Social Choice with Approximate Interpersonal Comparisons of Well-being

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- In fact, we often choose between *different* future selves (e.g. ‘I will be happier if I go to university and get an education.’). How can we make such choices without interpersonal comparisons?

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- (2) Use these approximate comparisons to construct 'approximate' social welfare orders.

Part I: Approximate interpersonal comparisons

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- $(\underset{\psi}{\succeq})$ on the rest of $\Psi \times \Phi$ is a ψ-type individual's (interpersonal) ranking of psychophysical states (which may be inaccurate).

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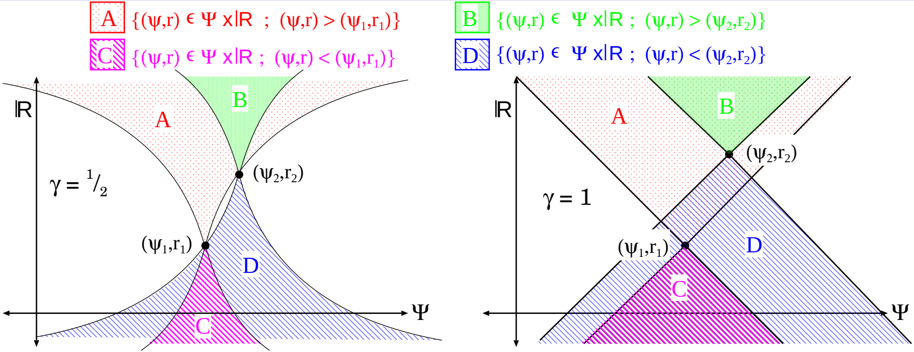
(These are both philosophically debatable. Actually, neither property is required for any of the later results on social choice. However, it happens that all of our examples satisfy these properties.)

Example: Approximate interpersonal comparisons of utility



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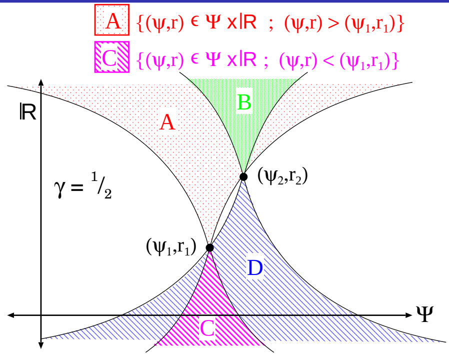

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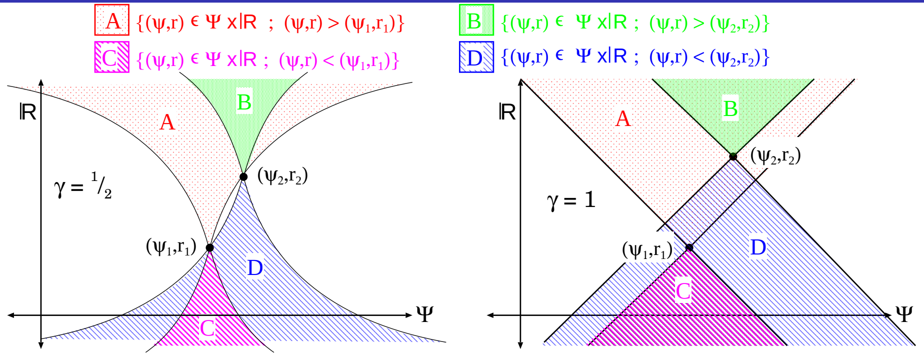

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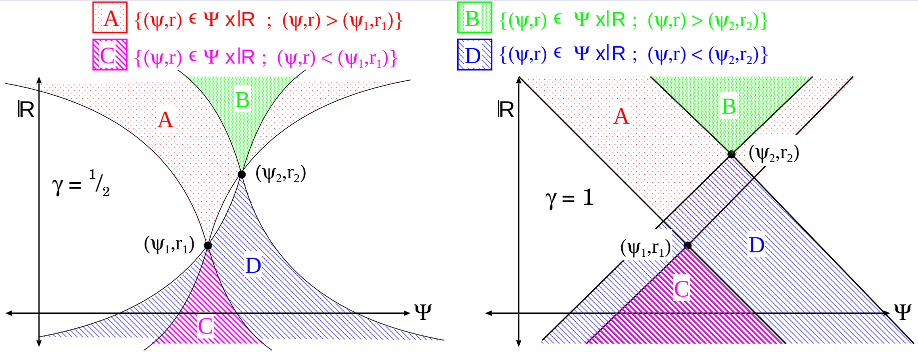

