Crystallographic Defects in Cellular Automata

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Cellular Automata

CA are the 'discrete analog' of partial differential equations. They are *spatially distributed* dynamical systems whose dynamics are driven by *local interactions* governed by *translationally equivariant* rules.

- **Space** is a lattice $\mathbb{Z}^D$ (for $D \geq 1$).
- The **local state** at each point in the lattice is an element of a finite alphabet, e.g. $\mathcal{A} := \{0, 1\}$.
- The **global state** is a $\mathbb{Z}^D$-indexed *configuration* $\mathbf{a} : \mathbb{Z}^D \longrightarrow \mathcal{A}$.
  The space of such configurations is denoted $\mathcal{A}^{\mathbb{Z}^D}$.
  A generic element of $\mathcal{A}^{\mathbb{Z}^D}$ will be denoted by $\mathbf{a} := \left[a_{\mathsf{z}}|_{\mathsf{z}\in\mathbb{Z}^D}\right]$.
- The evolution is governed by a map $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$, computed by applying a '**local rule**' ϕ at every point in space.

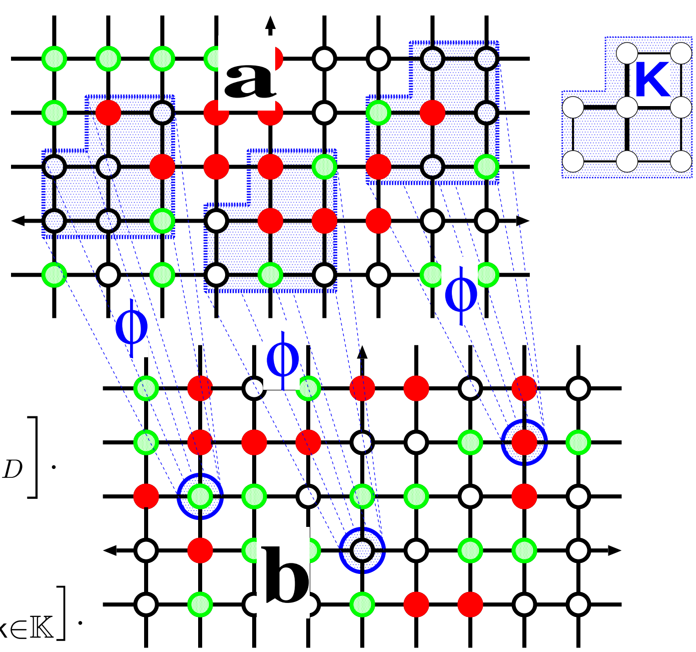
Neighbourhood:

$\mathbb{K} \subset \mathbb{Z}^D$ (finite set)

Local rule: ϕ: $\mathcal{A}^{\mathbb{K}} \longrightarrow \mathcal{A}$

Let $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$, $\quad \mathbf{a} := \left[a_{\mathsf{z}}|_{\mathsf{z}\in\mathbb{Z}^D}\right]$.

$\forall \mathsf{z} \in \mathbb{Z}^D$, let $b_{\mathsf{z}} := \phi\left[a_{(\mathsf{k}+\mathsf{z})}|_{\mathsf{k}\in\mathbb{K}}\right]$.



This defines new configuration $\mathbf{b} := \left[b_{\mathsf{z}}|_{\mathsf{z}\in\mathbb{Z}^D}\right]$.

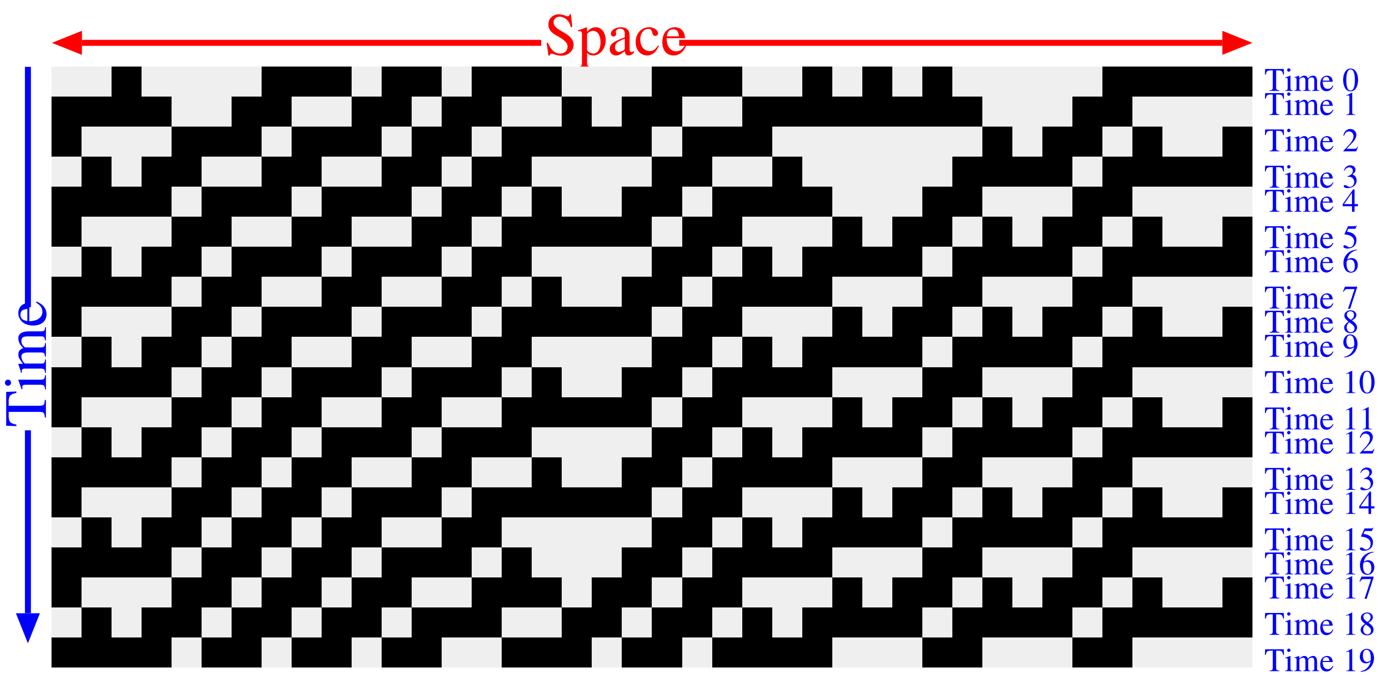
The CA **induced by** ϕ is function Φ: $\mathcal{A}^{\mathbb{Z}^D} \circlearrowleft$ defined: $\Phi(\mathbf{a}) := \mathbf{b}$.

Example: Elementary Cellular Automaton #62

Let $D := 1$, $\mathbb{K} := \{-1,0,1\}$, and $\mathcal{A} := \{0,1\}$.

Define $\phi_{62} : \{0,1\}^{\{-1,0,1\}} \longrightarrow \{0,1\}$ by:

$$\begin{array}{rclrcl} \phi_{62}(0,0,1) &=& 1; & \phi_{62}(0,0,0) &=& 0; \\ \phi_{62}(0,1,0) &=& 1; & \phi_{62}(1,1,0) &=& 0; \\ \phi_{62}(0,1,1) &=& 1; & \phi_{62}(1,1,1) &=& 0; \\ \phi_{62}(1,0,0) &=& 1; & & & \\ \phi_{62}(1,0,1) &=& 1. & & & \end{array}$$



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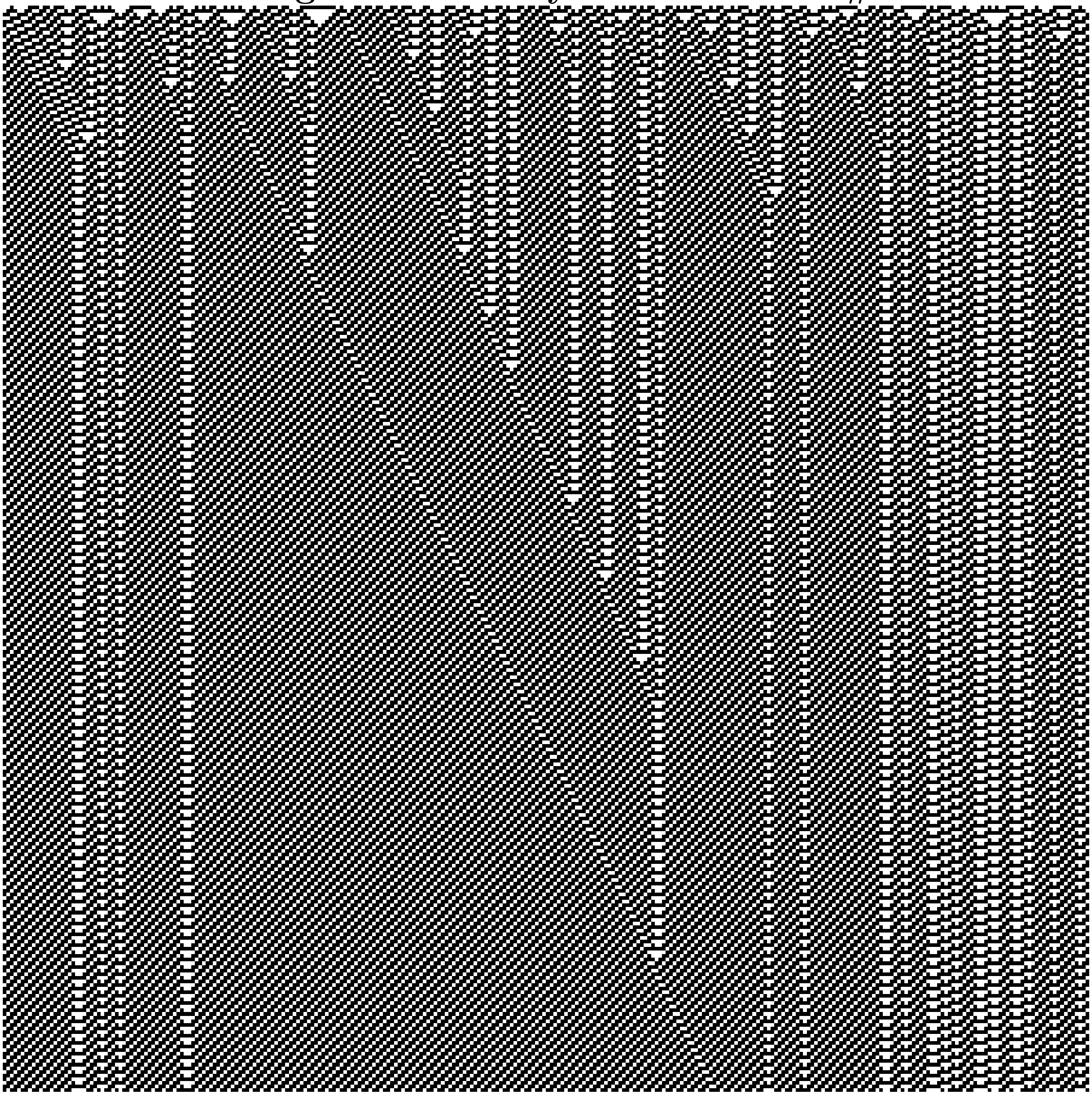
Such a nearest-neighbour CA on $\{0,1\}^{\mathbb{Z}}$ is called an **Elementary Cellular Automaton**. Each ECA is described by an 8-bit binary number (i.e. a number between 0 and 255) as follows:

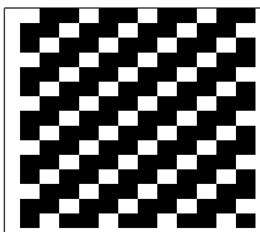
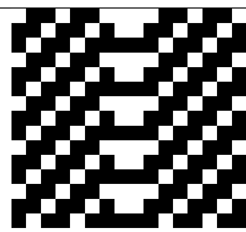
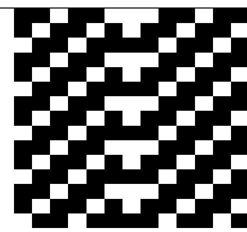
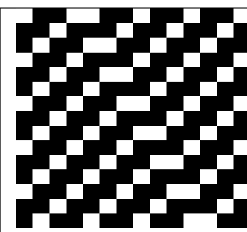
If $N = n_0 + 2n_1 + 2^2 n_2 + 2^3 n_3 + 2^4 n_4 + 2^5 n_5 + 2^6 n_6 + 2^7 n_7 \in [0...255]$

then $\phi_N(a_0, a_1, a_2) := n_k$, where $k := a_0 + 2a_1 + 4a_2 \in [0...7]$.

For example, the CA here is ECA#62, because $2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 62$.

Emergent Defect Dynamics in ECA#62



			
$(*)$	$(\boldsymbol{\alpha})$	$(\boldsymbol{\beta})$	$(\boldsymbol{\gamma})$

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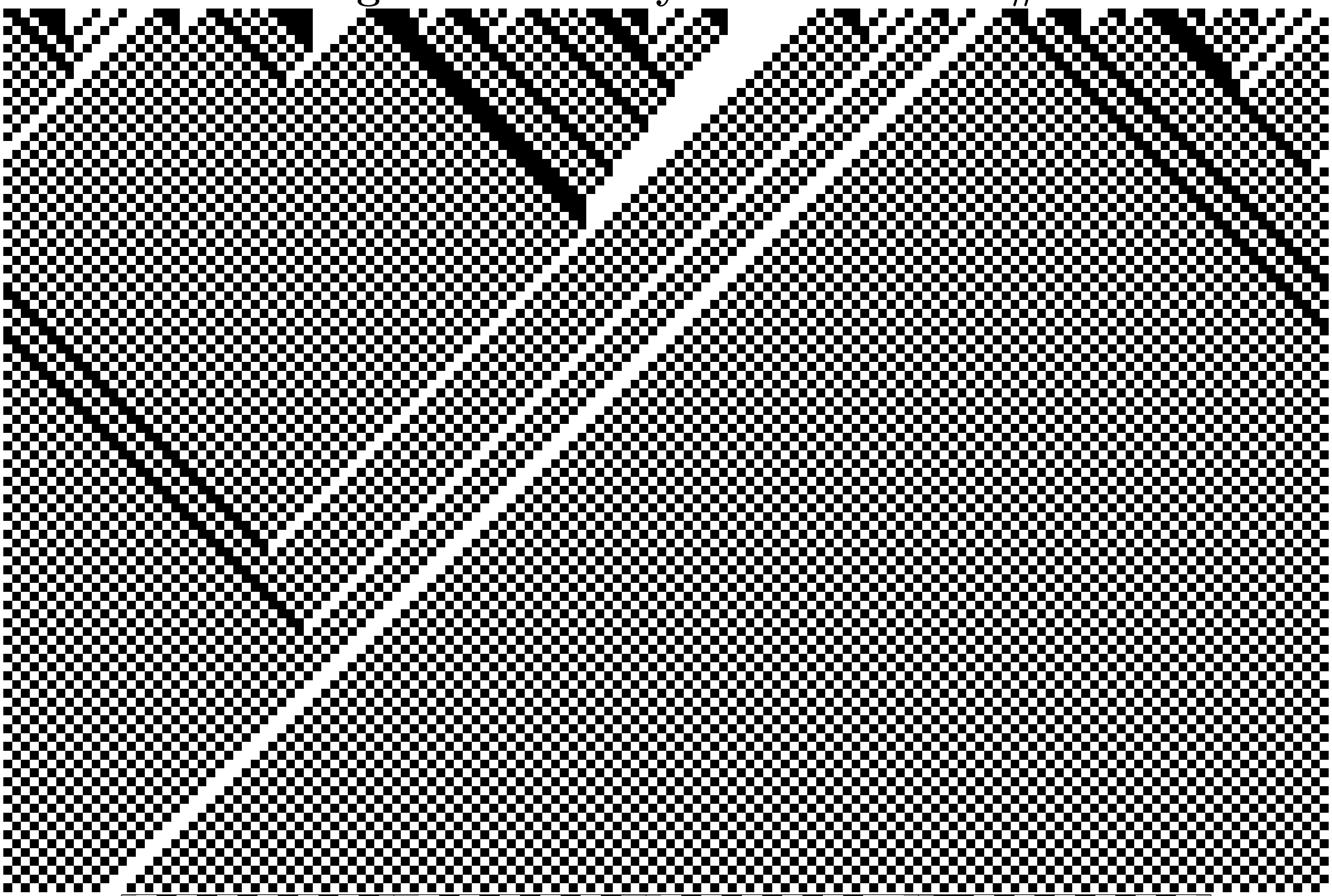
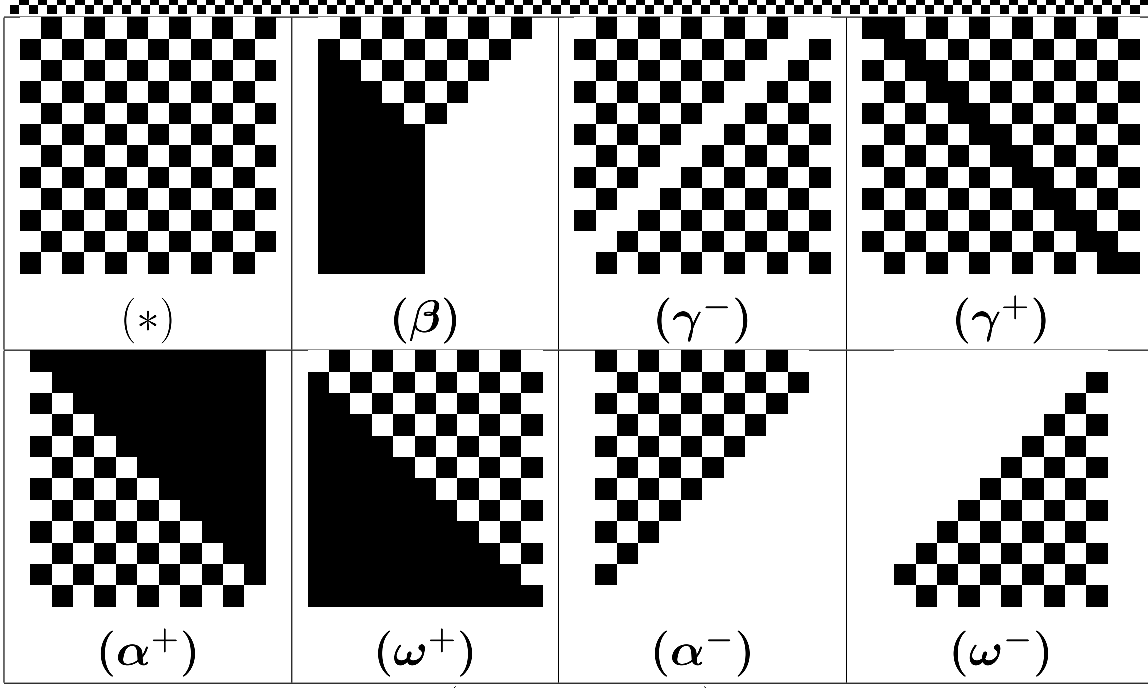
Emergent Defect Dynamics in ECA#184





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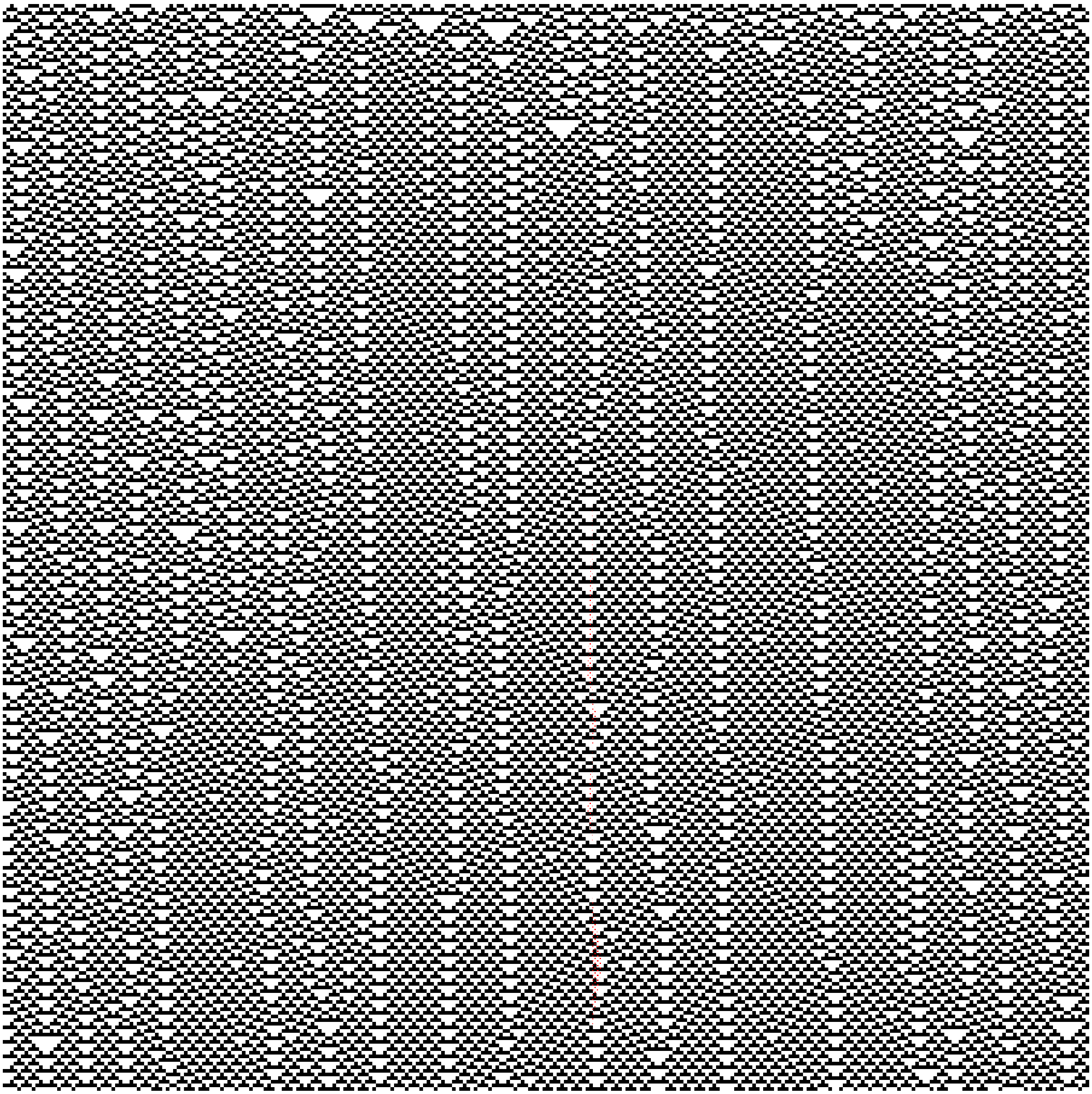
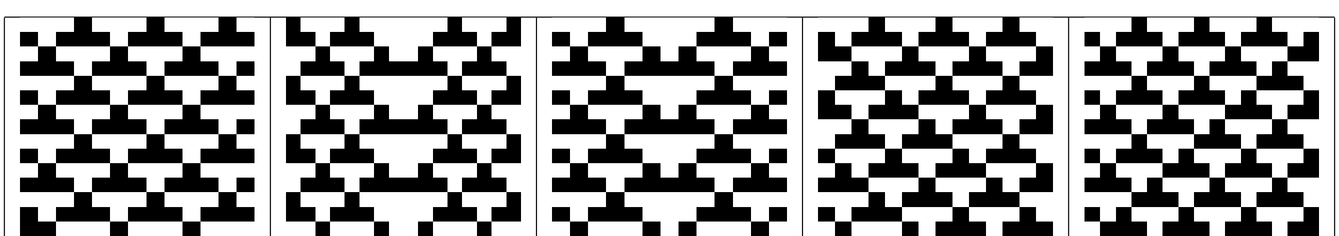
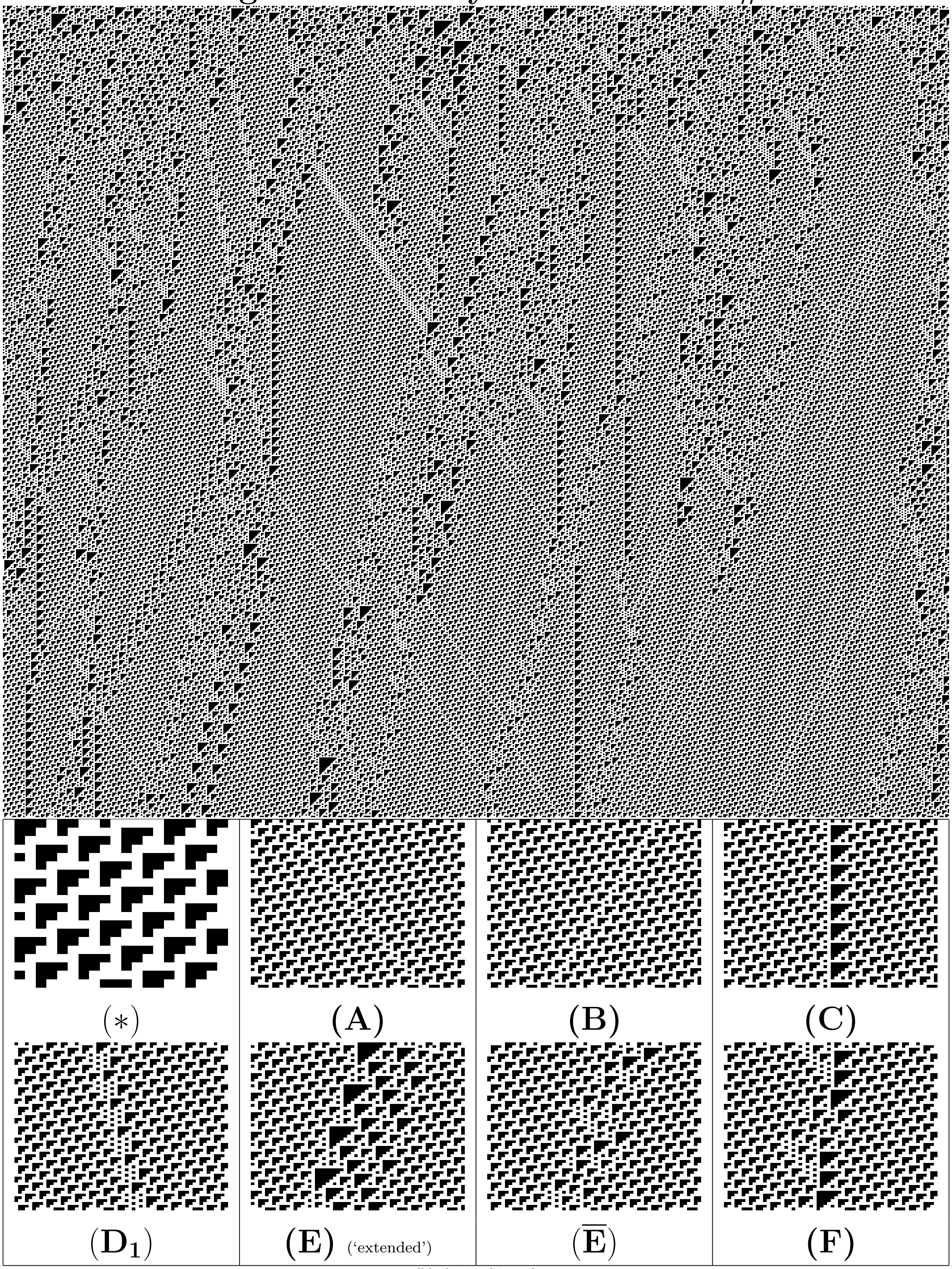




