Social choice with approximate interpersonal comparison of welfare gains

Informal Microeconomics Seminar
Université de Montréal

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—William Stanley Jevons

"*...il n'y a aucun rapport entre l'ophélimité du vin pour le vigneron et pour le laboureur, ni entre l'ophélimité du ble pour le vigneron et pour le laboureur. Il faut toujours se rapeller ce caractére subjectif de l'ophélimité.*"

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Consider the following statements.

(A) “A bowl of rice will benefit a starving man more than a well-fed man.”

(B) “Take the last slice of apple cake. You will enjoy it more than I will.”

(C) “The marginal utility of one dollar for someone on minimum wage is greater than the marginal utility of one dollar for a typical billionaire.”

(D) “If Alice and Bob are both healthy, the same age, each has no dependents and a net worth of \$100,000, then the marginal utility of one dollar is slightly greater for Alice (salary: \$50,000/year) than it is for Bob (salary: \$51,000/year).”

(E) “I am saving this money because I will need to consume it more next year than I need to consume it right now.”

These statements all involve interpersonal comparisons, not of welfare *levels*, but rather, of welfare *gains*.

Statements (A), (C) and (E) would command almost universal agreement. But statements (B) and (D) are more dubious. They could be true, they could be false, or they could just be meaningless.

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Upshot: Some interpersonal comparisons of welfare gains are obvious and uncontroversial. But others are obscure; we lack sufficient information to make them in practice, or they are impossible even in *principle*, because they involve hidden intricacies of human psychology.

Idea: Represent approximate interpersonal comparisons with a *difference preorder*: an incomplete preorder on the space of personal state transitions.

Now consider the following (grossly oversimplified) policy problems.

- "Suppose interest rates determines unemployment and inflation, and have no other effects on society. Then we should raise the interest rate if and only if the aggregate welfare loss due to increased unemployment is outweighed by the aggregate welfare gain due to lower inflation."
- "Suppose a system of taxes and subsidies results in a net transfer of wealth from the rich to the poor (and has no other effects on society). This is justifiable if and only if the aggregate welfare gain (to the poor) outweighs the aggregate welfare loss (to the wealthy)."

To find best policy, must aggregate welfare gains/losses of different people.
Problem: Given the approximate interpersonal comparisons encoded in the difference preorder, how can we compare the aggregate welfare gains or losses for society of different policies?

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Let $\mathcal{X}$ be a space of 'personal states'.

For example: an element of $\mathcal{X}$ could encode information about a person's *psychology* (personality, mood, knowledge, beliefs, memories, values, desires, etc.) and also about her *physical state* (health, wealth, personal property, physical location, consumption bundle, sense-data, etc.).

Any person, at any time, resides at some point in $\mathcal{X}$. Assume this entirely determines her level of wellbeing.

Perhaps 'precise' interpersonal comparisons of well-being are impossible, or even meaningless. But we can sometimes make approximate interpersonal comparisons of changes in well-being. In short, we can (sometimes) make sense of the statement:

> *"The welfare gain in moving from state x_1 to state x_2 is greater than the welfare gain in moving from state y_1 to y_2."*

We will show that even such approximate interpersonal comparisons allow us to (partially) rank social state changes (i.e. policies). Under plausible conditions, this yields a generalization of utilitarian ethics.

Framework: the space of personal states

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Difference preorders

Represent a *personal state change* "$x_1 \rightsquigarrow x_2$" as an element of $\mathcal{X} \times \mathcal{X}$.

Thus, a (partial) ordering of the welfare gains/losses induced by personal state changes can be represented by a preorder (a reflexive, transitive, possibly incomplete binary relation) "$\succeq$" on $\mathcal{X} \times \mathcal{X}$. The statement

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is represented by the formula "$(x_1 \rightsquigarrow x_2) \succ (y_1 \rightsquigarrow y_2)$".

We call $(\succeq)$ a *difference preorder* on $\mathcal{X}$ if it satisfies 4 axioms:

- (DP0) For all $x, y \in \mathcal{X}$, we have $(x \rightsquigarrow x) \approx (y \rightsquigarrow y)$.
- (DP1) For all $x_1, x_2, y_1, y_2 \in \mathcal{X}$, if $(x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)$, then $(x_2 \rightsquigarrow x_1) \preceq (y_2 \rightsquigarrow y_1)$.
- (DP2) For all x_0, x_1, x_2 and $y_0, y_1, y_2 \in \mathcal{X}$, if $(x_0 \rightsquigarrow x_1) \succeq (y_0 \rightsquigarrow y_1)$ and $(x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)$, then $(x_0 \rightsquigarrow x_2) \succeq (y_0 \rightsquigarrow y_2)$.
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Loags and utility functions

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Idea: This is 'minimum structure' needed to define cardinal utility function.

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(b) $\mathbb{R}^n$ is a loag under vector addition and the lexicographic order.

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A *linearly ordered abelian group* (loag) is a triple $(\mathcal{R}, +, >)$, where $\mathcal{R}$ is a set, $+$ is an abelian group operation, and $>$ is a complete, antisymmetric, transitive binary relation such that, for all $r, s \in \mathcal{R}$, if $r > 0$, then $r + s > s$.
Idea: This is 'minimum structure' needed to define cardinal utility function.
Example. (a) The additive group $\mathbb{R}$ of real numbers is a loag.
(b) $\mathbb{R}^n$ is a loag under vector addition and the lexicographic order.

An $\mathcal{R}$-valued *weak utility function* for $(\succeq)$ is a function $u : \mathcal{X} \longrightarrow \mathcal{R}$ such that, for all $x_1, x_2, y_1, y_2 \in \mathcal{X}$,

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Loags and utility functions

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Let $\mathcal{U}(\succeq)$ be the set of all weak utility functions for $(\succeq)$.

Definition: $(\succeq)$ has a *multiutility representation* if there is some subset $\mathcal{U}' \subseteq \mathcal{U}(\succeq)$ such that, for all $x_1, x_2, y_1, y_2 \in \mathcal{X}$,

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Social difference preorders

Let $\mathcal{I}$ be a finite set, indexing a population.

A *social state* is an element $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, which assigns a personal state $x_i \in \mathcal{X}$ to each $i \in \mathcal{I}$. Suppose the current social state is $\mathbf{x}^0$.

Any policy will change $\mathbf{x}^0$ to some other social state. To select the best policy we must compare the social value of one *social state change* $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1)$ with another social state change $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^2)$.

Now suppose a country has two provinces Ex and Wy, with equal populations (both indexed by $\mathcal{I}$), which are initially in states $\mathbf{x}^0$ and $\mathbf{y}^0$ respectively.

- Policy A will change Ex to state $\mathbf{x}^1$ and leave Wy unchanged.
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Which policy is better? We must compare $(\mathbf{x}^0 \rightsquigarrow \mathbf{x}^1)$ to $(\mathbf{y}^0 \rightsquigarrow \mathbf{y}^1)$.

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Upshot: To select the best policy, we need a difference preorder on $\mathcal{X}^{\mathcal{I}}$.

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Let Π be the group of all permutations (i.e. self-bijections) of $\mathcal{I}$.

For any $\pi \in \Pi$ and $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, we define $\pi(\mathbf{x}) := [x_{\pi(i)}]_{i\in\mathcal{I}} \in \mathcal{X}^{\mathcal{I}}$.

A $(\succeq)$-*social difference preorder* (SDP) is a preorder $(\trianglerighteq)$ on $\mathcal{X}^{\mathcal{I}} \times \mathcal{X}^{\mathcal{I}}$ satisfying six axioms:

(WPar) For any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, if $(x_i^1 \leadsto x_i^2) \succeq (y_i^1 \leadsto y_i^2)$ for all $i \in \mathcal{I}$, then $(\mathbf{x}^1 \leadsto \mathbf{x}^2) \trianglerighteq (\mathbf{y}^1 \leadsto \mathbf{y}^2)$.

