymbol{\lambda})$ for any $n, m, \ell \in [1...N]$.

(c) If two of the components $\mathbb{Y}_n$ and $\mathbb{Y}_m$ are projective, and δ_{nm} is nontrivial, then there is an essential domain boundary between $\mathbb{Y}_n$ and $\mathbb{Y}_m$

Proof: **(a)** follows from **(b)**, because if $\mathbb{G}_r(\mathbf{a})$ had only one connected component $\mathbb{Y}_1$, then **(b)**[ii] implies that $\delta_{11} \equiv 1$, and then **(b)**[i] says that $\gamma_{\mathsf{y},\mathsf{z}}(\boldsymbol{\lambda}) = 1$ for all $\mathsf{y}, \mathsf{z} \in \mathbb{Y}_1$, which contradicts the hypothesis that $\mathbf{a}$ has a $\boldsymbol{\lambda}$-dislocation. To prove **(b)** we need the following:

CLAIM 1: [a] $\gamma_{\mathsf{x},\mathsf{z}}(\boldsymbol{\lambda}) = 1$ *if* x, z *are in the same connected component of* $\mathbb{G}_r(\mathbf{a})$.

[b] *For any* $\mathsf{x}, \mathsf{y}, \mathsf{z} \in \mathbb{G}_r(\mathbf{a})$, *we have* $\gamma_{\mathsf{x},\mathsf{z}}(\boldsymbol{\lambda}) = \gamma_{\mathsf{x},\mathsf{y}}(\boldsymbol{\lambda}) \cdot \gamma_{\mathsf{y},\mathsf{z}}(\boldsymbol{\lambda})$.

[c] *If* $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \Phi, \sigma)$ *have radius* r, *then* $\boldsymbol{\lambda} := \boldsymbol{\lambda}_1 \cdot \boldsymbol{\lambda}_2$ *also has radius* r, *and* $\gamma_{\mathsf{x},\mathsf{y}}(\boldsymbol{\lambda}) = \gamma_{\mathsf{x},\mathsf{y}}(\boldsymbol{\lambda}_1) \cdot \gamma_{\mathsf{x},\mathsf{y}}(\boldsymbol{\lambda}_2)$.

Proof: [a] Let $\mathsf{x}, \mathsf{z} \in \mathbb{Y}_n$ for some $n \in [1...N]$. Let $\mathsf{x} = \mathsf{y}_0 \rightsquigarrow \mathsf{y}_1 \rightsquigarrow \cdots \rightsquigarrow \mathsf{y}_J = \mathsf{z}$ be a trail in $\mathbb{G}_r$ from x to z. Then for all $j \in [1...J]$, $\kappa_{\mathsf{y}_j}(\mathbf{a}) = \boldsymbol{\lambda}^{\mathsf{y}_j - \mathsf{y}_{j-1}} \kappa_{\mathsf{y}_{j-1}}(\mathbf{a})$ (*Proof:* Similar to Claim 1 of Proposition 2.1). Inductively, we have $f_{\mathsf{z}}(\mathbf{a}) = \boldsymbol{\lambda}^{\mathsf{z}-\mathsf{x}} f_{\mathsf{x}}(\mathbf{a})$. Thus, $\gamma_{\mathsf{x},\mathsf{z}}(\boldsymbol{\lambda}) = 1$.

[b] follows from eqn.(1). For [c], let $f_1 \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}_1}$ and $f_2 \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}_2}$, and let $f := f_1 \cdot f_2$. Lemma 3.1(a) says that $f \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}$; hence radius$(\boldsymbol{\lambda}) = r$. Now substitute f_1, f_2, and f respectively into eqn.(1) to compute $\gamma_{\mathsf{y},\mathsf{z}}(\boldsymbol{\lambda}_1)$, $\gamma_{\mathsf{y},\mathsf{z}}(\boldsymbol{\lambda}_2)$ and $\gamma_{\mathsf{y},\mathsf{z}}(\boldsymbol{\lambda})$. ◇ `Claim 1`

(b)[i] follows Claim 1[a,b]. Then **(b)**[ii] is by Claim 1[b]. Finally, Claim 1[c] implies that $\delta_{n,m} : \mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \Phi, \sigma) \longrightarrow \mathbb{T}$ is a homomorphism (i.e. a displacement).

(c) If $\mathbb{Y}_1$ and $\mathbb{Y}_2$ are projective, then we cannot eliminate the dislocation by changing $\mathbf{a}$ inside $\mathbb{G}_R(\mathbf{a})$ for any $R \in \mathbb{N}$, so the defect is essential. □

Theorem 3.3(b) allows us to speak of a *rational* $(\mathfrak{A}, \Phi)$*-dislocation* rather than of a "$\boldsymbol{\lambda}$-dislocation". (When $\mathfrak{A}$ and Φ are clear, we will just say "rational dislocation"). The cocycle property means: **(a)** All entries of the displacement matrix $\Delta_{\mathbf{a}}$ can be reconstructed from one row, and **(b)** $\Delta_{\mathbf{a}}$ is 'antisymmetric', i.e. $\delta_{nm}(\boldsymbol{\lambda}) = \delta_{mn}(\boldsymbol{\lambda})^{-1}$ and $\delta_{nn} \equiv 1, \forall\, n, m \in [1...N]$. If there are only two connected components, $\mathbb{Y}_1$ and $\mathbb{Y}_2$, then we define the *displacement* of $\mathbf{a}$ to be $\delta_{\mathbf{a}} := \delta_{12}$.

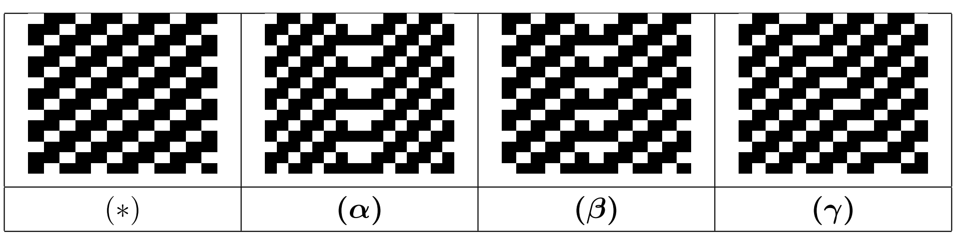


Figure 5. (∗) The periodic background generated by ${}_{\varepsilon}\Phi_{62}$ acting on $\mathfrak{D}$; $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$: Three rational dislocations; see Examples 3.4(a) and 3.7(a). (In [1, §III(C)], α, β, and γ are "g_e", "g_o" and "w" respectively.)

Example 3.4: (a) (ECA#62) Let $\mathcal{A} = \{■, □\}$, and let $\mathfrak{D}$ be the three-element σ-orbit of $\overline{■■□}$. Let $\lambda := e^{2\pi \mathbf{i}/3}$. Then $\mathfrak{D}$ is σ-transitive, and $\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{D}, \sigma) = \{1, \lambda, \lambda^2\} \subset \mathbb{T}$ [because $\mathfrak{D}$ has period $P = 3$; see Example 3.2(a)]. If ${}_{\varepsilon}\Phi_{62}$ is ECA#62, then ${}_{\varepsilon}\Phi_{62}|_{\mathfrak{D}} = \sigma$ [see Figure 5(∗)], so the homomorphism $\tau : \mathrm{S}^{\mathrm{pec}}(\mathfrak{D}, \sigma) \longrightarrow \mathbb{T}$ in Lemma 3.1(d) is the identity: $\tau(\lambda^p) = \lambda^p$ [see Example 3.2(a)]. Thus,

$$\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{D}, {}_{\varepsilon}\Phi_{62}, \sigma) \quad = \quad \{(1,1), (\lambda, \lambda), (\lambda^2, \lambda^2)\} \quad \subset \quad \mathbb{T}^2.$$

The homomorphism $\mathbb{Z} \ni \mathsf{z} \mapsto \delta_{\overrightarrow{(0;\,\mathsf{z})}} \in \widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{D}, {}_{\varepsilon}\Phi_{62}, \sigma) \cong \mathbb{Z}_{/3}$ is surjective, with kernel $3\mathbb{Z}$. Hence we identify displacements with elements of $\mathbb{Z}_{/3}$. Below are three rational dislocations in $\mathfrak{D}$ and their displacements.

α	$\boxed{■■□■■□}\overrightarrow{□□□}\boxed{■■□■■□}$	$\delta = 3 \equiv 0 \in \mathbb{Z}_{/3}$
β	$\boxed{■■□■■□}\overrightarrow{□□}\boxed{■■□■■□■}$	$\delta = 2 \in \mathbb{Z}_{/3}$
γ	$\boxed{■■□■■□}\overrightarrow{□}\boxed{■■□■■□■■}$	$\delta = 1 \in \mathbb{Z}_{/3}$

[compare to the top rows in Figure 5(α, β, γ).] The γ and β dislocations have nontrivial displacements, so they are are essential. The α dislocation is not essential (it can be removed by replacing the middle three blocks with ■■□).

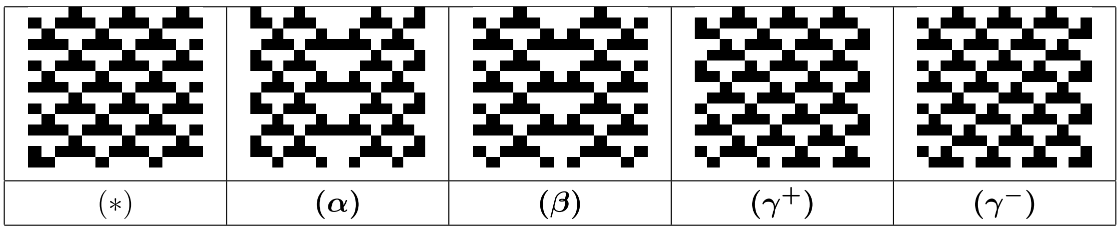
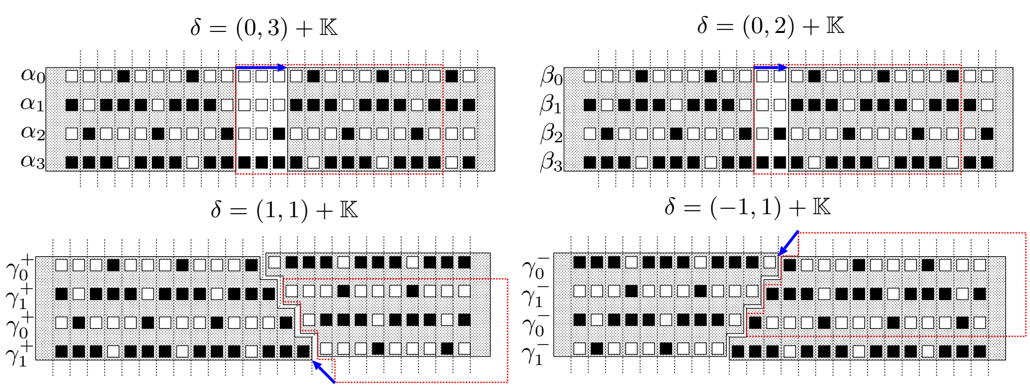


Figure 6. $(*)$ The periodic background generated by ${}_{\varepsilon}\Phi_{54}$ acting on $\mathfrak{B}$. $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}^{\pm})$: Four rational dislocations; see Examples 3.4(b) and 3.7(b). (This nomenclature is due to [7, Fig.8]. In the nomenclature of [1, §III(C)], $\alpha = g_o$, $\beta = g_e$, $\gamma^+ = \vec{w}$ and $\gamma^- = \overleftarrow{w}$.) See also [48, Example 1.2(b)].