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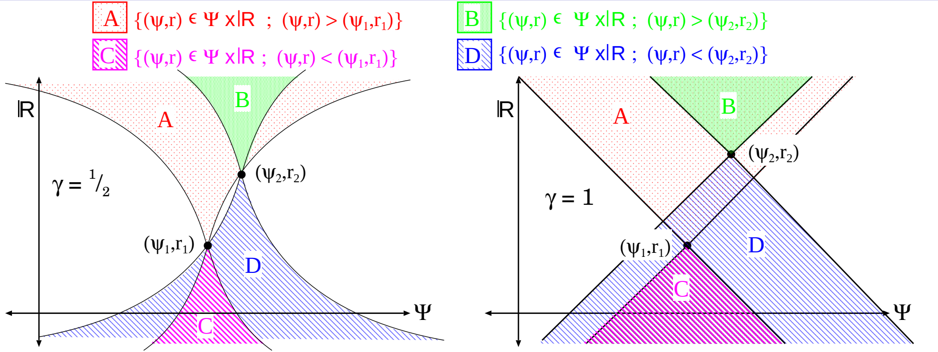

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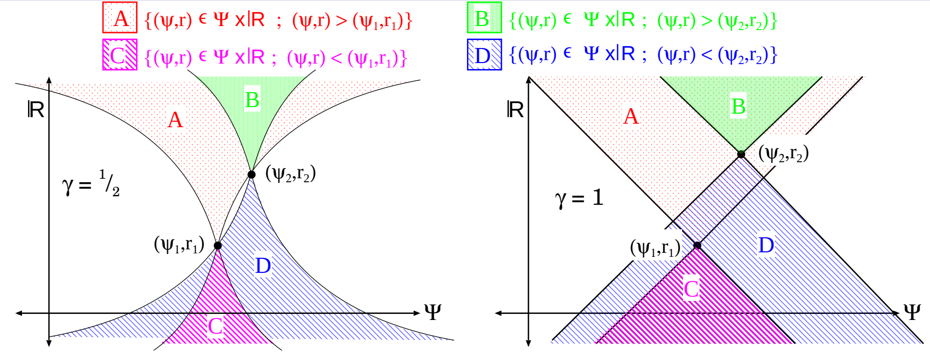

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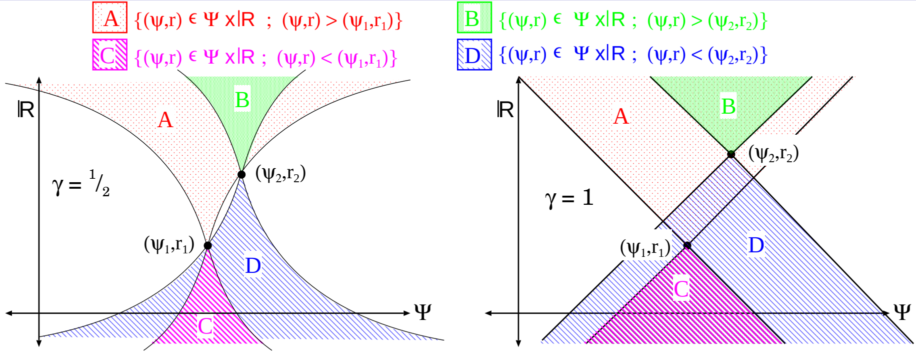

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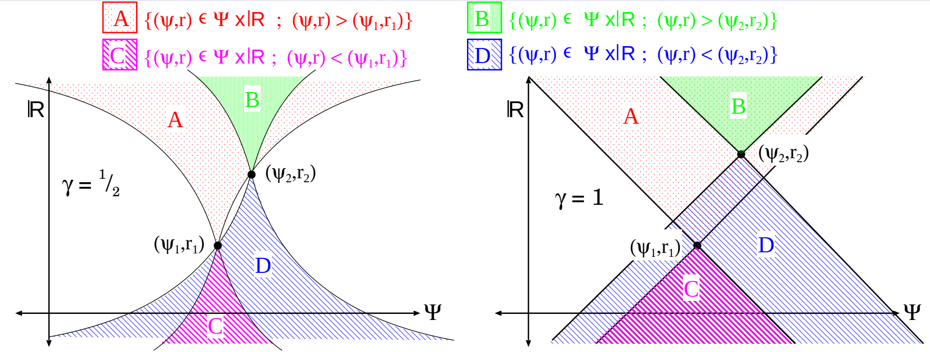