$(*)$	$(\boldsymbol{\alpha})$	$(\boldsymbol{\beta})$	$(\boldsymbol{\gamma}^{+})$	$(\boldsymbol{\gamma}^{-})$

Emergent Defect Dynamics in ECA#110



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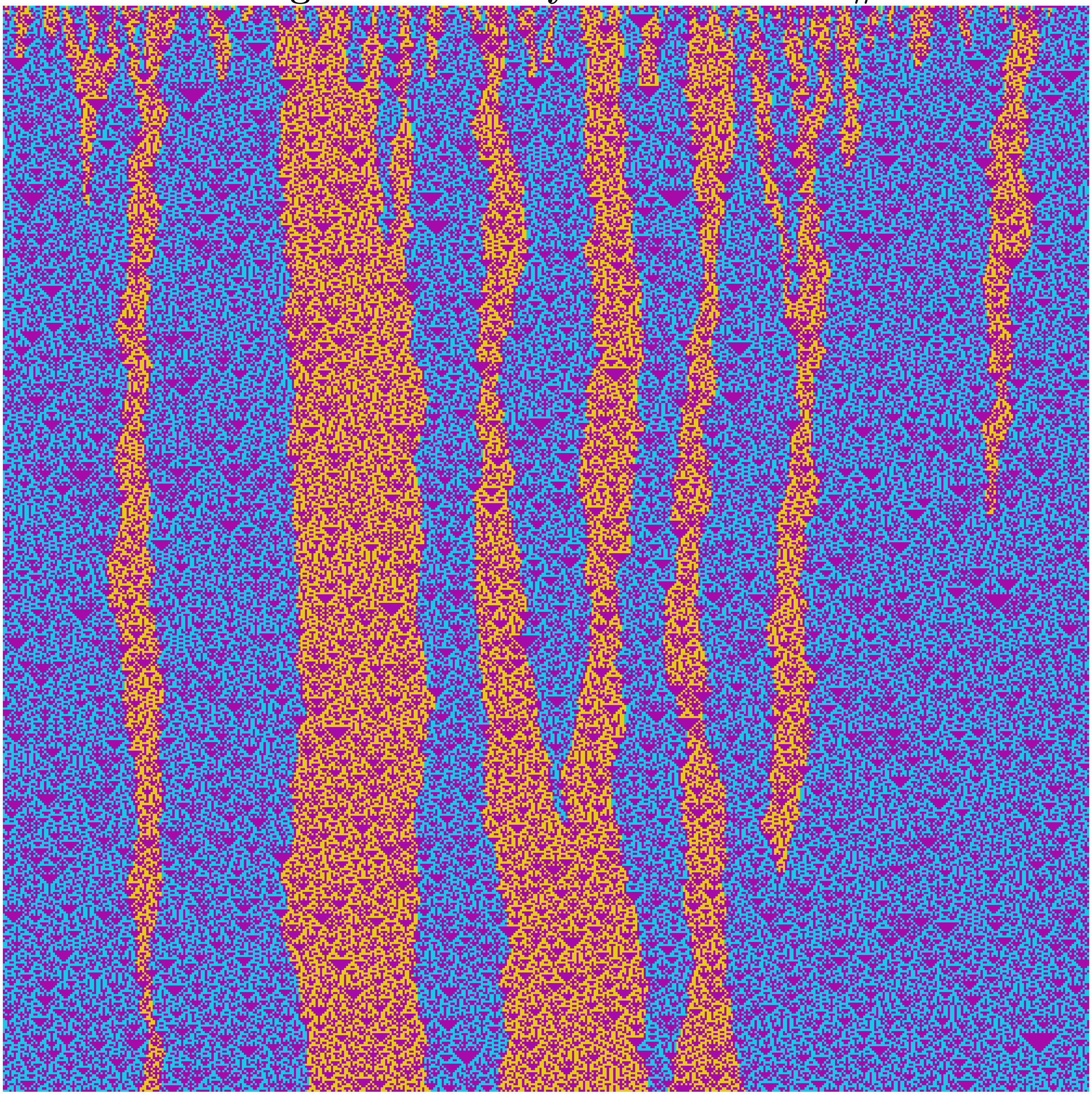
Emergent Defect Dynamics in ECA#18

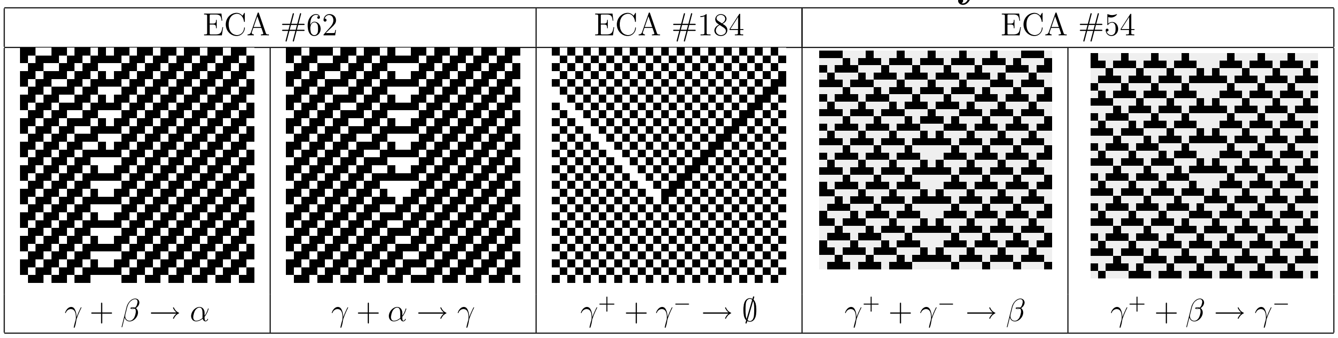


Invariant sofic subshift: $\textcircled{1} \leftrightarrows \textcircled{0} \leftrightarrows \textcircled{0}$ (the *Odd Shift*).

Defects are 'phase slips':

$$[\ldots \underbrace{00\ 01\ 00\ 01\ 01}_{\text{orange}}\ \ \underbrace{00\ 00\ 00\ 00\ 00\ 00\ 00\ 00\ 00}_{\text{even \# of zeroes}}\ \ \underbrace{10\ 00\ 10\ 00\ 00\ 10}_{\text{blue}} \ldots].$$

Defect Particle 'Chemistry'



Empirical Work: • P. Grassberger [1983, 1984].

• Steven Wolfram [1983-2005]. (Mainly ECA #110).

• S. Wolfram and Doug Lind [1986]. (Classified defects of ECA #110).

• N. Boccara, J. Naser, M. Rogers [1991]. (ECAs 18, 54, 62, 184).

• James Crutchfield and James Hanson's 'Computational Mechanics' [1992-2001]. (Also Cosma Shalizi, Wim Hordijk, Melanie Mitchell).

• Harold V. McIntosh [1999, 2000].

Theoretical Work: • Doug Lind [1984] conjectured:

(i) *Defects in* ECA#18 *perform random walks.*

(ii) *Defect density decays to zero through annihilations. Thus,* ECA#18 *converges 'in measure' to the 'odd' sofic shift* $1 \leftrightarrows 0 \leftrightarrows 0$.

• Kari Eloranta [1993-1995] proved Lind's conjecture **(i)**; studied quasirandom defect motion in 'partially permutive' CA.

• Petr Kůrka and Alejandro Maass [2000, 2002] studied CA convergence to limit sets through 'defect annihilation'. Kůrka [2003] proved Lind's conjecture **(ii)**.

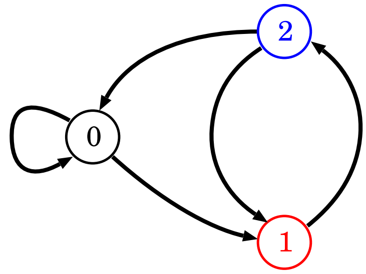
• S. Wolfram and Matthew Cook [2002, 2004]: ECA #110 *is computationally universal* (used 'defect physics' to engineer universal computer).

Questions:

- *What is a 'defect'? What is a 'regular background pattern'?*
- *Is there an 'algebraic structure' governing defect 'chemistry'?*
- *Why do defects 'persist' over time instead of disappearing? Is this related to aforementioned 'algebraic structure'?*
- *What is the 'kinematics' by which defects propagate through space?*

A **subshift** is a subset $\mathfrak{A} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ of configurations, defined by stipulating which 'local patterns' may or may not occur around each point in $\mathbb{Z}^D$.

Topological Markov Shifts: Let $D = 1$. Let $\mathcal{A} :=$ the vertices of a directed graph. A sequence $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}}$ is admissible iff it describes an infinite directed path through the graph.



A = {0,1,2}

a = [...0,1,2,1,2,0,0,0,0,1,2,0,0,1,2,1,2,1,2,0,0,...]

Sofic Shift: Let $D = 1$. Like a topological Markov shift, but now several vertices might be labelled with the same letter in $\mathcal{A}$.

Example: $1 \leftrightarrows 0 \leftrightarrows 0$ (the *Odd Shift* from ECA#18).

$[\ldots 00\ 01\ 00\ 01\ 01\ 00\ 00\ 00\ 00\ 01\ 00\ 00\ 00\ 00\ 01\ 0100\ 01\ 00\ 00\ 01 \ldots]$.

Let $\mathfrak{A}_{(r)}$:= set of $\mathfrak{A}$-admissible 'local patterns' seen in $\mathbb{B}(r) := [-r...r]^D$

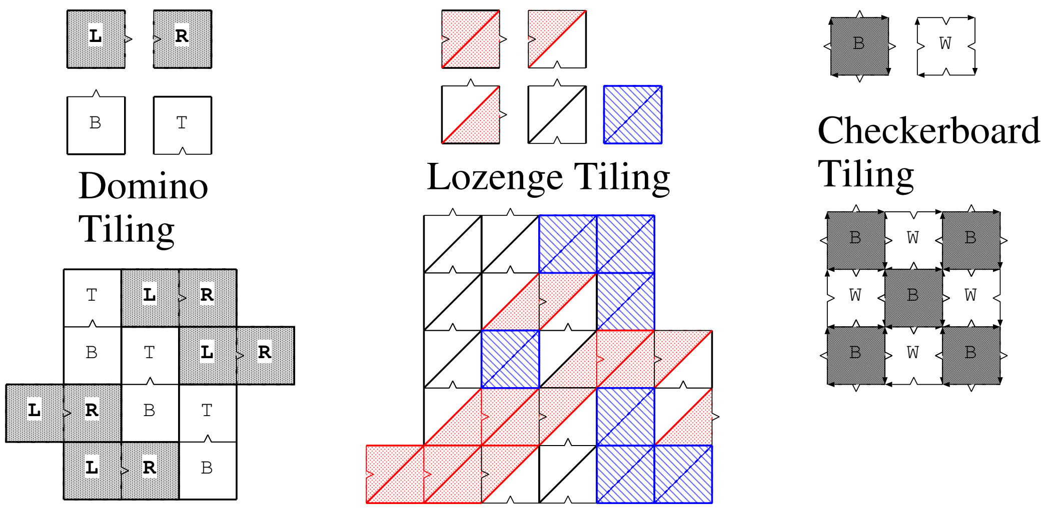
A configuration $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ is **defective** if there are points in $\mathbb{Z}^D$ where the local pattern in $\mathbf{a}$ is *inadmissible* —i.e. *not* in $\mathfrak{A}_{(r)}$. These points are called **defects**. Let $\mathbb{D}(\mathbf{a}) \subset \mathbb{Z}^D$ be the set of these 'defect points' in $\mathbf{a}$.

Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA. We say $\mathfrak{A}$ is **Φ-invariant** if $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$. Empirically, if $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ has defects, then so does $\Phi(\mathbf{a})$.

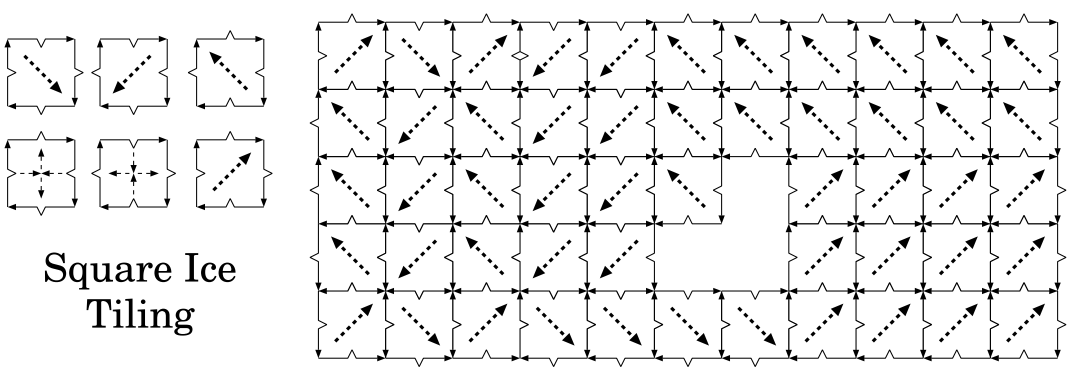
Let $\widetilde{\mathfrak{A}}$:= {configurations with 'finite' defects}. Then $\Phi(\widetilde{\mathfrak{A}}) \subseteq \widetilde{\mathfrak{A}}$.

Wang tilings

Let $D = 2$. Let $\mathcal{A}$:= set of square tiles, with notches on their edges which dictate how the tiles can be assembled. These **edge-matching constraints** determine a subshift $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^2}$, called a **Wang tiling**.



A defect corresponds to a 'hole' in the tiling:



Remark: Wang tilings and topological Markov shifts are **subshifts of finite type** (**SFT**s), meaning they are determined entirely by 'local constraints'. Sofic shifts are a broader class, which may have 'nonlocal' constraints. (Defect theory more complicated, but still possible.)

Generalization to $\mathbb{Z}^D$: Idea: $\mathcal{A}$ = set of 'atoms', with certain admissible 'chemical bonds' between them. Thus, an admissible configuration corresponds to a 'crystalline solid'. Defects are 'flaws' in crystal structure.

Questions:

- *Is there an 'algebraic structure' governing defect 'chemistry'?*
- *Why do defects 'persist' over time instead of disappearing? Is this related to aforementioned 'algebraic structure'?*
- *What is the 'kinematics' by which defects propagate through space?*

Formalism: Fix $D \in \mathbb{N}$. For any $r > 0$, let $\mathbb{B}(r) := [-r...r]^D \subset \mathbb{Z}^D$. Fix $r > 0$. Let $\mathfrak{A}_{(r)} \subset \mathcal{A}^{\mathbb{B}(r)}$ be a set of of **admissible r-blocks**.

The **subshift of finite type (SFT)** determined by $\mathfrak{A}_{(r)}$ is the set

$$\mathfrak{A} \quad := \quad \left\{ \mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D} \; ; \; \mathbf{a}_{\mathsf{z}+\mathbb{B}(r)} \in \mathfrak{A}_{(r)}, \; \forall \; \mathsf{z} \in \mathbb{Z}^D \right\}$$

For any $R > 0$, let $\mathfrak{A}_{(R)}$ be the projection of $\mathfrak{A}$ to $\mathcal{A}^{\mathbb{B}(R)}$.

If $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ and $\mathsf{z} \in \mathbb{Z}^D$, then $\mathbf{a}$ is **defective** at z if $\mathbf{a}_{\mathsf{z}+\mathbb{B}(r)} \not\in \mathfrak{A}_{(r)}$. The **defect set** of $\mathbf{a}$ is the set $\mathbb{D}(\mathbf{a})$ of all such z.

Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA. We say $\mathfrak{A}$ is **Φ-invariant** if $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$.

Empirically, if $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}$ has defects, then so does $\Phi(\mathbf{a})$.

We say $\mathbf{a}$ is **finitely defective** if, $\forall R > 0$, $\exists \, \mathsf{z} \in \mathbb{Z}^D$ with $\mathbf{a}_{\mathbb{B}(\mathsf{z},R)} \in \mathfrak{A}_{(R)}$.

Idea: $\mathbf{a}$ may have infinitely large defects, but $\mathbf{a}$ also has infinitely large 'nondefective' regions. Let $\widetilde{\mathfrak{A}} := \{$finitely defective $\mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D}\}$. $(\mathfrak{A} \subset \widetilde{\mathfrak{A}})$

Lemma: *If* $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$, *then* $\Phi(\widetilde{\mathfrak{A}}) \subseteq \widetilde{\mathfrak{A}}$.

Also, if $\mathbf{a} \in \widetilde{\mathfrak{A}}$ *and* $\mathbf{a}' = \Phi(\mathbf{a})$, *then the any defects in* $\mathbf{a}'$ *are 'close' to corresponding defects in* $\mathbf{a}$. □

The Fine Print: To extend the definition of 'defect' to other subshifts (not of finite type), it is necessary to introduce a 'detection range' $R > 0$. We must then talk about 'defects of range R'.

Domain Boundaries

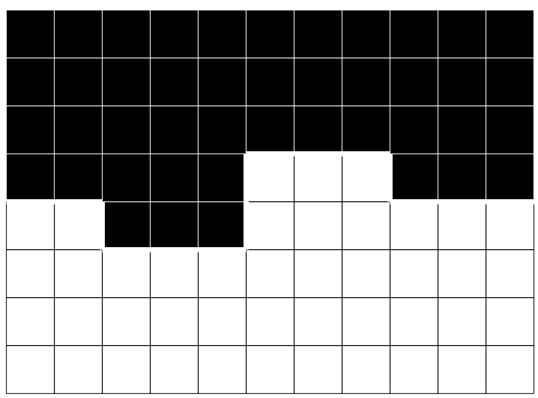
Let $\mathbb{G}(\mathbf{a}) := \{\mathsf{z} \in \mathbb{Z}^D\ ;\ \mathbf{a}$ is not defective at $\mathsf{z}\}$. Let $\mathbf{G}(\mathbf{a}) \subset \mathbb{R}^D$ be the union of all unit cubes whose corner vertices are all in $\mathbb{G}(\mathbf{a})$.

The defect in $\mathbf{a}$ is a **domain boundary*** if $\mathbf{G}(\mathbf{a})$ is disconnected.

Examples: (a) If $D = 1$, then all defects are domain boundaries.

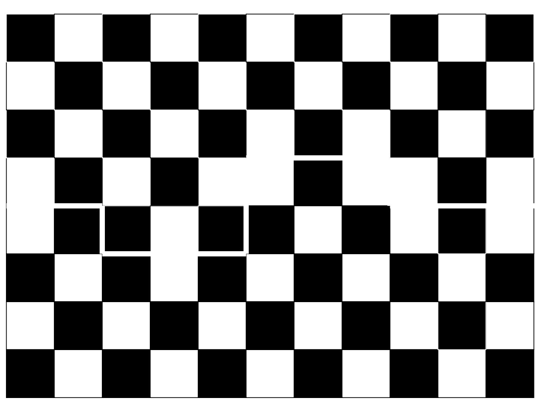
(b) (*Monochromatic*) Let $\mathcal{A} := \{\blacksquare, \square\}$. Let $\mathfrak{Mo} \subset \mathcal{A}^{\mathbb{Z}^2}$ be SFT such that no $\blacksquare$ can be adjacent to a $\square$.

The following configuration has a domain boundary defect:

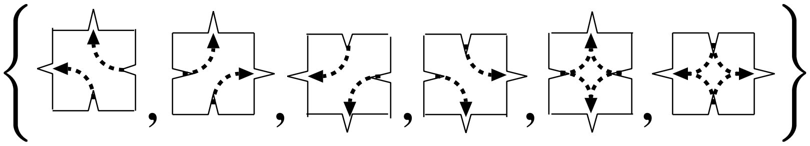


(c) (*Checkerboard*) Let $\mathcal{A} := \{\blacksquare, \square\}$. Let $\mathfrak{Ch} \subset \mathcal{A}^{\mathbb{Z}^2}$ be SFT where no $\blacksquare$ can be adjacent to a $\blacksquare$, and no $\square$ can be adjacent to a $\square$.

The following configuration has a domain boundary defect:

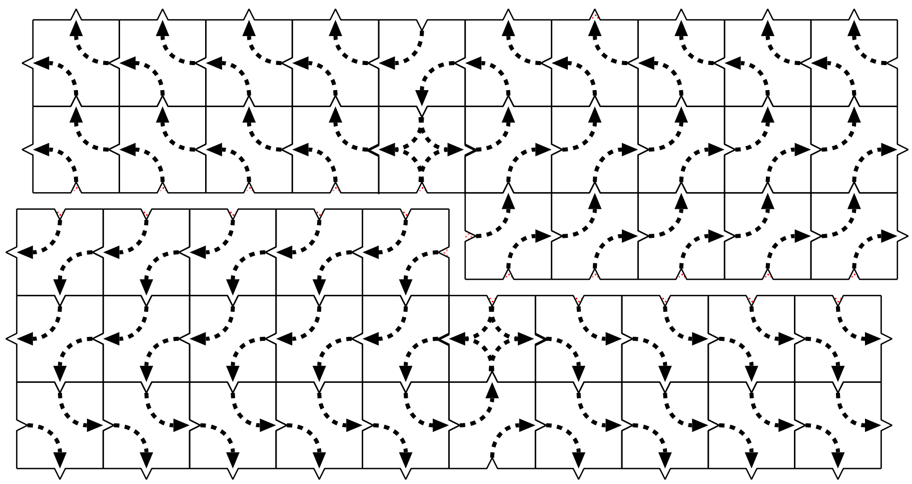


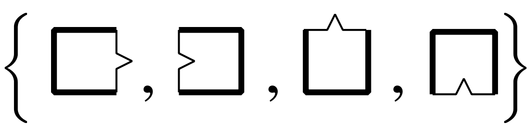
(∗) If we considering a defect of range $R > 0$, then technically this is a *domain boundary of range* R.

(d) (*Square ice*) Let $\mathcal{I} :=$ .

Let $\mathfrak{Ice} \subset \mathcal{I}^{\mathbb{Z}^2}$ be the SFT defined by obvious edge-matching conditions.

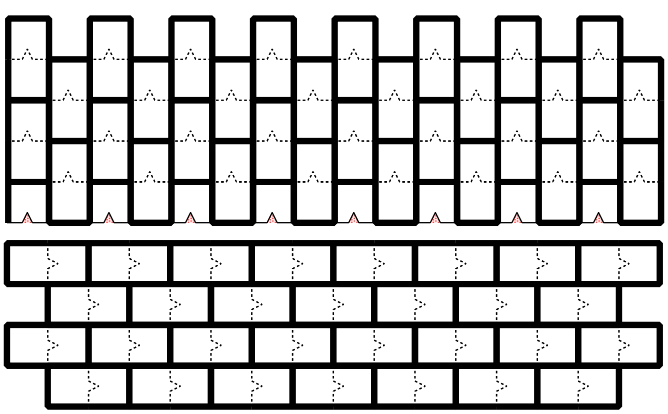
The following configuration has a domain boundary defect:

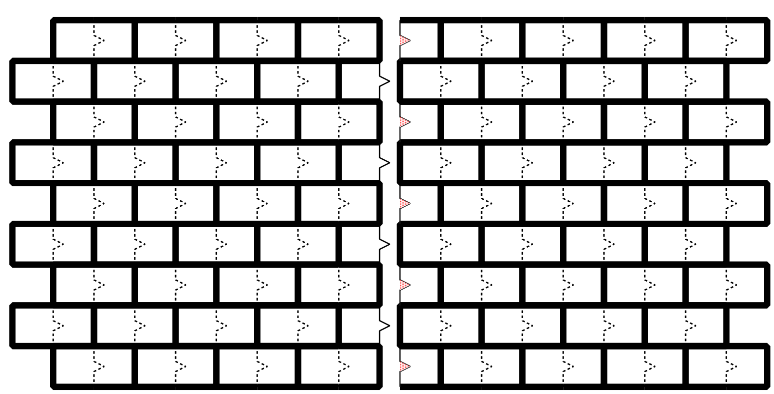


(e) (*Domino Tiling*) Let $\mathcal{D} :=$ .

Let $\mathfrak{Dom} \subset \mathcal{D}^{\mathbb{Z}^2}$ be the SFT defined by obvious edge-matching conditions.

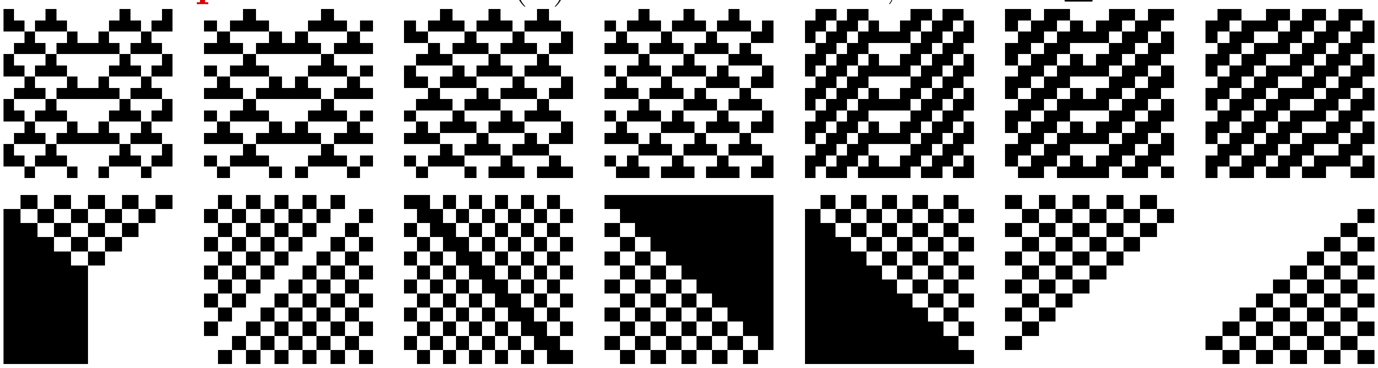
The following configurations have domain boundary defects:





Persistent Defects

Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA, with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$. Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$. The defect in $\mathbf{a}$ is **Φ-persistent** if $\Phi^t(\mathbf{a})$ also has a defect, for all $t \geq 0$.

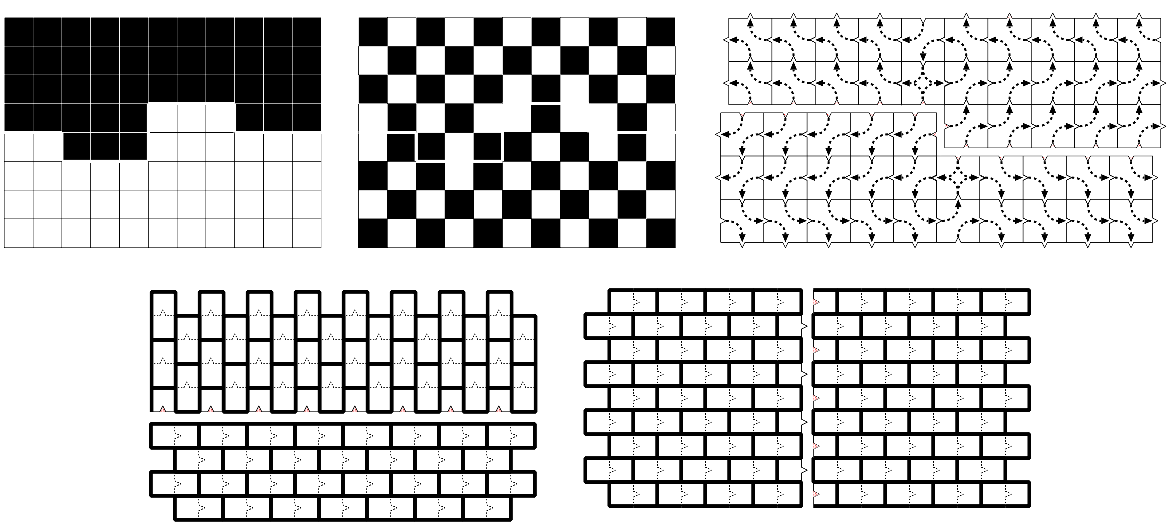


Question: These defects seem to be persistent. Are they? Why?