(b) (ECA#54) Let $\mathcal{A} = \{■, □\}$, and let $\mathfrak{B} := \mathfrak{B}_0 \sqcup \mathfrak{B}_1$, where $\mathfrak{B}_0$ is the four-element σ-orbit of $\overline{■■■□}$ and $\mathfrak{B}_1$ is the four-element σ-orbit of $\overline{□□□■}$. Then $\mathfrak{B}$ is not σ-transitive, and

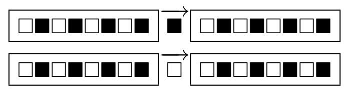
$$\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{B}_0, \sigma) \quad = \quad \mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{B}_1, \sigma) \quad = \quad \{\mathbf{i}^p\}_{p=0}^{3} \quad = \quad \{1, \mathbf{i}, -1, -\mathbf{i}\} \quad \subset \quad \mathbb{T}$$

[because $\mathfrak{B}$ has period $P = 4$; see Example 3.2(a)]. If ${}_{\varepsilon}\Phi_{54}$ is ECA#54, then ${}_{\varepsilon}\Phi_{54}(\mathfrak{B}_0) = \mathfrak{B}_1$ and ${}_{\varepsilon}\Phi_{54}(\mathfrak{B}_1) = \mathfrak{B}_0$, so $\mathfrak{B}$ is $({}_{\varepsilon}\Phi_{54}, \sigma)$-transitive. Also, ${}_{\varepsilon}\Phi^2_{54}|_{\mathfrak{B}} = \sigma^2$ [see Figure 6$(*)$], so the epimorphism $\mathbb{Z} \times \mathbb{Z} \ni (t; \mathsf{z}) \mapsto \delta_{\overrightarrow{(t;\,\mathsf{z})}} \in \widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{B}, {}_{\varepsilon}\Phi_{54}, \sigma)$ has kernel $\mathbb{K} := \mathbb{Z}(2, 2) \oplus \mathbb{Z}(0, 4)$. Hence we identify displacements with elements of $\mathbb{Z}^2/\mathbb{K}$. Below are four rational dislocations and their displacements [compare to the top rows in Fig.6(α, β), or middle rows in Fig.6$(\gamma^{\pm})$].



(c) (ECA#184) Let $\mathcal{A} = \{■, □\}$, and let $\mathfrak{G}_* \subset \mathcal{A}^{\mathbb{Z}}$ be as in Example 2.2(a). Then $\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{G}_*, \sigma) = \{\pm 1\}$ [because $\mathfrak{G}_*$ has period 2; see Example 3.2(a)]. Also, ${}_{\varepsilon}\Phi_{184}|_{\mathfrak{G}_*} = \sigma$ [see Figure 4$(*)$], so the homomorphism
$\tau : \mathrm{S}^{\mathrm{pec}}(\mathfrak{G}_*, \sigma) \longrightarrow \mathbb{T}$ in Lemma 3.1(d) is the identity [see Example 3.2(a)]. Thus, $\mathrm{S}^{\mathrm{pec}}(\mathfrak{B}_*, {}_{\varepsilon}\Phi_{184}, \sigma) =$

$\{(1,1),(-1,-1)\}$, and the homomorphism $\mathbb{Z} \ni \mathsf{z} \mapsto \delta_{\overrightarrow{(0;\,\mathsf{z})}} \in \widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{G}_*, {}_{\varepsilon}\Phi_{184}, \sigma) \cong \mathbb{Z}_{/2}$ is surjective, with kernel $2\mathbb{Z}$, so we identify displacements with elements of $\mathbb{Z}_{/2}$. Below are two rational dislocations and their displacements [compare to any rows in Figure 4($\gamma^{\pm}$)]:

γ^+  $\delta = 1 \in \mathbb{Z}_{/2}$
γ^- $\delta = 1 \in \mathbb{Z}_{/2}$

(d) (ECA #110) Let $\mathcal{A} := \{■, □\}$, and let $\mathfrak{E} \subset \mathcal{A}^{\mathbb{Z}}$ be the 14-element σ-orbit of the 14-periodic sequence $\overline{■■■□■■□□■□□□□□}$. Let $\lambda := e^{\pi \mathbf{i}/7}$. Then $\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{E}, \sigma) = \{1, \lambda, \ldots, \lambda^{13}\} \subset \mathbb{T}$ [because $\mathfrak{E}$ has period 14; see Example 3.2(a)]. If ${}_{\varepsilon}\Phi_{110}$ is ECA#110, then ${}_{\varepsilon}\Phi_{110}|_{\mathfrak{E}} = \sigma^4$ [see Figure 8($*$)], so the homomorphism $\tau : \mathrm{S}^{\mathrm{pec}}(\mathfrak{E}, \sigma) \longrightarrow \mathbb{T}$ in Lemma 3.1(d) is given $\tau(\lambda^p) = \lambda^{4p}$ [see Example 3.2(a)]. Thus,

$$\mathrm{S}^{\mathrm{pec}}(\mathfrak{D}, {}_{\varepsilon}\Phi_{62}, \sigma) \quad = \quad \{(1,1), (\lambda^4, \lambda), (\lambda^8, \lambda^2), \ldots, (\lambda^{10}, \lambda^{13})\},$$

and the homomorphism $\mathbb{Z} \ni \mathsf{z} \mapsto \delta_{\overrightarrow{(0;\,\mathsf{z})}} \in \widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{E}, {}_{\varepsilon}\Phi_{110}, \sigma) \cong \mathbb{Z}_{/14}$ is surjective, with kernel $14\mathbb{Z}$, so we identify displacements with elements of $\mathbb{Z}_{/14}$. Figure 7 shows seven essential rational dislocations in $\mathfrak{E}$ with nontrivial displacements. Figure 8 shows their ${}_{\varepsilon}\Phi_{110}$-evolution.

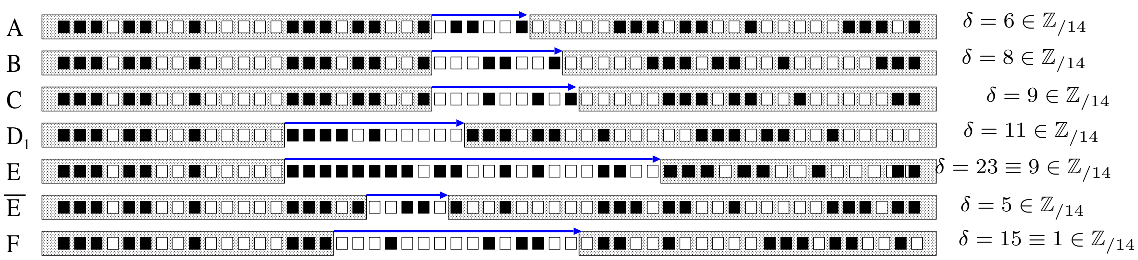


Figure 7. Seven essential rational dislocations in $\mathfrak{E}$ with nontrivial displacements; see Examples 3.4(d) and 3.7(d).

(e) Let $\mathcal{A} = \{■, □\}$ and let $\mathfrak{C}\mathfrak{h} \subset \mathcal{A}^{\mathbb{Z}^2}$ be as in Example 1.3(b). Then $\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{C}\mathfrak{h}, \sigma) = \{\pm 1\}^2 \subset \mathbb{T}^2$, by Example 3.2(b) [because $\mathfrak{C}\mathfrak{h}$ is $\mathbb{P}$-periodic, where $\mathbb{P} := \mathbb{Z}(1,1) \oplus \mathbb{Z}(1,-1)$]. Let Φ be a CA with $\mathfrak{C}\mathfrak{h} \subseteq \mathsf{Fix}[\Phi]$ [e.g. Example 1.8(b)]. Then $\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{C}\mathfrak{h}, \Phi, \sigma) = \{(1; \pm 1, \pm 1)\} \subset \mathbb{T}^3$ and $\widehat{\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}}(\mathfrak{C}\mathfrak{h}, \Phi, \sigma) \cong \mathbb{Z}^2/\mathbb{P}$. [by Lemma 3.1(d) and Example 3.2(b)]. Figure 2(B) shows a domain boundary in $\mathfrak{C}\mathfrak{h}$. If $\mathbb{Y}_0$ is the northern connected component and $\mathbb{Y}_1$ is the southern component, then we have $\delta_{12} = (1,0) + \mathbb{P}$. Both components are projective, so this is an essential rational dislocation. ♢

Proposition 3.5. Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ be a one-dimensional SFT. Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$.

(a) If $\mathfrak{A}$ is σ-mixing, then every finite defect of $\mathbf{a}$ is removable.

(b) If $\mathfrak{A}$ is (Φ, σ)-transitive (but not σ-mixing), then any finite essential defect in $\mathbf{a}$ is a rational dislocation.

(c) If $\mathfrak{A}$ is not (Φ, σ)-transitive, then any finite essential defect is either a rational dislocation or an interface.

Proof: **(a)** is Example 1.5(a). **(c)** follows from **(b)**, which follows from Example 3.1. □

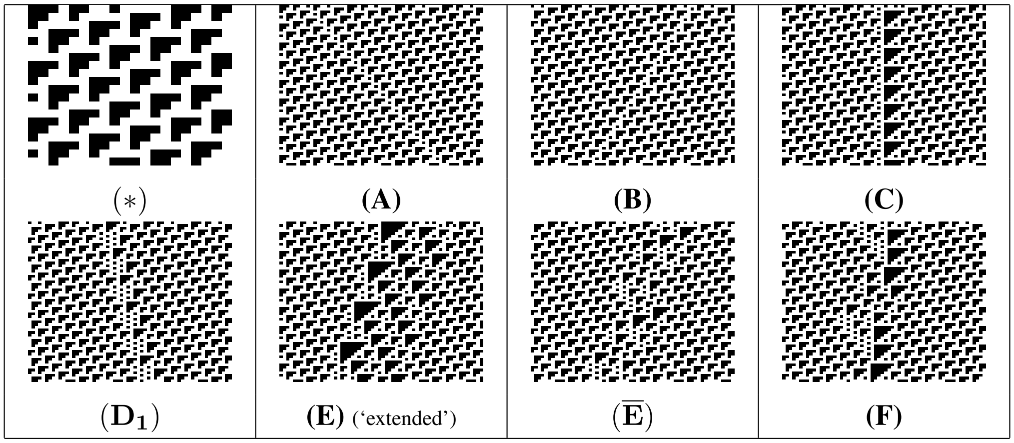


Figure 8. $(*)$ A 30×30 image of the periodic spacetime diagram of ${}_{\varepsilon}\Phi_{110}$ acting on $\mathfrak{C}$; **(A,B,C,$\mathbf{D_1}$,E,$\overline{\mathbf{E}}$,F):** 60×60 images of the ${}_{\varepsilon}\Phi_{110}$-evolution of seven dislocations in $\mathfrak{C}$. See Examples 3.4(d) and 3.7(d). (This nomenclature is due to [3]. In the nomenclature of [8] $A=\omega_{\mathrm{right}}$, $B=\omega_{\mathrm{left}}$, $C=\alpha$, $E=\beta$, $F=\eta$ etc.). See also [48, Example 1.2(c)].