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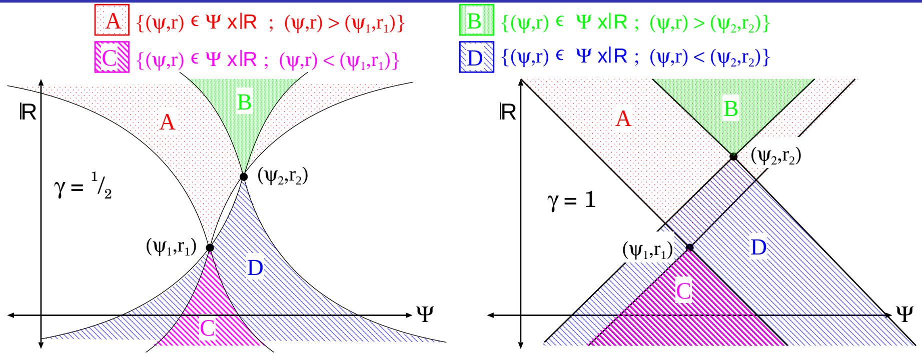

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$$\Big((\psi_1, r_1) \prec (\psi_2, r_2)\Big) \iff \Big(r_1 + d(\psi_1, \psi_2)^\gamma < r_2\Big).$$

Meanwhile, set $(\psi_1, r_1) \approx (\psi_2, r_2)$ if and only if $(\psi_1, r_1) = (\psi_2, r_2)$.

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Theorem 1. *Suppose $\mathcal{U}(\succeq) \neq \emptyset$. Then for all (ψ_1, ϕ_1) and (ψ_2, ϕ_2) in $\Psi \times \Phi$, we have* $\Big((\psi_1, \phi_1) \succeq (\psi_2, \phi_2)\Big) \iff \Big(u(\psi_1, \phi_1) \geq u(\psi_2, \phi_2)$*, for every* $u \in \mathcal{U}(\succeq)\Big)$.

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Part II:
Social preorders

Social preorders: Definition

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Example: the Suppes-Sen social preorder

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, define $\mathbf{x} \underset{s}{\trianglelefteq} \mathbf{y}$ if and only if there is a permutation $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ such that, for all $i \in \mathcal{I}$, we have $x_i \preceq y_{\sigma(i)}$.

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Example: (*Cost-benefit analysis*)

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- $\exists$ bijection $\beta : \mathcal{I}_0 \longrightarrow \mathcal{I}_0$ such that $x_i \approx y_{\beta(i)}$ for all $i \in \mathcal{I}_0$; and

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- $\exists$ injection $\alpha : \mathcal{I}_{\downarrow} \longrightarrow \mathcal{I}_{\uparrow}$ such that, for every $i \in \mathcal{I}_{\downarrow}$,

$$x_{\alpha(i)} \preceq y_i \prec x_i \preceq y_{\alpha(i)}.$$

(i.e. we can match every 'loser' $i \in \mathcal{I}_{\downarrow}$ with some 'winner' $\alpha(i) \in \mathcal{I}_{\uparrow}$ such that the gains for $\alpha(i)$ outweigh the losses for i).

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, define $\mathbf{x} \underset{s}{\trianglelefteq} \mathbf{y}$ if and only if there is a permutation $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ such that, for all $i \in \mathcal{I}$, we have $x_i \preceq y_{\sigma(i)}$.