Essential Defects

A defect is **essential** if it can't be removed through a local change. That is, $\forall\ R > 0$, if $\mathbf{a}' \in \mathcal{A}^{\mathbb{Z}^D}$ is obtained by modifying $\mathbf{a}$ in an R-neighbourhood of defect, then $\mathbf{a}'$ is also defective.

Proposition: *If* $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}$ *is bijective* (e.g. if $\mathfrak{A} \subseteq \mathsf{Fix}[\Phi]$ or $\mathfrak{A} \subseteq \mathsf{Fix}[\Phi^p]$ or $\mathfrak{A} \subseteq \mathsf{Fix}[\Phi^p \circ \sigma^q]$), *then any essential defect is Φ-persistent.* □



Question: These defects to be seem essential. Are they? Why?

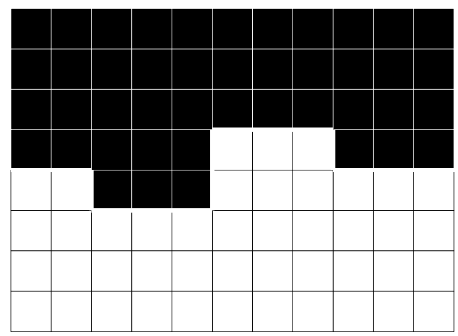
Interfaces (intuitive version)

Suppose $\mathfrak{A}_{(r)}$ breaks into two (or more) disjoint subsets $\mathfrak{A}_{(r)} = \mathfrak{B}_{(r)} \sqcup \mathfrak{C}_{(r)}$ (called (F,σ)-*transitive components*), such that, for each $\mathbf{a} \in \mathfrak{A}$,

either **a** is totally covered by $\mathfrak{B}_{(r)}$-blocks,
or **a** is totally covered by $\mathfrak{C}_{(r)}$-blocks,
but **a** *cannot* have a mixture of $\mathfrak{B}_{(r)}$-blocks and $\mathfrak{C}_{(r)}$-blocks.

An **interface** is a domain boundary between a $\mathfrak{B}_{(r)}$-covered region and a $\mathfrak{C}_{(r)}$-covered region. Such a boundary is necessarily an essential defect.

Example: Let $\mathfrak{M}$ be the *monochromatic* shift. Then $\mathfrak{M}_{(1)} := \mathfrak{B}_{(1)} \sqcup \mathfrak{W}_{(1)}$, where $\mathfrak{B}_{(1)} := \left\{ \begin{smallmatrix} \blacksquare\blacksquare\blacksquare \\ \blacksquare\blacksquare\blacksquare \\ \blacksquare\blacksquare\blacksquare \end{smallmatrix} \right\}$ and $\mathfrak{W}_{(1)} := \left\{ \begin{smallmatrix} \square\square\square \\ \square\square\square \\ \square\square\square \end{smallmatrix} \right\}$. The defect at right is an interface.



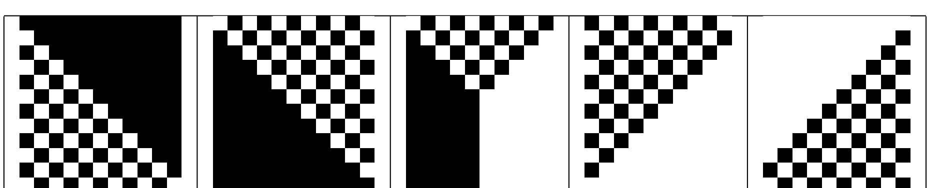
Example: (ECA #184) Let $\mathcal{A} = \{\square, \blacksquare\}$. Let $\mathfrak{G}_{(1)} := \mathfrak{B}_{(1)} \sqcup \mathfrak{W}_{(1)} \sqcup \mathfrak{C}_{(1)}$, where $\mathfrak{B}_{(1)} := \{\blacksquare\blacksquare\blacksquare\}$, $\mathfrak{W}_{(1)} := \{\square\square\square\}$, and $\mathfrak{C}_{(1)} := \{\blacksquare\square\blacksquare,\ \square\blacksquare\square\}$. This yields 6 possible interfaces:

$\alpha^+ :\ \mathfrak{C}_{(1)}\ \boxed{\ldots\blacksquare\square\blacksquare\square\blacksquare\square}\ \boxed{\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\ldots}\ \mathfrak{B}_{(1)}$ $\qquad \alpha^- :\ \mathfrak{C}_{(1)}\ \boxed{\ldots\blacksquare\square\blacksquare\square\blacksquare\square}\ \boxed{\square\square\square\square\square\square\ldots}\ \mathfrak{W}_{(1)}$

$\omega^+ :\ \mathfrak{B}_{(1)}\ \boxed{\ldots\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare}\ \boxed{\blacksquare\square\blacksquare\square\blacksquare\square\ldots}\ \mathfrak{C}_{(1)}$ $\qquad \omega^- :\ \mathfrak{W}_{(1)}\ \boxed{\ldots\square\square\square\square\square\square}\ \boxed{\blacksquare\square\blacksquare\square\blacksquare\square\ldots}\ \mathfrak{C}_{(1)}$

$\beta :\ \mathfrak{B}_{(1)}\ \boxed{\ldots\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare}\ \boxed{\square\square\square\square\square\square\ldots}\ \mathfrak{W}_{(1)}$ $\qquad \epsilon :\ \mathfrak{B}_{(1)}\ \boxed{\ldots\square\square\square\square\square\square}\ \boxed{\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\ldots}\ \mathfrak{W}_{(1)}$

$\Phi_{184}(\mathfrak{G}) \subseteq \mathfrak{G}$, and the Φ_{184}-propagation of these interfaces is as follows:



$(\boldsymbol{\alpha}^+)$ $(\boldsymbol{\omega}^+)$ $(\boldsymbol{\beta})$ $(\boldsymbol{\alpha}^-)$ $(\boldsymbol{\omega}^-)$

Theorem: *If* $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}$ *is surjective, then all interfaces are* Φ-*persistent defects.* $\square$

Interfaces (formal version)

$\mathfrak{A}$ is **(Φ, σ)-transitive** if $\bigcup_{t\in\mathbb{N}} \bigcup_{\mathsf{z}\in\mathbb{Z}^D} \Phi^{-t}\sigma^{-\mathsf{z}}(\mathfrak{O})$ is dense in $\mathfrak{A}$, for any nonempty open $\mathfrak{O} \subset \mathfrak{A}$. (Equivalent: most (Φ, σ)-orbits are dense in $\mathfrak{A}$).

Suppose $\mathfrak{A}$ is not transitive, but $\mathfrak{A} = \mathfrak{A}_1 \sqcup \cdots \sqcup \mathfrak{A}_K$, where $\mathfrak{A}_1, \ldots, \mathfrak{A}_K$ are clopen (Φ, σ)-transitive components.

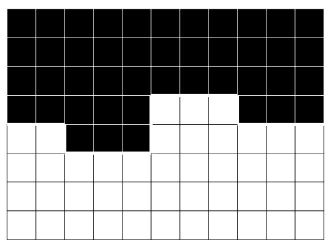
$\left(\mathfrak{A}_1, \ldots, \mathfrak{A}_K \text{ are clopen}\right) \Rightarrow \left(\text{indicator functions are } \textbf{locally determined}\right)$
i.e. $\exists r > 0$, and function $\kappa : \mathfrak{A}_{(r)} \longrightarrow [1...K]$ such that, $\forall\ \mathbf{a} \in \mathfrak{A}$,

$$\left(\mathbf{a} \in \mathfrak{A}_k\right) \quad \Longleftrightarrow \quad \left(\kappa(\mathbf{a}_{\mathbb{B}(r)}) = k\right).$$

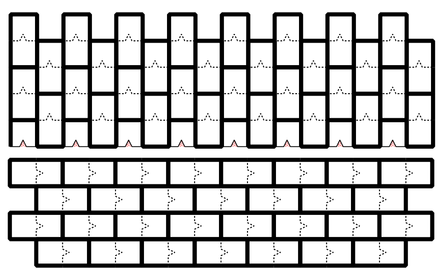
$\forall\ \mathsf{z} \in \mathbb{Z}^D$, let $\kappa_{\mathsf{z}}(\mathbf{a}) := \kappa(\mathbf{a}_{\mathbb{B}(\mathsf{z},r)})$. Then $\kappa_{\mathsf{z}}(\mathbf{a})$ is also well-defined for any $\mathbf{a} \in \widetilde{\mathfrak{A}}$ such that $\mathbf{a}_{\mathbb{B}(\mathsf{z},r)}$ is $\mathfrak{A}$-admissible.

If $\mathsf{y}, \mathsf{z} \in \mathbb{Z}^D$, then $\mathbf{a}$ has an **interface**† between y and z if $\kappa_{\mathsf{y}}(\mathbf{a}) \neq \kappa_{\mathsf{z}}(\mathbf{a})$.

Example: $\mathfrak{Mo}$ has two σ-transitive components: $\mathfrak{M}_0 :=$ all-black, and $\mathfrak{M}_1 :=$ all-white. This defect is an interface.



Nonexample: This is *not* an interface, because $\mathfrak{Dom}$ is σ-transitive [Einsiedler, 2001]. Instead this is a 'gap' defect.



Interfaces always form domain boundaries. Let $\mathbb{Y}_1, \ldots, \mathbb{Y}_N$ be the connected components of $\mathbb{G}(\mathbf{a})$. There is a function $\mathcal{K}$: $[1...N] \longrightarrow [1...K]$ such that for any $n \in [1...N]$ and any $\mathsf{y} \in \mathbb{Y}_n$, $\kappa_{\mathsf{y}}(\mathbf{a}) = \mathcal{K}(n)$.

(†) Technically, this is an interface *of range* r, and this concept only makes sense for domain boundaries of range $R \geq r$.

Persistence of Interfaces

A connected component $\mathbb{Y}_n$ of $\mathbb{G}$ is **projective** if, for all $R > 0$, $\exists\, \mathsf{y} \in \mathbb{Y}_n$ with $\mathbf{a}_{\mathbb{B}(\mathsf{y},R)} \in \mathfrak{A}_{(R)}$. (i.e. $\mathbb{Y}_n$ contains arbitrarily large $\mathfrak{A}$-admissible patches.)

Lemma: *The interface in* $\mathbf{a}$ *is essential if there are two projective components* $\mathbb{Y}_n$ *and* $\mathbb{Y}_m$ *with* $\mathcal{K}(n) \neq \mathcal{K}(m)$. $\square$

Signature of the interface := restriction of $\mathcal{K}$ to projective components.

Example: Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$. Suppose $\mathbf{a} \in \widetilde{\mathfrak{A}}$ has defects $\mathbf{d}_1, \ldots, \mathbf{d}_N$ with $\mathbb{Y}_0, \ldots, \mathbb{Y}_N$ being the $\mathfrak{A}$-admissible intervals between these defects:

$$\cdots \text{---} \mathbb{Y}_0 \longrightarrow \ \mathbf{d}_1 \longleftarrow \mathbb{Y}_1 \longrightarrow \ \mathbf{d}_2 \ \longleftarrow \mathbb{Y}_2 \longrightarrow \cdots \longleftarrow \mathbb{Y}_{N-1} \longrightarrow \ \mathbf{d}_N \ \longleftarrow \mathbb{Y}_N \text{---} \cdots$$

Projective components: $\mathbb{Y}_0$ & $\mathbb{Y}_N$. $\therefore$ Interface is essential if $\mathcal{K}(0) \neq \mathcal{K}(N)$.

Theorem: *If* $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}$ *is surjective, then all essential interfaces are* Φ*-persistent. If* $\mathbf{a} \in \widetilde{\mathfrak{A}}$ *has an essential interface, then* $\Phi(\mathbf{a})$ *also has an essential interface, with the same signature as* $\mathbf{a}$. $\square$

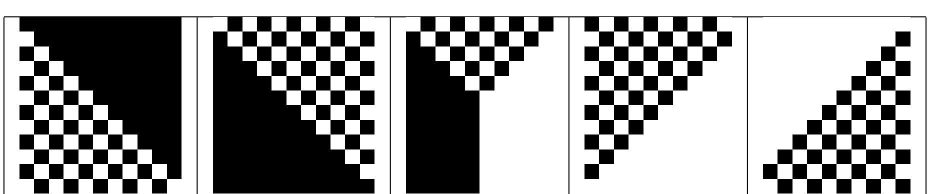
Example: (ECA #184) Let $\mathcal{A} = \{\square, \blacksquare\}$. Let $\mathfrak{G} := \mathfrak{G}_0 \sqcup \mathfrak{G}_1 \sqcup \mathfrak{G}_*$, where $\mathfrak{G}_0 := \{\overline{\blacksquare}\}$, $\mathfrak{G}_1 := \{\overline{\square}\}$, and $\mathfrak{G}_* := \{\overline{\blacksquare\square},\ \overline{\square\blacksquare}\}$. (Here, $\overline{\blacksquare} := [\ldots \blacksquare\underline{\blacksquare}\blacksquare\blacksquare \ldots]$ and $\overline{\blacksquare\square} := [\ldots \blacksquare\square\underline{\blacksquare}\square\blacksquare\square \ldots]$, etc.

Then $\mathfrak{G}_0 \cup \mathfrak{G}_1 \subset \mathsf{Fix}\,[\Phi_{184}]$, while $\Phi_{184}|_{\mathfrak{G}_*} = \sigma$.

$\mathfrak{G}$ has three (Φ_{184}, σ)-transitive components, so $\exists$ 6 possible interfaces:

$\alpha^+ :$	$\mathfrak{G}_*$	$\ldots\blacksquare\square\blacksquare\square\blacksquare\square$	$\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\ldots$	$\mathfrak{G}_0$	$\alpha^- :$	$\mathfrak{G}_*$	$\ldots\blacksquare\square\blacksquare\square\blacksquare\square$	$\square\square\square\square\square\square\ldots$	$\mathfrak{G}_1$
$\omega^+ :$	$\mathfrak{G}_0$	$\ldots\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare$	$\blacksquare\square\blacksquare\square\blacksquare\square\ldots$	$\mathfrak{G}_*$	$\omega^- :$	$\mathfrak{G}_1$	$\ldots\square\square\square\square\square\square$	$\blacksquare\square\blacksquare\square\blacksquare\square\ldots$	$\mathfrak{G}_*$
$\beta :$	$\mathfrak{G}_0$	$\ldots\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare$	$\square\square\square\square\square\square\ldots$	$\mathfrak{G}_1$	$\epsilon :$	$\mathfrak{G}_0$	$\ldots\square\square\square\square\square\square$	$\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\ldots$	$\mathfrak{G}_1$

The Φ_{184}-propagation of these defects is as follows:



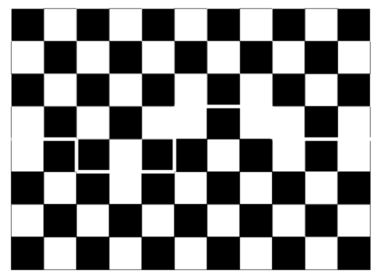
$(\boldsymbol{\alpha}^+)$ $(\boldsymbol{\omega}^+)$ $(\boldsymbol{\beta})$ $(\boldsymbol{\alpha}^-)$ $(\boldsymbol{\omega}^-)$

Dislocations (intuitive version)

Suppose $\mathfrak{A}$ has a *spatiotemporally periodic* structure. In any $\mathfrak{A}$-admissible configuration, certain patterns must recur periodically in space and time.

A **dislocation** is a domain boundary between two regions which are 'out of phase' with respect to this periodic structure. Such a domain boundary is necessarily an essential defect.

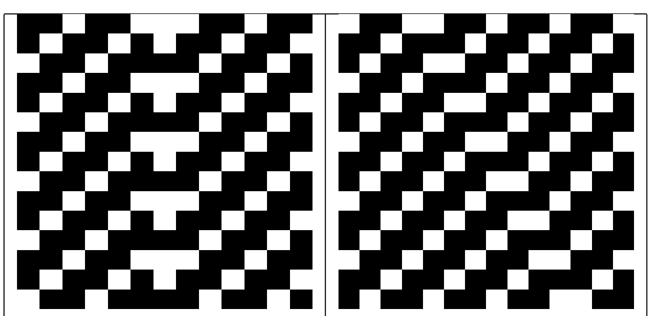
Example: The checkerboard shift $\mathfrak{Ch}$ is both vertically and horizontally 2-periodic in space. The domain boundary at right is a dislocation.



The spatiotemporally periodic structure of $\mathfrak{A}$ is described by a subgroup $\mathbb{K} \subset \mathbb{Z}^{D+1}$. Each dislocation is characterized by a **displacement** $\delta \in \Delta$, where $\Delta := \mathbb{Z}^{D+1}/\mathbb{K}$ is the quotient group.

Example: (ECA#62) Let $\mathfrak{D}$ = orbit of [. . . ■■□ ■■□ ■■□ . . .]. Then $\Phi_{62}|_{\mathfrak{D}} = \sigma$, so $(\mathfrak{D}, \Phi_{62})$ is 3-periodic in both space and time, and $\Delta \cong \mathbb{Z}_{/3}$.

Here are two dislocations in $\mathfrak{D}$ and their displacements:



β | γ

$\beta \quad \boxed{□■■□} \xrightarrow[□□]{2} \boxed{■■□}$

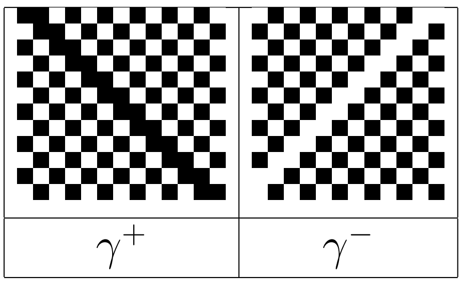
$\gamma \quad \boxed{■■□} \xrightarrow[□]{1} \boxed{■■□■■}$

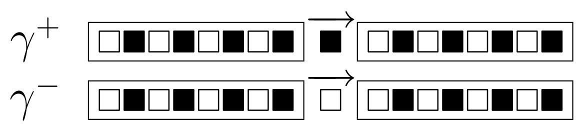
Theorem: *If $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}$ is surjective, then any nontrivial dislocation is a Φ-persistent defect. Futhermore the displacement of each dislocation is constant over time.* □

Dislocations in ECA#184 (intuitive version)

Let $\mathfrak{G}_*$ = orbit of [... □■ □■ □■ ...]. Then $\Phi_{184}|_{\mathfrak{G}_*} = \sigma$, so $(\mathfrak{G}_*, \Phi_{184})$ is 2-periodic in both space and time, and $\Delta \cong \mathbb{Z}_{/2}$.

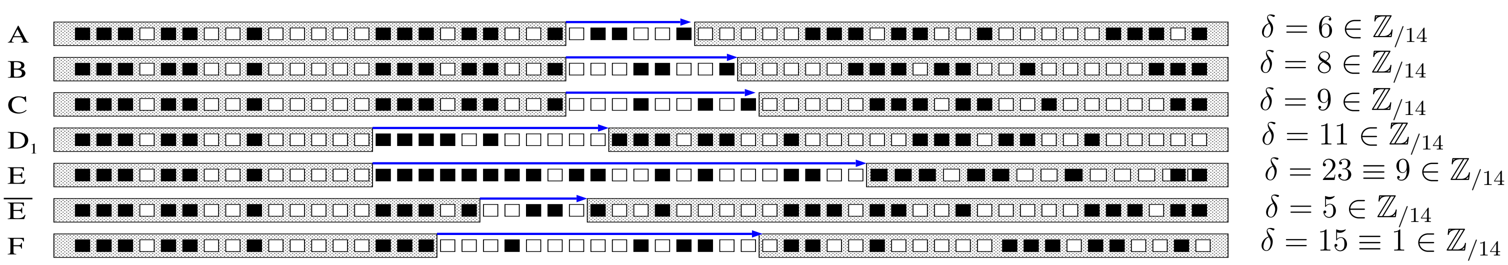
Here are two dislocations, both with displacement $1 \in \mathbb{Z}_{/2}$:

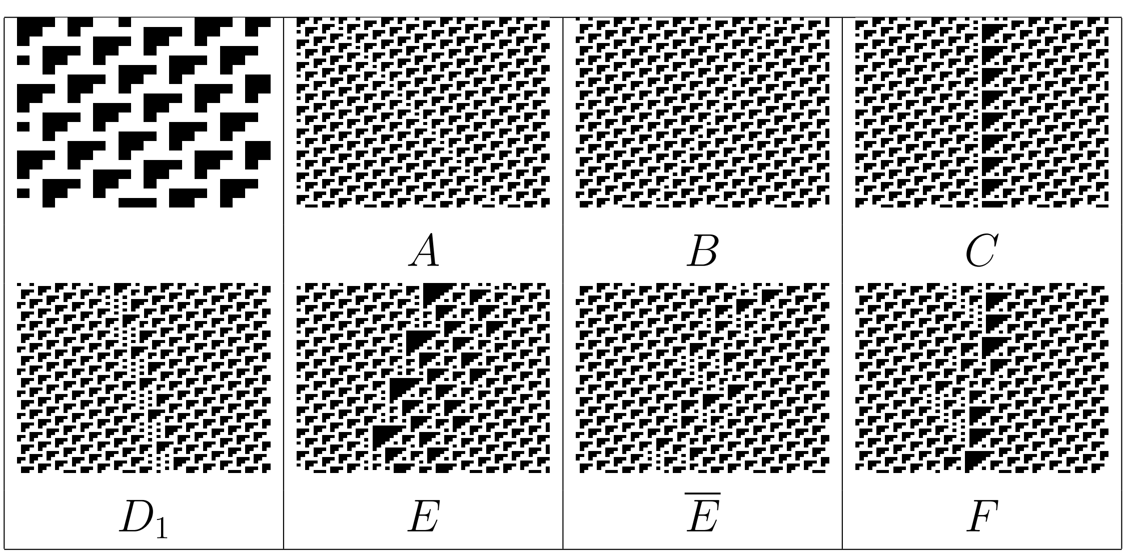




Dislocations in ECA#110

Let $\mathfrak{E}$ = orbit of [...■■■□■■□□■□□□□□ ■■■□■■□□■□□□□□...]. Then $\Phi_{110}|_{\mathfrak{E}} = \sigma^4$, so $(\mathfrak{E}, \Phi_{110})$ is spatiotemporally periodic, and $\Delta \cong \mathbb{Z}_{/14}$. Here are seven dislocations in $\mathfrak{E}$:

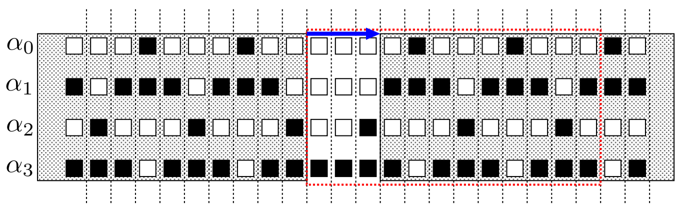




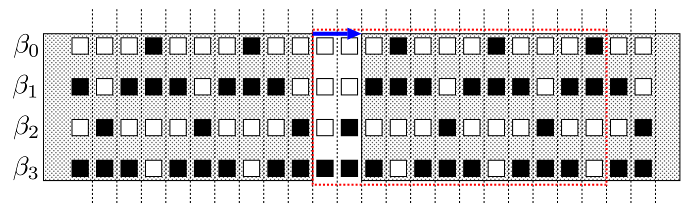
Dislocations in ECA#54 (intuitive version)

Let $\mathfrak{B} := \mathfrak{B}_0 \sqcup \mathfrak{B}_1$, where $\mathfrak{B}_0$ is the σ-orbit of $[\ldots ■■■□■■■□■■■□\ldots]$ and $\mathfrak{B}_1$ is the σ-orbit of $[\ldots □□□■□□□■□□□■\ldots]$. Then $\Phi_{54}(\mathfrak{B}_0) = \mathfrak{B}_1$, $\Phi_{54}(\mathfrak{B}_1) = \mathfrak{B}_0$, and $\Phi_{54}^2|_{\mathfrak{B}} = \sigma^2$. Thus, $(\mathfrak{B}, \Phi_{54})$ is spatiotemporally periodic, and $\Delta = \mathbb{Z}^2/\mathbb{K}$, where $\mathbb{K} := \mathbb{Z}(2,2) \oplus \mathbb{Z}(0,4)$, Here are four dislocations in ECA#54 and their displacements:

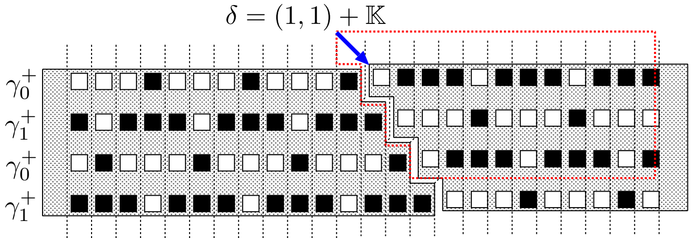
$\delta = (0,3) + \mathbb{K}$



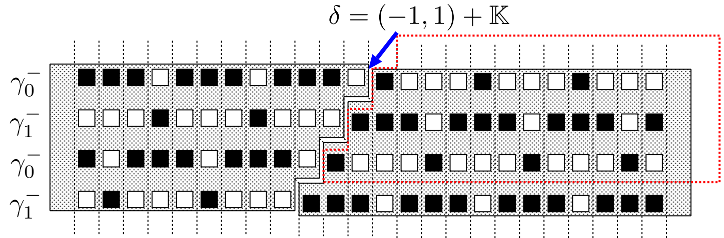
$\delta = (0,2) + \mathbb{K}$

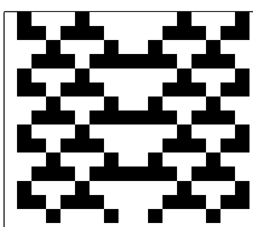
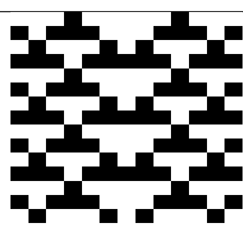
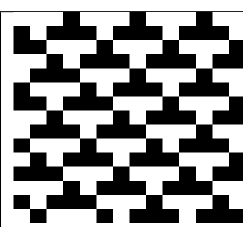
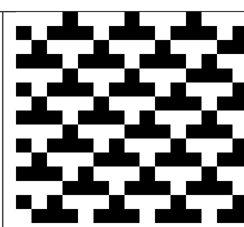


$\delta = (1,1) + \mathbb{K}$



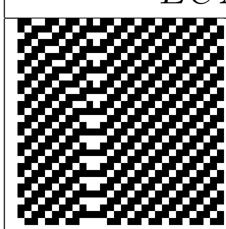
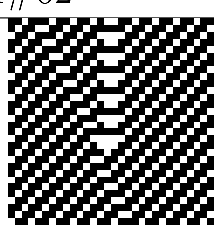
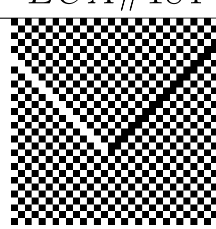
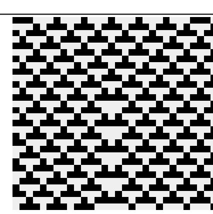
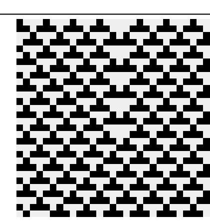
$\delta = (-1,1) + \mathbb{K}$



			
α	β	γ^+	γ^-

Displacement Algebra and Defect Chemistry

When two displacement defects collide, the outcome can be partially predicted by the algebra of the displacement group Δ.