(Anon) For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $\pi \in \Pi$, $(\mathbf{x} \leadsto \mathbf{x}) \triangleq (\mathbf{x} \leadsto \pi(\mathbf{x}))$.

(DP0$^{\trianglerighteq}$) For all $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $(\mathbf{x} \leadsto \mathbf{x}) \triangleq (\mathbf{y} \leadsto \mathbf{y})$.

(DP1$^{\trianglerighteq}$) For all $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, if $(\mathbf{x}^1 \leadsto \mathbf{x}^2) \trianglerighteq (\mathbf{y}^1 \leadsto \mathbf{y}^2)$, then $(\mathbf{x}^2 \leadsto \mathbf{x}^1) \trianglelefteq (\mathbf{y}^2 \leadsto \mathbf{y}^1)$.

(DP2$^{\trianglerighteq}$) For all $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2$ and $\mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, if $(\mathbf{x}^0 \leadsto \mathbf{x}^1) \trianglerighteq (\mathbf{y}^0 \leadsto \mathbf{y}^1)$ and $(\mathbf{x}^1 \leadsto \mathbf{x}^2) \trianglerighteq (\mathbf{y}^1 \leadsto \mathbf{y}^2)$ then $(\mathbf{x}^0 \leadsto \mathbf{x}^2) \trianglerighteq (\mathbf{y}^0 \leadsto \mathbf{y}^2)$.

(DP3$^{\trianglerighteq}$) For all $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2$ and $\mathbf{y}^0, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, if $(\mathbf{x}^0 \leadsto \mathbf{x}^1) \trianglerighteq (\mathbf{y}^1 \leadsto \mathbf{y}^2)$ and $(\mathbf{x}^1 \leadsto \mathbf{x}^2) \trianglerighteq (\mathbf{y}^0 \leadsto \mathbf{y}^1)$ then $(\mathbf{x}^0 \leadsto \mathbf{x}^2) \trianglerighteq (\mathbf{y}^0 \leadsto \mathbf{y}^2)$.

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Axiom (WPar) is a weak form of the Pareto axiom.
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(WPar) For any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, if $(x_i^1 \rightsquigarrow x_i^2) \succeq (y_i^1 \rightsquigarrow y_i^2)$ for all $i \in \mathcal{I}$, then $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \trianglerighteq (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$.

(Anon) For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $\pi \in \Pi$, $(\mathbf{x} \rightsquigarrow \mathbf{x}) \triangleq (\mathbf{x} \rightsquigarrow \pi(\mathbf{x}))$.
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Suppose $\mathcal{X} := \mathcal{P} \times \mathbb{R}_+$, where $\mathcal{P}$:= set of 'personality types'.
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Take social state $(\mathbf{p}, \mathbf{r}) \in \mathcal{P}^{\mathcal{I}} \times \mathbb{R}^{\mathcal{I}}$, where $r_1 < r_2$ (Juan is poorer than Sue).
A *redistributive transfer* is a change $(\mathbf{p}, \mathbf{r}) \rightsquigarrow (\mathbf{p}, \mathbf{s})$, where $r_1 \leq s_1 \leq s_2 \leq r_2$, and where $s_1 + s_2 \leq r_1 + r_2$.
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Application: Redistributive transfers (2)

$$\left(\frac{\beta(s_1)-\beta(r_1)}{\beta(s_2)-\beta(r_2)} \; > \; C\right) \implies \Big(((p_1,r_1) \rightsquigarrow (p_1,s_1)) \succ ((p_2,r_2) \rightsquigarrow (p_2,s_2))\Big).$$

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(So, average slope of β is decreasing. Declining marginal benefits from wealth.)

If $(\trianglerighteq)$ *is any* $(\succeq)$*-SDP on* $\mathcal{X}^{\mathcal{I}}$, *then* $((\mathbf{p},\mathbf{r}) \rightsquigarrow (\mathbf{p},\mathbf{s})) \trianglerighteq ((\mathbf{p},\mathbf{r}) \rightsquigarrow (\mathbf{p},\mathbf{r}))$.

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Quasiutilitarian SDPs

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Let $\mathcal{V} \subseteq \mathcal{U}(\succeq)$ be a nonempty set of weak utility functions for $(\succeq)$.

We define the $\mathcal{V}$-*quasiutilitarian* SDP $(\underset{\mathcal{V}}{\trianglerighteq})$ as follows. For any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, set $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{V}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ if,

$$\text{for all } v \in \mathcal{V}, \qquad \sum_{i \in \mathcal{I}} \Big(v(x_i^2) - v(x_i^1) \Big) \quad \geq \quad \sum_{i \in \mathcal{I}} \Big(v(y_i^2) - v(y_i^1) \Big).$$

Proposition 2. *Let $(\succeq)$ be a difference preorder on $\mathcal{X}$, and let $\mathcal{V} \subseteq \mathcal{U}(\succeq)$.*
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Let $(\trianglerighteq)$ be a $(\succeq)$-SDP, and let $(\mathcal{R}, +, >)$ be a loag.

Definition: An $\mathcal{R}$-valued *social welfare function* (SWF) for $(\trianglerighteq)$ is a function $W : \mathcal{X}^{\mathcal{I}} \longrightarrow \mathcal{R}$ such that, for any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, we have

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Definition: A collection $\mathcal{W}$ of SWFs yields a *multiwelfare representation* for $(\trianglerighteq)$ if, for any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, we have

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Here is our first major result:

Theorem A. *Let $(\trianglerighteq)$ be a $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$.*

$$\Big((\trianglerighteq) \text{ admits a multiwelfare representation}\Big) \iff \Big((\trianglerighteq) \text{ is quasiutilitarian}\Big).$$

When is an SDP quasiutilitarian?

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Let $(\underset{1}{\trianglerighteq})$ and $(\underset{2}{\trianglerighteq})$ be two $(\succeq)$-SDPs on $\mathcal{X}^{\mathcal{I}}$.

We say $(\underset{2}{\trianglerighteq})$ *extends* $(\underset{1}{\trianglerighteq})$ if, for all $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, we have

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The minimal SDP

Let $\mathrm{SDP}(\succeq)$ be the set of all $(\succeq)$-social difference preorders on $\mathcal{X}^{\mathcal{I}}$.

Define the *minimal* SDP as follows: for any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$,

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Proposition. *Let* $(\trianglerighteq)$ *be any* $(\succeq)$*-SDP.*

(a) $(\trianglerighteq)$ *extends* $(\underset{*}{\trianglerighteq})$.

(b) $(\trianglerighteq)$ *satisfies* $(\mathrm{SPar}) \iff (\trianglerighteq)$ *refines* $(\underset{*}{\trianglerighteq})$ *and* $(\underset{*}{\trianglerighteq})$ *satisfies* (SPar).

Upshot: $(\underset{*}{\trianglerighteq})$ is the 'core' of every $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$.

Definition: A difference preorder $(\succeq)$ is *empathic* if, for any $x_1, x_2, y_1 \in \mathcal{X}$, there exists $y_2 \in \mathcal{X}$ such that $(x_1 \rightsquigarrow x_2) \approx (y_1 \rightsquigarrow y_2)$.

Idea: For any possible state transition facing a person in state x_1, a person in state y_1 can imagine an exactly analogous transition for herself.