Theorem 2: $(\underset{s}{\trianglerighteq})$ *is a subrelation of every other social preorder.*

Example: (*Cost-benefit analysis*) Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$. Define:

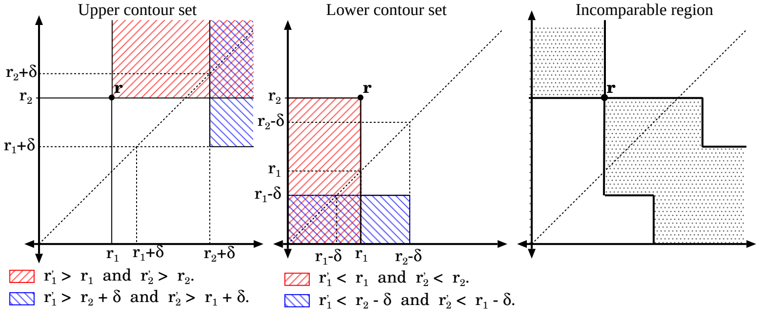
$$\begin{aligned}
\mathcal{I}_{\downarrow} &:= \{i \in \mathcal{I} \,;\, x_i \succ y_i\} = \text{'losers' in the transition from } \mathbf{x} \text{ to } \mathbf{y}; \\
\mathcal{I}_{\uparrow} &:= \{i \in \mathcal{I} \,;\, x_i \prec y_i\} = \text{'winners' in this transition; and} \\
\mathcal{I}_{0} &:= \mathcal{I} \setminus (\mathcal{I}_{\downarrow} \sqcup \mathcal{I}_{\uparrow}) = \text{everyone else.}
\end{aligned}$$

Suppose that:

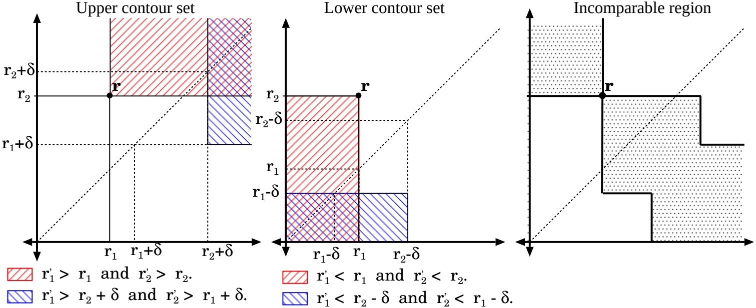
- $\exists$ bijection $\beta : \mathcal{I}_0 \longrightarrow \mathcal{I}_0$ such that $x_i \approx y_{\beta(i)}$ for all $i \in \mathcal{I}_0$; and
- $\exists$ injection $\alpha : \mathcal{I}_{\downarrow} \longrightarrow \mathcal{I}_{\uparrow}$ such that, for every $i \in \mathcal{I}_{\downarrow}$,

$$x_{\alpha(i)} \preceq y_i \prec x_i \preceq y_{\alpha(i)}.$$

(i.e. we can match every 'loser' $i \in \mathcal{I}_{\downarrow}$ with some 'winner' $\alpha(i) \in \mathcal{I}_{\uparrow}$ such that the gains for $\alpha(i)$ outweigh the losses for i). Then $\mathbf{x} \underset{s}{\trianglelefteq} \mathbf{y}$.



Let $\mathcal{I} := \{1, 2\}$ ('Juan' and 'Sue'). Then $\mathbf{y} \underset{s}{\trianglelefteq} \mathbf{x}$ if and only if



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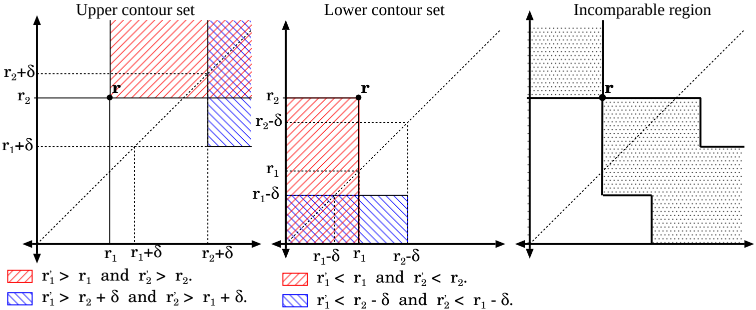
- *or* $y_1 \preceq x_2$ and $y_2 \preceq x_1$. (Pareto + permutation)