$ECA\#62$		$ECA\#184$	$ECA\#54$	
				
$\gamma + \beta \to \alpha$	$\gamma + \alpha \to \gamma$	$\gamma^+ + \gamma^- \to \emptyset$	$\gamma^+ + \gamma^- \to \beta$	$\gamma^+ + \beta \to \gamma^-$
$2 + 1 \equiv 0$	$2 + 0 \equiv 2$	$1 + 1 \equiv 0$	$(1,1) + (-1,1) = (0,2)$	$(1,1) + (0,2) \equiv (-1,1)$

Dislocations (fomal version)

Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a Φ-invariant subshift. Let $\boldsymbol{\lambda} := (\lambda_0; \lambda_1, \ldots, \lambda_D)$ be a $(D+1)$-tuple of complex roots of unity. A **rational eigenfunction** of $\mathfrak{A}$ with **eigenvalue** $\boldsymbol{\lambda}$ is a function $F : \mathfrak{A} \longrightarrow \mathbb{C}$ such that:

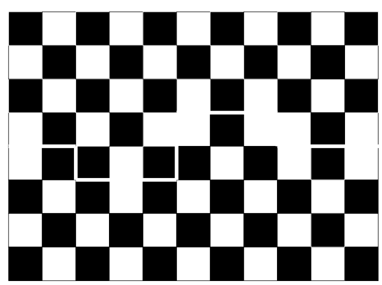
$$F \circ \Phi \;=\; \lambda_0 F, \qquad \text{and} \qquad F \circ \sigma^{\mathsf{z}} \;=\; \boldsymbol{\lambda}^{\mathsf{z}} F, \;\; \forall\, \mathsf{z} \in \mathbb{Z}^D.$$

Here, if $\mathsf{z} = (z_1, \ldots, z_D)$, then we define $\boldsymbol{\lambda}^{\mathsf{z}} \;:=\; \lambda_1^{z_1} \cdots \lambda_D^{z_D}$.

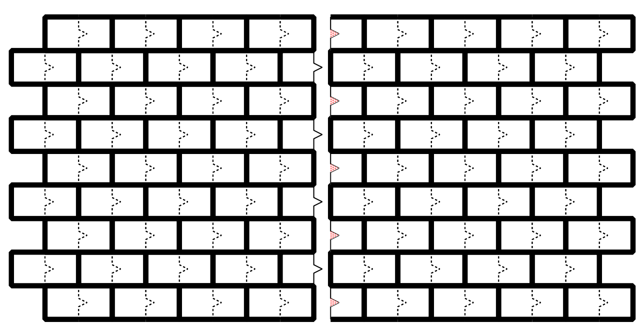
Any rational eigenfunction is **locally determined** i.e. $\exists r > 0$, and function $f : \mathfrak{A}_{(r)} \longrightarrow \mathbb{C}$ such that, $\forall\, \mathbf{a} \in \mathfrak{A}$, $F(\mathbf{a}) \;=\; f(\mathbf{a}_{\mathbb{B}(r)})$.

$\forall\, \mathsf{z} \in \mathbb{Z}^D$, let $f_{\mathsf{z}}(\mathbf{a}) \;:=\; f(\mathbf{a}_{\mathbb{B}(\mathsf{z},r)})$. Then $f_{\mathsf{z}}(\mathbf{a})$ is also well-defined for any $\mathbf{a} \in \widetilde{\mathfrak{A}}$ such that $\mathbf{a}_{\mathbb{B}(\mathsf{z},r)}$ is $\mathfrak{A}$-admissible. If $\mathsf{x}, \mathsf{y} \in \mathbb{Z}^D$, then $\mathbf{a}$ has an **$(\mathfrak{A}, \Phi)$-dislocation**‡ between x and y if $f_{\mathsf{x}}(\mathbf{a})/f_{\mathsf{y}}(\mathbf{a}) \neq \boldsymbol{\lambda}^{\mathsf{x}-\mathsf{y}}$.

Example: Define $F : \mathfrak{Ch} \longrightarrow \{\pm 1\}$ by local rule $f : \{\blacksquare, \square\} \longrightarrow \{\pm 1\}$ where $f(\blacksquare) = 1$ and $f(\square) = -1$. Then F is σ-eigenfunction with eigenvalue $(-1, -1)$.



Nonexample: This is *not* a dislocation, because $\mathfrak{Dom}$ is σ-mixing [Einsiedler, 2001], and thus, has no nontrivial eigenfunctions [Keynes & Robertson, 1969].
Instead this is a 'gap' defect.



Dislocations always form domain boundaries. Let $\mathbb{K} := \left\{\mathsf{k} \in \mathbb{Z}^D \;;\; \boldsymbol{\lambda}^{\mathsf{k}} = 1\right\}$. For any connected components $\mathbb{X}, \mathbb{Y}$ of $\mathbb{G}(\mathbf{a})$, $\exists$ unique **displacement** $\boldsymbol{\delta} \in \mathbb{Z}^{D+1}/\mathbb{K}$ such that, for any $\mathsf{x} \in \mathbb{X}$ and $\mathsf{y} \in \mathbb{Y}$, $\dfrac{f_{\mathsf{x}}(\mathbf{a})}{\boldsymbol{\lambda}^{\mathsf{x}-\mathsf{y}}\, f_{\mathsf{y}}(\mathbf{a})} \;=\; \boldsymbol{\lambda}^{\boldsymbol{\delta}}$.

(‡) Technically, this is a dislocation *of range r*, and this concept only makes sense for domain boundaries of range $R \geq r$.

Persistence of Dislocations

Lemma: *The dislocation in* $\mathbf{a}$ *is essential if* $\exists$ *two projective components* $\mathbb{X}$ *and* $\mathbb{Y}$ *with a nontrivial displacement between them.* $\square$

If $\mathbf{a}$ has N projective components, then the **displacement matrix** is the antisymmetric $N \times N$ matrix of $(\mathbb{Z}^{D+1}/\mathbb{K})$-valued displacements between them. Essential dislocations are persistent:

Theorem: *If* $\Phi : \mathfrak{A} \longrightarrow \mathfrak{A}$ *is surjective, then all essential dislocations are* Φ*-persistent. If* $\mathbf{a} \in \widetilde{\mathfrak{A}}$ *has essential dislocation, then* $\Phi(\mathbf{a})$ *also has essential dislocation, with the same displacement matrix as* $\mathbf{a}$. $\square$

Example: (ECA#62) Let $\mathcal{A} = \{\blacksquare, \square\}$. Let $\mathfrak{D}$ be the three-periodic σ-orbit of $\overline{\blacksquare\blacksquare\square}$. Then $\Phi_{62}|_{\mathfrak{D}} = \sigma$.

Let $\lambda := e^{2\pi\mathbf{i}/3}$. Define $F : \mathfrak{D} \longrightarrow \mathbb{C}$ by $F(\overline{\blacksquare\blacksquare\square}) = \square$, $F(\overline{\blacksquare\square\blacksquare}) = \lambda$, and $F(\overline{\square\blacksquare\blacksquare}) = \lambda^2$. Then $F \circ \sigma = \lambda F = F \circ \Phi_{62}$, so F is eigenfunction with eigenvalue (λ, λ).

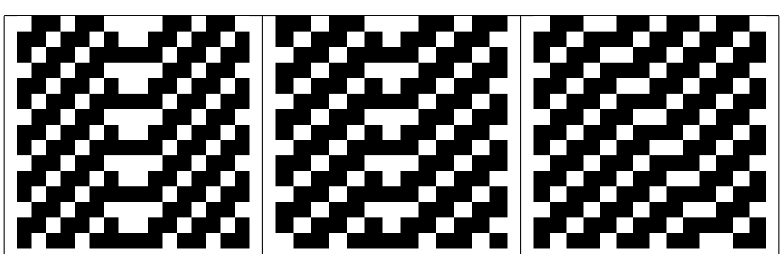
$\mathbb{K} = \mathbb{Z}(3,0) \oplus \mathbb{Z}(1,2)$, so displacements are elements of $\Delta \cong \mathbb{Z}_{/3}$.

Below are three rational dislocations in $\mathfrak{D}$ and their displacements.

$$\begin{array}{lll} \alpha & \boxed{\blacksquare\blacksquare\square\blacksquare\blacksquare\square}\overrightarrow{\square\square\square}\boxed{\blacksquare\blacksquare\square\blacksquare\blacksquare\square} & \delta = 3 \equiv 0 \in \mathbb{Z}_{/3} \\ \beta & \boxed{\blacksquare\blacksquare\square\blacksquare\blacksquare\square}\overrightarrow{\square\square}\boxed{\blacksquare\blacksquare\square\blacksquare\blacksquare\square\blacksquare} & \delta = 2 \in \mathbb{Z}_{/3} \\ \gamma & \boxed{\blacksquare\blacksquare\square\blacksquare\blacksquare\square}\overrightarrow{\square}\boxed{\blacksquare\blacksquare\square\blacksquare\blacksquare\square\blacksquare\blacksquare} & \delta = 1 \in \mathbb{Z}_{/3} \end{array}$$

The β and γ defects are essential, hence persistent by the theorem.

The α defect is *not* essential, but is still persistent (not because of the theorem).



α	β	γ

Persistence of Dislocations in ECA #54

Let $\mathfrak{B} := \mathfrak{B}_0 \sqcup \mathfrak{B}_1$, where $\mathfrak{B}_0$ is the 4-periodic σ-orbit of $\overline{\blacksquare\blacksquare\square\blacksquare}$ and $\mathfrak{B}_1$ is the 4-periodic σ-orbit of $\overline{\square\square\blacksquare\square}$.

Then $\Phi_{54}(\mathfrak{B}_0) = \mathfrak{B}_1, \quad \Phi_{54}(\mathfrak{B}_1) = \mathfrak{B}_0, \quad$ and $\Phi^2_{54}|_{\mathfrak{B}} = \sigma^2$.

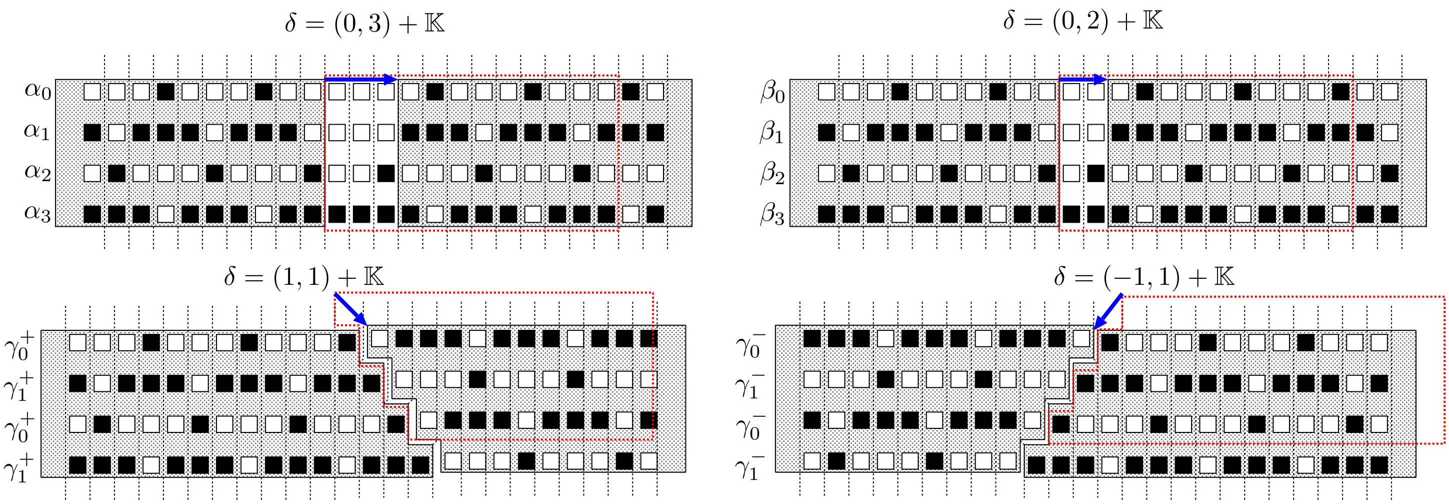
Define $F : \mathfrak{B} \longrightarrow \{\pm 1, \pm \mathbf{i}\}$ by

$$\begin{aligned}
F(\overline{\blacksquare\blacksquare\square\blacksquare}) &= F(\overline{\square\blacksquare\square\square}) &= \quad 1;\\
F(\overline{\blacksquare\square\blacksquare\blacksquare}) &= F(\overline{\blacksquare\square\square\square}) &= \quad \mathbf{i};\\
F(\overline{\square\blacksquare\blacksquare\blacksquare}) &= F(\overline{\square\square\square\blacksquare}) &= -1;\\
F(\overline{\blacksquare\blacksquare\blacksquare\square}) &= F(\overline{\square\square\blacksquare\square}) &= -\mathbf{i}.
\end{aligned}$$

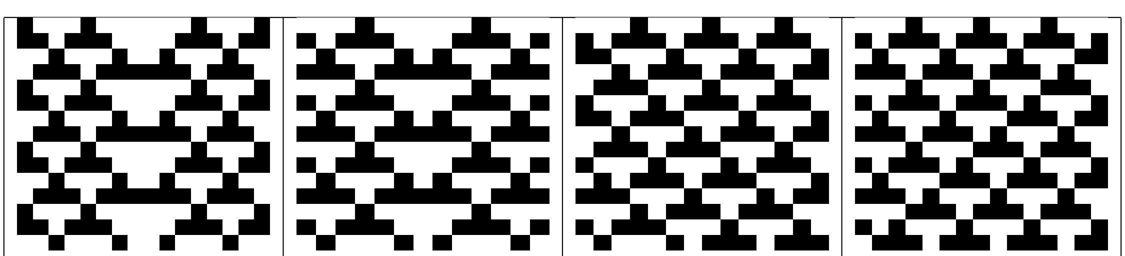
Then $F \circ \sigma = \mathbf{i}F = F \circ \Phi_{54}$, so F is eigenfunction with eigenvalue $(\mathbf{i}, \mathbf{i})$.

$\mathbb{K} := \mathbb{Z}(2,2) \oplus \mathbb{Z}(0,4)$, so displacements are elements of $\mathbb{Z}^2/\mathbb{K}$.

Here are four rational dislocations in ECA#54 and their displacements:



All four have nontrivial displacement, so they are essential, $\therefore$ Φ_{54}-persistent.



α	β	γ^+	γ^-

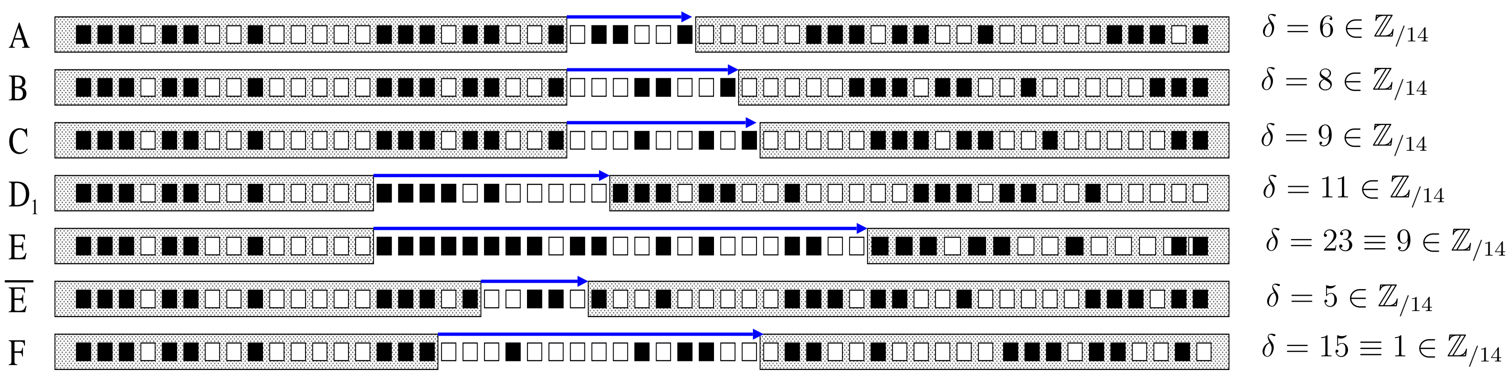
Persistence of Dislocations in ECA #110

Let $\mathfrak{E} \subset \mathcal{A}^{\mathbb{Z}}$ be the 14-periodic σ-orbit of $\overline{\blacksquare\blacksquare\blacksquare\square\blacksquare\blacksquare\square\square\blacksquare\square\square\square\square\square}$. Then $\Phi_{110}|_{\mathfrak{E}} = \sigma^4$.

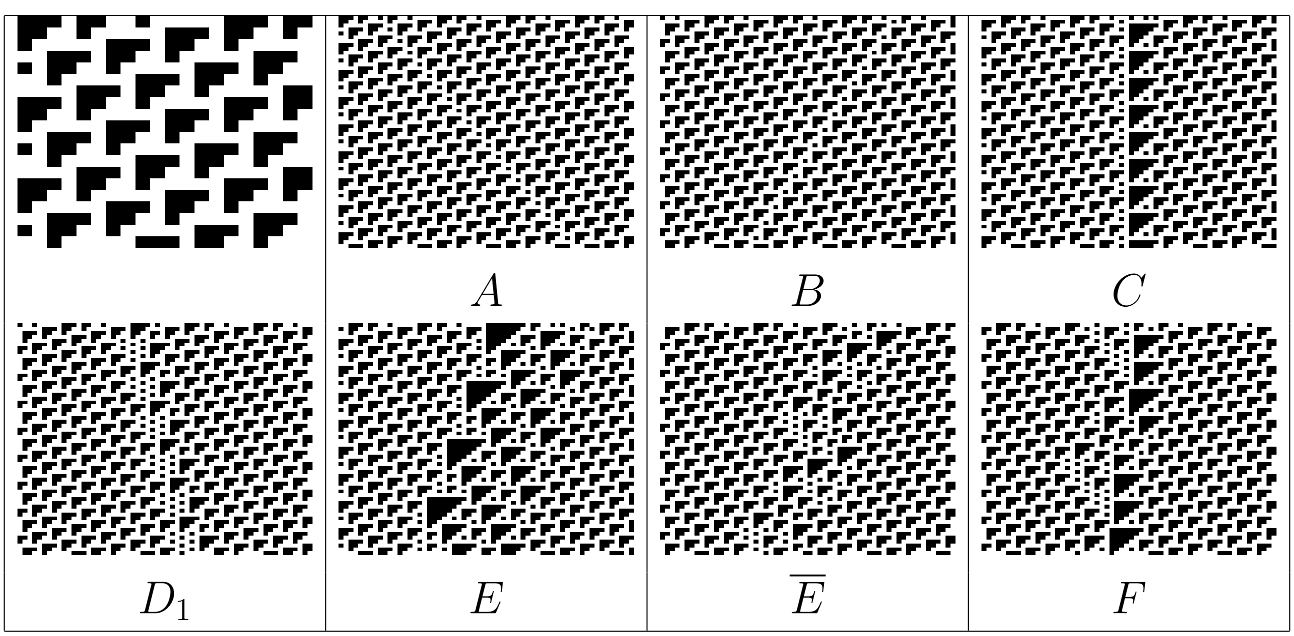
Let $\lambda := e^{\pi \mathbf{i}/7}$. Let $F : \mathfrak{E} \longrightarrow \{\lambda^k\}_{k=0}^{13}$ be a σ-eigenfunction with $F \circ \sigma = \lambda F$. Then $F \circ \Phi_{110} = \lambda^4 F$, so F is a (Φ_{184}, σ)-eigenfunction with eigenvalue $(\lambda^4; \lambda)$.

$\mathbb{K} = \mathbb{Z}(0,14) \oplus \mathbb{Z}(1,10)$, so displacements are elements of $\mathbb{Z}^2/\mathbb{K} \cong \mathbb{Z}_{/14}$.

Here are seven rational dislocations in $\mathfrak{E}$:



All have nontrivial displacement, so they are essential and Φ_{110}-persistent.



Persistence of Dislocations in ECA #184

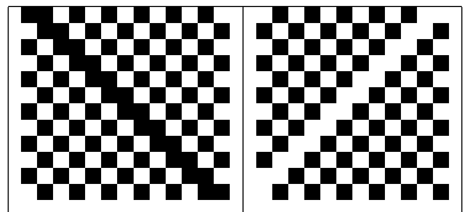
Let $\mathfrak{G}_* = \{$□■, ■□$\}$. Then $\Phi_{184}|_{\mathfrak{G}_*} = \sigma$.

Define $F : \mathfrak{G}_* \longrightarrow \{\pm 1\}$ by $F($□■$) = 1$ and $F($■□$) = -1$. Then

$F \circ \sigma = -F = F \circ \Phi_{184}$, so F is eigenfunction with eigenvalue $(-1, -1)$.

$\mathbb{K} = \mathbb{Z}(2,0) \oplus \mathbb{Z}(1,1)$, so displacements are elements of $\mathbb{Z}^2/\mathbb{K} \cong \mathbb{Z}_{/2}$.

Here are two dislocations and their displacements:



γ^+	γ^-

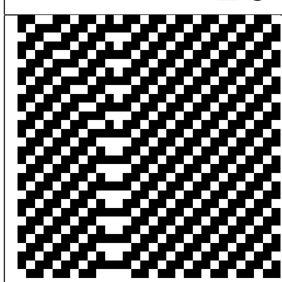
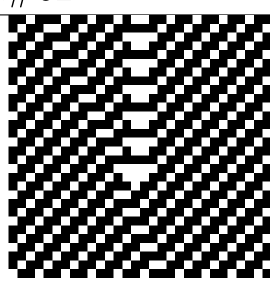
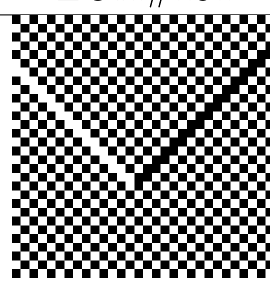
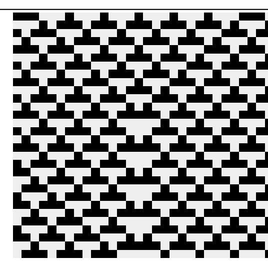
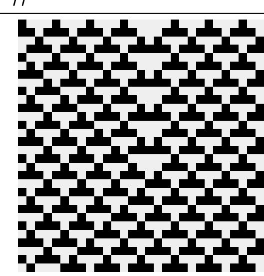
γ^+ [□■□■□■□■] $\overrightarrow{■}$ [□■□■□■□■] $\delta = 1 \in \mathbb{Z}_{/2}$

γ^- [□■□■□■□■] $\overrightarrow{□}$ [□■□■□■□■] $\delta = 1 \in \mathbb{Z}_{/2}$

Both have nontrivial displacement, so they are essential and Φ_{184}-persistent.

Displacement Algebra and Defect Chemistry

When two displacement defects collide, the outcome can be partially predicted by the algebra of the displacement group $\mathbb{Z}^{D+1}/\mathbb{K}$.

ECA#62		*ECA#184*	*ECA#54*	
				
$\gamma + \beta \to \alpha$ $2 + 1 \equiv 0 \pmod 3$	$\gamma + \alpha \to \gamma$ $2 + 0 \equiv 2 \pmod 3$	$\gamma^+ + \gamma^- \to \emptyset$ $1 + 1 \equiv 0 \pmod 2$	$\gamma^+ + \gamma^- \to \beta$ $(1,1) + (-1,1) = (0,2) \in \mathbb{Z}^2/\mathbb{K}$	$\gamma^+ + \beta \to \gamma^-$ $(1,1) + (0,2) = (1,3) \equiv (-1,1) \in \mathbb{Z}^2/\mathbb{K}$

The Fine Print: Our definition of 'displacement' here is somewhat oversimplified. The 'real' definition is that a displacement is a *character* on the spectral group of $(\mathfrak{A}, \Phi, \sigma)$. This is necessary to extend the theory of dislocations to *irrational* eigenvalues (e.g. in Sturmian shifts or multidimensional SFTS) or *discontinuous* eigenfunctions (e.g. on sofic shifts, as in ECA#18).

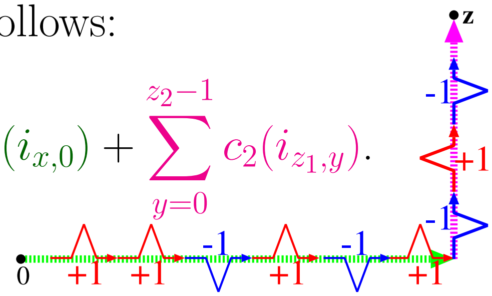
Cocycles

Let $\mathfrak{A} \subseteq \mathcal{A}^{\mathbb{Z}^D}$ be a subshift. Let $(\mathcal{G}, \cdot)$ be a (discrete) group. A $\mathcal{G}$-valued **cocycle** is continuous function $C : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ satisfying **cocycle equation:**

$$C(\mathsf{y}+\mathsf{z}, \mathbf{a}) \quad = \quad C(\mathsf{y}, \sigma^{\mathsf{z}}(\mathbf{a})) \cdot C(\mathsf{z}, \mathbf{a}), \qquad \forall\, \mathbf{a} \in \mathcal{A}^{\mathbb{Z}^D} \text{ and } \forall\, \mathsf{y}, \mathsf{z} \in \mathbb{Z}^D.$$

Examples: (a) Let $\mathfrak{Ice} \subset \mathcal{I}^{\mathbb{Z}^2}$ be square ice. Define $c_1, c_2 : \mathcal{I} \longrightarrow \{\pm 1\}$ by $c_1(\;) := +1 =: c_2(\;)$ and $c_1(\;) := -1 =: c_2(\;)$ ('$*$' means 'anything'). Define cocycle $C : \mathbb{Z}^2 \times \mathfrak{Ice} \longrightarrow \mathbb{Z}$ as follows:

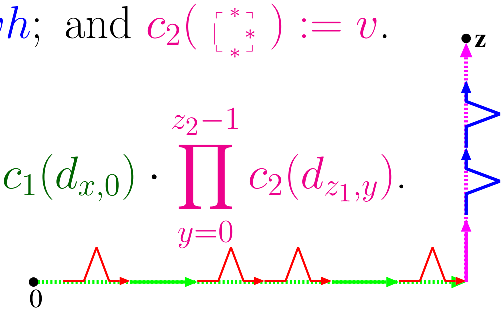
$$\forall\, \mathbf{i} \in \mathfrak{Ice},\ \forall\, \mathsf{z} = (z_1, z_2) \in \mathbb{Z}^2,\ C(\mathsf{z}, \mathbf{i}) \ := \ \sum_{x=0}^{z_1-1} c_1(i_{x,0}) + \sum_{y=0}^{z_2-1} c_2(i_{z_1,y}).$$



This is a **height function** (a $\mathbb{Z}$-valued cocycle). These arise in tilings [e.g. K. Eloranta 1999-2005, H.Cohn & J.Propp] and statistical mechanics [R.Baxter 1989].

(b) Let $\mathfrak{Dom} \subset \mathcal{D}^{\mathbb{Z}^2}$ be dominoes. Let $\mathcal{G} := \mathbb{Z}_{/2} * \mathbb{Z}_{/2}$ be group of finite products $vhvhv \cdots vhv$, where v and h are noncommuting generators with $v^2 = e = h^2$. Define $c_1, c_2 : \mathcal{I} \longrightarrow \mathcal{G}$ by
$c_1(\;) := vhv$; $\quad c_1(\;) := h$; $\quad c_2(\;) := hvh$; and $c_2(\;) := v$.

$$\forall\, \mathbf{d} \in \mathfrak{Dom},\ \forall\, \mathsf{z} = (z_1, z_2) \in \mathbb{Z}^2,\ C(\mathsf{z}, \mathbf{d}) \ := \ \prod_{x=0}^{z_1-1} c_1(d_{x,0}) \cdot \prod_{y=0}^{z_2-1} c_2(d_{z_1,y}).$$



(c) If $b : \mathfrak{A} \longrightarrow \mathcal{G}$ is continuous, then function $C(\mathsf{z}, \mathbf{a}) := b(\sigma^{\mathsf{z}}(\mathbf{a})) \cdot b(\mathbf{a})^{-1}$ is a cocycle, called a **coboundary**.