Theorem B. *If* $(\succeq)$ *is empathic, and has multiutility representation given by some subset* $\mathcal{V} \subseteq \mathcal{U}(\succeq)$*, then* $(\underset{*}{\trianglerighteq}) = (\underset{\mathrm{u}}{\trianglerighteq}) = (\underset{\mathcal{V}}{\trianglerighteq})$.

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Example. Let $\mathcal{X} := \mathbb{R}^N$, where each coordinate is some quantitative measure of well-being (e.g. a functioning, consumption level, health, etc.). For any $x_1, x_2, y_1, y_2 \in \mathcal{X}$, suppose $(x_1 \rightsquigarrow x_2) \succeq (y_1 \rightsquigarrow y_2)$ if and only if $(x_2 - x_1) \geq (y_2 - y_1)$ (where "$\geq$" means coordinatewise dominance). Then $(\succeq)$ is empathic. Furthermore, the N coordinate projections on $\mathbb{R}^N$ provide a multiutility representation for $(\succeq)$.

Thus, Theorem B says that every $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$ is built on a foundation of utilitarian principles.

In particular, if $N = 1$ (i.e. $\mathcal{X} = \mathbb{R}$), then $(\succeq)$ is a *complete* order on $\mathcal{X} \times \mathcal{X}$, and $\mathcal{U}(\succeq) = \{$all affine increasing functions from $\mathbb{R}$ to itself$\}$.

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Minimality of the approximate utilitarian SDP: example

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Example. Let $\mathcal{X} := \mathbb{R}^N$, where each coordinate is some quantitative measure of well-being (e.g. a functioning, consumption level, health, etc.). For any $\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_1, \mathrm{y}_2 \in \mathcal{X}$, suppose $(\mathrm{x}_1 \rightsquigarrow \mathrm{x}_2) \succeq (\mathrm{y}_1 \rightsquigarrow \mathrm{y}_2)$ if and only if $(\mathrm{x}_2 - \mathrm{x}_1) \geq (\mathrm{y}_2 - \mathrm{y}_1)$ (where "$\geq$" means coordinatewise dominance). Then $(\succeq)$ is empathic. Furthermore, the N coordinate projections on $\mathbb{R}^N$ provide a multiutility representation for $(\succeq)$.

Thus, Theorem B says that every $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$ is built on a foundation of utilitarian principles.

In particular, if $N = 1$ (i.e. $\mathcal{X} = \mathbb{R}$), then $(\succeq)$ is a *complete* order on $\mathcal{X} \times \mathcal{X}$, and $\mathcal{U}(\succeq) = \{$all affine increasing functions from $\mathbb{R}$ to itself$\}$.

In this case, $(\underset{\mathrm{u}}{\trianglerighteq})$ is equivalent to the classic utilitarian SWO:

$$\Big((\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{u}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)\Big) \iff \left(\sum_{i \in \mathcal{I}} \left(x_i^2 - x_i^1\right) \ \geq \ \sum_{i \in \mathcal{I}} \left(y_i^2 - y_i^1\right)\right).$$

Then Theorem B implies that $(\underset{\mathrm{u}}{\trianglerighteq})$ is the *unique* $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$.

For any $x \in \mathcal{X}$, $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$, and $j \in \mathcal{I}$, we define $\binom{x_j}{\mathbf{z}_{-j}} \in \mathcal{X}^{\mathcal{I}}$ by setting $\binom{x_j}{\mathbf{z}_{-j}}_j := x$, while $\binom{x_j}{\mathbf{z}_{-j}}_i := z_i$ for all $i \in \mathcal{I} \setminus \{j\}$.

Let $(\trianglerighteq)$ be a $(\succeq)$-SDP. We say that $(\trianglerighteq)$ exhibits *no extra hidden interpersonal comparisons* if the following holds:

(NEHIC) For all $x, x', y, y' \in \mathcal{X}$ and $\mathbf{z} \in \mathcal{X}^{\mathcal{I}}$,

$$\Big((x \rightsquigarrow x') \succeq (y \rightsquigarrow x')\Big) \iff \left(\left(\binom{x_j}{\mathbf{z}_{-j}} \rightsquigarrow \binom{x'_j}{\mathbf{z}_{-j}}\right) \trianglerighteq \left(\binom{y_j}{\mathbf{z}_{-j}} \rightsquigarrow \binom{y'_j}{\mathbf{z}_{-j}}\right)\right).$$

(*Note:* "$\Longrightarrow$" follows immediately from axiom (WPar). The real content of (NEHIC) lies in the "$\Longleftarrow$" direction.)

Idea: If $(\binom{x_j}{\mathbf{z}_{-j}} \rightsquigarrow \binom{x'_j}{\mathbf{z}_{-j}}) \trianglerighteq (\binom{y_j}{\mathbf{z}_{-j}} \rightsquigarrow \binom{y'_j}{\mathbf{z}_{-j}})$, then $(\trianglerighteq)$ is implicitly judging that $(x \rightsquigarrow x')$ is a greater welfare gain than $(y \rightsquigarrow y')$.

(NEHIC) says that $(\trianglerighteq)$ can only make such interpersonal comparisons when they are justified by the underlying difference preorder $(\succeq)$.

Theorem C. *Suppose $(\succeq)$ is empathic. Let $(\trianglerighteq)$ be an $(\succeq)$-SDP. If $(\trianglerighteq)$ has a multiwelfare representation and satisfies* (NEHIC), *then* $(\trianglerighteq) = (\trianglerighteq_{\mathrm{u}}) = (\trianglerighteq_{*})$.

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No extra hidden interpersonal comparisons [Skip to end]

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Let $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$. Let $\mathcal{J}, \mathcal{K} \subseteq \mathcal{I}$ with $J := |\mathcal{J}|$ and $K := |\mathcal{K}|$. Write "$(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J},\mathcal{K}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$" if there exist $w_0, w_1, \ldots, w_J \in \mathcal{X}$ and $z_0, z_1, \ldots, z_K \in \mathcal{X}$ and bijections $\alpha : \mathcal{J} \longrightarrow [1 \ldots J]$ and $\beta : \mathcal{K} \longrightarrow [1 \ldots K]$ such that:

(JK1) $(x_j^1 \rightsquigarrow x_j^2) \succeq (w_{\alpha(j)-1} \rightsquigarrow w_{\alpha(j)})$ for all $j \in \mathcal{J}$;

(JK2) $(z_{\beta(k)-1} \rightsquigarrow z_{\beta(k)}) \succeq (y_k^1 \rightsquigarrow y_k^2)$, for all $k \in \mathcal{K}$; and

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Idea. $w_0 \rightsquigarrow w_J$ aggregates the net welfare gain of the chain

$$w_0 \rightsquigarrow w_1 \rightsquigarrow w_2 \rightsquigarrow \cdots \rightsquigarrow w_J.$$

Thus, (JK1) implies that net welfare gain for the $\mathcal{J}$-population induced by the change $\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2$ is at least as large as the net welfare gain of $w_0 \rightsquigarrow w_J$. Meanwhile, (JK2) implies that the net welfare gain for the $\mathcal{K}$-population induced by $\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2$ is at *most* as large as $z_0 \rightsquigarrow z_K$.
Thus, if (JK3) holds, then the $\mathcal{J}$-population, in aggregate, gains more welfare from $\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2$ than the $\mathcal{K}$-population gains from $\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2$.

The $(\mathcal{J},\mathcal{K})$-preorder: definition

Let $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$. Let $\mathcal{J}, \mathcal{K} \subseteq \mathcal{I}$ with $J := |\mathcal{J}|$ and $K := |\mathcal{K}|$.