Next we show that any essential, rational $(\mathfrak{A},\Phi)$-dislocation is Φ-persistent:

Theorem 3.6. Let $\Phi:\mathcal{A}^{\mathbb{Z}^D}\longrightarrow\mathcal{A}^{\mathbb{Z}^D}$ be a CA and let $\mathfrak{A}\subset\mathcal{A}^{\mathbb{Z}^D}$ be a subshift with $\Phi(\mathfrak{A})=\mathfrak{A}$. If $\mathbf{a}\in\widetilde{\mathfrak{A}}$ has an essential rational dislocation, then $\Phi(\mathbf{a})$ also has an essential rational dislocation, with the same displacement matrix as $\mathbf{a}$.

Proof: Let $\mathbf{a}':=\Phi(\mathbf{a})$, and suppose Φ has radius $R>0$. Then each projective component of $\mathbb{G}_{r+R}(\mathbf{a})$ is contained in a projective component of $\mathbb{G}_r(\mathbf{a}')$, because $\mathbb{G}_{r+R}(\mathbf{a})\subseteq\mathbb{G}_r(\mathbf{a}')$ by Lemma 1.2(b). Let $\mathbb{G}_{R+r}(\mathbf{a})$ have projective components $\mathbb{Y}_1,\ldots,\mathbb{Y}_N$, and let $\mathbb{G}_r(\mathbf{a}')$ have projective components $\mathbb{Y}'_1,\ldots,\mathbb{Y}'_N$, where $\mathbb{Y}_n\subset\mathbb{Y}'_n$ for all $n\in[1...N]$. Let $\Delta_{\mathbf{a}}=[\delta_{nm}]$ and $\Delta_{\mathbf{a}'}=[\delta'_{nm}]$. We must show $\delta_{nm}=\delta'_{nm}$ for all $n,m\in[1...N]$. If $\lambda\in\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A},\Phi,\sigma)$ and $f\in\mathcal{E}^{\mathit{igen}}_{\boldsymbol{\lambda}}$, then $f\circ\Phi\underset{\overline{\overline{\mathrm{ae}}}}{}\lambda_0 f$. Let $\mathsf{y}\in\mathbb{Y}_n\subset\mathbb{Y}'_n$ and let $\mathsf{z}\in\mathbb{Y}_m\subset\mathbb{Y}'_m$. Then

$$\delta_{nm}(\boldsymbol{\lambda})\ \underset{(*)}{=\!=}\ \boldsymbol{\lambda}^{\mathsf{y}-\mathsf{z}}\cdot\frac{f_{\mathsf{z}}(\mathbf{a})}{f_{\mathsf{y}}(\mathbf{a})}\ =\ \boldsymbol{\lambda}^{\mathsf{y}-\mathsf{z}}\cdot\frac{\lambda_0 f_{\mathsf{z}}(\mathbf{a})}{\lambda_0 f_{\mathsf{y}}(\mathbf{a})}\ \underset{(\dagger)}{=\!=}\ \boldsymbol{\lambda}^{\mathsf{y}-\mathsf{z}}\cdot\frac{f_{\mathsf{z}}(\mathbf{a}')}{f_{\mathsf{y}}(\mathbf{a}')}\ \underset{(*)}{=\!=}\ \delta'_{nm}(\boldsymbol{\lambda}).$$

Here, $(*)$ is by eqn.(1) and Theorem 3.3(b)[i], and $(\dagger)$ is because $\lambda_0 f(\mathbf{a})=(f\circ\Phi)_{\mathsf{z}}(\mathbf{a})=f_{\mathsf{z}}(\mathbf{a}')$. □

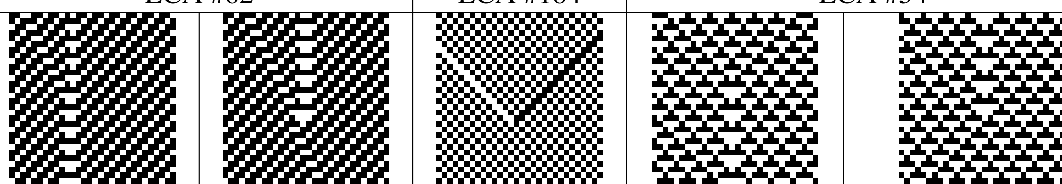
Example 3.7: (a) The γ and β dislocations of Example 3.4(a) are ${}_{\varepsilon}\Phi_{62}$-persistent, by Theorem 3.6. (The α dislocation is also ${}_{\varepsilon}\Phi_{62}$-persistent, but not because of Theorem 3.6). The ${}_{\varepsilon}\Phi_{62}$-evolution of all three dislocations is shown in Figure $5(\alpha,\beta,\gamma)$. Their large-scale ${}_{\varepsilon}\Phi_{62}$-dynamics were shown in Figure 1(C).

(b) All four dislocations in Example 3.4(b) are ${}_{\varepsilon}\Phi_{54}$-persistent; their ${}_{\varepsilon}\Phi_{54}$-evolution is shown in Figure $6(\alpha,\beta,\gamma^{\pm})$. See also Figure 1(B).

(c) Both dislocations in Example 3.4(c) are ${}_{\varepsilon}\Phi_{184}$-persistent; their ${}_{\varepsilon}\Phi_{184}$-evolution is shown in Figure $4(\gamma^{\pm})$. See also Figure 1(D).

(d) All seven dislocations in Example 3.4(d) are ${}_{\varepsilon}\Phi_{110}$-persistent. Figure 8 shows their ${}_{\varepsilon}\Phi_{110}$-evolution. These are only some of the plethora of defect particles of ECA #110 (see [39, 43, 8] or [27, §3.1.4.4]), whose complex interactions can support universal computation; see [3], [44] or [55, Ch.11].

(e) If $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ is any one-dimensional SFT, then every essential defect is persistent. (Combine Propositions 2.3 and 3.5(c) with Theorem 3.6). $\diamondsuit$



ECA #62		ECA #184	ECA #54	
(a) $\gamma + \beta \to \alpha$ $2 + 1 \equiv 0 \in \mathbb{Z}_{/3}$	**(b)** $\gamma + \alpha \to \gamma$ $2 + 0 \equiv 2 \in \mathbb{Z}_{/3}$	**(c)** $\gamma^+ + \gamma^- \to \emptyset$ $1 + 1 \equiv 0 \in \mathbb{Z}_{/2}$	**(d)** $\gamma^+ + \gamma^- \to \beta$ $(1,1) + (-1,1) =$ $(0,2) \in \mathbb{Z}^2/\mathbb{K}$	**(e)** $\gamma^+ + \beta \to \gamma^-$ $(1,1) + (0,2) =$ $(1,3) \equiv (-1,1) \in \mathbb{Z}^2/\mathbb{K}$

Figure 9. Dislocation coalescence and the algebra of $\widehat{\mathrm{Spec}_{\text{æ}}}(\mathfrak{A}, \Phi, \sigma)$.

Defect coalescence: If $D = 1$, then dislocations can be thought of as 'particles'; see [48]. If two such 'dislocation particles' x and y coalesce to form z, then $\delta_z = \delta_x \cdot \delta_y$. In particular, x and y can annihilate only if $\delta_x \cdot \delta_y \equiv 1$. Thus, the algebra of $\widehat{\mathrm{Spec}_{\text{æ}}}(\mathfrak{A}, \Phi, \sigma)$ yields 'conservation laws' which helps to determine the 'chemistry' of dislocation particles. This partially answers Question #3 from the introduction. See Figure 9 for some examples.

3.2. Projective Dislocations

If $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ is a one-dimensional subshift of finite type, then Proposition 3.5 completely characterizes its essential defects. However, if $\mathfrak{A}$ is *not* of finite type, or even if $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ *is* of finite type, but $D \geq 2$, then Proposition 3.5 fails, because some σ-eigenvalues of $\mathfrak{A}$ may be irrational[1], and even rational eigenvalues may only have discontinuous eigenfunctions. Theorem 3.3 is not applicable to such eigenfunctions, because they are *not* locally determined. We can extend the methods of §3.1 to irrational or æ-continuous eigenfunctions, but only for projective domain boundaries, and only if $\mathfrak{A}$ satisfies certain transitivity and extendibility conditions.

A ***meager*** subset of $\mathfrak{A}$ is any countable union of closed, nowhere-dense sets. A ***comeager*** (or ***residual***) subset of $\mathfrak{A}$ is the compliment of a meager subset; the family of comeager subsets of $\mathfrak{A}$ is thus closed under countable intersections and arbitrary unions. The Baire Category Theorem [17, Thm.5.8] says that any comeager set is dense in $\mathfrak{A}$. A bounded function $f\colon \mathfrak{A} \to \mathbb{C}$ is ***almost everywhere*** ("æ") ***continuous*** if f is continuous at each point in a comeager subset of $\mathfrak{A}$. If $f, g\colon \mathfrak{A} \to \mathbb{C}$, then we say that $f = g$ ***almost everywhere*** ("$f \underset{\text{æ}}{\equiv} g$") if $\{\mathbf{a} \in \mathfrak{A} \;;\; f(\mathbf{a}) = g(\mathbf{a})\}$ is comeager in $\mathfrak{A}$. This is an equivalence relation. Let $\mathcal{C}_{\text{æ}}(\mathfrak{A})$ be the $\mathbb{C}$-vector space of æ-equivalence classes of æ-continuous

[1] This is can occur in higher dimensional SFTs which are conjugate to substitution shifts [45]; see [49, §3(d)] or [50, §7.1] for a discussion.

functions from $\mathfrak{A}$ into $\mathbb{C}$. An element $f \in \mathcal{C}_{æ}(\mathfrak{A})$ is an (æ) (Φ, σ)-*eigenfunction* with (æ) *eigenvalue* $\boldsymbol{\lambda} = (\lambda_0; \lambda_1, ..., \lambda_D) \in \mathbb{T}^{D+1}$ if:

(a) $f \circ \Phi \underset{\overline{\text{æ}}}{=} \lambda_0 f$.