(d) Let $\mathbf{X}$ = topological space. Let $\mathcal{H}$ =homeo($\mathbf{X}$). Then $\mathcal{H}$-valued cocycles are the fibre-wise maps of a skew product extension of the σ-action on $\mathfrak{A}$ to a $\mathbb{Z}^D$-action on $\mathfrak{A} \times \mathbf{X}$. [R.Zimmer 1976-80, J.Kammeyer 1990-93]

Cohomology

Two cocycles C and C' are **cohomologous** ($C \approx C'$) if $\exists$ continuous **transfer function** $b : \mathfrak{A} \longrightarrow \mathcal{G}$ such that

$$C'(\mathsf{z}, \mathbf{a}) \;=\; b(\sigma^{\mathsf{z}}(\mathbf{a})) \cdot C(\mathsf{z}, \mathbf{a}) \cdot b(\mathbf{a})^{-1}, \qquad \forall\, \mathsf{z} \in \mathbb{Z}^D, \text{ and } \mathbf{a} \in \mathfrak{A}.$$

Let $\underline{C}$:= cohomology equivalence class of the cocycle C.

$\mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$:= $\{\mathcal{G}$-valued cocycles$\}$.

$\mathcal{H}^1(\mathfrak{A}, \mathcal{G})$:= $\{$cohomology equivalence classes in $\mathcal{Z}^1(\mathfrak{A}, \mathcal{G})\}$.

If $(\mathcal{G}, \cdot)$ is abelian, then $\mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$ is a group (under pointwise multipication), and $\mathcal{H}^1(\mathfrak{A}, \mathcal{G})$ is a quotient group, called the **1st cohomology group** of $\mathfrak{A}$ (with coefficients in $\mathcal{G}$). [see e.g. K.Schmidt (1995, 1998) for discussion]

Trails and locally determined cocycles

Let $\mathbb{E} := \left\{\mathsf{z} \in \mathbb{Z}^D \,;\, \mathsf{z} = (0, ..., 0, \pm 1, 0, ..., 0)\right\}$. A **trail** is a sequence $\zeta = (\mathsf{z}_0, \mathsf{z}_1, \ldots, \mathsf{z}_N) \subset \mathbb{Z}^D$, where, $\forall n \in [1...N]$, $\mathsf{z}'_n := (\mathsf{z}_n - \mathsf{z}_{n-1}) \in \mathbb{E}$.

Let $r > 0$. Let $c : \mathbb{E} \times \mathfrak{A}_{(r)} \longrightarrow \mathcal{G}$ be such that, $\forall\, \mathsf{e}, \mathsf{e}' \in \mathbb{E}$, $\forall\, \mathbf{a} \in \mathfrak{A}$,

$$\begin{array}{llll} \textbf{(a)} & c(\mathsf{e}', \mathbf{a}_{\mathbb{B}(\mathsf{e},r)}) \cdot c(\mathsf{e}, \mathbf{a}_{\mathbb{B}(r)}) \;=\; c(\mathsf{e}, \mathbf{a}_{\mathbb{B}(\mathsf{e}',r)}) \cdot c(\mathsf{e}', \mathbf{a}_{\mathbb{B}(r)}). & \text{i.e. } c\left(\uparrow\rightarrow\right) \;=\; c\left(\rightarrow\uparrow\right) \\ \textbf{(b)} & c(-\mathsf{e}, \mathbf{a}_{\mathbb{B}(\mathsf{e},r)}) \;=\; c(\mathsf{e}, \mathbf{a}_{\mathbb{B}(r)})^{-1}. & \text{i.e. } c\left(\downarrow\right) \;=\; c\left(\uparrow\right)^{-1} \end{array}$$

Then $c(\zeta, \mathbf{a}) \;:=\; \displaystyle\prod_{n=1}^{N} c(\mathsf{z}'_n, \mathbf{a}_{\mathbb{B}(\mathsf{z}_{n-1},r)})$ depends only on z_0 and z_N, not ζ.

Example: If ζ is **closed** (i.e. $\mathsf{z}_N = \mathsf{z}_0$) then $c(\zeta, \mathbf{a}) = e_{\mathcal{G}}$.

Define cocycle $C : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ as follows: $\forall\, \mathbf{a} \in \mathfrak{A}$, $\mathsf{z} \in \mathbb{Z}^D$, $C(\mathsf{z}, \mathbf{a}) := c(\zeta, \mathbf{a})$, (where ζ is *any* trail from 0 to z). We say C is **locally determined** with **local rule** c of **radius** r.

If $\mathcal{G}$ is discrete, then $\forall$ continuous $\mathcal{G}$-valued cocycles are locally determined. For any $r > 0$, let $\mathcal{Z}^1_r(\mathfrak{A}, \mathcal{G})$:= radius-$r$ cocycles on $\mathfrak{A}$.

Cocycles and Cellular Automata

Proposition: *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift. Let $\Phi : \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a cellular automaton with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$. Let $\mathcal{G}$ be a group.*

(a) *Let $C \in \mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$ be cocycle. Define $\Phi_* C : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ by $\Phi_* C(\mathsf{z}, \mathbf{a}) = C(\mathsf{z}, \Phi(\mathbf{a}))$. Then $\Phi_* C$ is also a cocycle on $\mathfrak{A}$.*

(b) *If Φ has radius R, and C is locally determined with radius r, then $\Phi_* C$ is locally determined with radius $r + R$.*

(c) *Let $C, C' \in \mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$. If $C \approx C'$, then $\Phi^* C \approx \Phi^* C'$. Thus, Φ induces a function $\Phi_* : \mathcal{H}^1(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^1(\mathfrak{A}, \mathcal{G})$.*

(d) *If $(\mathcal{G}, \cdot)$ is abelian, then Φ_* is a group endomorphism.* $\square$

We will see that the Φ-persistence of certain kinds of defects depends critically on the surjectivity of the endomorphism Φ_*.

Question: *When is Φ_* surjective?*

Gap Defects: Definition

Some domain boundaries exhibit divergence in cocycle asymptotics.

Let $C \in \mathcal{Z}^1_r(\mathfrak{A}, \mathbb{Z})$ be a range-r cocycle (i.e. 'height function').

Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$. Let $\mathbb{X}$ be an infinite, simply-connected component of $\mathbb{G}_r(\mathbf{a})$. Fix $\mathsf{x}^* \in \mathbb{X}$. For any $\mathsf{x} \in \mathbb{X}$, we define the **height difference**:

$$\mathbf{C_a}(\mathsf{x}^*, \mathsf{x}) \quad := \quad c(\zeta, \mathbf{a}),$$

where $c : \mathfrak{A}_{(r)} \longrightarrow \mathbb{Z}$ is 'local rule', and ζ is any trail in $\mathbb{X}$ from x^* to x.

(Well-defined independent of ζ because $\mathbb{X}$ is a simply-connected.) Note:

$$|C_{\mathbf{a}}(\mathsf{x}^*, \mathsf{x})| \quad \leq \quad K \cdot d_{\mathbb{X}}(\mathsf{x}^*, \mathsf{x}),$$

where $K := \max_{\mathbf{a} \in \mathfrak{A}_{(r)}} |c(\mathbf{a})|$, and $d_{\mathbb{X}}(\mathsf{x}^*, \mathsf{x}) :=$ min length ($\mathbb{X}$-trail from x^* to x).

Let $\mathbb{Y}$ be another infinite connected component of $\mathbb{G}_r(\mathbf{a})$. Fix $\mathsf{y}^* \in \mathbb{Y}$. For any $\mathsf{y} \in \mathbb{Y}$, define $C_{\mathbf{a}}(\mathsf{y}, \mathsf{y}^*)$ in the same way as $C_{\mathbf{a}}(\mathsf{x}^*, \mathsf{x})$ above. We then define

$$\mathbf{C}(\mathsf{y}, \mathsf{x}) \quad := \quad C(\mathsf{y}, \mathsf{y}^*) + C(\mathsf{x}^*, \mathsf{x}).$$

If $\mathbb{X}$ and $\mathbb{Y}$ were the same connected component (or if we could remove the defect in $\mathbf{a}$ so that they were), then we expect

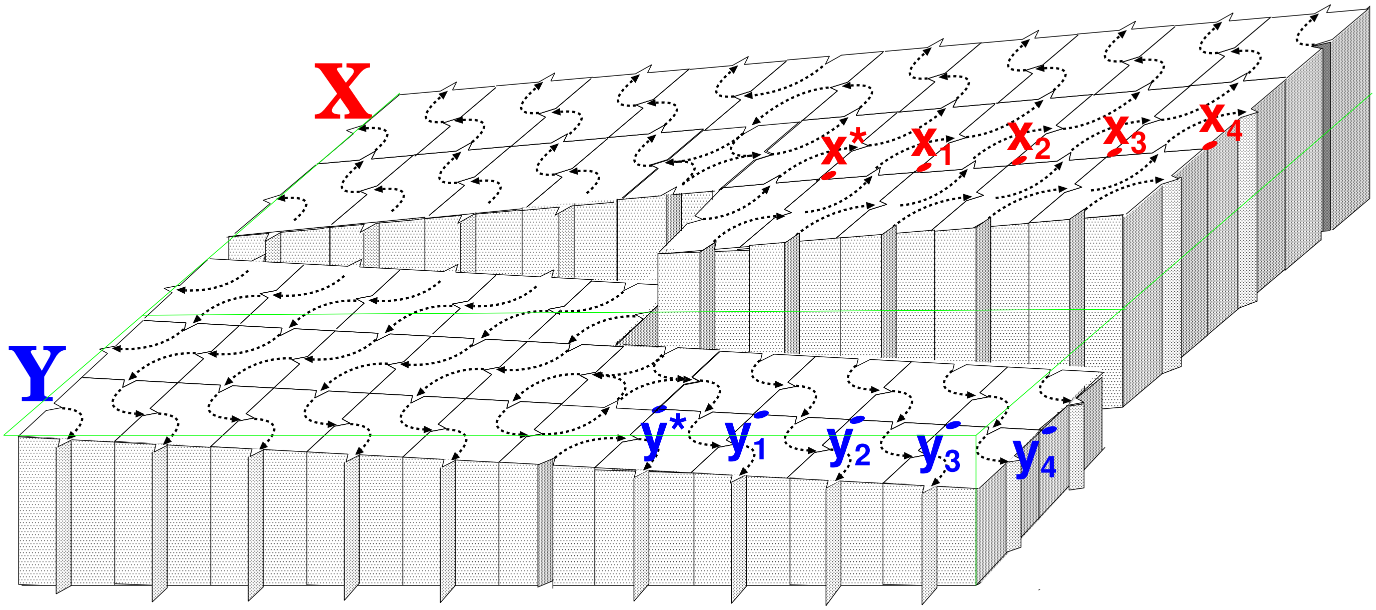
$$C(\mathsf{y}, \mathsf{x}) \quad \leq \quad K \cdot d_{\mathbb{X}}(\mathsf{y}, \mathsf{x}) + \text{const.} \quad \approx \quad K|\mathsf{y} - \mathsf{x}| + \text{const.}$$

We say there is a ***C*-gap** between $\mathbb{X}$ and $\mathbb{Y}$ if $\sup_{\mathsf{y} \in \mathbb{Y},\ \mathsf{x} \in \mathbb{X}} \dfrac{|C(\mathsf{y}, \mathsf{x})|}{|\mathsf{y} - \mathsf{x}|} = \infty$.

(This suggests that the defect separating $\mathbb{X}$ and $\mathbb{Y}$ is essential.)

Fine print: If $\mathcal{G} \neq \mathbb{Z}$, we can also define gaps for $\mathcal{G}$-valued cocycles, by first defining an appropriate *pseudonorm* $\|\bullet\| : \mathcal{G} \longrightarrow \mathbb{R}$ which satisfies the triangle inequality and is invariant under conjugation.

Gaps in the Ice



Example: Consider the defective configuration in $\widetilde{\mathfrak{Ice}}$ shown above, and let $\{\mathsf{x}^*, \mathsf{x}_1, \mathsf{x}_2, \ldots\} \subset \mathbb{X}$ and $\{\mathsf{y}^*, \mathsf{y}_1, \mathsf{y}_2, \ldots\} \subset \mathbb{Y}$ be as shown. Let $C \in \mathcal{Z}^1(\mathfrak{Ice}, \mathbb{Z})$ be the cocycle with local rule

$$c_1(\quad) := +1 =: c_2(\quad) \text{ and } c_1(\quad) := -1 =: c_2(\quad).$$

Then $C(\mathsf{x}^*, \mathsf{x}_n) = n$ and $C(\mathsf{y}^*, \mathsf{y}_n) = -n$, so $C(\mathsf{x}_n, \mathsf{y}_n) = 2n, \ \forall\, n \in \mathbb{N}$.

But $|\mathsf{x}_n - \mathsf{y}_n| = 2, \ \forall\ n \in \mathbb{N}$, so $\displaystyle\lim_{n\to\infty} \frac{|C(\mathsf{x}_n, \mathsf{y}_n)|}{|\mathsf{x} - \mathsf{y}|} = \lim_{n\to\infty} \frac{2n}{2} = \infty$; hence there is a gap between $\mathbb{X}$ and $\mathbb{Y}$.

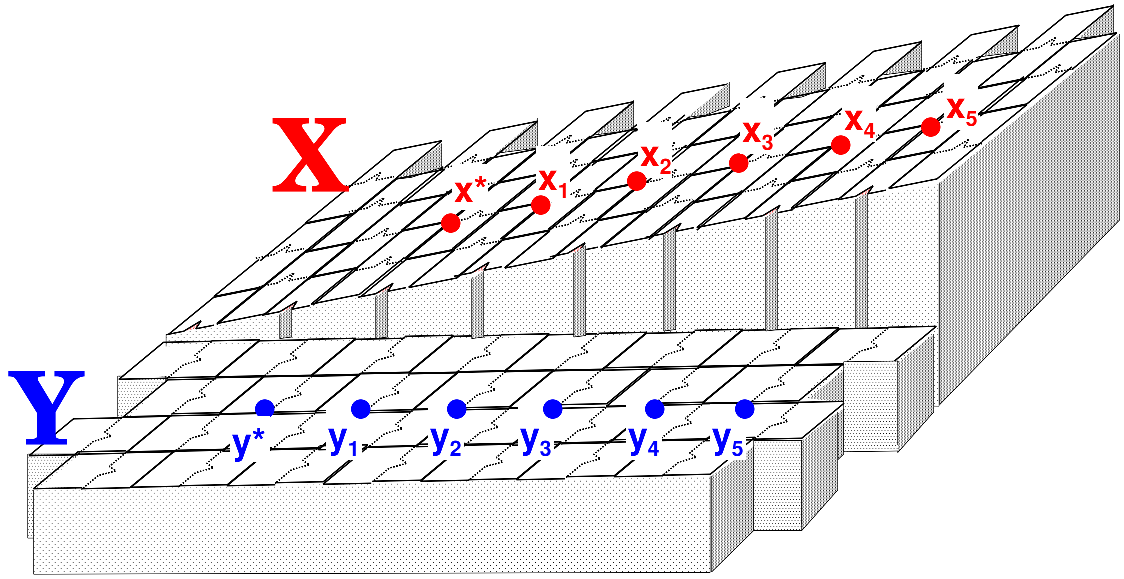
Example: Let $C : \mathbb{Z}^2 \times \mathfrak{Dom} \longrightarrow \mathcal{G} := \mathbb{Z}_{/2} * \mathbb{Z}_{/2}$ have local rule:

$$c_1(\quad) := vhv; \quad c_1(\quad) := h; \quad c_2(\quad) := hvh; \text{ and } c_2(\quad) := v.$$

Let $\mathcal{Z} := \{\text{cyclic subgroup generated by } vh\} \subset \mathcal{G}$. Then $(\mathcal{Z}, \cdot) \cong (\mathbb{Z}, +)$, and for all $\mathbf{d} \in \mathfrak{Dom}$ and $2\mathsf{z} \in 2\mathbb{Z}^2$, $C(2\mathsf{z}, \mathbf{d}) \in \mathcal{Z}$.

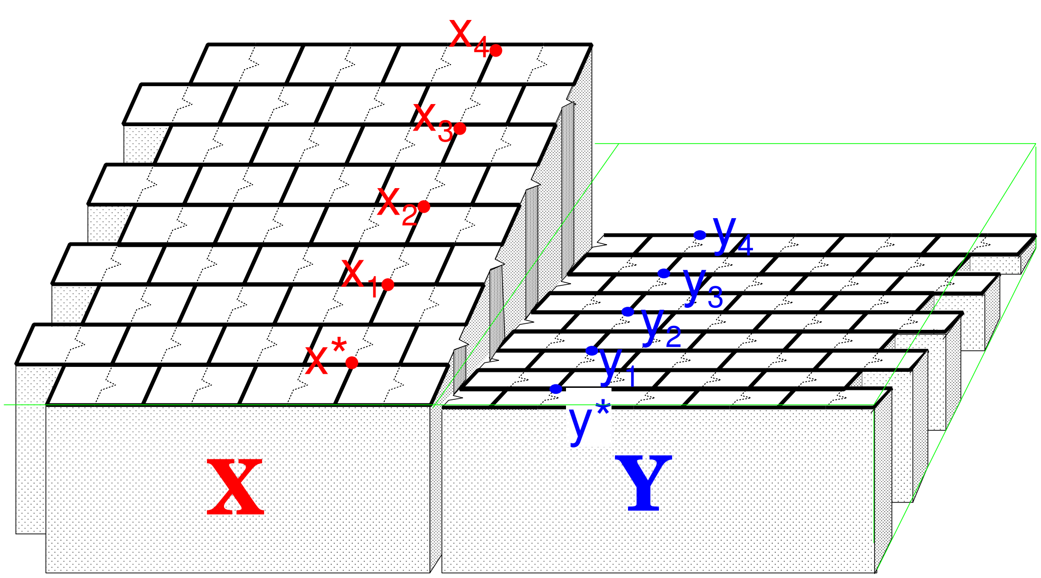
Let $\mathcal{D}_2 \subset \mathcal{D}^{2\times 2}$ be the alphabet of $\mathfrak{Dom}$-admissible 2×2 blocks. Let $\mathfrak{D}_2 \subset \mathcal{D}_2^{\mathbb{Z}^2}$ be 'recoding' of $\mathfrak{Dom}$ in this alphabet. Then $2\mathbb{Z}^2$ acts on $\mathfrak{D}_2$ in the obvious way, and C yields a cocycle $C' : 2\mathbb{Z}^2 \times \mathfrak{D}_2 \longrightarrow \mathcal{Z} \cong \mathbb{Z}$.

Gaps in Dominoes



In the $\widetilde{\mathfrak{Dom}}$-configuration shown above, $C'(\mathsf{x}^*, \mathsf{x}_n) = (vhvh)^n \cong 2n$, while $C'(\mathsf{y}^*, \mathsf{y}_n) = h^{2n} \cong 0$, so $C'(\mathsf{y}_n, \mathsf{x}_n) = n$, for all $n \in \mathbb{N}$.

But $|\mathsf{x}_n - \mathsf{y}_n| = 4$, $\forall\ n \in \mathbb{N}$, so $\displaystyle\lim_{n\to\infty} \frac{|C'(\mathsf{x}_n, \mathsf{y}_n)|}{|\mathsf{x} - \mathsf{y}|} = \lim_{n\to\infty} \frac{n}{4} = \infty.$



In the $\widetilde{\mathfrak{Dom}}$-configuration shown above, $C'(\mathsf{x}^*, \mathsf{x}_n) = (vhvh)^n \cong 2n$, while $C'(\mathsf{y}^*, \mathsf{y}_n) = (hvhv)^n \cong -2n$, so $C'(\mathsf{y}_n, \mathsf{x}_n) = -4n$, $\forall\ n \in \mathbb{N}$.

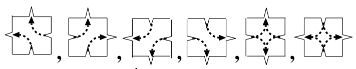
But $|\mathsf{x}_n - \mathsf{y}_n| = 4$, $\forall\ n \in \mathbb{N}$, so $\displaystyle\lim \frac{|C'(\mathsf{x}_n, \mathsf{y}_n)|}{|\mathsf{x} - \mathsf{y}|} = \lim \frac{-4n}{4} = -\infty.$

Persistence of Gaps

Theorem: *If* $\Phi \colon \mathcal{A}^{\mathbb{Z}^D} \to \mathcal{A}^{\mathbb{Z}^D}$ *is a CA,* $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$*, and endomorphism*

$$\Phi_* : \mathcal{H}^1(\mathfrak{A}, \mathbb{Z}) \ni C \mapsto C \circ \Phi \in \mathcal{H}^1(\mathfrak{A}, \mathbb{Z})$$

is surjective, then any gap is Φ*-persistent.*

Example: If $\mathcal{I} := \{$  $\}$, and $\Phi : \mathcal{I}^{\mathbb{Z}^2} \longrightarrow \mathcal{I}^{\mathbb{Z}^2}$ is CA with $\Phi(\mathfrak{Ice}) \subseteq \mathfrak{Ice}$, and $\Phi_* : \mathcal{H}^1(\mathfrak{Ice}, \mathbb{Z}) \longrightarrow \mathcal{H}^1(\mathfrak{Ice}, \mathbb{Z})$ is surjective, then Φ cannot destroy the ice gap (or even change the 'difference in slope').

Proof idea: First show that C-gaps depend only on cohomology class of C, i.e.:

Lemma: *If* $C \approx C'$*, then any* C*-gap is also a* C'*-gap.* $\diamondsuit$

Now suppose $\mathbf{a}$ has C-gap. Now Φ_* is surjective, so find $C' \in \mathcal{Z}^1$ such that $\Phi_* C' \approx C$. Then $\mathbf{a}$ also has $(\Phi_* C')$-gap. But this implies that $\Phi(\mathbf{a})$ has C' gap. $\Box$

Sharp Gaps are Essential

A gap in $\mathbb{G}_r(\mathbf{a})$ is **sharp** if, for all $R \geq r \geq 0$, there exists constant $K = K(R, r) \in \mathbb{N}$ such that, for any $\mathsf{y} \in \mathbb{G}_r(\mathbf{a})$, $\exists\, \mathsf{x} \in \mathbb{G}_R(\mathbf{a})$ in same connected component $\mathbb{X}$ of $\mathbb{G}_r(\mathbf{a})$ as y, with $d_{\mathbb{X}}(\mathsf{x}, \mathsf{y}) \leq K$.

Idea: The gap does not ramify into lots of 'tributaries'.

Example: If $\mathfrak{A}$ is a subshift of finite type, and defect set $\mathbb{D}(\mathbf{a})$ is confined to a thickened hyperplane [as in previous three examples] then the gap is sharp.

Theorem: *Sharp gaps are essential defects.*

Proof idea: First show:

Lemma: *The existence of a gap does not depend on the choice of reference points* $\mathsf{x}^* \in \mathbb{X}$ *and* $\mathsf{y}^* \in \mathbb{Y}$*.* $\diamondsuit$

Thus, we can always move our basepoint x^* and 'gap-detection' sequence $\{\mathsf{x}_1, \mathsf{x}_2, \ldots\}$ far away from gap. Thus, a gap is 'detectable' from any distance; hence it cannot be removed by locally changing $\mathbf{a}$. $\Box$

Defect Codimension

A domain boundary is a defect of **codimension 1**.

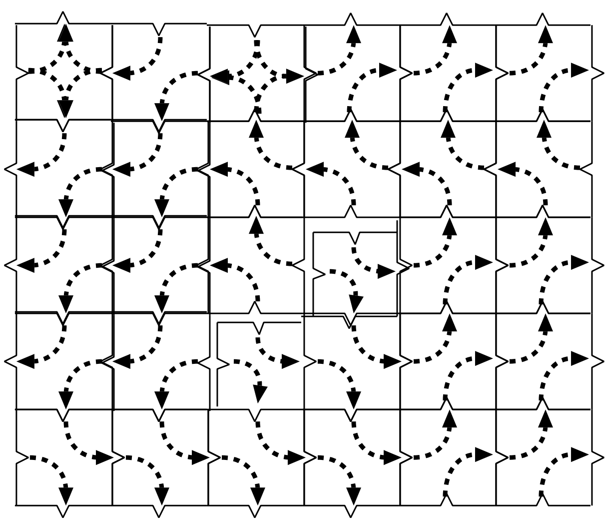
Fix $r \in \mathbb{N}$. Let $\mathbb{G}_r(\mathbf{a}) := \left\{ \mathsf{z} \in \mathbb{Z}^D \; ; \; \mathbf{a}_{\mathbb{B}(\mathsf{z},r)} \in \mathfrak{A}_{(r)} \right\}$. (Loosely, this is the complement of a radius-r neighbourhood around the defects in $\mathbf{a}$.)

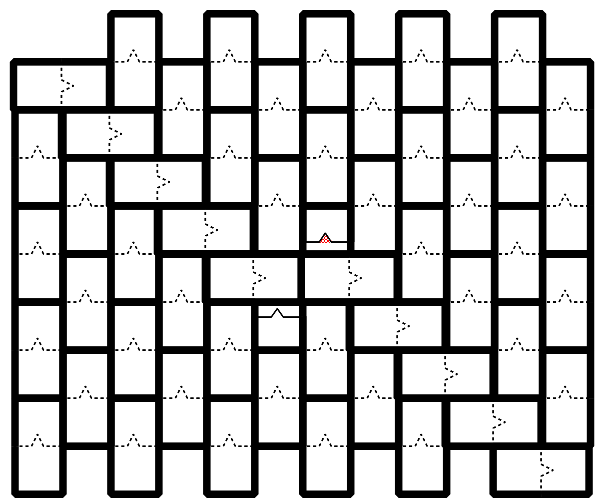
Let $\mathbf{G_r}(\mathbf{a})$:= union of all unit cubes whose corners are all in $\mathbb{G}_r(\mathbf{a})$.

We say $\mathbf{a}$ has a (range r) **codimension** $(k+1)$ defect if the kth homotopy group $\pi_k\left[\mathbf{G}_r(\mathbf{a})\right]$ is nontrivial$^{(*)}$.

Examples of Codimension-Two Defects:

In $\mathfrak{Ice}$: In $\mathfrak{Dom}$:





[due to S. Lightwood, via M. Einsiedler, 2001]

The sequence of inclusions $\mathbb{G}_1(\mathbf{a}) \supseteq \mathbb{G}_2(\mathbf{a}) \supseteq \mathbb{G}_3(\mathbf{a}) \supseteq \cdots$ yields sequence of homomorphisms

$$\pi_k\left[\mathbf{G}_1(\mathbf{a})\right] \longleftarrow \pi_k\left[\mathbf{G}_2(\mathbf{a})\right] \longleftarrow \pi_k\left[\mathbf{G}_3(\mathbf{a})\right] \longleftarrow \cdots$$

Define $\pi_k\left[\mathbf{G}_\infty(\mathbf{a})\right]$:= inverse limit of this sequence$^{(\dagger)}$ (detects 'extremely large scale' homotopy properties).