Write "$(\mathbf{x}^1 \leadsto \mathbf{x}^2) \underset{\mathcal{J},\mathcal{K}}{\trianglerighteq} (\mathbf{y}^1 \leadsto \mathbf{y}^2)$" if there exist $w_0, w_1, \ldots, w_J \in \mathcal{X}$ and $z_0, z_1, \ldots, z_K \in \mathcal{X}$ and bijections $\alpha : \mathcal{J} \longrightarrow [1 \ldots J]$ and $\beta : \mathcal{K} \longrightarrow [1 \ldots K]$ such that:

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Meanwhile, (JK2) implies that the net welfare gain for the $\mathcal{K}$-population induced by $\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2$ is at *most* as large as $z_0 \rightsquigarrow z_K$.
Thus, if (JK3) holds, then the $\mathcal{J}$-population, in aggregate, gains more welfare from $\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2$ than the $\mathcal{K}$-population gains from $\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2$.

Let $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$. Let $\mathcal{J}, \mathcal{K} \subseteq \mathcal{I}$ with $J := |\mathcal{J}|$ and $K := |\mathcal{K}|$. Write "$(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J},\mathcal{K}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$" if there exist $w_0, w_1, \ldots, w_J \in \mathcal{X}$ and $z_0, z_1, \ldots, z_K \in \mathcal{X}$ and bijections $\alpha : \mathcal{J} \longrightarrow [1 \ldots J]$ and $\beta : \mathcal{K} \longrightarrow [1 \ldots K]$ such that:

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Idea. $w_0 \rightsquigarrow w_J$ aggregates the net welfare gain of the chain

$$w_0 \rightsquigarrow w_1 \rightsquigarrow w_2 \rightsquigarrow \cdots \rightsquigarrow w_J.$$

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The $(\mathcal{J},\mathcal{K})$-preorder: definition

Let $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$. Let $\mathcal{J}, \mathcal{K} \subseteq \mathcal{I}$ with $J := |\mathcal{J}|$ and $K := |\mathcal{K}|$. Write "$(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J},\mathcal{K}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$" if there exist $w_0, w_1, \ldots, w_J \in \mathcal{X}$ and $z_0, z_1, \ldots, z_K \in \mathcal{X}$ and bijections $\alpha : \mathcal{J} \longrightarrow [1 \ldots J]$ and $\beta : \mathcal{K} \longrightarrow [1 \ldots K]$ such that:

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Net Gain: definition

A *partition* of $\mathcal{I}$ is a collection $\{\mathcal{J}_\ell\}_{\ell\in\mathcal{L}}$ of disjoint subsets of $\mathcal{I}$ (where $\mathcal{L} :=$ some indexing set), such that $\mathcal{I} = \bigsqcup_{\ell\in\mathcal{L}} \mathcal{J}_\ell$.

Define the *net gain* relation as follows: For any $\mathbf{x}^1, \mathbf{x}^2, \mathbf{y}^1, \mathbf{y}^2 \in \mathcal{X}^{\mathcal{I}}$, say $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ if there exist two partitions $\{\mathcal{J}_\ell\}_{\ell\in\mathcal{L}}$ and $\{\mathcal{K}_\ell\}_{\ell\in\mathcal{L}}$ of $\mathcal{I}$ (with the *same* indexing set $\mathcal{L}$), such that,

$$\text{For all } \ell \in \mathcal{L}, \qquad (\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathcal{J}_\ell,\mathcal{K}_\ell}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2).$$

Idea. We can split up $\mathcal{I}$ into disjoint subsets such that, for each $\ell \in \mathcal{L}$, the 'net welfare gain' induced by $\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2$ for $\mathcal{J}_\ell$ is larger than the 'net welfare gain' induced by $\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2$ for $\mathcal{K}_\ell$.
Thus, if we aggregate over all $\ell \in \mathcal{L}$, then the 'net welfare gain' over all of $\mathcal{I}$ must be greater for $\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2$ than it is for $\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2$.

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Net Gain is the minimal SDP

Recall: $(\mathbf{x}^1 \rightsquigarrow \mathbf{x}^2) \underset{\mathrm{ng}}{\trianglerighteq} (\mathbf{y}^1 \rightsquigarrow \mathbf{y}^2)$ if there exist two partitions $\{\mathcal{J}_\ell\}_{\ell\in\mathcal{L}}$ and $\{\mathcal{K}_\ell\}_{\ell\in\mathcal{L}}$ of $\mathcal{I}$ (with the *same* indexing set $\mathcal{L}$), such that,

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We now come to our fourth major result:

Theorem D. *If* $(\succeq)$ *is empathic, then* $(\underset{\mathrm{ng}}{\trianglerighteq}) = (\underset{*}{\trianglerighteq})$, *and satisfies* (SPar).

In general, if $(\succeq)$ is not empathic, then $(\underset{\mathrm{ng}}{\trianglerighteq})$ might not be an SDP; however it will still be the case that every SDP extends $(\underset{\mathrm{ng}}{\trianglerighteq})$.

Proposition.

- *The relation* $(\underset{\mathrm{ng}}{\trianglerighteq})$ *is reflexive, and satisfies axioms* (WPar), (Anon), $(\mathrm{DP0}^{\trianglerighteq})$, *and* $(\mathrm{DP1}^{\trianglerighteq})$.
- *If* $(\trianglerighteq)$ *is any* $(\succeq)$*-SDP on* $\mathcal{X}^{\mathcal{I}}$, *then* $(\trianglerighteq)$ *extends* $(\underset{\mathrm{ng}}{\trianglerighteq})$. *Furthermore, if* $(\trianglerighteq)$ *also satisfies* (SPar), *then* $(\trianglerighteq)$ *also refines* $(\underset{\mathrm{ng}}{\trianglerighteq})$.

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From SDPs to social preorders, and back again

Social Preorders

Let $(\succcurlyeq)$ be an (ordinary) preorder on $\mathcal{X}$, encoding approximate interpersonal comparisons of welfare *levels*.

Thus, the formula "$x \succcurlyeq y$" means "a person in state x has greater well-being than a person in state y".

A $(\succcurlyeq)$-*social preorder* is a preorder $(\trianglerighteq)$ on $\mathcal{X}^{\mathcal{I}}$ satisfying two axioms:

(WPar′) For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, if $x_i \succcurlyeq y_i$ for all $i \in \mathcal{I}$, then $\mathbf{x} \trianglerighteq \mathbf{y}$.

(Anon′) For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$ and $\pi \in \Pi$, $\mathbf{x} \triangleq \pi(\mathbf{x})$.

Thus, a social preorder is the analog of a *social welfare order* (SWO) in the context of the approximate interpersonal comparisons encoded by $(\succcurlyeq)$.

In particular, if $\mathcal{X} = \mathbb{R}$ with the standard (complete linear) ordering $(\succcurlyeq)$, then a SWO is just a *complete* social preorder on $\mathbb{R}^{\mathcal{I}}$.

Let $\mathcal{R}$ be a loag. A function $W : \mathcal{X}^{\mathcal{I}} \longrightarrow \mathcal{R}$ is a *social welfare function* (SWF) for $(\trianglerighteq)$ if, for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $\Big(\mathbf{x} \trianglerighteq \mathbf{y}\Big) \Rightarrow \Big(W(\mathbf{x}) \geq W(\mathbf{y})\Big)$.

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Goal: Method to convert a social preorder on $\mathcal{X}^{\mathcal{I}}$ into a social difference preorder, or vice versa.

Two applications.

1. Define and/or axiomatically characterize new SDPs, starting from social preorders. (Thus, can leverage previous results on 'approximate interpersonal comparisons', e.g. from Pivato (2011,2012).)
2. Define and/or axiomatically characterize a social preorder, starting from an SDP (e.g. apply Theorems A-D to social preorders).