(b) For any $\mathsf{z} = (z_1, \ldots, z_D) \in \mathbb{Z}^D$, $f \circ \sigma^{\mathsf{z}} \underset{\overline{\text{æ}}}{=} \boldsymbol{\lambda}^{\mathsf{z}} f$.

Let $\mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \Phi, \sigma) \subset \mathbb{T}^{D+1}$ be the set of all such eigenvalues. For any $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \Phi, \sigma)$, let

$$\mathcal{E}^{\mathit{igen}}_{\boldsymbol{\lambda}} \quad := \quad \left\{ f \in \mathcal{C}_{æ}(\mathfrak{A})\ ;\ f \circ \Phi \underset{\overline{\text{æ}}}{=} \lambda_0 f \ \text{ and } \ f \circ \sigma^{\mathsf{z}} \underset{\overline{\text{æ}}}{=} \boldsymbol{\lambda}^{\mathsf{z}} f,\ \forall \mathsf{z} \in \mathbb{Z}^D \right\}$$

be the *eigenspace* of $\boldsymbol{\lambda}$.

Example 3.8: (a) Let $\mathfrak{S}$ be the sofic shift of Example 1.1(c). Let

$$\mathfrak{E} \ := \ \{\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}\ ;\ a_z = 0,\ \forall \text{ even } z \in \mathbb{Z}\} \quad \text{and} \quad \mathfrak{O} \ := \ \{\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}\ ;\ a_z = 0,\ \forall \text{ odd } z \in \mathbb{Z}\}.$$

Then $\mathfrak{S} := \mathfrak{E} \cup \mathfrak{O}$, and $\mathfrak{E} \cap \mathfrak{O} = \{\bar{0}\}$. Let $\mathfrak{E}^* := \mathfrak{E} \setminus \{\bar{0}\}$ and $\mathfrak{O}^* := \mathfrak{O} \setminus \{\bar{0}\}$. Then $\sigma(\mathfrak{E}^*) = \mathfrak{O}^*$ and $\sigma(\mathfrak{O}^*) = \mathfrak{E}^*$. Define $f : \mathfrak{S} \longrightarrow \{-1, 0, 1\}$ by $f_{|_{\mathfrak{O}^*}} \equiv -1,\ f(\bar{0}) = 0,$ and $f_{|_{\mathfrak{E}^*}} \equiv 1$. Then $f \in \mathcal{C}_{æ}(\mathfrak{S})$ and is a σ-eigenfunction with eigenvalue -1. However, $\mathcal{E}^{\mathit{igen}}_{-1}(\mathfrak{S}, \sigma)$ has no *continuous* eigenfunctions, because $\mathfrak{S}$ contains a dense subset of points which are σ-homoclinic to $\bar{0}$ (namely sequences with only finitely many ones). Hence $-1 \in \mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{S}, \sigma) \setminus \mathrm{S}^{\mathrm{pec}}(\mathfrak{S}, \sigma)$.

If ${}_{\varepsilon}\Phi_{18}$ is ECA#18, then ${}_{\varepsilon}\Phi_{18}(\mathfrak{E}) = \mathfrak{O}$ and ${}_{\varepsilon}\Phi_{18}(\mathfrak{O}) = \mathfrak{E}$. Hence $f \in \mathcal{E}^{\mathit{igen}}_{(-1;-1)}(\mathfrak{S}, \Phi, \sigma)$, and $(-1; -1) \in \mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{S}, {}_{\varepsilon}\Phi_{18}, \sigma)$. In this case,

$$\mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{S}, {}_{\varepsilon}\Phi_{18}, \sigma) \quad = \quad \{(1,1), (-1,-1)\} \quad \subset \quad \mathbb{T}^2,$$

and the function $\mathbb{Z} \ni \mathsf{z} \mapsto \delta_{\overrightarrow{(0;\,\mathsf{z})}} \in \widehat{\mathrm{S}^{\mathrm{pec}}_{æ}}(\mathfrak{S}, {}_{\varepsilon}\Phi_{18}, \sigma) \cong \mathbb{Z}_{/2}$ is a surjection with kernel $2\mathbb{Z}$; hence we identify displacements with elements of $\mathbb{Z}_{/2}$.

(b) Let $\mathcal{A} = \{0, 1\}$, let $\lambda \in [0, 1)$ be irrational, and define $\tau : [0, 1) \ni x \mapsto (x + \lambda \bmod 1) \in [0, 1)$ (i.e. the rotation system induced by λ). Let $\mathfrak{T} \subset \mathcal{A}^{\mathbb{Z}}$ be the Sturmian subshift [16, Ch.6] obtained by projecting τ-orbits through the partition $\mathcal{P} := \{[0, 1-\lambda),\ [1-\lambda, 1)\}$. Then $\mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \sigma) = \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \sigma) = \{e^{2n\pi\mathbf{i}\lambda}\}_{n\in\mathbb{Z}} \subset \mathbb{T}$, so $\mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \sigma)$ is trivial. In this case, the function $\mathbb{Z} \ni \mathsf{z} \mapsto \delta_{\vec{\mathsf{z}}} \in \widehat{\mathrm{S}^{\mathrm{pec}}_{æ}}(\mathfrak{A}, \sigma)$ is an isomorphism, so we can identify all displacements with integers. $\diamondsuit$

The system $(\mathfrak{A}, \Phi, \sigma)$ is *topologically weakly mixing* if the Cartesian product $(\mathfrak{A} \times \mathfrak{A}, \Phi \times \Phi, \sigma \times \sigma)$ is topologically transitive.

Lemma 3.2. [i] $\mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma) \subseteq \mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \Phi, \sigma)$. If $(\mathfrak{A}, \Phi, \sigma)$ is transitive, then Lemma 3.1**(a-e)** [but not **(f)**] are still true if we replace "$\mathcal{C}(\mathfrak{A})$" with "$\mathcal{C}_{æ}(\mathfrak{A})$" and "$\mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma)$" with "$\mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \Phi, \sigma)$".

[ii] If $(\mathfrak{A}, \Phi, \sigma)$ is topologically weakly mixing, then $\mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \Phi, \sigma) = \{\mathbf{1}\}$.

[iii] If $(\mathfrak{A}, \Phi, \sigma)$ is not topologically weakly mixing, and there is a (Φ, σ)-ergodic measure with full support on $\mathfrak{A}$, then $\mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \Phi, \sigma)$ is nontrivial.

Proof: [i] The proofs of **(a,c,d,e)** are exactly as in Lemma 3.1. To see **(b)**, suppose $f_1, f_2 \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}$. Then $g := f_1/f_2$ is æ-well-defined (because $|f_2| \underset{\text{æ}}{=} 1$) and $g \in \mathcal{E}^{igen}_{\mathbf{1}}$, where $\mathbf{1} := (1, ..., 1)$ [by part **(a)**]. But if $(\mathfrak{A}, \Phi, \sigma)$ is transitive, then any element of $\mathcal{E}^{igen}_{\mathbf{1}}$ is æ-constant by [31, Thm 2.2]. So $g \underset{\text{æ}}{=} c$ for some constant $c \in \mathbb{C}$; hence $f_1 \underset{\text{æ}}{=} c \cdot f_2$.

[ii] is [31, Thm 2.3], and [iii] is [31, Thm 2.5 and Prop 2.6]. □

An ***eigenset*** for $\mathfrak{A}$ is a collection $\{f_{\boldsymbol{\lambda}}\ ;\ \boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)\}$ containing exactly one eigenfunction $f_{\boldsymbol{\lambda}} \in \mathcal{C}_{\text{æ}}(\mathfrak{A})$ for each $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)$.

Lemma 3.3. **(a)** $\mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)$ is countable (hence any eigenset is countable).

(b) Let $\mathcal{F} = \{f_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)}$ be an eigenset for $\mathfrak{A}$. There is a comeager, (Φ, σ)-invariant subset $\underline{\mathfrak{A}} = \underline{\mathfrak{A}}(\mathcal{F}) \subseteq \mathfrak{A}$ such that, for any $\underline{\mathbf{a}} \in \underline{\mathfrak{A}}$, any $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)$, any $\mathsf{z} \in \mathbb{Z}^D$, and $n \in \mathbb{N}$,

$$f_{\boldsymbol{\lambda}} \circ \Phi^n \circ \sigma^{\mathsf{z}}(\underline{\mathbf{a}}) \quad = \quad \lambda_0^n \boldsymbol{\lambda}^{\mathsf{z}} f_{\boldsymbol{\lambda}}(\underline{\mathbf{a}}).$$

(c) If $\mathcal{F} \subset \mathcal{C}(\mathfrak{A})$ [e.g. if $\mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma) = \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma)$] then $\underline{\mathfrak{A}} = \mathfrak{A}$.

Proof: **(a)** Any eigenset $\{f_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)}$ defines an æ-continuous factor mapping from $(\mathfrak{A}, \sigma)$ into a $\mathbb{Z}^{D+1}$-system $(\mathbf{T}, \rho)$, where $\mathbf{T}$ is a compact abelian group and ρ is a $\mathbb{Z}^{D+1}$-action by rotations of $\mathbf{T}$. We have $\mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma) = \mathrm{S}^{\mathrm{pec}}(\mathbf{T}, \rho)$. Choosing one eigenfunction for each $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}(\mathbf{T}, \rho)$, we get an orthogonal basis of $\mathbf{L}^2(\mathbf{T}, \mu)$ (where μ is the Haar measure on $\mathbf{T}$). But $\mathbf{L}^2(\mathbf{T}, \mu)$ is separable, so $\mathrm{S}^{\mathrm{pec}}(\mathbf{T}, \rho)$ is countable.