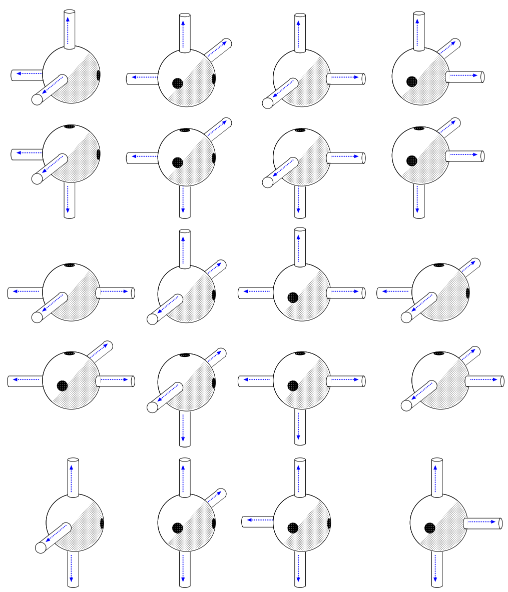
Say $\mathbf{a}$ has a **projective** codimension $(k+1)$ defect if $\pi_k\left[\mathbf{G}_\infty(\mathbf{a})\right] \neq \{0\}$.

$(*)$ Strictly speaking, we must fix a basepoint and a connected component of $\mathbf{G}_r$.

$(\dagger)$ We must fix a *proper base ray*, and assume $\mathbf{G}_r$ has unique connected component for large r.

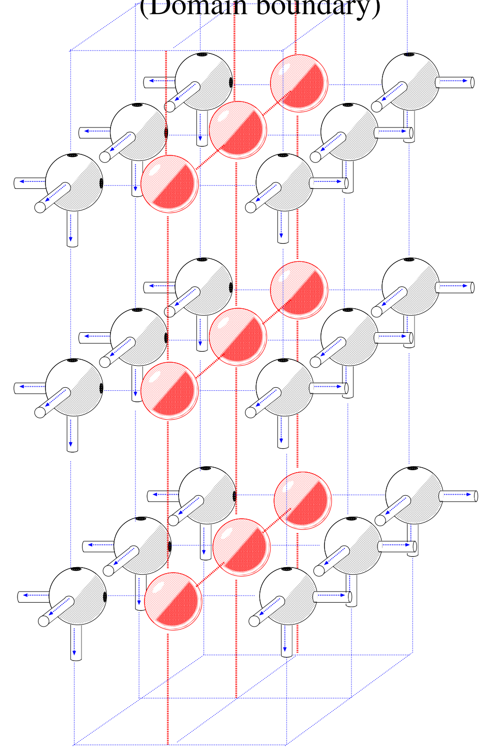
Defect Codimension in 3D

The 'Ice Cube' Shift:

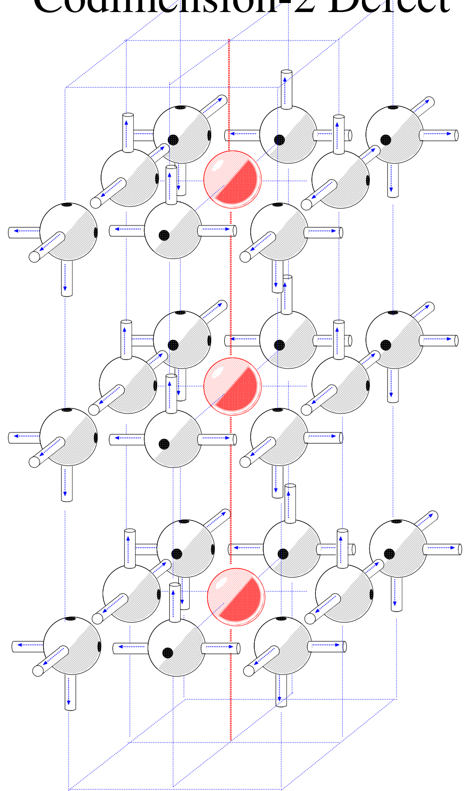


Codimension-1 Defect

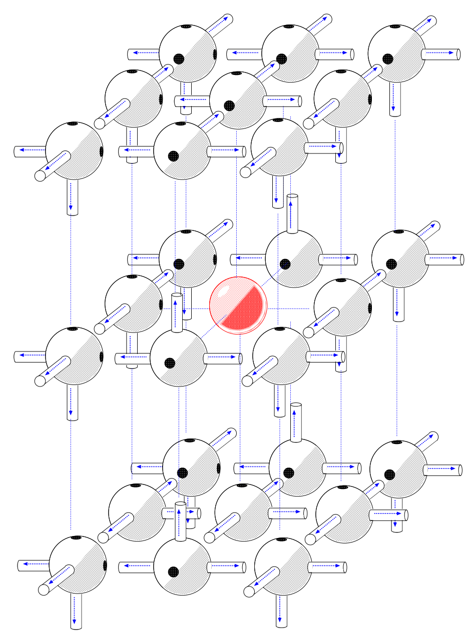
(Domain boundary)



Codimension-2 Defect



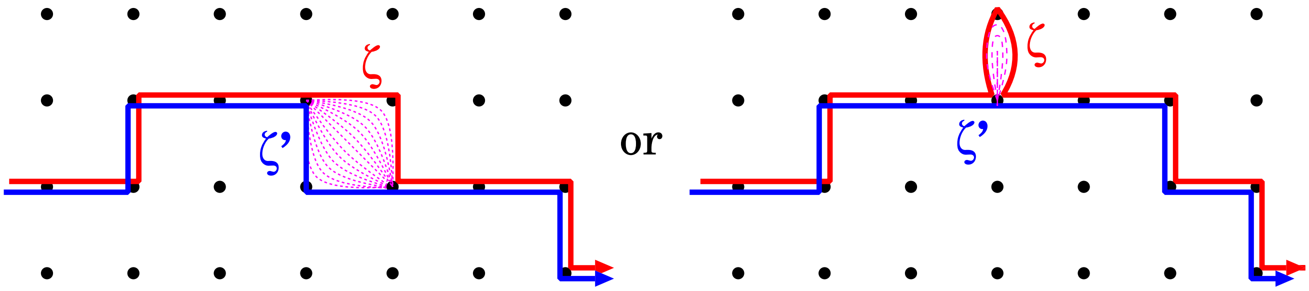
Codimension-3 Defect



Trail Homotopy

Let $\mathbb{Y} \subseteq \mathbb{Z}^D$ and let ζ and ζ' be trails in $\mathbb{Y}$.

ζ and ζ' are **homotopic in $\mathbb{Y}$** (notation: $\zeta \approx \zeta'$) if we can move from ζ to ζ' through a sequence of transformations like:



If $\mathbf{Y}$ is connected, then every homotopy class of $\pi_1(\mathbf{Y})$ can be represented as a (trail) homotopy class of trails in $\mathbb{Y}$.

Hence regard $\pi_1(\mathbb{Y}) = \{$group of $\mathbb{Y}$-homotopy classes of $\mathbb{Y}$-trails$\}$.

Lemma: *Let $C \in \mathcal{Z}^1_r(\mathfrak{A}, \mathcal{G})$. Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$. Let ζ be closed trail in $\mathbb{G}_r(\mathbf{a})$.*

(a) *If $\zeta \approx \zeta'$ in $\mathbb{G}_r(\mathbf{a})$, then $C(\zeta, \mathbf{a}) = C(\zeta', \mathbf{a})$.*

(e.g. If ζ is nullhomotopic in $\mathbb{G}_r(\mathbf{a})$, then $C(\zeta, \mathbf{a}) = e_{\mathcal{G}}$.)

(b) *Suppose $(\mathcal{G}, \cdot)$ is abelian. If $C \approx C'$ then $C(\zeta, \mathbf{a}) = C'(\zeta, \mathbf{a})$.* $\square$

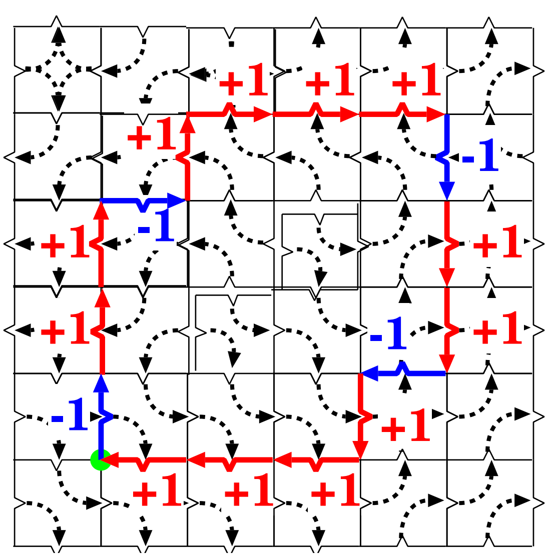
We say that $\mathbf{a}$ has a ***C*-pole** if $C(\zeta, \mathbf{a}) \neq e_{\mathcal{G}}$ for some closed trail $\zeta \in \pi_1[\mathbb{G}_r(\mathbf{a})]$.

Example: Recall $C : \mathfrak{Ice} \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}$

$c_1(\quad) := +1 =: c_2(\quad)$

$c_1(\quad) := -1 =: c_2(\quad)$

If ζ is the clockwise trail around the defect, then $C(\zeta, \mathbf{a}) = 8$. Thus, $\mathbf{a}$ has a pole.



12 x (+1) + 4 x (-1) = 8

Poles and Residues

Proposition: *Let* $\mathbf{a} \in \widetilde{\mathfrak{A}}$. *Let* $C \in \mathcal{Z}^1_r(\mathfrak{A}, \mathcal{G})$.

(a) $\mathrm{Res}_{\mathbf{a}} C : \pi_1[\mathbb{G}_r(\mathbf{a})] \ni \underline{\zeta} \mapsto C(\zeta, a) \in \mathcal{G}$ *is a group homomorphism.*

(b) *If* $(\mathcal{G}, \cdot)$ *is abelian, and* $C \approx C'$ *then* $\mathrm{Res}_{\mathbf{a}} C = \mathrm{Res}_{\mathbf{a}} C'$. *Thus, we get group homomorphism*

$$\mathrm{Res}_{\mathbf{a}} : \mathcal{H}_{\mathrm{dy}}(\mathfrak{A}, \mathcal{G}) \times \pi_1[\mathbb{G}_\infty(\mathbf{a})] \times \ni (\underline{C}, \underline{\zeta}) \mapsto C(\zeta, a) \in \mathcal{G}. \qquad \Box$$

The configuration $\mathbf{a}$ has a **$\mathcal{G}$-pole** if $\mathrm{Res}_{\mathbf{a}}$ is nontrivial homomorphism. The function $\mathrm{Res}_{\mathbf{a}}$ acts as an algebraic 'signature' of the defect in $\mathbf{a}$.

Theorem: *$\mathcal{G}$-poles are essential defects.* $\Box$

Persistence of Poles

Theorem: *If the function* $\Phi_* : \mathcal{H}^1(\mathfrak{A}, \mathcal{G}) \ni C \mapsto (C \circ \Phi) \in \mathcal{H}^1(\mathfrak{A}, \mathcal{G})$ *is surjective, then all $\mathcal{G}$-poles are Φ-persistent.*

Example: If $\Phi : \mathcal{I}^{\mathbb{Z}^2} \longrightarrow \mathcal{I}^{\mathbb{Z}^2}$ was a CA with $\Phi(\mathfrak{Ice}) \subseteq \Phi(\mathfrak{Ice})$, and Φ_* was surjective, then the ice pole would persist under Φ. $\diamondsuit$

Proof idea: Let R :=radius(Φ). If $\mathbf{a} \in \widetilde{\mathfrak{A}}$ and $\mathbf{a}' := \Phi(\mathbf{a})$, then $\mathbb{G}_{r+R}(\mathbf{a}) \subseteq \mathbb{G}_r(\mathbf{a}')$.

This yields homomorphisms $\Phi_\dagger : \pi_1[\mathbb{G}_{r+R}(\mathbf{a})] \longrightarrow \pi_1[\mathbb{G}_r(\mathbf{b})]$, for all $r \in \mathbb{N}$.

Lemma: *For all* $\zeta \in \pi_1[\mathbb{G}_{r+R}(\mathbf{a})]$ *and* $C' \in \mathcal{Z}^1_r(\mathfrak{A}, \mathcal{G})$, *if* $\zeta' := \Phi_\dagger(\zeta)$ *and* $C \approx \Phi_*(C')$, *then* $C'(\mathbf{a}', \zeta') = C(\mathbf{a}, \zeta)$. $\diamondsuit$

Now, if $\mathbf{a}$ has a C-pole for some $C \in \mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$, then there exists $\zeta \in \pi_1[\mathbb{G}_{r+R}(\mathbf{a})]$ with $C(\mathbf{a}, \zeta)$ nontrivial.

Φ_* is surjective, so $\exists\ C' \in \mathcal{Z}^1(\mathfrak{A}, \mathcal{G})$ with $\Phi_* C' \approx C$. Let $\zeta' := \Phi_\dagger(\zeta) \in \pi_1[\mathbb{G}_r(\mathbf{a}')]$. Then $C'(\mathbf{a}', \zeta') = C(\mathbf{a}, \zeta)$ is nontrivial. Thus $\mathbf{a}'$ has a C'-pole. $\Box$

Remark: We can also characterize poles using the *fundamental cocycles* of [K.Schmidt, 1998].

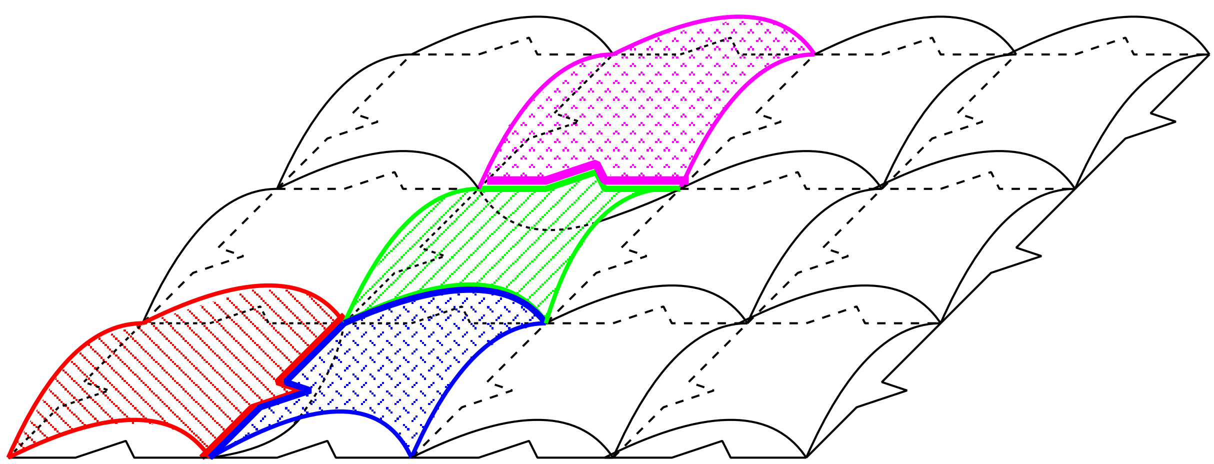
The Conway-Lagarias Tiling Group

Let $\mathcal{W}$ be a (finite) set of notched square prototiles (to tile $\mathbb{R}^2$). The **tile complex** of $\mathcal{W}$ is a 2-dimensional cell complex $\mathbf{X}$ defined as follows:

• For each $\mathsf{z} \in \mathbb{Z}^D$ and each $w \in \mathcal{W}$, there is a w-shaped 2-cell in $\mathbf{X}$, positioned in space 'over' z. Each notched edge of w is a 1-cell in $\mathbf{X}$.

• If z and z' are adjacent in $\mathbb{Z}^2$, and tiles w and w' 'match' along the corresponding edge, then glue together tiles (w, z) and (w', z') in $\mathbf{X}$.

Example: (Piece of tile-complex for $\mathfrak{Dom}$). Each square contains four 2-cells $\Big\{ \;,\;,\;,\; \Big\}$. Between each vertex-pair $\exists$ two edges $\{|, \succ\}$.



$\exists$ natural projection $\Pi : \mathbf{X} \longrightarrow \mathbb{R}^2$ (sending the vertices of $\mathbf{X}^0$ into $\mathbb{Z}^2$).

$$\Big(\text{Admissible } \mathcal{W}\text{-tiling } \mathbf{w} \text{ of } \mathbb{R}^2\Big) \cong \Big(\text{Continuous } \Pi\text{-section } \varsigma_{\mathbf{w}} : \mathbb{R}^2 \longrightarrow \mathbf{X}\Big)$$

$$\Big(\text{'Partial' } \mathcal{W}\text{-tiling } \mathbf{w} \text{ of } \mathbf{U} \subset \mathbb{R}^2\Big) \cong \Big(\text{'Partial' } \Pi\text{-section } \varsigma_{\mathbf{w}} : \mathbf{U} \longrightarrow \mathbf{X}\Big)$$

In the second case, $\varsigma_{\mathbf{w}}$ defines homomorphism $\varsigma_{\mathbf{w}}^* : \pi_1(\mathbf{U}) \longrightarrow \pi_1(\mathbf{X})$. Then:

$$\Big(\mathbf{U}^{\complement}\text{-hole in } \mathbf{w} \text{ can be admissibly filled}\Big) \Longrightarrow$$

$$\Big(\varsigma_{\mathbf{w}}^*\text{-image of any loop in } \mathbf{U} \text{ is nullhomotopic}\Big) \iff \Big(\varsigma_{\mathbf{w}}^* \text{ is trivial}\Big).$$

$\pi_1(\mathbf{X})$ = '**tile homotopy group**' [J.H.Conway & J.C.Lagarias, 1990; W.Thurston, 1990]

Higher homotopy/homology groups for Wang tiles

Let $\mathcal{W}$ be a (finite) set of D-dimensional notched hypercubic **Wang tiles** (to tile $\mathbb{R}^D$). Build a D-dimensional cell complex $\mathbf{X}$ analogous to before. Get projection $\Pi : \mathbf{X} \longrightarrow \mathbb{R}^D$ such that $\Pi(\mathbf{X}^0) = \mathbb{Z}^D$.

$$\left(\text{Admissible } \mathcal{W}\text{-tiling } \mathbf{w} \text{ of } \mathbb{R}^D\right) \cong \left(\text{Continuous } \Pi\text{-section } \varsigma_{\mathbf{w}} : \mathbb{R}^D \longrightarrow \mathbf{X}\right).$$

$$\left(\text{`Partial' } \mathcal{W}\text{-tiling } \mathbf{w} \text{ of } \mathbf{U} \subset \mathbb{R}^D\right) \cong \left(\text{`Partial' } \Pi\text{-section } \varsigma_{\mathbf{w}} : \mathbf{U} \longrightarrow \mathbf{X}\right).$$

In this case, for all $k \in \mathbb{N}$, the section $\varsigma_{\mathbf{w}}$ defines homomorphisms:

$$\begin{array}{rcll}
\pi_{\mathbf{k}}\varsigma_{\mathbf{w}} : \pi_k(\mathbf{U}, u) & \longrightarrow & \pi_k(\mathbf{X}, x); & \quad (x, u = \text{suitable basepoints}) \\
\mathcal{H}_{\mathbf{k}}\varsigma_{\mathbf{w}} : \mathcal{H}_k(\mathbf{U}, \mathcal{G}) & \longrightarrow & \mathcal{H}_k(\mathbf{X}, \mathcal{G}); & \quad ((\mathcal{G}, +) = \text{some coefficient group, e.g. } \mathcal{G} = \mathbb{Z}) \\
\mathcal{H}^{\mathbf{k}}\varsigma_{\mathbf{w}} : \mathcal{H}^k(\mathbf{U}, \mathcal{G}) & \longrightarrow & \mathcal{H}^k(\mathbf{X}, \mathcal{G}) &
\end{array}$$

$$\left(\text{Hole in } \mathbf{w} \text{ is fillable}\right) \Longrightarrow \left(\pi_k\varsigma_{\mathbf{w}},\ \mathcal{H}_k\varsigma_{\mathbf{w}} \text{ and } \mathcal{H}^k\varsigma_{\mathbf{w}} \text{ are trivial, } \ \forall\, k \in \mathbb{N}\right).$$

Homotopy/homology groups for subshifts of finite type

Let $\mathcal{A}$ be a finite alphabet. Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be a subshift of finite type of radius $r > 0$. Fix $R \geq r$. Treat $\mathcal{W} := \mathfrak{A}_{(R)}$ as Wang tiles with obvious edge-matching conditions. Get tile complex $\mathbf{X}_R$. Then:

$$\left(\mathbf{a} \in \mathfrak{A}\right) \cong \left(\mathcal{W}\text{-admissible tiling of } \mathbb{R}^D\right) \cong \left(\Pi\text{-section } \varsigma_{\mathbf{a}} : \mathbb{R}^D \longrightarrow \mathbf{X}_R\right).$$

Idea: Use homotopy/(co)homology groups of $\mathbf{X}_R$ as invariant for $\mathfrak{A}$ (and get algebraic invariants for codimension-$(k+1)$ defects in $\widetilde{\mathfrak{A}}$).

Problems:

[i] There $\exists$ many different Wang representations for $\mathfrak{A}$. None is 'canonical'. Different Wang representations may yield non-isomorphic groups.

[ii] Wang representations (and hence, their homotopy/homology groups) do not behave well under subshift homomorphisms (i.e. CA).

The Geller-Propp Projective Fundamental Group

Solution: There are natural surjections $\mathbf{X}_r \leftarrow \mathbf{X}_{r+1} \leftarrow \mathbf{X}_{r+2} \leftarrow \cdots$

Get homomorphisms $\pi_k(\mathbf{X}_r, x_r) \leftarrow \pi_k(\mathbf{X}_{r+1}, x_{r+1}) \leftarrow \pi_k(\mathbf{X}_{r+2}, x_{r+2}) \leftarrow \cdots$

(Here, $\{x_k\}$ are basepoints determined by some fixed $\mathbf{a} \in \mathfrak{A}$.)

Define kth **projective homotopy group** $\pi_k(\mathfrak{A}, \mathbf{a})$:= inverse limit of this sequence. (If $k = 1$ this is the *projective fundamental group* of W.Geller & J.Propp, 1995).

Likewise, we define kth **projective (co)homology groups**

$$\begin{aligned} \mathcal{H}_{\mathbf{k}}(\mathfrak{A}, \mathcal{G}) &:= \varprojlim \left(\mathcal{H}_k(\mathbf{X}_r, \mathcal{G}) \leftarrow \mathcal{H}_k(\mathbf{X}_{r+1}, \mathcal{G}) \leftarrow \mathcal{H}_k(\mathbf{X}_{r+2}, \mathcal{G}) \leftarrow \cdots\right) \\ \mathcal{H}^{\mathbf{k}}(\mathfrak{A}, \mathcal{G}) &:= \varinjlim \left(\mathcal{H}^k(\mathbf{X}_r, \mathcal{G}) \rightarrow \mathcal{H}^k(\mathbf{X}_{r+1}, \mathcal{G}) \rightarrow \mathcal{H}^k(\mathbf{X}_{r+2}, \mathcal{G}) \rightarrow \cdots\right) \end{aligned}$$

- Isomorphism invariants of $\mathfrak{A}$.
- Detects codimension $(k+1)$ defects.

Basepoint Freedom

The definition of $\pi_k(\mathfrak{A})$ depends upon a chosen 'basepoint' $\mathbf{a} \in \mathfrak{A}$.

We say $\mathfrak{A}$ is **basepoint free** in dimension k if, for any $\mathbf{a}, \mathbf{a}' \in \mathfrak{A}$, there is a canonical isomorphism $\pi_k(\mathfrak{A}, \mathbf{a}) \cong \pi_k(\mathfrak{A}, \mathbf{a}')$.

Proposition:

(a) *Suppose* $\Pi_r^0 : \mathbf{X}_r^0 \longrightarrow \mathbb{Z}^D$ *is injective for all large enough* $r \in \mathbb{N}$. *Then* $\mathfrak{A}$ *is basepoint-free in all dimensions.*

Suppose $(\mathfrak{A}, \sigma)$ *is topologically weakly mixing* [i.e. the Cartesian product $(\mathfrak{A} \times \mathfrak{A}, \sigma \times \sigma)$ is topologically transitive]. *Then:*

(b) *If* $\pi_1(\mathfrak{A}, \mathbf{a})$ *is abelian, then* $\mathfrak{A}$ *is basepoint free in dimension* 1.

(c) *If* $\pi_1(\mathfrak{A}, \mathbf{a})$ *is trivial, then* $\mathfrak{A}$ *is basepoint free in all dimensions.* $\square$

Projective Groups and Cellular Automata

Proposition: *Let* $\Phi\colon \mathcal{A}^{\mathbb{Z}^D}\longrightarrow\mathcal{A}^{\mathbb{Z}^D}$ *be a CA with* $\Phi(\mathfrak{A})\subseteq\mathfrak{A}$. *Then* Φ *induces group endomorphisms:*

$$\begin{array}{rcll} \pi_{\mathbf{d}}\boldsymbol{\Phi}\colon \pi_d(\mathfrak{A},\mathbf{a}) & \longrightarrow & \pi_d(\mathfrak{A},\mathbf{a}') & (\cong\ \pi_d(\mathfrak{A},\mathbf{a}) \textit{ if basepoint free})\\ \mathcal{H}_{\mathbf{d}}\boldsymbol{\Phi}\colon \mathcal{H}_d(\mathfrak{A},\mathcal{G}) & \longrightarrow & \mathcal{H}_d(\mathfrak{A},\mathcal{G}) & \\ \mathcal{H}^{\mathbf{d}}\boldsymbol{\Phi}\colon \mathcal{H}^d(\mathfrak{A},\mathcal{G}) & \longrightarrow & \mathcal{H}^d(\mathfrak{A},\mathcal{G}). & \end{array}$$

Proof: (Idea) If Φ has radius q, then Φ induces a cellular map $\Phi_*:\mathbf{X}_{R+q}\longrightarrow\mathbf{X}_R$ for all $R\geq r$, which yields corresponding homotopy/(co)homology homomorphisms. The resulting infinite commuting ladder of homomorphisms defines a homomorphism of the inverse/direct limit groups. □

Recall that $\pi_{\mathbf{k}}[\mathbb{G}_\infty(\mathbf{a})]$:= inverse limit of $\pi_k[\mathbb{G}_r(\mathbf{a})]$ as $r\to\infty$.

Likewise define $\mathcal{H}^k[\mathbb{G}_\infty(\mathbf{a})]$ (direct limit) and $\mathcal{H}_k[\mathbb{G}_\infty(\mathbf{a})]$ (inverse limit), $\forall\, k\in\mathbb{N}$.

If $\mathbf{a}\in\widetilde{\mathfrak{A}}$, then $\mathbf{a}$ defines 'partial' Π-section $\varsigma_{\mathbf{a}}:\mathbf{G}_R(\mathbf{a})\longrightarrow\mathbf{X}_R$ for all $R\geq r$. This induces group homomorphisms:

$$\begin{array}{rcl} \mathcal{H}_k\mathbf{a}\colon \mathcal{H}_k[\mathbb{G}_R(\mathbf{a}),\mathcal{G}] & \longrightarrow & \mathcal{H}_k(\mathbf{X}_R,\mathcal{G});\\ \mathcal{H}^k\mathbf{a}\colon \mathcal{H}^k(\mathbf{X}_R,\mathcal{G}) & \longrightarrow & \mathcal{H}^k[\mathbb{G}_R(\mathbf{a}),\mathcal{G}];\\ \pi_k\mathbf{a}\colon \pi_k[\mathbb{G}_R(\mathbf{a})] & \longrightarrow & \pi_k(\mathbf{X}_R). \end{array}$$

The resulting infinite commuting ladders of homomorphisms define homomorphisms of the inverse/direct limit groups. Thus, we have:

Theorem: (a) *Any* $\mathbf{a}\in\widetilde{\mathfrak{A}}$ *induces group homomorphisms:*
$\mathcal{H}_{\mathbf{k}}\mathbf{a}\colon \mathcal{H}_k[\mathbb{G}_\infty(\mathbf{a}),\mathcal{G}]\longrightarrow\mathcal{H}_k(\mathfrak{A},\mathcal{G})$ *and* $\mathcal{H}^{\mathbf{k}}\mathbf{a}\colon \mathcal{H}^k(\mathfrak{A},\mathcal{G})\longrightarrow\mathcal{H}^k[\mathbb{G}_\infty(\mathbf{a}),\mathcal{G}]$.
(b) *If* $\mathfrak{A}$ *is basepoint-free in dimension* k, *then* $\mathbf{a}$ *also induces a group homomorphism* $\pi_k\mathbf{a}:\pi_k[\mathbb{G}_\infty(\mathbf{a})]\longrightarrow\pi_k(\mathfrak{A})$.