Motivation. Sometimes we only needs to choose an optimal social *state* (rather than an optimal social state *transition*).

For this, we only needs a social preorder ($\trianglerighteq$), not an SDP.

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A preorder $(\trianglerighteq)$ on $\mathcal{X}^{\mathcal{I}}$ is *separable* if, for any $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in \mathcal{X}^{\mathcal{I}}$, and any disjoint $\mathcal{J}, \mathcal{K} \subset \mathcal{I}$ with $\mathcal{J} \sqcup \mathcal{K} = \mathcal{I}$, we have

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Separability

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Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, and let $\mathcal{J}, \mathcal{K} \subseteq \mathcal{I}$ be disjoint subsets such that $\mathcal{J} \sqcup \mathcal{K} = \mathcal{I}$. Define $\binom{\mathbf{x}_{\mathcal{J}}}{\mathbf{y}_{\mathcal{K}}} \in \mathcal{X}^{\mathcal{I}}$ by setting $\binom{\mathbf{x}_{\mathcal{J}}}{\mathbf{y}_{\mathcal{K}}}_j := x_j$ for all $j \in \mathcal{J}$, while $\binom{\mathbf{x}_{\mathcal{J}}}{\mathbf{y}_{\mathcal{K}}}_k := y_k$ for all $k \in \mathcal{K}$.

A preorder $(\trianglerighteq)$ on $\mathcal{X}^{\mathcal{I}}$ is *separable* if, for any $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in \mathcal{X}^{\mathcal{I}}$, and any disjoint $\mathcal{J}, \mathcal{K} \subset \mathcal{I}$ with $\mathcal{J} \sqcup \mathcal{K} = \mathcal{I}$, we have

$$\left(\begin{pmatrix} \mathbf{x}_{\mathcal{J}} \\ \mathbf{z}_{\mathcal{K}} \end{pmatrix} \trianglerighteq \begin{pmatrix} \mathbf{y}_{\mathcal{J}} \\ \mathbf{z}_{\mathcal{K}} \end{pmatrix} \right) \iff \left(\begin{pmatrix} \mathbf{x}_{\mathcal{J}} \\ \mathbf{w}_{\mathcal{K}} \end{pmatrix} \trianglerighteq \begin{pmatrix} \mathbf{y}_{\mathcal{J}} \\ \mathbf{w}_{\mathcal{K}} \end{pmatrix} \right).$$

(For example, if $\mathcal{X} = \mathbb{R}$, then the classic utilitarian SWO, the leximin SWO, and the Nash SWO are all separable preorders on $\mathbb{R}^{\mathcal{I}}$.)

Idea. Since $\binom{\mathbf{x}_{\mathcal{J}}}{\mathbf{z}_{\mathcal{K}}}$ and $\binom{\mathbf{y}_{\mathcal{J}}}{\mathbf{z}_{\mathcal{K}}}$ have identical $\mathcal{K}$ coordinates, the ordering between them 'should' be decided only by comparing $\mathbf{x}_{\mathcal{J}}$ with $\mathbf{y}_{\mathcal{J}}$.
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Derivatives of difference preorders

Now, let $(\succeq)$ be a difference preorder on $\mathcal{X}$ (or $\mathcal{X}^{\mathcal{I}}$).

The *derivative* of $(\succeq)$ is the binary relation $(\mathrel{\succ\!\!\!\succeq})$ on $\mathcal{X}$ defined as follows:

for any $x, y \in \mathcal{X}$, $\Big(x \mathrel{\succ\!\!\!\succeq} y\Big) \iff \Big((y \rightsquigarrow x) \succeq (y \rightsquigarrow y)\Big)$.

Proposition. (a) *If $(\succeq)$ is any difference preorder on $\mathcal{X}$, then its derivative $(\mathrel{\succ\!\!\!\succeq})$ is a preorder on $\mathcal{X}$.*

(b) *If $(\trianglerighteq)$ is an $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$, then its derivative $(\underline{\blacktriangleright})$ is a separable $(\mathrel{\succ\!\!\!\succeq})$-social preorder on $\mathcal{X}^{\mathcal{I}}$.*

Example. Let $\mathcal{V} \subseteq \mathcal{U}(\succeq)$, and let $(\underset{\mathcal{V}}{\trianglerighteq})$ be the $\mathcal{V}$-quasiutilitarian SDP.

The derivative of $(\underset{\mathcal{V}}{\trianglerighteq})$ is the $\mathcal{V}$-*quasiutilitarian* social preorder $(\underset{\mathcal{V}}{\blacktriangleright})$. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $\mathbf{x} \underset{\mathcal{V}}{\blacktriangleright} \mathbf{y} \iff \sum_{i \in \mathcal{I}} v(x_i) \geq \sum_{i \in \mathcal{I}} v(y_i)$, for all $v \in \mathcal{V}$.

Remark. It is also possible to 'antidifferentiate' a preorder on $\mathcal{X}$, to obtain a difference preorder. But the antiderivative of a separable social preorder is not necessarily an SDP (it could violate (WPar)).

Now, let $(\succeq)$ be a difference preorder on $\mathcal{X}$ (or $\mathcal{X}^{\mathcal{I}}$).

The *derivative* of $(\succeq)$ is the binary relation $(\succeq\!\!\!\!\!\rightharpoonup)$ on $\mathcal{X}$ defined as follows:

for any $x, y \in \mathcal{X}$, $\left(x \succeq\!\!\!\!\!\rightharpoonup y\right) \iff \left((y \rightsquigarrow x) \succeq (y \rightsquigarrow y)\right)$.

Proposition. (a) *If $(\succeq)$ is any difference preorder on $\mathcal{X}$, then its derivative $(\succeq\!\!\!\!\!\rightharpoonup)$ is a preorder on $\mathcal{X}$.*

(b) *If $(\trianglerighteq)$ is an $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$, then its derivative $(\blacktriangleright)$ is a separable $(\succeq\!\!\!\!\!\rightharpoonup)$-social preorder on $\mathcal{X}^{\mathcal{I}}$.*

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The derivative of $(\underset{\mathcal{V}}{\trianglerighteq})$ is the $\mathcal{V}$-*quasiutilitarian* social preorder $(\underset{\mathcal{V}}{\blacktriangleright})$. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $\mathbf{x} \underset{\mathcal{V}}{\blacktriangleright} \mathbf{y} \iff \sum_{i \in \mathcal{I}} v(x_i) \geq \sum_{i \in \mathcal{I}} v(y_i)$, for all $v \in \mathcal{V}$.

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Derivatives of difference preorders

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The *derivative* of $(\succeq)$ is the binary relation $(\succcurlyeq)$ on $\mathcal{X}$ defined as follows:

for any $x, y \in \mathcal{X}$, $\Big(x \succcurlyeq y\Big) \iff \Big((y \rightsquigarrow x) \succeq (y \rightsquigarrow y)\Big)$.

Proposition. (a) *If $(\succeq)$ is any difference preorder on $\mathcal{X}$, then its derivative $(\succcurlyeq)$ is a preorder on $\mathcal{X}$.*

(b) *If $(\trianglerighteq)$ is an $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$, then its derivative $(\blacktriangleright)$ is a separable $(\succcurlyeq)$-social preorder on $\mathcal{X}^{\mathcal{I}}$.*

Example. Let $\mathcal{V} \subseteq \mathcal{U}(\succeq)$, and let $(\underset{\mathcal{V}}{\trianglerighteq})$ be the $\mathcal{V}$-quasiutilitarian SDP. The derivative of $(\underset{\mathcal{V}}{\trianglerighteq})$ is the $\mathcal{V}$-*quasiutilitarian* social preorder $(\underset{\mathcal{V}}{\blacktriangleright})$. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $\mathbf{x} \underset{\mathcal{V}}{\blacktriangleright} \mathbf{y} \iff \sum_{i \in \mathcal{I}} v(x_i) \geq \sum_{i \in \mathcal{I}} v(y_i)$, for all $v \in \mathcal{V}$.