(b) Fix $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)$. For each $n \in \mathbb{N}$ and $\mathsf{z} \in \mathbb{Z}^D$, there is a comeager set $\mathfrak{E}_{(n;\mathsf{z})} \subseteq \mathfrak{A}$ such that, for any $\mathbf{e} \in \mathfrak{E}_{(n;\mathsf{z})}$, $f_{\boldsymbol{\lambda}} \circ \Phi^n \circ \sigma^{\mathsf{z}}(\mathbf{e}) = \lambda_0^n \boldsymbol{\lambda}^{\mathsf{z}} f_{\boldsymbol{\lambda}}(\mathbf{e})$. Let $\mathfrak{C}_{\boldsymbol{\lambda}} := \bigcap_{(n;\mathsf{z}) \in \mathbb{N} \times \mathbb{Z}^D} \mathfrak{E}_{(n;\mathsf{z})}$. Then $\mathfrak{C}_{\boldsymbol{\lambda}}$ is comeager in $\mathfrak{A}$, and for every $n \in \mathbb{N}$, $\mathsf{z} \in \mathbb{Z}^D$, and $\mathbf{c} \in \mathfrak{C}_{\boldsymbol{\lambda}}$, we have $f_{\boldsymbol{\lambda}} \circ \Phi^n \circ \sigma^{\mathsf{z}}(\mathbf{c}) = \lambda_0^n \boldsymbol{\lambda}^{\mathsf{z}} f_{\boldsymbol{\lambda}}(\mathbf{c})$. Let $\mathfrak{B}_{\boldsymbol{\lambda}} := \bigcap_{(n;\mathsf{z}) \in \mathbb{N} \times \mathbb{Z}^D} \Phi^{-n} \sigma^{-\mathsf{z}}(\mathfrak{C}_{\boldsymbol{\lambda}})$; then $\mathfrak{B}_{\boldsymbol{\lambda}}$ is comeager, and also $\Phi^n(\mathfrak{B}_{\boldsymbol{\lambda}}) \subseteq \mathfrak{B}_{\boldsymbol{\lambda}}$ and $\sigma^{\mathsf{z}}(\mathfrak{B}_{\boldsymbol{\lambda}}) = \mathfrak{B}_{\boldsymbol{\lambda}}$, for any $n \in \mathbb{N}$ and $\mathsf{z} \in \mathbb{Z}^D$.

Finally, let $\underline{\mathfrak{A}} = \bigcap_{\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)} \mathfrak{B}_{\boldsymbol{\lambda}}$; then $\underline{\mathfrak{A}}$ is a countable intersection [by **(a)**] of comeager sets, and thus also comeager. Also, $\underline{\mathfrak{A}}$ is (Φ, σ)-invariant because each $\mathfrak{B}_{\boldsymbol{\lambda}}$ is (Φ, σ)-invariant.

(c) If $f_{\boldsymbol{\lambda}} \in \mathcal{C}(\mathfrak{A})$, then $\mathfrak{B}_{\boldsymbol{\lambda}} = \mathfrak{A}$, because $\mathfrak{C}_{\boldsymbol{\lambda}} = \mathfrak{A}$, because $\mathfrak{E}_{(n;\mathsf{z})} = \mathfrak{A}$ for all $n \in \mathbb{N}$ and $\mathsf{z} \in \mathbb{Z}^D$. Thus, $\underline{\mathfrak{A}} = \mathfrak{A}$. □

Example 3.9: (a) Let $\mathfrak{S}$ be as in Example 1.1(c). Let $\mathcal{F} := \{f_1, f_{-1}\}$, where $f_1 := \mathbf{1}$ and where $f_{-1} := f \in \mathcal{E}^{igen}_{-1}(\mathfrak{S}, \sigma)$ is from Example 3.8(a). Then $\underline{\mathfrak{S}}(\mathcal{F}) = \mathfrak{S} \setminus \{\bar{0}\}$.

(b) If $\mathfrak{T}$ is as in Example 3.8(b), then $\underline{\mathfrak{T}} = \mathfrak{T}$ [because $\mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{T}) = \mathrm{S}^{\mathrm{pec}}(\mathfrak{T})$]. ◇

For any $\mathbb{Y} \subset \mathbb{Z}^D$ and $r \in \mathbb{N}$, let $\mathbb{Y}(r) := \{\mathsf{y} \in \mathbb{Y}\ ;\ \mathbb{B}(\mathsf{y}, r) \subset \mathbb{Y}\}$. Recall that $\mathbb{Y}$ is ***spacious*** if $\mathbb{Y}(r) \neq \emptyset$ for all $r \in \mathbb{N}$. A subshift $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ is ***projectively transitive*** if, for any spacious $\mathbb{Y} \subset \mathbb{Z}^D$ and open $\mathfrak{O} \subset \mathfrak{A}$, the set $\bigcup_{\mathsf{y} \in \mathbb{Y}} \sigma^{-\mathsf{y}}(\mathfrak{O})$ is dense in $\mathfrak{A}$.

Example 3.10: (a) If $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$, then $\mathfrak{A}$ is projectively transitive iff both σ^1 and σ^{-1} are forward-transitive on $\mathfrak{A}$. [To see this, use Example 1.4(a).]

(b) Suppose $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ has a σ-ergodic measure μ with full support (i.e. $\mu[\mathfrak{O}] > 0$ for any open $\mathfrak{O} \subseteq \mathfrak{A}$). Then $\mathfrak{A}$ is projectively transitive. To see this, let $\mathbb{Y} \subseteq \mathbb{Z}^D$ be spacious and let $\mathfrak{O} \subseteq \mathfrak{A}$ be open.

CLAIM 1: *There is a subset* $\mathfrak{X} \subseteq \bigcup_{\mathsf{y} \in \mathbb{Y}} \sigma^{-\mathsf{y}}(\mathfrak{O})$ *with* $\mu[\mathfrak{X}] = 1$.

Proof: For each $r \in \mathbb{N}$, find $\mathsf{y}_r \in \mathbb{Y}$ with $\mathbb{F}_r := \mathbb{B}(\mathsf{y}_r, r) \subset \mathbb{Y}$, and define the function

$$\alpha_r \quad := \quad \frac{1}{\#(\mathbb{F}_r)} \sum_{\mathsf{f} \in \mathbb{F}_r} \mathbb{1}_{\mathfrak{O}} \circ \sigma^{\mathsf{f}}.$$

The sequence of sets $\{\mathbb{F}_r\}_{r=1}^{\infty}$ is a Følner sequence, so the generalized Mean Ergodic Theorem [46, 51] says that the sequence $\{\alpha_r\}_{r=1}^{\infty}$ converges to the constant function $\mu[\mathfrak{O}]$ in $\mathbf{L}^2(\mathfrak{A}, \mu)$. Thus, there is a subsequence $\{r_n\}_{n=1}^{\infty}$ and a set $\mathfrak{X} \subseteq \mathfrak{A}$ with $\mu[\mathfrak{X}] = 1$ such that $\lim_{n \to \infty} \alpha_{r_n}(\mathbf{x}) = \mu[\mathfrak{O}]$ for all $\mathbf{x} \in \mathfrak{X}$ [17, Corol.2.32]. But $\mathfrak{O}$ is open, so $\mu[\mathfrak{O}] > 0$ (because μ has full support on $\mathfrak{A}$). Thus, for any $\mathbf{x} \in \mathfrak{X}$, there exists $n \in \mathbb{N}$ such that $\alpha_{r_n}(\mathbf{x}) > 0$ (indeed, infinitely many such $n \in \mathbb{N}$), which means there exists $\mathsf{f} \in \mathbb{F}_{r_n}$ such that $\mathbb{1}_{\mathfrak{O}} \circ \sigma^{\mathsf{f}}(\mathbf{x}) = 1$ which means $\sigma^{\mathsf{f}}(\mathbf{x}) \in \mathfrak{O}$. But $\mathsf{f} \in \mathbb{F}_{r_n} \subset \mathbb{Y}$, so this means $\mathbf{x} \in \bigcup_{\mathsf{y} \in \mathbb{Y}} \sigma^{-\mathsf{y}}(\mathfrak{O})$. ◇ `Claim 1`

Finally, $\mathfrak{X}$ is dense in $\mathfrak{A}$, because $\mu[\mathfrak{X}] = 1$ and μ has full support on $\mathfrak{A}$. ◇

Heuristically speaking, an eigenfunction f detects some underlying ‘rigidity’ in the structure of $\mathfrak{A}$. Thus, we don’t need to know every coordinate of $\mathbf{a} \in \mathfrak{A}$ to evaluate $f(\mathbf{a})$; it suffices to have information about some ‘large enough fragment’ of $\mathbf{a}$. This is the idea of the next lemma, where ‘large enough’ means ‘spacious’:

Lemma 3.4. Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a projectively transitive subshift. Let $\mathcal{F} := \{f_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \Phi, \sigma)}$ be an eigenset and let $\underline{\mathfrak{A}} := \underline{\mathfrak{A}}(\mathcal{F})$. Let $\mathbb{Y} \subset \mathbb{Z}^D$ be spacious.

(a) There is a (Φ, σ)-invariant, comeager subset $\widehat{\mathfrak{A}} = \widehat{\mathfrak{A}}(\mathbb{Y}) \subseteq \underline{\mathfrak{A}}$ with the following Extension Property: For any $\mathbf{a} \in \widehat{\mathfrak{A}}$ and $\underline{\mathbf{a}} \in \underline{\mathfrak{A}}$, if $\underline{\mathbf{a}}_{\mathbb{Y}} = \mathbf{a}_{\mathbb{Y}}$, then for every $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \Phi, \sigma)$, $f_{\boldsymbol{\lambda}}(\underline{\mathbf{a}}) = f_{\boldsymbol{\lambda}}(\mathbf{a})$.

(b) If $\mathcal{F} \subset \mathcal{C}(\mathfrak{A})$ [e.g. if $\mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \Phi, \sigma) = \mathrm{S}^{\mathrm{pec}}(\mathfrak{A}, \Phi, \sigma)$] then $\widehat{\mathfrak{A}} = \underline{\mathfrak{A}} = \mathfrak{A}$.

(c) If $\mathbb{Y}' \subset \mathbb{Y}$, then $\widehat{\mathfrak{A}}(\mathbb{Y}') \subseteq \widehat{\mathfrak{A}}(\mathbb{Y})$.

(d) If $\mathbb{Y}' := \mathbb{Y}(R)$ for some $R \in \mathbb{N}$, then $\widehat{\mathfrak{A}}(\mathbb{Y}') = \widehat{\mathfrak{A}}(\mathbb{Y})$.

(e) If Φ has radius R, and $\mathbb{Y}' := \mathbb{Y}(R)$, then $\Phi[\widehat{\mathfrak{A}}(\mathbb{Y})] \subseteq \widehat{\mathfrak{A}}(\mathbb{Y}')$.

Proof: **(a)** Fix $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \Phi, \sigma)$.