We call $\pi_k\mathbf{a}$ (resp. $\mathcal{H}_k\mathbf{a}$ or $\mathcal{H}^k\mathbf{a}$) the **kth homotopy** (resp. **(co)homology**) **signature** of $\mathbf{a}$; if it is nontrivial, we say $\mathbf{a}$ has a **homotopy** (resp. (**co**)**homology**) **defect** of codimension $(k+1)$.

Persistence of Homotopy/(co)homology Defects

Theorem: *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be SFT. Let $\Phi\colon \mathcal{A}^{\mathbb{Z}^D} \to \mathcal{A}^{\mathbb{Z}^D}$ be CA with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$.*

(a) *Suppose $\mathfrak{A}$ is basepoint-free in dimension k. If $\pi_k\Phi : \pi_k(\mathfrak{A}) \longrightarrow \pi_k(\mathfrak{A})$ is injective, then every homotopy defect of codimension $(k+1)$ is Φ-persistent.*

(b) *If $\mathcal{H}_k\Phi : \mathcal{H}_k(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}_k(\mathfrak{A}, \mathcal{G})$ is injective, then every homology defect of codimension $(k+1)$ is Φ-persistent.*

(c) *If $\mathcal{H}^k\Phi : \mathcal{H}^k(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^k(\mathfrak{A}, \mathcal{G})$ is surjective, then every cohomology defect of codimension $(k+1)$ is Φ-persistent.* □

This follows from:

Theorem: *Let $\Phi\colon \mathcal{A}^{\mathbb{Z}^D} \longrightarrow \mathcal{A}^{\mathbb{Z}^D}$ be a CA with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$. Let $\mathbf{a} \in \widetilde{\mathfrak{A}}$ and let $\Phi(\mathbf{a}) = \mathbf{b}$. Then we have commuting diagrams:*

$$\begin{tikzcd}
\mathcal{H}_k[\mathbb{G}_\infty(\mathbf{a}), \mathcal{G}] \arrow[r, "\mathcal{H}_k\iota"] \arrow[d, "\mathcal{H}_k\mathbf{a}"'] & \mathcal{H}_k[\mathbb{G}_\infty(\mathbf{b}), \mathcal{G}] \arrow[d, "\mathcal{H}_k\mathbf{b}"] \\
\mathcal{H}_k(\mathfrak{A}, \mathcal{G}) \arrow[r, "\mathcal{H}_k\Phi"] & \mathcal{H}_k(\mathfrak{A}, \mathcal{G})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{H}^k[\mathbb{G}_\infty(\mathbf{a}), \mathcal{G}] & \mathcal{H}^k[\mathbb{G}_\infty(\mathbf{b}), \mathcal{G}] \arrow[l, "\mathcal{H}^k\iota"'] \\
\mathcal{H}^k(\mathfrak{A}, \mathcal{G}) \arrow[u, "\mathcal{H}^k\mathbf{a}"] & \mathcal{H}^k(\mathfrak{A}, \mathcal{G}) \arrow[l, "\mathcal{H}^k\Phi"'] \arrow[u, "\mathcal{H}^k\mathbf{b}"']
\end{tikzcd}$$

If $\mathfrak{A}$ is basepoint-free, we also get a commuting diagram:

$$\begin{tikzcd}
\pi_k[\mathbb{G}_\infty(\mathbf{a}), \omega] \arrow[r, "\pi_k\iota"] \arrow[d, "\pi_k\mathbf{a}"'] & \pi_k[\mathbb{G}_\infty(\mathbf{b}), \omega] \arrow[d, "\pi_k\mathbf{b}"] \\
\pi_k(\mathfrak{A}) \arrow[r, "\pi_k\Phi"] & \pi_k(\mathfrak{A})
\end{tikzcd}$$

Proof: (Idea) Stick together all the aforementioned infinite commuting ladders to get infinite commuting ‘girder’, which yields commuting square of inverse limit homomorphisms. □

$$
\begin{tikzcd}[column sep=small]
& \pi_d[\mathbb{G}_{q+R}(\mathbf{a})] \arrow[dl, "\iota_R^*"'] \arrow[dd, "\pi_d^{q+R}\mathbf{a}" near start] & & \pi_d[\mathbb{G}_{q+R+1}(\mathbf{a})] \arrow[ll, "\alpha^*_{q+R}"'] \arrow[dl, "\iota_{R+1}^*"'] \arrow[dd, "\pi_d^{q+R+1}\mathbf{a}" near start] & & \pi_d[\mathbb{G}_{q+R+2}(\mathbf{a})] \arrow[ll, "\alpha^*_{q+R+1}"'] \arrow[dl, "\iota_{R+2}^*"'] \arrow[dd, "\pi_d^{q+R+2}\mathbf{a}" near start] & & \cdots \arrow[ll, "\alpha^*_{q+R+2}"'] & & \pi_d[\mathbb{G}_\infty(\mathbf{a})] \arrow[dl, "\iota^*"'] \arrow[dd, "\pi_d\mathbf{a}"] \\
\pi_d[\mathbb{G}_R(\mathbf{b})] \arrow[dd] & & \pi_d[\mathbb{G}_{R+1}(\mathbf{b})] \arrow[ll, "\beta^*_R"'] \arrow[dd, "\pi_d^{R+1}\mathbf{b}"' near end] & & \pi_d[\mathbb{G}_{R+2}(\mathbf{b})] \arrow[ll, "\beta^*_{R+1}"'] \arrow[dd, "\pi_d^{R+2}\mathbf{b}"' near end] & & \cdots \arrow[ll, "\beta^*_{R+2}"'] & & \pi_d[\mathbb{G}_\infty(\mathbf{b})] \arrow[dd, "\pi_d\mathbf{b}"'] & \\
& \pi_d^{q+R}(\mathfrak{A}) \arrow[dl, "\pi_d^R\Phi"] & & \pi_d^{q+R+1}(\mathfrak{A}) \arrow[ll, "\zeta^*_{q+R}"] \arrow[dl, "\pi_d^{R+1}\Phi"] & & \pi_d^{q+R+2}(\mathfrak{A}) \arrow[ll, "\zeta^*_{q+R+1}"] \arrow[dl, "\pi_d^{R+2}\Phi"] & & \cdots \arrow[ll, "\zeta^*_{q+R+2}"] & & \pi_d(\mathfrak{A}) \arrow[dl, "\pi_d\Phi"] \\
\pi_d^R(\mathfrak{A}) & & \pi_d^{R+1}(\mathfrak{A}) \arrow[ll, "\zeta^*_R"] & & \pi_d^{R+2}(\mathfrak{A}) \arrow[ll, "\zeta^*_{R+1}"] & & \cdots \arrow[ll, "\zeta^*_{R+2}"] & & \pi_d(\mathfrak{A}) &
\end{tikzcd}
$$

Equivariant (co)Homology

Question: *Is there a higher-codimension analog to the codimension-2 'poles' from dynamical cohomology?*

Let $k \in \mathbb{N}$. A (cubic) **k-chain** is a formal 'sum' of k-dimensional cubes in $\mathbb{R}^D$ with vertices in $\mathbb{Z}^D$ (combinatorial analog of 'k-dimensional submanifold'). Fix an abelian group $(\mathcal{G}, +)$. Define $\mathcal{C}_{\mathbf{k}} :=$ {free abelian group of cubic k-chains}.

$$\mathcal{C}^{\mathbf{k}}(\mathcal{G}) \quad := \quad \{(\text{cubic}) \ k\text{-cochains}\} \ = \ \{\text{homomorphisms } c : \mathcal{C}_k \longrightarrow \mathcal{G}\}.$$

(combinatorial analog of 'k-dimensional differential forms').

$\mathbb{Z}^D$ acts on $\mathbb{R}^D$ by shifts. This induces $\mathbb{Z}^D$-action on $\mathcal{C}_k$, and thus on $\mathcal{C}^k$.

Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ be subshift. An **equivariant cochain** on $\mathfrak{A}$ is a continuous function $C : \mathfrak{A} \longrightarrow \mathcal{Z}^k(\mathcal{G})$ which commutes with all $\mathbb{Z}^D$-shifts.

Idea: For any $\mathbf{a} \in \mathfrak{A}$, $C(\mathbf{a})$ is a cochain. If $\zeta \in \mathcal{C}_k$ is any chain, then

$$C\left(\sigma^{\mathsf{z}}(\mathbf{a})\right)[\zeta] \ = \ C(\mathbf{a})\left[\sigma^{\mathsf{z}}(\zeta)\right].$$

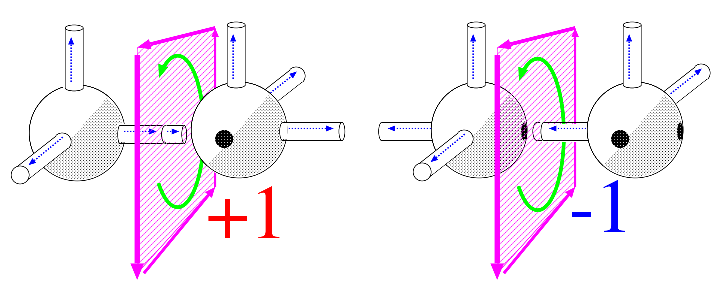
Let $\mathcal{C}^{\mathbf{k}}_{\mathrm{eq}}(\mathfrak{A}, \mathcal{G}) :=$ {equivariant k-chains}. There is a natural **coboundary** operator $\delta^{\mathbf{k}} : \mathcal{C}^k_{\mathrm{eq}} \longrightarrow \mathcal{C}^{k+1}_{\mathrm{eq}}$. Let $\mathcal{Z}^{\mathbf{k}}_{\mathrm{eq}} := \ker(\delta^k)$ be the group of **equivariant cocycles**.

Examples: (a) Recall that a 'dynamical' cocycle is a function $c : \mathbb{Z}^D \times \mathfrak{A} \longrightarrow \mathcal{G}$ such that

$$c(\mathsf{y} + \mathsf{z}, \mathbf{a}) \ = \ c[\mathsf{y}, \sigma^{\mathsf{z}}(\mathbf{a})] + c(\mathsf{z}, \mathbf{a}).$$

Any dynamical cocycle defines an equivariant cocycle $C \in \mathcal{Z}^1_{\mathrm{eq}}$ as follows: for any chain $\zeta \in \mathcal{C}_k$, treat ζ as a 'trail' and define $C(\zeta, \mathbf{a})$ as before.

(b) (*Equivariant cocycle $C \in \mathcal{Z}^2_{\mathrm{eq}}$ on 'ice cube' shift*) This picture shows how to evaluate C on a single 2-cell (i.e. oriented square). To evaluate C on 2-chain, sum values on all constituent 2-cells.



Equivariant Cohomology vs. Dynamical Cohomology

Let $\mathcal{B}_{\text{eq}}^{\mathbf{k}} := \text{image}(\delta^{k-1})$ (equivariant coboundaries).

Define **equivariant cohomology group** $\mathcal{H}_{\text{eq}}^{k}(\mathfrak{A}, \mathcal{G}) := \mathcal{Z}_{\text{eq}}^{k}/\mathcal{B}_{\text{eq}}^{k}$.

$\mathcal{Z}_{\text{eq}}^{k}$ and $\mathcal{B}_{\text{eq}}^{k}$ are σ-invariant. Thus, σ induces $\mathbb{Z}^D$-action on $\mathcal{H}_{\text{eq}}^{k}$. Let

$$\begin{aligned}\mathcal{Z}_{\text{dy}}^{\mathbf{1}}(\mathfrak{A}, \mathcal{G}) &:= \{\text{dynamical cocycles}\};\\ \mathcal{H}_{\text{dy}}^{\mathbf{1}}(\mathfrak{A}, \mathcal{G}) &:= \text{`dynamical' cohomology group.}\end{aligned}$$

Theorem: *Let $(\mathcal{G}, +)$ be abelian. There are canonical isomorphisms:*

$$\mathcal{Z}_{\text{eq}}^{1}(\mathfrak{A}, \mathcal{G}) \cong \mathcal{Z}_{\text{dy}}^{1}(\mathfrak{A}, \mathcal{G}) \quad \textit{and} \quad \mathcal{H}_{\text{eq}}^{1}(\mathfrak{A}, \mathcal{G}) \cong \mathcal{H}_{\text{dy}}^{1}(\mathfrak{A}, \mathcal{G}).$$

Proof idea: Given $C \in \mathcal{Z}_{\text{dy}}^{1}$, define $C' \in \mathcal{Z}_{\text{eq}}^{1}$ as follows: for any chain $\zeta \in \mathcal{C}_k$, represent ζ with (sum of) trails ζ', and then define $C'(\zeta, \mathbf{a}) := C(\zeta', \mathbf{a})$. This sends cocycles to cocycles because $\left(\delta^1 C' \equiv 0\right) \iff \left(C'(\partial_2 \xi, \mathbf{a}) = 0 \text{ for all } \xi \in \mathcal{C}_2\right) \iff \left(C(\zeta', \mathbf{a}) = 0 \text{ for any closed trail } \zeta' \text{ in } \mathbb{Z}^D\right)$. □

Codimension-k poles

Let $\partial_k : \mathcal{C}_k \longrightarrow \mathcal{C}_{k-1}$ be combinatorial 'boundary' operator

Let $\mathcal{Z}_{\mathbf{k}} := \ker(\partial_k) = \{k\text{-dimensional cycles}\}$ ('submanifolds without boundary').

Example: $\mathcal{Z}_1 := \{\text{(sums of) closed trails}\}$.

If $C \in \mathcal{Z}_{\text{eq}}^{k}(\mathfrak{A}, \mathcal{G})$, and $\mathbf{a} \in \mathfrak{A}$, and $\zeta \in \mathcal{Z}_k$, then $C(\mathbf{a}, \zeta) = 0$.

If $\mathcal{G}$ is discrete, then C is 'locally determined' by rule of radius $R > 0$.

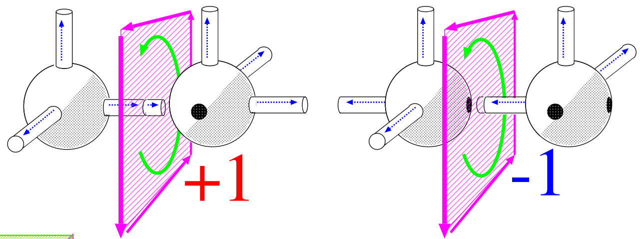
If $\mathbf{a} \in \widetilde{\mathfrak{A}}$, and ζ stays inside $\mathbb{G}_r(\mathbf{a})$ (for some $r \geq R$), then $C(\mathbf{a}, \zeta)$ is still well-defined.

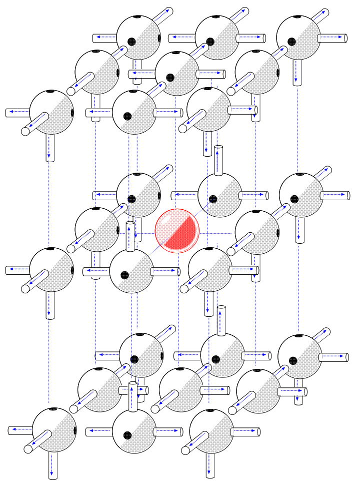
$\mathbf{a}$ has a **C-pole** (of radius r) if there is some cycle ζ such that $C(\mathbf{a}, \zeta) \neq 0$.

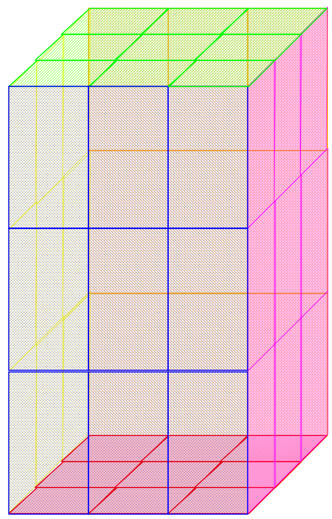
$\mathbf{a}$ has a **projective C-pole** if $\mathbf{a}$ has a radius-r pole for all large $r \in \mathbb{N}$.

Example: Codimension-3 pole in Ice Cube shift

Let $\mathfrak{Q}$ be the 'ice cube' shift. Recall the equivariant 2-cocycle $C \in \mathcal{Z}^2_{\text{eq}}(\mathfrak{Q})$ defined:





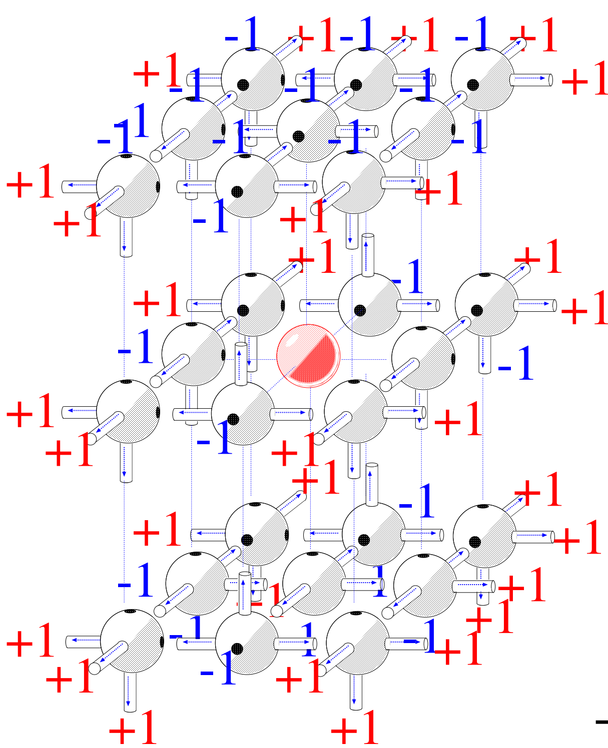


Let $\mathbf{a}$ be the defective configuration at left.

Let $\zeta \in \mathcal{Z}_2$ be the 2-cycle on right (the oriented boundary of a $3 \times 3 \times 3$ cube).

Then
$C(\mathbf{a}, \zeta)$
$= 30 - 24$
$= 6.$

Thus, the defect in $\mathbf{a}$ is a C-pole with residue 6.



-9	-9
+9	-3
+8	-4
+9	-3
+4	-5
+30	-24=6

Persistence of Poles

Theorem: *Projective poles are essential defects.*

Proof idea: Similar to 'dynamical' cohomology proof for codimension-2 poles. □

Theorem: *Let* $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ *be an SFT. Let* $\Phi : \mathcal{A}^{\mathbb{Z}^D} \to \mathcal{A}^{\mathbb{Z}^D}$ *be a CA with* $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$*. Fix* $d \in [1...D]$.

(a) *Define* $\Phi_* : \mathcal{C}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{C}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$ *by* $\Phi_* C(\mathbf{a}, \zeta) := C[\Phi(\mathbf{a}), \zeta]$.
This induces endomorphism $\mathcal{H}^d_{\text{eq}}\Phi : \mathcal{H}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G}) \longrightarrow \mathcal{H}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$.

(b) *Suppose* $\mathcal{H}^d_{\text{eq}}\Phi$ *is an epimorphism.*

[i] *If* $\mathcal{G}$ *is the additive group of a field* (e.g. $\mathcal{G} = \mathbb{Z}_{/p}$ for p prime), *then all projective* $\mathcal{G}$*-poles are* Φ*-persistent.*

[ii] *If* $d = 1$ *or* D*, then any projective* d*-pole is* Φ*-persistent.* □

Invariant Cohomology

Questions: (a) *What is relationship between the (dynamical) cocycles of* $\mathfrak{A}$ *and the (co)homology groups of Wang tile cell complex of* $\mathfrak{A}$*?*

(b) *What is relationship between poles and (co)homology defects?*

$\forall\ r \geq R :=$ radius($\mathfrak{A}$), let $\mathbf{X}_r :=$ radius-r Wang tile cell complex for $\mathfrak{A}$.

The σ-action on $\mathfrak{A}$ induces natural $\mathbb{Z}^D$-action on $\mathbf{X}_r$; hence on $\mathcal{H}^k(\mathbf{X}_r, \mathcal{G})$.

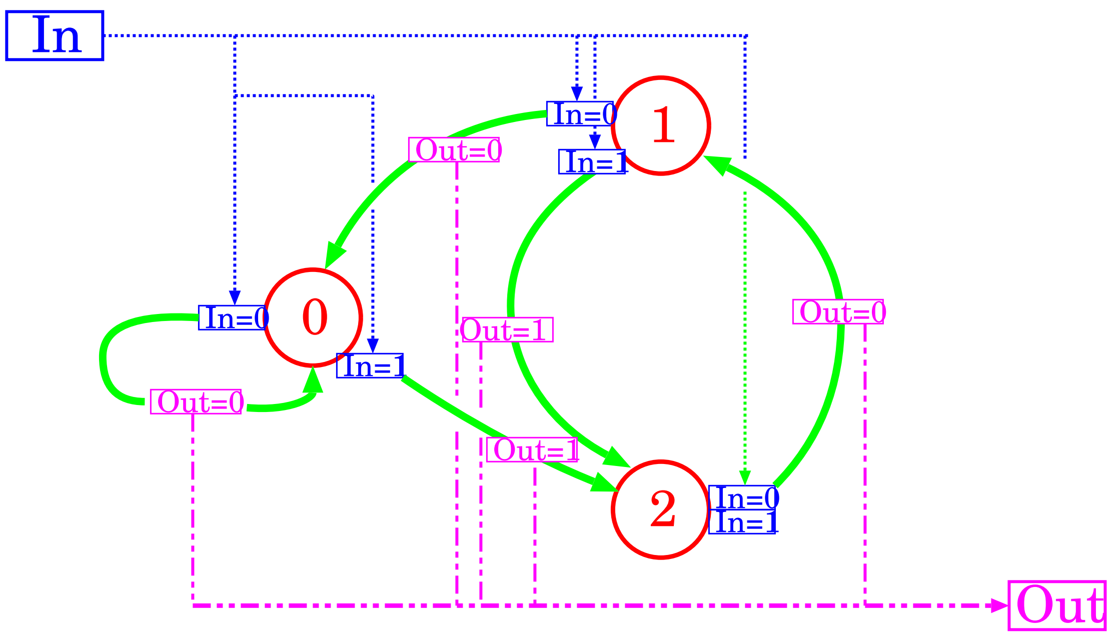
Let $\mathcal{H}^k_{\text{inv}}(\mathbf{X}_r, \mathcal{G})$:= group of $\mathbb{Z}^D$-fixed elements of $\mathcal{H}^k(\mathbf{X}_r, \mathcal{G})$. We define the kth **invariant cohomology group** of $\mathfrak{A}$:

$$\mathcal{H}^{\mathbf{k}}_{\text{inv}}(\mathfrak{A}, \mathcal{G}) := \varinjlim \left(\mathcal{H}^k_{\text{inv}}(\mathbf{X}_{R+1}, \mathcal{G}) \to \mathcal{H}^k_{\text{inv}}(\mathbf{X}_{R+2}, \mathcal{G}) \to \mathcal{H}^k_{\text{inv}}(\mathbf{X}_{R+3}, \mathcal{G}) \to \cdots \right)$$

Theorem: *Let* $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}^D}$ *be SFT. Let* $(\mathcal{G}, +)$ *be discrete abelian group. There is a natural isomorphism* $\mathcal{H}^d_{\text{inv}}(\mathfrak{A}, \mathcal{G}) \cong \mathcal{H}^d_{\text{eq}}(\mathfrak{A}, \mathcal{G})$.
In particular, $\mathcal{H}^1_{\text{inv}}(\mathfrak{A}, \mathcal{G}) \cong \mathcal{H}^1_{\text{dy}}(\mathfrak{A}, \mathcal{G})$. □

Thus, poles are $\mathcal{H}^d_{\text{inv}}(\mathfrak{A}, \mathcal{G})$-cohomology defects.

Finite State Machines



A **finite state machine** (FSM) has a finite set of internal *states* $\mathcal{S}$, finite *input alphabet* $\mathcal{I}$ and *output alphabet* $\mathcal{O}$, and *update rule*

$$\Upsilon : \mathcal{I} \times \mathcal{S} \longrightarrow \mathcal{S} \times \mathcal{O}$$

If FSM begins in state s_0, and receives input stream $i_0, i_1, i_2, \ldots, i_{N-1}$, then it proceeds through states $s_1, s_2, \ldots, s_N$ and produces output $o_1, o_1, \ldots, o_N$, where, for every $n \in [0...N)$,

$$\Upsilon(i_n, s_n) \quad = \quad (s_{n+1}, o_{n+1})$$

Diagramatically:

$$\begin{array}{ccccccccccccccc}
i_0 & & i_1 & & i_2 & & i_3 & \ldots & \ldots & \ldots & i_{N-1} & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & & & \\
s_0 & \Longrightarrow & s_1 & \Longrightarrow & s_2 & \Longrightarrow & s_3 & \Longrightarrow & \ldots & \Longrightarrow & s_{N-1} & \Longrightarrow & s_N & & \\
& \searrow & & \searrow & & \searrow & & \searrow & & & & \searrow & & & \\
& o_1 & & o_2 & & o_3 & & o_4 & \ldots & \ldots & \ldots & o_N & & &
\end{array}$$

Defect Particle Kinematics

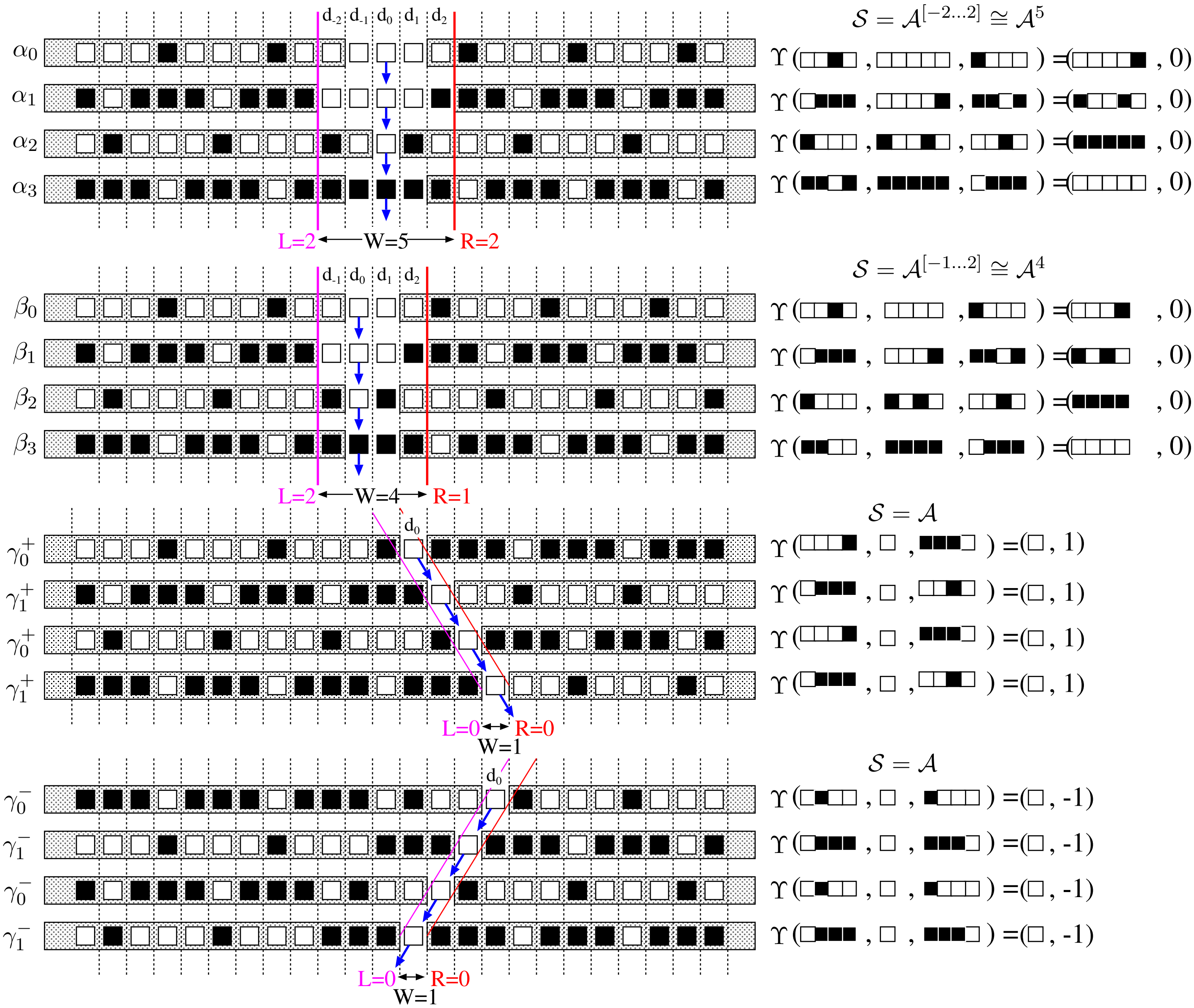
A **defect particle** in **a** is a defect which is finite in size and whose size in $\Phi^t(\mathbf{a})$ remains bounded for all $t > 0$. Defect particles act like FSM:

Internal state = $\mathfrak{A}$-inadmissible symbol-sequence inside defect.