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Proposition. (a) *If $(\succeq)$ is any difference preorder on $\mathcal{X}$, then its derivative $(\succeq')$ is a preorder on $\mathcal{X}$.*

(b) *If $(\trianglerighteq)$ is an $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$, then its derivative $(\blacktriangleright)$ is a separable $(\succeq')$-social preorder on $\mathcal{X}^{\mathcal{I}}$.*

Example. Let $\mathcal{V} \subseteq \mathcal{U}(\succeq)$, and let $(\underset{\mathcal{V}}{\trianglerighteq})$ be the $\mathcal{V}$-quasiutilitarian SDP.

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The *derivative* of $(\succeq)$ is the binary relation $(\succeq\!\!\!\!\!\succ)$ on $\mathcal{X}$ defined as follows:

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Proposition. (a) *If $(\succeq)$ is any difference preorder on $\mathcal{X}$, then its derivative $(\succeq\!\!\!\!\!\succ)$ is a preorder on $\mathcal{X}$.*

(b) *If $(\trianglerighteq)$ is an $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{I}}$, then its derivative $(\blacktriangleright\!\!\!\!\underline{\ })$ is a separable $(\succeq\!\!\!\!\!\succ)$-social preorder on $\mathcal{X}^{\mathcal{I}}$.*

Example. Let $\mathcal{V} \subseteq \mathcal{U}(\succeq)$, and let $(\underset{\mathcal{V}}{\trianglerighteq})$ be the $\mathcal{V}$-quasiutilitarian SDP.

The derivative of $(\underset{\mathcal{V}}{\trianglerighteq})$ is the *$\mathcal{V}$-quasiutilitarian* social preorder $(\underset{\mathcal{V}}{\blacktriangleright})$. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have $\mathbf{x} \underset{\mathcal{V}}{\blacktriangleright} \mathbf{y} \iff \sum_{i \in \mathcal{I}} v(x_i) \geq \sum_{i \in \mathcal{I}} v(y_i)$, for all $v \in \mathcal{V}$.

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Factors of separable social preorders

Let $(\succeq)$ be a preorder on $\mathcal{X}$.

Let $(\trianglerighteq)$ be a separable $(\succeq)$-social preorder on $\mathcal{X}^{\mathcal{I}}$.

For any $\mathcal{J} \subseteq \mathcal{I}$, define separable $(\succeq)$-social preorder $(\underset{\mathcal{J}}{\trianglerighteq})$ on $\mathcal{X}^{\mathcal{J}}$ as follows.

Let $\mathcal{K} := \mathcal{I} \setminus \mathcal{J}$, and fix $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$. Then for any $\mathbf{x}_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}} \in \mathcal{X}^{\mathcal{J}}$, we define

$$\left(\mathbf{x}_{\mathcal{J}} \underset{\mathcal{J}}{\trianglerighteq} \mathbf{y}_{\mathcal{J}}\right) \iff \left(\begin{pmatrix} \mathbf{x}_{\mathcal{J}} \\ \mathbf{z}_{\mathcal{K}} \end{pmatrix} \trianglerighteq \begin{pmatrix} \mathbf{y}_{\mathcal{J}} \\ \mathbf{z}_{\mathcal{K}} \end{pmatrix}\right).$$

By separability, $(\underset{\mathcal{J}}{\trianglerighteq})$ is well-defined independent of the choice of $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$.

Now let $\mathcal{J}$ be any set with $|\mathcal{J}| \leq |\mathcal{I}|$.
Define $(\underset{\mathcal{J}}{\trianglerighteq})$ on $\mathcal{X}^{\mathcal{J}}$ by bijectively identifying $\mathcal{J}$ with some subset $\mathcal{J}' \subseteq \mathcal{I}$.

Because $(\trianglerighteq)$ satisfies (Anon$'$), the resulting preorder is well-defined independent of the choice of $\mathcal{J}'$ and the choice of bijection.

We call $(\underset{\mathcal{J}}{\trianglerighteq})$ the $\mathcal{J}$-*factor* of $(\trianglerighteq)$.

Let $(\succeq)$ be a preorder on $\mathcal{X}$.

Let $(\underline{\blacktriangleright})$ be a separable $(\succeq)$-social preorder on $\mathcal{X}^{\mathcal{I}}$.

For any $\mathcal{J} \subseteq \mathcal{I}$, define separable $(\succeq)$-social preorder $(\underset{\mathcal{J}}{\underline{\blacktriangleright}})$ on $\mathcal{X}^{\mathcal{J}}$ as follows.

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By separability, $(\underset{\mathcal{J}}{\underline{\blacktriangleright}})$ is well-defined independent of the choice of $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$.

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Define $(\underset{\mathcal{J}}{\underline{\blacktriangleright}})$ on $\mathcal{X}^{\mathcal{J}}$ by bijectively identifying $\mathcal{J}$ with some subset $\mathcal{J}' \subseteq \mathcal{I}$.

Because $(\underline{\blacktriangleright})$ satisfies (Anon$'$), the resulting preorder is well-defined independent of the choice of $\mathcal{J}'$ and the choice of bijection.

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Factors of separable social preorders

Let $(\succeq)$ be a preorder on $\mathcal{X}$.

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We call $(\underset{\mathcal{J}}{\trianglerighteq})$ the *$\mathcal{J}$-factor* of $(\trianglerighteq)$.

Factors of separable social preorders

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For any $\mathcal{J} \subseteq \mathcal{I}$, define separable $(\succeq)$-social preorder $(\underset{\mathcal{J}}{\trianglerighteq})$ on $\mathcal{X}^{\mathcal{J}}$ as follows.

Let $\mathcal{K} := \mathcal{I} \setminus \mathcal{J}$, and fix $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$. Then for any $\mathbf{x}_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}} \in \mathcal{X}^{\mathcal{J}}$, we define

$$\left(\mathbf{x}_{\mathcal{J}} \underset{\mathcal{J}}{\trianglerighteq} \mathbf{y}_{\mathcal{J}}\right) \iff \left(\begin{pmatrix}\mathbf{x}_{\mathcal{J}} \\ \mathbf{z}_{\mathcal{K}}\end{pmatrix} \trianglerighteq \begin{pmatrix}\mathbf{y}_{\mathcal{J}} \\ \mathbf{z}_{\mathcal{K}}\end{pmatrix}\right).$$

By separability, $(\underset{\mathcal{J}}{\trianglerighteq})$ is well-defined independent of the choice of $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$.

Now let $\mathcal{J}$ be any set with $|\mathcal{J}| \leq |\mathcal{I}|$.
Define $(\underset{\mathcal{J}}{\trianglerighteq})$ on $\mathcal{X}^{\mathcal{J}}$ by bijectively identifying $\mathcal{J}$ with some subset $\mathcal{J}' \subseteq \mathcal{I}$.

Because $(\trianglerighteq)$ satisfies (Anon′), the resulting preorder is well-defined independent of the choice of $\mathcal{J}'$ and the choice of bijection.

We call $(\underset{\mathcal{J}}{\trianglerighteq})$ the *$\mathcal{J}$-factor* of $(\trianglerighteq)$.

Let $(\succeq)$ be a preorder on $\mathcal{X}$.

Let $(\trianglerighteq)$ be a separable $(\succeq)$-social preorder on $\mathcal{X}^{\mathcal{I}}$.