CLAIM 1: *For any* $\epsilon > 0$, *there exists* $r_\epsilon > 0$ *and a* (Φ, σ)*-invariant, comeager subset* $\mathfrak{G}_\epsilon \subseteq \mathfrak{A}$ *with the following property: for any* $\mathbf{g} \in \mathfrak{G}_\epsilon$, *there is some* $\mathsf{y} = \mathsf{y}(\mathbf{g}) \in \mathbb{Y}$ *with* $\mathbb{B}(\mathsf{y}, r_\epsilon) \subset \mathbb{Y}$ *such that:*

$$\text{For any } \mathbf{a} \in \mathfrak{A}, \qquad \left(\mathbf{a}_{\mathbb{B}(\mathsf{y}, r_\epsilon)} = \mathbf{g}_{\mathbb{B}(\mathsf{y}, r_\epsilon)}\right) \Longrightarrow \left(f_{\boldsymbol{\lambda}}\left(\sigma^{\mathsf{y}}[\mathbf{a}]\right) \underset{\epsilon}{\sim} f_{\boldsymbol{\lambda}}\left(\sigma^{\mathsf{y}}[\mathbf{g}]\right)\right) \qquad (2)$$

Proof: For all $r > 0$, let

$$\mathfrak{W}_r(\epsilon) := \left\{\mathbf{w} \in \mathfrak{A} \; ; \; \forall \, \mathbf{a} \in \mathfrak{A}, \; \left(\mathbf{a}_{\mathbb{B}(r)} = \mathbf{w}_{\mathbb{B}(r)}\right) \Longrightarrow \left(f_{\boldsymbol{\lambda}}(\mathbf{a}) \underset{\epsilon}{\sim} f_{\boldsymbol{\lambda}}(\mathbf{w})\right)\right\}.$$

CLAIM 1.1: *If r is large enough, then $\mathfrak{W}_r(\epsilon)$ has nonempty interior.*

Proof: Let $\mathbf{w} \in \mathfrak{W}_r(\epsilon/2)$, and let $\mathfrak{C} := \left\{\mathbf{c} \in \mathfrak{A} \; ; \; \mathbf{c}_{\mathbb{B}(r)} = \mathbf{w}_{\mathbb{B}(r)}\right\}$ be a cylinder neighbourhood around $\mathbf{w}$; we will show that $\mathfrak{C} \subseteq \mathfrak{W}_r(\epsilon)$. For any $\mathbf{c}, \mathbf{c}' \in \mathfrak{C}$, we have $f_{\boldsymbol{\lambda}}(\mathbf{c}) \underset{\epsilon}{\sim} f_{\boldsymbol{\lambda}}(\mathbf{c}')$, because $f_{\boldsymbol{\lambda}}(\mathbf{c}) \underset{\epsilon/2}{\sim} f_{\boldsymbol{\lambda}}(\mathbf{w}) \underset{\epsilon/2}{\sim} f_{\boldsymbol{\lambda}}(\mathbf{c}')$, because $\mathbf{c}_{\mathbb{B}(r)} = \mathbf{w}_{\mathbb{B}(r)} = \mathbf{c}'_{\mathbb{B}(r)}$. Thus, $\mathbf{c} \in \mathfrak{W}_r(\epsilon)$ for all $\mathbf{c} \in \mathfrak{C}$, so $\mathfrak{C} \subseteq \mathfrak{W}_r(\epsilon)$.

It remains to show that, if r is large enough, then $\mathfrak{W}_r(\epsilon/2)$ is nonempty. To see this, recall that $f_{\boldsymbol{\lambda}} \in \mathcal{C}_{æ}$, so $f_{\boldsymbol{\lambda}}$ has continuity points. If $\mathbf{a} \in \mathfrak{A}$ is any such continuity point, then $\mathbf{a} \in \mathfrak{W}_r(\epsilon/2)$ if r is large enough. $\bigtriangledown$ `Claim 1.1`

Let $r_\epsilon := r$; then $\mathbb{Y}(r)$ is nonempty and is also spacious. Let $\mathfrak{U}_\epsilon$ be the (nonempty) interior of $\mathfrak{W}_r(\epsilon)$. Let $\mathfrak{O}_\epsilon := \displaystyle\bigcup_{\mathsf{y} \in \mathbb{Y}(r)} \sigma^{-\mathsf{y}}(\mathfrak{U}_\epsilon)$. For any $\mathbf{g} \in \mathfrak{O}_\epsilon$, there exists $\mathsf{y} \in \mathbb{Y}(r)$ with $\sigma^{\mathsf{y}}(\mathbf{g}) \in \mathfrak{U}_\epsilon \subseteq \mathfrak{W}_r(\epsilon)$. Then $\mathbf{g}$ and y satisfy eqn.(2).

Let $\mathfrak{G}_\epsilon := \displaystyle\bigcap_{\mathsf{z} \in \mathbb{Z}^D} \bigcap_{n=0}^{\infty} \sigma^{-\mathsf{z}} \Phi^{-n}(\mathfrak{O}_\epsilon)$. Then $\mathfrak{G}_\epsilon$ is (Φ, σ)-invariant. Also, $\mathfrak{G}_\epsilon \subseteq \mathfrak{O}_\epsilon$, so for every $\mathbf{g} \in \mathfrak{G}_\epsilon$ there is some $\mathsf{y} \in \mathbb{Y}$ satisfying eqn.(2). To see that $\mathfrak{G}_\epsilon$ is comeager, let $\mathfrak{U}_\epsilon^{n;\mathsf{z}} := \sigma^{-\mathsf{z}} \Phi^{-n}(\mathfrak{U}_\epsilon)$ for each $n \in \mathbb{N}$ and $\mathsf{z} \in \mathbb{Z}^D$. Then $\mathfrak{U}_\epsilon^{n;\mathsf{z}}$ is open because Φ^n and σ^{z} are continuous and $\mathfrak{U}$ is open. Thus,

$$\sigma^{-\mathsf{z}} \Phi^{-n}(\mathfrak{O}_\epsilon) = \bigcup_{\mathsf{y} \in \mathbb{Y}(r)} \sigma^{-\mathsf{z}} \Phi^{-n} \sigma^{-\mathsf{y}}(\mathfrak{U}_\epsilon) = \bigcup_{\mathsf{y} \in \mathbb{Y}(r)} \sigma^{-\mathsf{y}} \sigma^{-\mathsf{z}} \Phi^{-n}(\mathfrak{U}_\epsilon) = \bigcup_{\mathsf{y} \in \mathbb{Y}(r)} \sigma^{-\mathsf{y}}(\mathfrak{U}_\epsilon^{n;\mathsf{z}})$$

is open and dense in $\mathfrak{A}$ (because $\mathfrak{A}$ is projectively transitive). Thus, $\mathfrak{G}_\epsilon$ is a countable intersection of dense open sets, hence $\mathfrak{G}_\epsilon$ is comeager. $\diamondsuit$ `Claim 1`

Now, let $\mathfrak{D}_{\boldsymbol{\lambda}} := \underline{\mathfrak{A}} \cap \displaystyle\bigcap_{n=1}^{\infty} \mathfrak{G}_{1/n}$. Then $\mathfrak{D}_{\boldsymbol{\lambda}}$ is (Φ, σ)-invariant and comeager in $\mathfrak{A}$.

CLAIM 2: *If $\mathbf{d} \in \mathfrak{D}_{\boldsymbol{\lambda}}$, $\underline{\mathbf{a}} \in \underline{\mathfrak{A}}$ and $\mathbf{d}_{\mathbb{Y}} = \underline{\mathbf{a}}_{\mathbb{Y}}$, then $f_{\boldsymbol{\lambda}}(\mathbf{d}) = f_{\boldsymbol{\lambda}}(\underline{\mathbf{a}})$.*

Proof: Fix $\epsilon > 0$. First, we claim that $f(\mathbf{d}) \underset{\epsilon}{\sim} f(\underline{\mathbf{a}})$. To see this, find $n \in \mathbb{N}$ with $1/n < \epsilon$. Let $r := r_{1/n}$ and $\mathsf{y} := \mathsf{y}(\mathbf{d})$ be as in Claim 1. Then

$$\boldsymbol{\lambda}^{\mathsf{y}} f(\mathbf{d}) \underset{(*)}{=} f\left(\sigma^{\mathsf{y}}[\mathbf{d}]\right) \underset{\epsilon}{\sim} f\left(\sigma^{\mathsf{y}}[\underline{\mathbf{a}}]\right) \underset{(\dagger)}{=} \boldsymbol{\lambda}^{\mathsf{y}} f(\underline{\mathbf{a}}). \tag{3}$$

$(*)$ is because $\mathbf{d} \in \underline{\mathfrak{A}}$ and $(\dagger)$ is because $\underline{\mathbf{a}} \in \underline{\mathfrak{A}}$. Finally, "$\underset{\epsilon}{\sim}$" is by Claim 1, because $\mathbf{d} \in \mathfrak{D}_{\boldsymbol{\lambda}} \subseteq \mathfrak{G}_{1/n}$, and because $\mathbf{d}_{\mathbb{B}(\mathsf{y},r)} = \underline{\mathbf{a}}_{\mathbb{B}(\mathsf{y},r)}$ because $\mathbf{d}_{\mathbb{Y}} = \underline{\mathbf{a}}_{\mathbb{Y}}$. But, $|\boldsymbol{\lambda}^{\mathsf{y}}| = 1$, so eqn.(3) implies $f(\mathbf{d}) \underset{\epsilon}{\sim} f(\underline{\mathbf{a}})$.

This argument works for any $\epsilon > 0$. Thus, $f(\mathbf{d}) = f(\underline{\mathbf{a}})$. $\diamondsuit$ `Claim 2`

Let $\mathfrak{C}_0 := \displaystyle\bigcap_{\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{æ}(\mathfrak{A}, \Phi, \sigma)} \mathfrak{D}_{\boldsymbol{\lambda}}$. Then $\mathfrak{C}_0$ has the Extension Property, and is (Φ, σ)-invariant. Finally, $\mathfrak{C}_0$ is comeager, because Lemma 3.3(a) implies that $\mathfrak{C}_0$ is a countable intersection of comeager sets.

Thus, $\mathfrak{C}_0$ satisfies all the requirements of **(a)**. However, in preparation for the proofs of **(d)** and **(e)** below, we must refine this construction somewhat. For each $r \in \mathbb{N}$, repeat the above construction to obtain a (Φ, σ)-invariant comeager set $\mathfrak{C}_r$ satisfying the Extension Property for $\mathbb{Y}(r)$. Let $\widehat{\mathfrak{A}} := \bigcap_{r=0}^{\infty} \mathfrak{C}_r$; then $\widehat{\mathfrak{A}}$ is also comeager and (Φ, σ)-invariant, and $\widehat{\mathfrak{A}}$ satisfies the Extension Property for $\mathbb{Y}$.

(b) Repeat the construction in **(a)** for $f_{\boldsymbol{\lambda}} \in \mathcal{C}(\mathfrak{A})$. In Claim 1, we have $\mathfrak{O}_\epsilon = \mathfrak{A}$ [because if f is continuous, then f is uniformly continuous (because $\mathfrak{A}$ is compact), so that $\mathfrak{U}_\epsilon = \mathfrak{W}_r(\epsilon) = \mathfrak{A}$ if r is large enough]. Thus, $\mathfrak{G}_\epsilon = \mathfrak{A}$. Also, $\underline{\mathfrak{A}} = \mathfrak{A}$ by Lemma 3.3(c). Thus, $\mathfrak{D}_{\boldsymbol{\lambda}} = \mathfrak{A}$ for each $\boldsymbol{\lambda} \in \mathrm{S^{pec}}(\mathfrak{A}, \Phi, \sigma)$. Thus, $\mathfrak{C}_0 := \bigcap_{\boldsymbol{\lambda} \in \mathrm{S^{pec}}(\mathfrak{A}, \Phi, \sigma)} \mathfrak{D}_{\boldsymbol{\lambda}} = \mathfrak{A}$. Likewise, $\mathfrak{C}_r = \mathfrak{A}$ for all $r \in \mathbb{N}$; hence $\widehat{\mathfrak{A}} = \mathfrak{A}$.