Input = $\mathfrak{A}$-admissible symbols on boundary of defect.

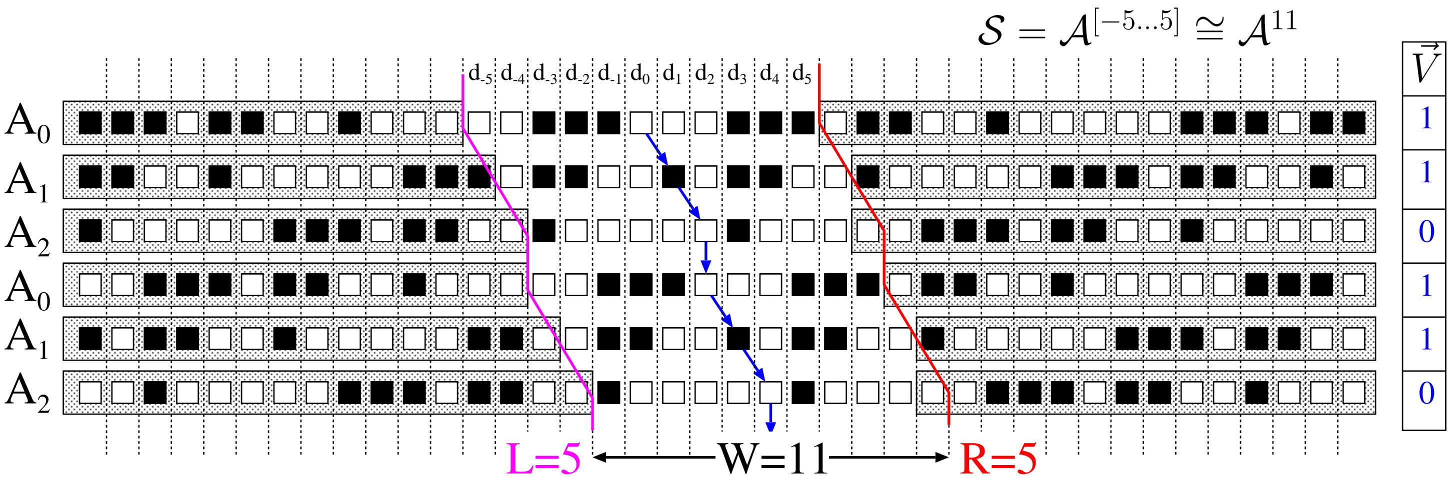
Output = Instantaneous velocity.

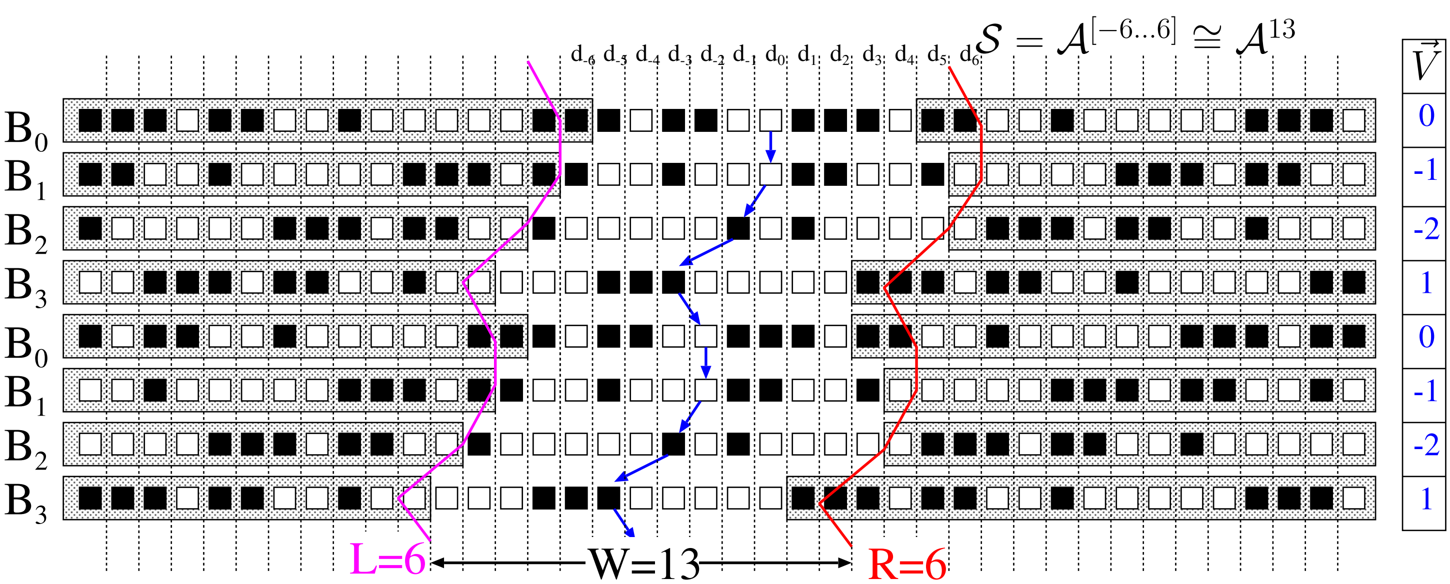
Example: Defect particles in ECA#54:



Defect Particle Kinematics

Example: The **A** and **B** defect particles of ECA#110:



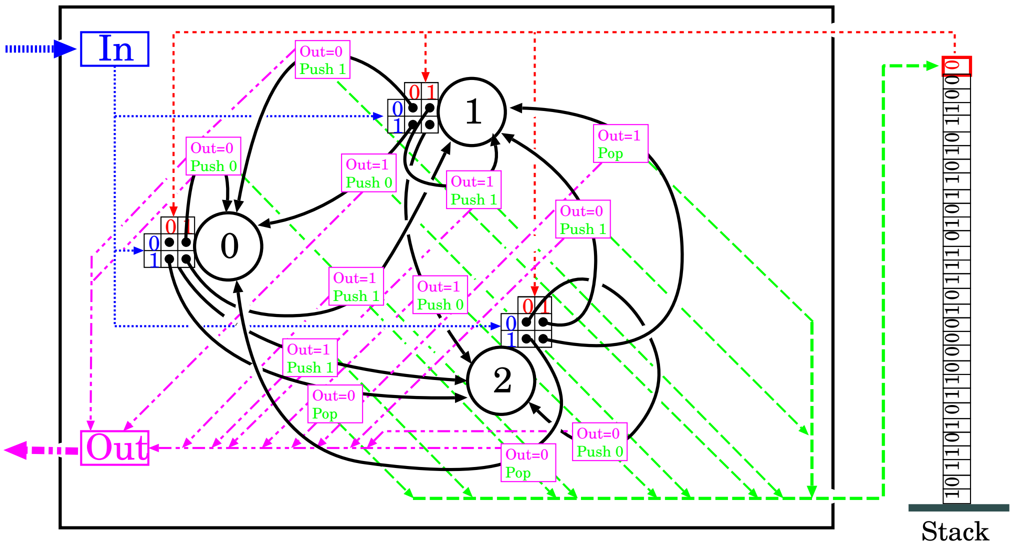


Remarks: • The width of inadmissible region fluctuates over time. We define the **width** of the defect to be the maximum width it ever obtains. This defines the effective 'state space' of the FSM representation.

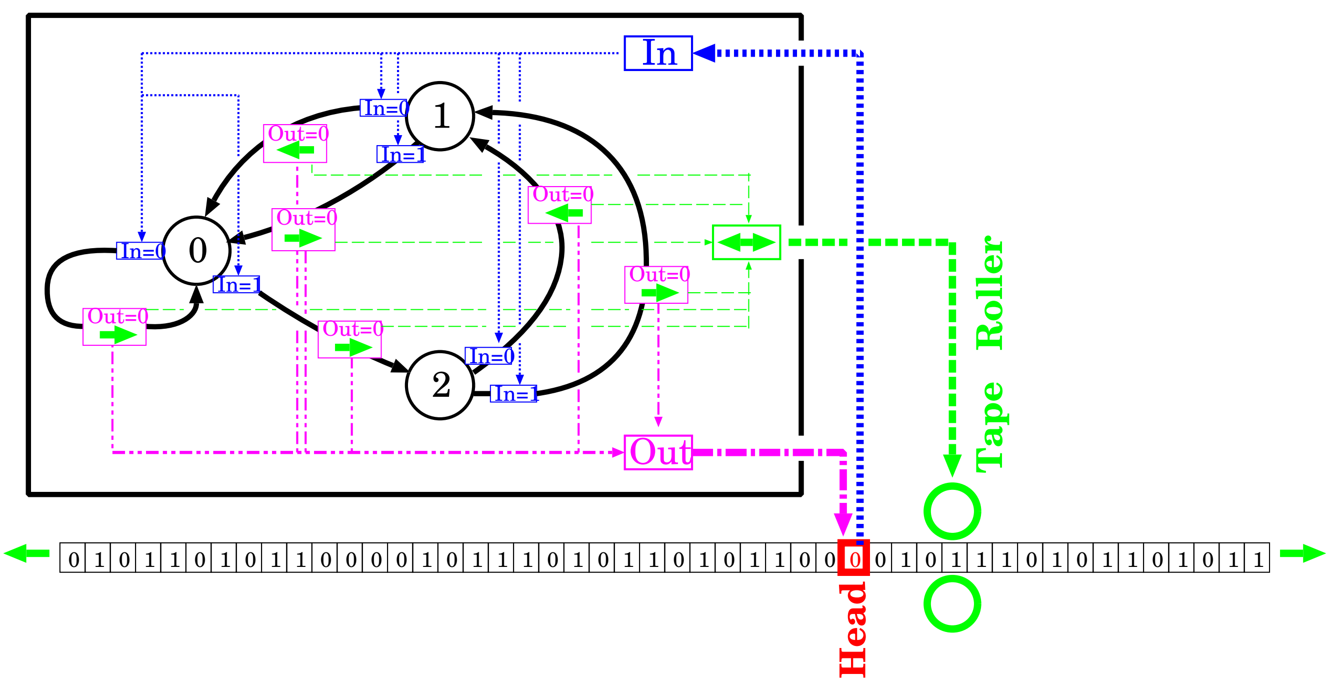
• If $\mathfrak{A}$ is (Φ, σ)-periodic (as in these examples), then the FSM is driven by periodic input, so its long-term behaviour is periodic.

• The defect velocity fluctuates over time, but there is a long-term 'average' velocity obtained by averaging over the period.

Pushdown Automata and Turing Machines



A **pushdown automaton** (PDA) is an FSM augmented with 'last in, first out' memory model called a **stack**. The machine can 'push' symbols onto the stack, and later 'pop' them off the stack in reverse order.



A **Turing machine** is an FSM augmented with a biinfinite random access memory model called a 'tape'. The FSM acts has a 'head' which can read/write symbols as it moves along the tape.

One-dimensional CA: Kinematic Regimes

In one-dimensional CA, the particle kinematics depends upon the kind of subshifts found to the right and left of the particle.

Defect Kinematic Regimes		Right Side (σ,Φ)-Dynamics			
	σ-dynamics	Zero Entropy, σ-periodic	Right-regular	Nonzero σ-Entropy, Not σ-periodic	
σ-dynamics	Φ-dynamics	⇓ Φ-Periodic or Φ-Fixed	Right-resolving	Φ-Periodic or Φ-Fixed	Anything else
Left Side (σ,Φ)-Dynamics: Zero Entropy, σ-periodic	⇒ Φ-Periodic or Φ-Fixed	**Ballistic**	**Diffusive**	**Autonomous PDA**	Complicated
Left-regular	Left-resolving	**Diffusive**	**Diffusive**	**Markov PDA**	Complicated
Nonzero σ-Entropy, Not σ-periodic	Φ-Periodic or Φ-Fixed	**Autonomous PDA**	**Markov PDA**	**Turing Machine**	Complicated
	Anything else	Complicated		Complicated	

Ballistic: Defect has (Φ, σ)-periodic subshifts on both sides. Acts like FSM driven by periodic input. Moves with constant average velocity through periodic background. Examples: ECAs 54, 62, 110, and 184

Diffusive: Regular, Φ-resolving subshifts on one or both sides. Acts like FSM driven by Markov process. Performs generalized random walk. Example: ECA #18.

Turing Machine: Defect moves through Φ-fixed, positive σ-entropy background, and modifies background as it goes. Acts like Turing machine: particle is the 'head', and inert background is the 'tape'.

Autonomous Pushdown Automaton: Φ-fixed, positive σ-entropy domain on one side (which acts as a 'stack' memory), and zero-entropy domain on the other side. Acts like a PDA without external input.

Markov PDA: Φ-fixed, positive σ-entropy domain on one side (acts as a 'stack'), and regular Φ-resolving subshift on the other. Acts like a PDA driven by a Markov process.

Regular Markov Subshifts & Resolving CA

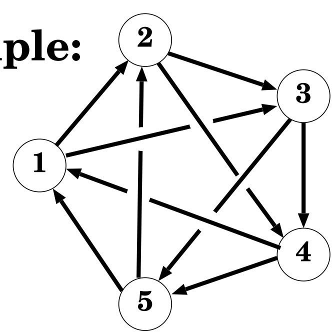
$\forall\ a \in \mathcal{A}$, let $\mathcal{F}(a) \subseteq \mathcal{A}$ be a set of 'admissible followers'. Write $a \rightsquigarrow b$ if $b \in \mathcal{F}(a)$.

The corresponding **Markov subshift** $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ is the set of all infinite sequences $[\cdots \rightsquigarrow a \rightsquigarrow b \rightsquigarrow c \rightsquigarrow d \rightsquigarrow \cdots]$ (Every SFT can be recoded thus.)

Let $\mathcal{P}(a) := \{b \in \mathcal{A}\ ;\ b \rightsquigarrow a\}$ be the set of admissible 'predecessors'.

$\mathfrak{A}$ is **regular** if $\exists F \in \mathbb{N}$ such that $\#[\mathcal{F}(a)] = F$ for all $a \in \mathcal{A}$, and $\exists P \in \mathbb{N}$ such that $\#[\mathcal{P}(a)] = P$ for all $a \in \mathcal{A}$.

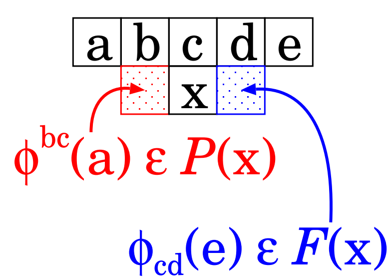
Example:



$$\begin{array}{ll} \mathcal{F}(1) = \{2,3\}; & \mathcal{P}(1) = \{4,5\} \\ \mathcal{F}(2) = \{3,4\}; & \mathcal{P}(2) = \{5,1\} \\ \mathcal{F}(3) = \{4,5\}; & \mathcal{P}(3) = \{1,2\} \\ \mathcal{F}(4) = \{5,1\}; & \mathcal{P}(4) = \{2,3\} \\ \mathcal{F}(5) = \{1,2\}; & \mathcal{P}(5) = \{3,4\} \end{array}$$

Let $\Phi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ be a CA with local rule $\phi : \mathcal{A}^3 \longrightarrow \mathcal{A}$. Suppose $\Phi(\mathfrak{A}) \subset \mathfrak{A}$. Let $(b \rightsquigarrow c \rightsquigarrow d)$ and let $x := \phi(b,c,d)$.

If $d \rightsquigarrow e$, then $x \rightsquigarrow \phi(c,d,e)$. Thus, we get function $\phi_{c,d} : \mathcal{F}(d) \longrightarrow \mathcal{F}(x)$. We say Φ is **right-resolving** if $\phi_{c,d}$ is bijective for all such (c,d).
If $a \rightsquigarrow b$, then $\phi(a,b,c) \rightsquigarrow x$. Thus, we get function $\phi^{b,c} : \mathcal{P}(b) \longrightarrow \mathcal{P}(x)$. We say Φ is **left-resolving** if $\phi^{b,c}$ is bijective for all such (b,c).



Φ is **resolving** if it is both left- and right- resolving.

Examples: (a) *Permutative* CA acting on full shift $\mathfrak{A} = \mathcal{A}^{\mathbb{Z}}$.

(b) *Linear* CA acting on Markov subgroup. (Here $\mathcal{A}$ is a group, so $\mathcal{A}^{\mathbb{Z}}$ is a group. $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ is a subgroup, and $\Phi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ is endomorphism.)

Diffusive Defect Particle Kinematics

The **Parry measure** μ is the measure of maximal entropy on $\mathfrak{A}$. It is Markov measure given equal transition probability to all $b \in \mathcal{F}(a)$.

Theorem: *Let $\mathfrak{A} \subset \mathcal{A}^{\mathbb{Z}}$ be regular Markov subshift. Let $\Phi : \mathcal{A}^{\mathbb{Z}} \longrightarrow \mathcal{A}^{\mathbb{Z}}$ be CA with $\Phi(\mathfrak{A}) \subseteq \mathfrak{A}$ and Φ resolving on $\mathfrak{A}$. Let $\mu =$ Parry measure on $\mathfrak{A}$. (Then $\Phi\mu = \mu$.)*

Let $\mathbf{l} \in \mathcal{A}^{(-\infty\ldots0)}$ be μ-random, left-infinite $\mathfrak{A}$-admissible sequence.

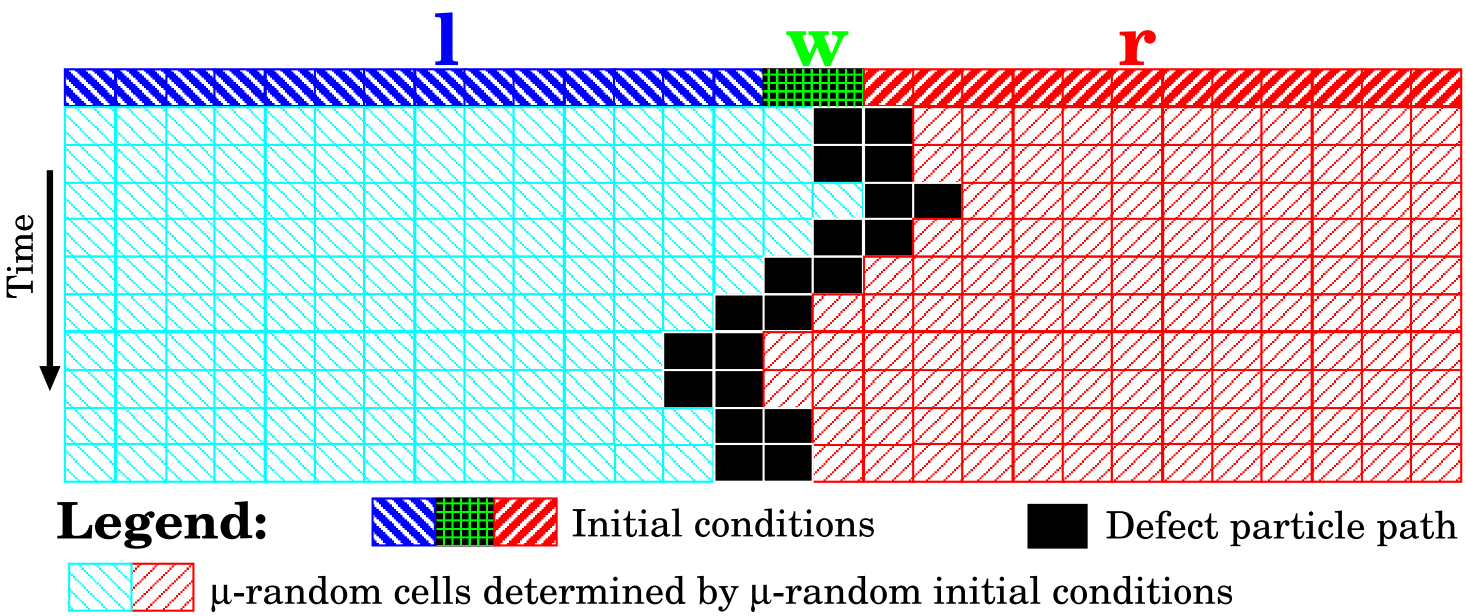
Let $\mathbf{r} \in \mathcal{A}^{[W\ldots\infty)}$ be μ-random, right-infinite $\mathfrak{A}$-admissible sequence.

Let $\mathbf{w} \in \mathcal{A}^{[0\ldots W)}$ be 'defect' word. Set initial condition: $\mathbf{a} := \mathbf{l}.\mathbf{w}.\mathbf{r}$.

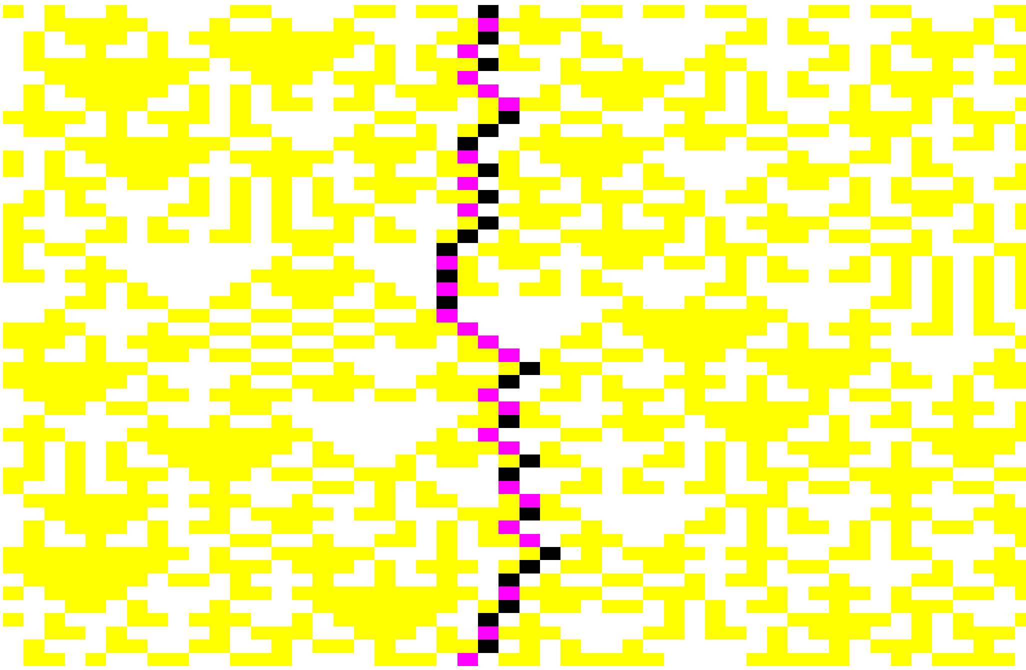
Define $\zeta : \mathbb{N} \longrightarrow \mathbb{Z}$ by $\zeta(t) :=$ position of defect in $\Phi^t(\mathbf{a})$. Then ζ is a generalized random walk. [i.e. increments of ζ are a hidden Markov process].

(Generalizes Eloranta [1993-1995]; similar result for 0-width defects in 'partially permutive' CA.)

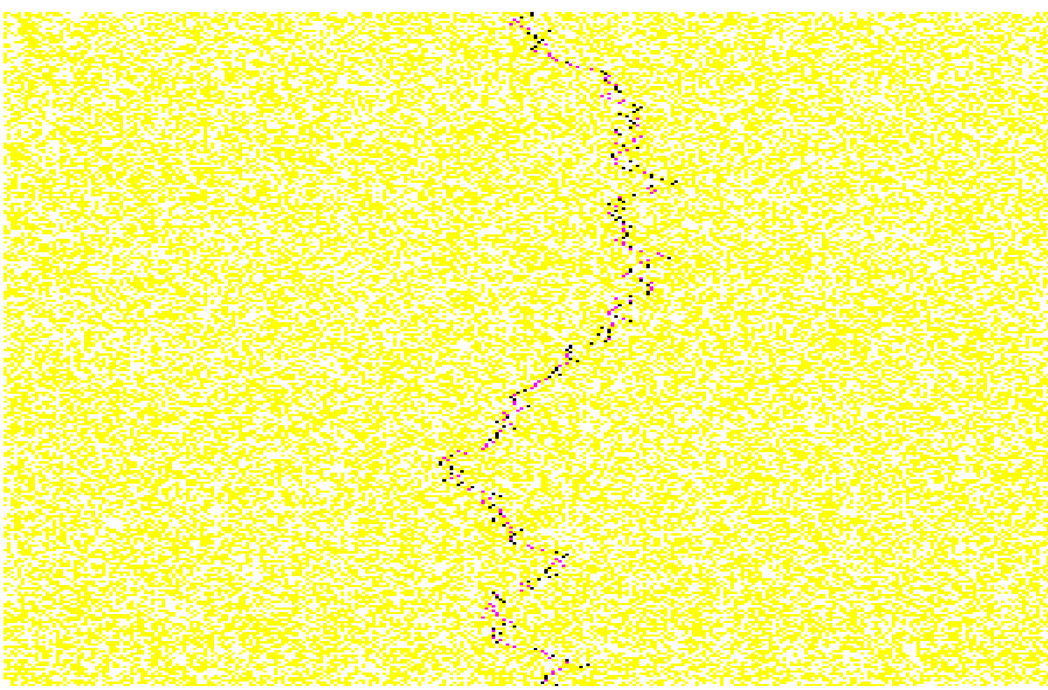
Proof idea: The defect motion is driven by 'μ-random information' coming in from the left and right, as follows:



Diffusive Defect Particle Kinematics



Scale: 50×50 (space $\times$ time)



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