For any $\mathcal{J} \subseteq \mathcal{I}$, define separable $(\succeq)$-social preorder $(\underset{\mathcal{J}}{\trianglerighteq})$ on $\mathcal{X}^{\mathcal{J}}$ as follows.

Let $\mathcal{K} := \mathcal{I} \setminus \mathcal{J}$, and fix $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$. Then for any $\mathbf{x}_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}} \in \mathcal{X}^{\mathcal{J}}$, we define

$$\left(\mathbf{x}_{\mathcal{J}} \underset{\mathcal{J}}{\trianglerighteq} \mathbf{y}_{\mathcal{J}}\right) \iff \left(\begin{pmatrix}\mathbf{x}_{\mathcal{J}}\\ \mathbf{z}_{\mathcal{K}}\end{pmatrix} \trianglerighteq \begin{pmatrix}\mathbf{y}_{\mathcal{J}}\\ \mathbf{z}_{\mathcal{K}}\end{pmatrix}\right).$$

By separability, $(\underset{\mathcal{J}}{\trianglerighteq})$ is well-defined independent of the choice of $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$.

Now let $\mathcal{J}$ be any set with $|\mathcal{J}| \leq |\mathcal{I}|$.
Define $(\underset{\mathcal{J}}{\trianglerighteq})$ on $\mathcal{X}^{\mathcal{J}}$ by bijectively identifying $\mathcal{J}$ with some subset $\mathcal{J}' \subseteq \mathcal{I}$.

Because $(\trianglerighteq)$ satisfies (Anon$'$), the resulting preorder is well-defined independent of the choice of $\mathcal{J}'$ and the choice of bijection.

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Because $(\trianglerighteq)$ satisfies (Anon$'$), the resulting preorder is well-defined independent of the choice of $\mathcal{J}'$ and the choice of bijection.

We call $(\underset{\mathcal{J}}{\trianglerighteq})$ the *$\mathcal{J}$-factor* of $(\trianglerighteq)$.

From social preorders to social difference preorders

Let $(\succeq)$ be preorder on $\mathcal{X}$. Let $(\trianglerighteq)$ be separable $(\succeq)$-social preorder on $\mathcal{X}^{\mathcal{I}}$.

Let $\mathcal{J}$ be any set with $|\mathcal{J}| \leq |\mathcal{I}|/3$.
We define a difference preorder $(\underset{\mathcal{J}}{\trianglerighteq})$ on $\mathcal{X}^{\mathcal{J}}$ as follows.

Let $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{I}$ be two disjoint subsets, with $|\mathcal{J}_1| = |\mathcal{J}_2| = |\mathcal{J}|$.
Let $\beta_1 : \mathcal{J}_1 \longrightarrow \mathcal{J}$ and $\beta_2 : \mathcal{J}_2 \longrightarrow \mathcal{J}$ be bijections.
Let $\mathcal{K} := \mathcal{I} \setminus (\mathcal{J}_1 \sqcup \mathcal{J}_2)$, and fix $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$.
For any $\mathbf{x}_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}} \in \mathcal{X}^{\mathcal{J}}$, let $(\mathbf{x}_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}, \mathbf{z}_{\mathcal{K}}) \in \mathcal{X}^{\mathcal{I}}$ be the unique $\mathbf{w} \in \mathcal{X}^{\mathcal{I}}$ with $w_j := x_{\beta_1(j)}, \forall\, j \in \mathcal{J}_1$, $w_j := y_{\beta_2(j)}, \forall\, j \in \mathcal{J}_2$, and $w_k := z_k, \forall\, k \in \mathcal{K}$.
Now, for any $\mathbf{x}_{\mathcal{J}}, \mathbf{x}'_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}, \mathbf{y}'_{\mathcal{J}} \in \mathcal{X}^{\mathcal{J}}$, we define

$$\Big((\mathbf{x}_{\mathcal{J}} \rightsquigarrow \mathbf{x}'_{\mathcal{J}}) \underset{\mathcal{J}}{\trianglerighteq} (\mathbf{y}_{\mathcal{J}} \rightsquigarrow \mathbf{y}'_{\mathcal{J}})\Big) \iff \Big((\mathbf{x}_{\mathcal{J}}, \mathbf{y}'_{\mathcal{J}}, \mathbf{z}_{\mathcal{K}}) \trianglelefteq (\mathbf{x}'_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}, \mathbf{z}_{\mathcal{K}})\Big).$$

This is independent of choice of $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$ by separability, and independent of choice of $\mathcal{J}_1$, $\mathcal{J}_2$, β_1, and β_2, by (Anon$'$).
Intuition: The social gain in moving from $\mathbf{x}_{\mathcal{J}}$ to $\mathbf{x}'_{\mathcal{J}}$ outweighs the social loss of moving from $\mathbf{y}'_{\mathcal{J}}$ to $\mathbf{y}_{\mathcal{J}}$.

Let $(\succeq)$ be preorder on $\mathcal{X}$. Let $(\trianglerighteq)$ be separable $(\succeq)$-social preorder on $\mathcal{X}^{\mathcal{I}}$.

Let $\mathcal{J}$ be any set with $|\mathcal{J}| \leq |\mathcal{I}|/3$.

We define a difference preorder $(\underset{\mathcal{J}}{\trianglerighteq})$ on $\mathcal{X}^{\mathcal{J}}$ as follows.

Let $\mathcal{J}_1, \mathcal{J}_2 \subseteq \mathcal{I}$ be two disjoint subsets, with $|\mathcal{J}_1| = |\mathcal{J}_2| = |\mathcal{J}|$.
Let $\beta_1 : \mathcal{J}_1 \longrightarrow \mathcal{J}$ and $\beta_2 : \mathcal{J}_2 \longrightarrow \mathcal{J}$ be bijections.
Let $\mathcal{K} := \mathcal{I} \setminus (\mathcal{J}_1 \sqcup \mathcal{J}_2)$, and fix $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$.
For any $\mathbf{x}_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}} \in \mathcal{X}^{\mathcal{J}}$, let $(\mathbf{x}_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}, \mathbf{z}_{\mathcal{K}}) \in \mathcal{X}^{\mathcal{I}}$ be the unique $\mathbf{w} \in \mathcal{X}^{\mathcal{I}}$ with $w_j := x_{\beta_1(j)}$, $\forall\, j \in \mathcal{J}_1$, $w_j := y_{\beta_2(j)}$, $\forall\, j \in \mathcal{J}_2$, and $w_k := z_k$, $\forall\, k \in \mathcal{K}$.
Now, for any $\mathbf{x}_{\mathcal{J}}, \mathbf{x}'_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}, \mathbf{y}'_{\mathcal{J}} \in \mathcal{X}^{\mathcal{J}}$, we define

$$\Big((\mathbf{x}_{\mathcal{J}} \rightsquigarrow \mathbf{x}'_{\mathcal{J}}) \underset{\mathcal{J}}{\trianglerighteq} (\mathbf{y}_{\mathcal{J}} \rightsquigarrow \mathbf{y}'_{\mathcal{J}}) \Big) \iff \Big((\mathbf{x}_{\mathcal{J}}, \mathbf{y}'_{\mathcal{J}}, \mathbf{z}_{\mathcal{K}}) \trianglelefteq (\mathbf{x}'_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}, \mathbf{z}_{\mathcal{K}}) \Big).$$

This is independent of choice of $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$ by separability, and independent of choice of $\mathcal{J}_1$, $\mathcal{J}_2$, β_1, and β_2, by (Anon$'$).
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Now, for any $\mathbf{x}_\mathcal{J}, \mathbf{x}'_\mathcal{J}, \mathbf{y}_\mathcal{J}, \mathbf{y}'_\mathcal{J} \in \mathcal{X}^{\mathcal{J}}$, we define

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For any $\mathbf{x}_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}} \in \mathcal{X}^{\mathcal{J}}$, let $(\mathbf{x}_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}, \mathbf{z}_{\mathcal{K}}) \in \mathcal{X}^{\mathcal{I}}$ be the unique $\mathbf{w} \in \mathcal{X}^{\mathcal{I}}$ with $w_j := x_{\beta_1(j)}$, $\forall\ j \in \mathcal{J}_1$, $w_j := y_{\beta_2(j)}$, $\forall\ j \in \mathcal{J}_2$, and $w_k := z_k$, $\forall\ k \in \mathcal{K}$.