(c) Suppose $\mathbb{Y}' \subset \mathbb{Y}$, and repeat the construction in **(a)** for $\mathbb{Y}'$. In Claim 1, $\mathfrak{O}'_\epsilon \subseteq \mathfrak{O}_\epsilon$, so $\mathfrak{G}'_\epsilon \subseteq \mathfrak{G}_\epsilon$. Thus, $\mathfrak{D}'_{\boldsymbol{\lambda}} \subseteq \mathfrak{D}_{\boldsymbol{\lambda}}$ for each $\boldsymbol{\lambda}$; thus $\mathfrak{C}'_0 \subseteq \mathfrak{C}_0$. Likewise, for each $r \in \mathbb{N}$, we have $\mathfrak{C}'_r \subseteq \mathfrak{C}_r$ because $\mathbb{Y}'(r) \subseteq \mathbb{Y}(r)$. Thus, $\widehat{\mathfrak{A}}(\mathbb{Y}') \subseteq \widehat{\mathfrak{A}}(\mathbb{Y})$.

(d) If $\mathbb{Y}' := \mathbb{Y}(R)$ for some $R \in \mathbb{N}$, then for all $r \in \mathbb{N}$, $\mathbb{Y}'(r) = \mathbb{Y}(r+R)$ [this is because $\mathbb{B}(r+R) = \mathbb{B}(r) + \mathbb{B}(R)$]. Thus, for every $r \in \mathbb{N}$, $\mathfrak{C}'_r = \mathfrak{C}_{r+R}$. Thus,

$$\widehat{\mathfrak{A}}(\mathbb{Y}') \quad = \quad \bigcap_{r=1}^{\infty} \mathfrak{C}'_r \quad = \quad \bigcap_{r=1}^{\infty} \mathfrak{C}_{r+R} \quad \supseteq \quad \bigcap_{r=1}^{\infty} \mathfrak{C}_r \quad = \quad \widehat{\mathfrak{A}}(\mathbb{Y}) \quad \underset{(*)}{\supseteq} \quad \widehat{\mathfrak{A}}(\mathbb{Y}'),$$

where $(*)$ is by **(c)**. Hence $\widehat{\mathfrak{A}}(\mathbb{Y}') = \widehat{\mathfrak{A}}(\mathbb{Y})$.

(e) We have $\Phi[\widehat{\mathfrak{A}}(\mathbb{Y})] \subseteq \widehat{\mathfrak{A}}(\mathbb{Y}) \underset{(*)}{=} \widehat{\mathfrak{A}}(\mathbb{Y}(R))$, where $(*)$ is by **(d)** $\square$

Example 3.11: Let $\mathcal{A} = \{0, 1\}$. Let $\mathbb{Y} \subset \mathbb{Z}$ be spacious [see Example 1.4(a)].

(a) If $\mathfrak{S} \subset \mathcal{A}^{\mathbb{Z}}$ and $\mathcal{F} = \{f_1, f_{-1}\}$ are as in Example 3.9(a), then $\widehat{\mathfrak{S}}(\mathbb{Y}) = \mathfrak{S} \setminus \{\bar{0}\}$.

(b) If $\mathfrak{T} \subset \mathcal{A}^{\mathbb{Z}}$ is as in Example 3.8(b), then $\widehat{\mathfrak{T}}(\mathbb{Y}) = \mathfrak{T}$, by Lemma 3.4(b). $\diamondsuit$

Remark: The constructions of $\underline{\mathfrak{A}}(\mathcal{F})$ in Lemma 3.3(a) and $\widehat{\mathfrak{A}}(\mathbb{Y})$ in Lemma 3.4(a) depend upon the eigenset $\mathcal{F}$, because any æ-equivalence class in $\mathcal{E}^{eigen}_{\boldsymbol{\lambda}}$ could contain uncountably many functions, each pair of which differ on a meager subset of $\mathfrak{A}$. Two eigenfunctions $f_{\boldsymbol{\lambda}}, f'_{\boldsymbol{\lambda}} \in \mathcal{E}^{eigen}_{\boldsymbol{\lambda}}$ could thus yield two sets $\widehat{\mathfrak{A}}(\mathbb{Y})$ and $\widehat{\mathfrak{A}}'(\mathbb{Y})$ whose symmetric difference was meager. $\diamondsuit$

Fix an eigenset $\mathcal{F}$, and let $\underline{\mathfrak{A}} := \underline{\mathfrak{A}}(\mathcal{F})$ be as in Lemma 3.3(a). Define

$$\underline{\widetilde{\mathfrak{A}}} := \left\{ \mathbf{a} \in \widetilde{\mathfrak{A}} \; ; \; \mathbf{a}_{\mathbb{X}} \in \underline{\mathfrak{A}}_{\mathbb{X}}, \text{ for every } r > 0 \text{ and projective component } \mathbb{X} \text{ of } \mathbb{G}_r(\mathbf{a}) \right\}.$$

Heuristically, 'almost all' elements of $\widetilde{\mathfrak{A}}$ are in $\underline{\widetilde{\mathfrak{A}}}$ (because almost all elements of $\mathfrak{A}$ are in $\underline{\mathfrak{A}}$). For any spacious $\mathbb{Y} \subseteq \mathbb{Z}^D$, we define $\widehat{\mathfrak{A}}_{\mathbb{Y}} := \left\{ \mathbf{a}_{\mathbb{Y}} \; ; \; \mathbf{a} \in \widehat{\mathfrak{A}}(\mathbb{Y}) \right\} \subset \mathcal{A}^{\mathbb{Y}}$, where $\widehat{\mathfrak{A}}(\mathbb{Y})$ is as in Lemma 3.4. For any collection $\mathbb{Y}_1, \ldots, \mathbb{Y}_N$ of disjoint spacious subsets of $\mathbb{Z}^D$, we define

$$\widehat{\mathfrak{A}}(\mathbb{Y}_1, \ldots, \mathbb{Y}_N) \;\; := \;\; \left\{ \mathbf{a} \in \underline{\widetilde{\mathfrak{A}}} \; ; \; \forall\, n \in [1...N], \; \mathbf{a}_{\mathbb{Y}_n} \in \widehat{\mathfrak{A}}_{\mathbb{Y}_n} \right\}.$$

If $\mathbf{a} \in \widehat{\mathfrak{A}}(\mathbb{Y}_1, \ldots, \mathbb{Y}_N)$, and $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)$, then for each $n \in [1...N]$, Lemma 3.4(a) says there is a unique value $c_n \in \mathbb{C}$ such that $f_{\boldsymbol{\lambda}}(\underline{\mathbf{a}}) = c_n$ for any 'extension' $\underline{\mathbf{a}} \in \underline{\mathfrak{A}}$ with $\underline{\mathbf{a}}_{\mathbb{Y}_n} = \mathbf{a}_{\mathbb{Y}_n}$. We can thus define $f_{\boldsymbol{\lambda}}(\mathbf{a}_{\mathbb{Y}_n}) := c_n$. For any $n, m \in [1...N]$, we then define $\delta_{n,m}(\boldsymbol{\lambda}) := f_{\boldsymbol{\lambda}}(\mathbf{a}_{\mathbb{Y}_n})/f_{\boldsymbol{\lambda}}(\mathbf{a}_{\mathbb{Y}_m})$. Note that we make no assumption here about the relationship between the sets $\mathbb{Y}_1, \ldots, \mathbb{Y}_N$ and the projective components of $\mathbb{G}_r(\mathbf{a})$. Clearly, if $\mathbf{a} \in \mathfrak{A}$, then $\delta_{n,m}(\boldsymbol{\lambda}) = 1$ for all $n, m \in [1..N]$. However, if $\delta_{n,m}(\boldsymbol{\lambda}) \neq 1$, then $\mathbf{a}$ must be defective, and $\mathbb{Y}_n$ and $\mathbb{Y}_m$ must lie in different projective components of $\mathbb{G}_r(\mathbf{a})$, by part **(d)** of the next result:

Theorem 3.12. Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA and let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a projectively transitive subshift with $\Phi(\mathfrak{A}) = \mathfrak{A}$. Fix an eigenset $\mathcal{F} = \{f_{\boldsymbol{\lambda}}\}_{\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)}$. Let $\mathbb{Y}_1, \ldots, \mathbb{Y}_N$ be disjoint spacious subsets of $\mathbb{Z}^D$ and let $\mathbf{a} \in \widehat{\mathfrak{A}}(\mathbb{Y}_1, \ldots, \mathbb{Y}_N)$. Then

(a) $\delta_{n,m}(\boldsymbol{\lambda})$ is well-defined independent of the choice of $f_{\boldsymbol{\lambda}} \in \mathcal{E}^{igen}_{\boldsymbol{\lambda}}$.

(b) $\delta_{n,m} : \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma) \longrightarrow \mathbb{T}$ is a group homomorphism; i.e. $\delta_{n,m} \in \widehat{\mathrm{S}^{\mathrm{pec}}_{\text{æ}}}(\mathfrak{A}, \Phi, \sigma)$.

(c) $\delta_{n\ell}(\boldsymbol{\lambda}) = \delta_{nm}(\boldsymbol{\lambda})\delta_{m\ell}(\boldsymbol{\lambda})$ for any $n, m, \ell \in [1...N]$.

(d) If $\delta_{nm} \neq \mathbf{1}$, then $\mathbf{a}$ has a projective domain boundary between $\mathbb{Y}_n$ and $\mathbb{Y}_m$.

(e) If $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\mathbb{Q}}(\mathfrak{A}, \Phi, \sigma)$, then this $\delta_{n,m}(\boldsymbol{\lambda})$ is the same as the one in Theorem 3.3.

Proof: **(a)** is by Lemma 3.2[i](b). Part **(b)** is by Lemma 3.2[i](a), as in Claim 1[c] of Theorem 3.3. Part **(c)** is true by definition of δ_{nm}, and **(e)** is straightforward.

For **(d)**, suppose by contradiction that $\mathbb{Y}_n$ and $\mathbb{Y}_m$ lie in the same projective component $\mathbb{X}$ of $\mathbb{G}_r(\mathbf{a})$. Find some extension $\underline{\mathbf{a}} \in \underline{\mathfrak{A}}$ such that $\underline{\mathbf{a}}_{\mathbb{X}} = \mathbf{a}_{\mathbb{X}}$ (this $\underline{\mathbf{a}}$ exists because $\mathbf{a} \in \widetilde{\mathfrak{A}}$). Note that $\underline{\mathbf{a}}$ is also an extension of $\mathbf{a}_{\mathbb{Y}_n}$ and of $\mathbf{a}_{\mathbb{Y}_m}$. Thus, for every $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)$, we have $f_{\boldsymbol{\lambda}}(\mathbf{a}_{\mathbb{Y}_n}) = f_{\boldsymbol{\lambda}}(\underline{\mathbf{a}}) = f_{\boldsymbol{\lambda}}(\mathbf{a}_{\mathbb{Y}_m})$ by definition, so $\delta_{nm}(\boldsymbol{\lambda}) = f_{\boldsymbol{\lambda}}(\mathbf{a}_{\mathbb{Y}_n})/f_{\boldsymbol{\lambda}}(\mathbf{a}_{\mathbb{Y}_m}) = 1$.