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Now, for any $\mathbf{x}_{\mathcal{J}}, \mathbf{x}'_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}, \mathbf{y}'_{\mathcal{J}} \in \mathcal{X}^{\mathcal{J}}$, we define

$$\Big((\mathbf{x}_{\mathcal{J}} \leadsto \mathbf{x}'_{\mathcal{J}}) \underset{\mathcal{J}}{\trianglerighteq} (\mathbf{y}_{\mathcal{J}} \leadsto \mathbf{y}'_{\mathcal{J}}) \Big) \iff \Big((\mathbf{x}_{\mathcal{J}}, \mathbf{y}'_{\mathcal{J}}, \mathbf{z}_{\mathcal{K}}) \trianglelefteq (\mathbf{x}'_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}, \mathbf{z}_{\mathcal{K}}) \Big).$$

This is independent of choice of $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$ by separability, and independent of choice of $\mathcal{J}_1$, $\mathcal{J}_2$, β_1, and β_2, by (Anon′).

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Now, for any $\mathbf{x}_{\mathcal{J}}, \mathbf{x}'_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}, \mathbf{y}'_{\mathcal{J}} \in \mathcal{X}^{\mathcal{J}}$, we define

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This is independent of choice of $\mathbf{z}_{\mathcal{K}} \in \mathcal{X}^{\mathcal{K}}$ by separability, and independent of choice of $\mathcal{J}_1$, $\mathcal{J}_2$, β_1, and β_2, by (Anon$'$).
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Recall. $\Big((\mathbf{x}_\mathcal{J} \rightsquigarrow \mathbf{x}'_\mathcal{J}) \underset{\mathcal{J}}{\trianglerighteq} (\mathbf{y}_\mathcal{J} \rightsquigarrow \mathbf{y}'_\mathcal{J})\Big) \iff \Big((\mathbf{x}_\mathcal{J}, \mathbf{y}'_\mathcal{J}, \mathbf{z}_\mathcal{K}) \blacktriangleleft\!\!\!\!\_ \,(\mathbf{x}'_\mathcal{J}, \mathbf{y}_\mathcal{J}, \mathbf{z}_\mathcal{K})\Big)$.

Example Let $\mathcal{V} \subseteq \mathcal{U}(\succeq)$, and let $(\underline{\blacktriangleright})$ be the $\mathcal{V}$-quasiutilitarian social preorder on $\mathcal{X}^\mathcal{I}$. This preorder is separable. For any finite $\mathcal{J} \subseteq \mathcal{I}$, the relation $(\underset{\mathcal{J}}{\trianglerighteq})$ is the $\mathcal{V}$-quasiutilitarian SDP on $\mathcal{X}^\mathcal{J}$.

Special case: If $|\mathcal{J}| = 1$ (so $\mathcal{X}^\mathcal{J} = \mathcal{X}$), then we obtain a difference preorder $(\succeq)$ on $\mathcal{X}$, as follows.

Fix $i, j \in \mathcal{I}$ and $\mathbf{z} \in \mathcal{X}^{\mathcal{I}\setminus\{i,j\}}$. For all $x, x', y, y' \in \mathcal{X}$, define
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Again, separability and (Anon′) make this independent of i, j, and $\mathbf{z}$.
We call this the difference preorder on $\mathcal{X}$ *induced* by $(\underline{\blacktriangleright})$.

Proposition.

(a) *$(\underset{\mathcal{J}}{\trianglerighteq})$ is a $(\succeq)$-social difference preorder on $\mathcal{X}^\mathcal{J}$.*

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From social preorders to social difference preorders

Recall. $\Big((\mathbf{x}_{\mathcal{J}} \rightsquigarrow \mathbf{x}'_{\mathcal{J}}) \underset{\mathcal{J}}{\trianglerighteq} (\mathbf{y}_{\mathcal{J}} \rightsquigarrow \mathbf{y}'_{\mathcal{J}}) \Big) \iff \Big((\mathbf{x}_{\mathcal{J}}, \mathbf{y}'_{\mathcal{J}}, \mathbf{z}_{\mathcal{K}}) \trianglelefteq (\mathbf{x}'_{\mathcal{J}}, \mathbf{y}_{\mathcal{J}}, \mathbf{z}_{\mathcal{K}}) \Big)$.

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We can reverse this construction, to derive a social preorder from an SDP.

Let $\mathcal{J}$ be a finite set, and let $(\underset{\mathcal{J}}{\trianglerighteq})$ be an $(\succeq)$-SDP on $\mathcal{X}^{\mathcal{J}}$.

Let $\mathcal{I}$ be another set. Suppose $\mathcal{I} = \mathcal{J}_1 \sqcup \mathcal{J}_2$, where $|\mathcal{J}_1| = |\mathcal{J}_2| = |\mathcal{J}|$. Let $\beta_1 : \mathcal{J} \longrightarrow \mathcal{J}_1$ and $\beta_2 : \mathcal{J} \longrightarrow \mathcal{J}_2$ be bijections. Use these to identify $\mathcal{X}^{\mathcal{J}_1}$ and $\mathcal{X}^{\mathcal{J}_2}$ with $\mathcal{X}^{\mathcal{J}}$. For any $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$, let $\mathbf{x}_{\mathcal{J}_1}$ and $\mathbf{x}_{\mathcal{J}_2}$ be the projections of $\mathbf{x}$ onto $\mathcal{X}^{\mathcal{J}_1}$ and $\mathcal{X}^{\mathcal{J}_2}$, identified with elements of $\mathcal{X}^{\mathcal{J}}$ via β_1 and β_2.

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, define $\left(\mathbf{x} \trianglelefteq \mathbf{y}\right) \iff \left((\mathbf{x}_{\mathcal{J}_1} \rightsquigarrow \mathbf{y}_{\mathcal{J}_1}) \underset{\mathcal{J}}{\trianglerighteq} (\mathbf{y}_{\mathcal{J}_2} \rightsquigarrow \mathbf{x}_{\mathcal{J}_2})\right)$.

If $W : \mathcal{X}^{\mathcal{J}} \longrightarrow \mathcal{R}$ is a SWF for $(\underset{\mathcal{J}}{\trianglerighteq})$, then define $\widetilde{W} : \mathcal{X}^{\mathcal{I}} \longrightarrow \mathcal{R}$ by setting $\widetilde{W}(\mathbf{x}) := W(\mathbf{x}_{\mathcal{J}_1}) + W(\mathbf{x}_{\mathcal{J}_2})$ for all $\mathbf{x} \in \mathcal{X}^{\mathcal{I}}$.

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From social difference preorders to social preorders

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From social difference preorders to social preorders

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- Many results extend to infinite populations (this is more technically complicated). This is useful e.g. for infinite-horizon intergenerational social choice, or social choice under uncertainty.
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Approximate interpersonal comparisons of well-being:

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Merci & Thank you.

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`<http://euclid.trentu.ca/pivato/Research/approx4.pdf>`

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