So, if $\delta_{nm}(\boldsymbol{\lambda}) \neq 1$ for any $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)$, then $\mathbb{Y}_n$ and $\mathbb{Y}_m$ must be in different projective components of $\mathbb{G}_r(\mathbf{a})$, which means that $\mathbf{a}$ has a projective domain boundary between $\mathbb{Y}_n$ and $\mathbb{Y}_m$. □

If $\delta_{n,m}(\boldsymbol{\lambda}) \neq 1$, then we say $\mathbf{a}$ has a *projective* $(\mathfrak{A}, \Phi)$*-dislocation* between $\mathbb{Y}_n$ and $\mathbb{Y}_m$. (When $\mathfrak{A}$ and Φ are clear, we will just call this a "projective dislocation"). The matrix $\Delta_{\mathbf{a}} := [\delta_{nm}]_{n,m=1}^N$ is called the *displacement matrix* of $\mathbf{a}$ with respect to the sets $(\mathbb{Y}_1, \ldots, \mathbb{Y}_N)$.

Example 3.13: (a) Let $\mathfrak{S} \subset \mathcal{A}^{\mathbb{Z}}$ and $\mathcal{F} = \{f_1, f_{-1}\}$ be as in Examples 3.9(a) and 3.11(a). If $\mathbf{s} \in \widetilde{\mathfrak{S}}$ is as in Example 1.1(c), then $\mathbf{s}$ has a dislocation, with displacement $\delta_{\mathbf{a}} = 1 \in \mathbb{Z}_{/2}$.

(b) Let $\mathcal{A} = \{0, 1\}$, let $\lambda := (\sqrt{5} - 1)/2$ and let $\mathfrak{T} \subset \mathcal{A}^{\mathbb{Z}}$ be the 'Fibonacci' Sturmian subshift [16, §5.4.3] generated by λ [see Example 3.8(b)].

$$\text{A typical point is } (\ldots 0100100101001010010010\,0100100101001010010010100\ldots).$$
$$\text{Instead, let } \mathbf{a} := (\ldots 0100100101001010010010.1001001010010100100101001\ldots).$$

Then $\mathbf{a}$ has a dislocation at the decimal point, with displacement $\delta_{\mathbf{a}} = 1 \in \mathbb{Z}$.

(c) Let $\mathfrak{Dom}$ be as in Example 1.3(c). Despite appearances, the domain boundary in Figure 2(D) is *not* a dislocation, because $(\mathfrak{Dom}, \sigma)$ is topologically mixing [10, Lemma 2.1], so $\mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{Dom}, \sigma)$ is trivial by Lemma 3.2[ii]. Instead, this is a 'gap' defect; see [47, Example 2.14(c)]. ◇

Our last major result, analogous to Theorem 3.6, is that projective dislocations are Φ-persistent defects:

Theorem 3.14. Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA of radius $r \geq 0$. Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a projectively transitive subshift with $\Phi(\mathfrak{A}) = \mathfrak{A}$. Let $\mathbb{Y}_1, \ldots, \mathbb{Y}_N \subset \mathbb{Z}^D$ be disjoint spacious sets. For all $n \in [1...N]$, let $\mathbb{Y}'_n := \mathbb{Y}_n(r)$. Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$ and $\mathbf{a}' := \Phi(\mathbf{a})$.

(a) If $\mathbf{a} \in \widehat{\mathfrak{A}}(\mathbb{Y}_1, \ldots, \mathbb{Y}_N)$, then $\mathbf{a}' \in \widehat{\mathfrak{A}}(\mathbb{Y}'_1, \ldots, \mathbb{Y}'_N)$.

(b) If $\mathbf{a}$ has a projective dislocation, then so does $\mathbf{a}'$. Indeed, the $(\mathbb{Y}'_1, ..., \mathbb{Y}'_N)$-displacement matrix of $\mathbf{a}'$ is equal to the $(\mathbb{Y}_1, ..., \mathbb{Y}_N)$-displacement matrix of $\mathbf{a}$.

Proof: **(a)** Follows immediately from Lemma 3.4(e).

(b) Fix $n, m \in [1..N]$. Let $\underline{\mathbf{a}}_n \in \underline{\mathfrak{A}}$ and $\underline{\mathbf{a}}_m \in \underline{\mathfrak{A}}$ be extensions of $\mathbf{a}_{\mathbb{Y}_n}$ and $\mathbf{a}_{\mathbb{Y}_m}$, respectively. Let $\underline{\mathbf{a}}'_n := \Phi(\underline{\mathbf{a}}_n)$ and $\underline{\mathbf{a}}'_m := \Phi(\underline{\mathbf{a}}_m)$; then $\underline{\mathbf{a}}'_n$ and $\underline{\mathbf{a}}'_m$ are also in $\underline{\mathfrak{A}}$ [because $\Phi(\underline{\mathfrak{A}}) \subseteq \underline{\mathfrak{A}}$ by Lemma 3.3(b)], and are extensions of $\mathbf{a}'_{\mathbb{Y}'_n}$ and $\mathbf{a}'_{\mathbb{Y}'_m}$, respectively. Thus, for any $\boldsymbol{\lambda} = (\lambda_0; \lambda_1, \ldots, \lambda_D) \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)$, we have

$$\delta'_{nm}(\boldsymbol{\lambda}) \underset{(\dagger)}{=} \frac{f(\underline{\mathbf{a}}'_n)}{f(\underline{\mathbf{a}}'_m)} \underset{(*)}{=} \frac{\lambda_0 f(\underline{\mathbf{a}}_n)}{\lambda_0 f(\underline{\mathbf{a}}_m)} = \frac{f(\underline{\mathbf{a}}_n)}{f(\underline{\mathbf{a}}_m)} \underset{(\ddagger)}{=} \delta_{nm}(\boldsymbol{\lambda}).$$

$(*)$ is because $f_{\boldsymbol{\lambda}}(\underline{\mathbf{a}}'_n) = f_{\boldsymbol{\lambda}} \circ \Phi(\underline{\mathbf{a}}_n) = \lambda_0 f_{\boldsymbol{\lambda}}(\underline{\mathbf{a}}_n)$, because $\underline{\mathbf{a}}_n \in \underline{\mathfrak{A}}$. Likewise $f_{\boldsymbol{\lambda}}(\underline{\mathbf{a}}'_m) = \lambda_0 f_{\boldsymbol{\lambda}}(\underline{\mathbf{a}}_m)$ because $\underline{\mathbf{a}}_m \in \underline{\mathfrak{A}}$. $(\ddagger)$ is by definition of $\delta_{nm}(\boldsymbol{\lambda})$, and is independent of the choice of $\underline{\mathbf{a}}_n$ and $\underline{\mathbf{a}}_m$, by Lemma 3.4(a). Likewise $(\dagger)$ is by definition of $\delta'_{nm}(\boldsymbol{\lambda})$.

This holds for all $n, m \in [1...N]$ and $\boldsymbol{\lambda} \in \mathrm{S}^{\mathrm{pec}}_{\text{æ}}(\mathfrak{A}, \Phi, \sigma)$. □

Example 3.15: The essential dislocation in Example 3.13(a) is ${}_{\varepsilon}\Phi_{18}$-persistent. Figure 1(a) shows the long-term evolution of such defects. ♢

Remark 3.16: For one-dimensional SFTs, topological weak mixing implies mixing. However, for other one-dimensional symbolic dynamical systems (e.g. rank one systems) this is not true. If $\mathfrak{A}$ is topologically weakly mixing, but not mixing, then $\mathfrak{A}$ admits no dislocations (by Lemma 3.2[ii]), but must still admit other essential defects [by Example 1.5(a)], which are undetectable by the spectral invariants developed here. Are there other spectral invariants which detect these defects? ♢

Conclusion

We have used spectral theory to explain the persistence and interaction of domain boundaries in cellular automata. However, many questions remain.

1. Domain boundaries also emerge in coupled-map lattices [29, 28, 30]; [27, §8.2.4]. Can analogous spectral invariants be developed in this context?

2. In most of our examples (e.g. ECAs #54, #62, #110, and #184), the defects remain bounded in size, and act like 'particles' [48]. In general, however, defects may grow over time like 'blights' which invade the whole lattice. What are necessary/sufficient conditions for the defect to remain bounded?

3. Aside from the aforementioned ECAs, there are relatively few known examples of 'naturally occuring' defect dynamics in CA, and none in $\mathbb{Z}^D$ with $D \geq 2$. It is easy to contrive artificial examples, but this generally does not yield any surprises. Are there nontrivial examples of defect dynamics in multidimensional cellular automata? Can we find them without blindly searching the (vast) space of possible rules?

4. If $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$, and there is a CA Φ and $n \in \mathbb{N}$ with $\Phi^n(\mathcal{A}^{\mathbb{Z}^D}) \subseteq \mathfrak{A} \subseteq \mathsf{Fix}[\Phi]$, then $\mathfrak{A}$ admits no essential defects. The converse is also true, when $\mathfrak{A}$ is a one-dimensional sofic shift with a σ-fixed point [42]. Is the converse true in higher dimensions?

5. Even when $\mathfrak{A}$ admits essential defects, Kůrka and Maass [37, 38, 34, 36] have described how a one-dimensional CA can 'converge in measure' to $\mathfrak{A}$ through a gradual process of defect coalescence/annihilation. Given a subshift $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$, is it possible to build a CA which converges to $\mathfrak{A}$ in this sense?

6. The defect dynamics in ECAs #18, #54, #62, #110, and #184 were easy to discover by accident, because each CA contains a 'condensing' subshift $\mathfrak{A}$, such that generic initial conditions rapidly 'condense' into sequences containing relatively few defects separated by long, $\mathfrak{A}$-admissible intervals. What are necessary/sufficient conditions for the existence of such a condensing subshift? (This rapid primordial condensation is not the same as the long-term convergence in question #5, but the two may be related.) This question is closely related to question #2, because condensation should prevent defects from growing. Also, it relates to question #3, because a characterization of CA with condensing subshifts might yield nontrivial examples of defect dynamics.

Finally, we remark that the spectral invariants in this paper are only applicable to defects of codimension one (i.e. domain boundaries). In a companion paper [47], we develop algebraic invariants for defects of codimensions two or more.

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