Two-person Suppes-Sen social preorder on $\mathbb{R}^2$



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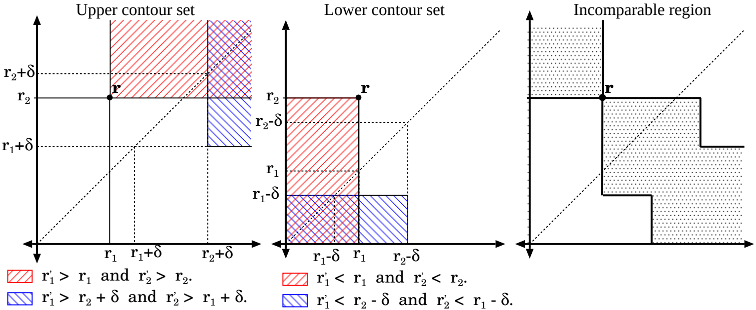

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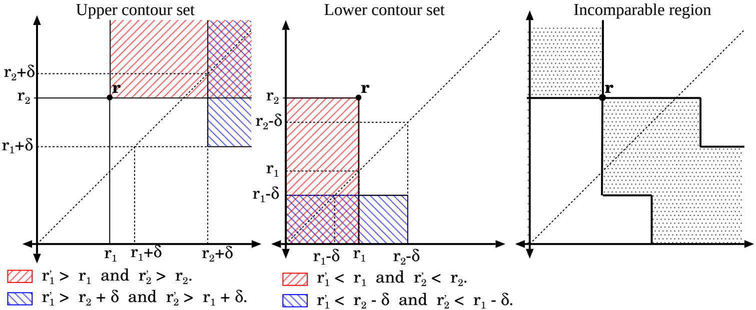
For example, suppose $\mathcal{X} := \Psi \times \mathbb{R}$, where Ψ=space of 'psychological types'.

Two-person Suppes-Sen social preorder on $\mathbb{R}^2$



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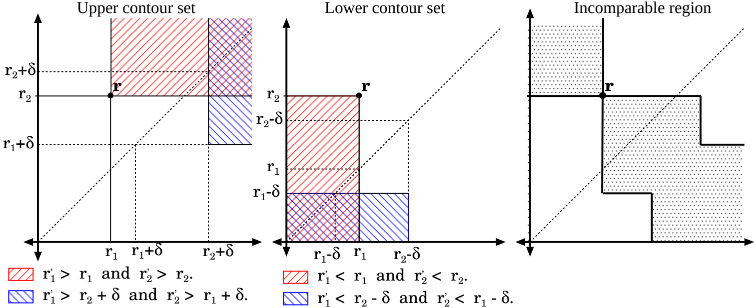

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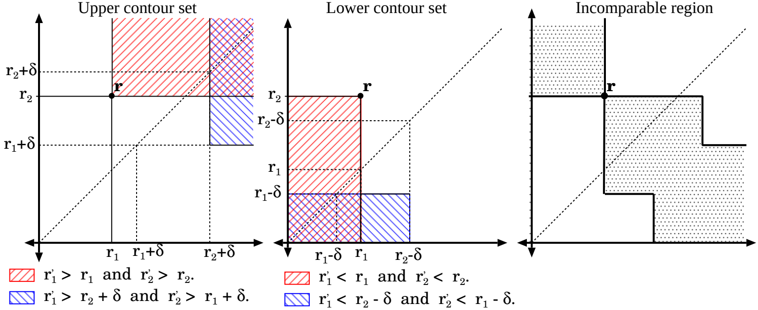

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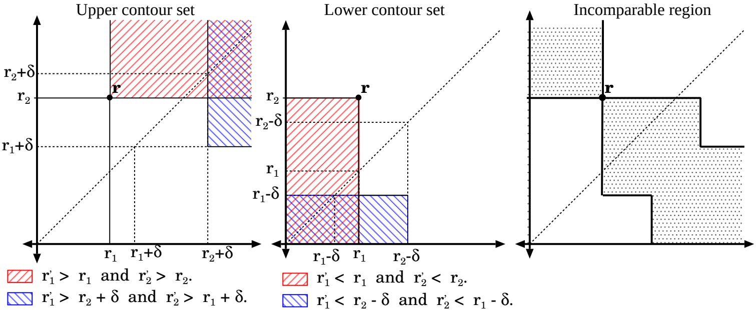
Then $(\underset{s}{\trianglerighteq})$ induces a preorder $(\underset{s,\boldsymbol{\psi}}{\blacktriangleright})$ on $\mathbb{R}^2$ as follows:

Two-person Suppes-Sen social preorder on $\mathbb{R}^2$



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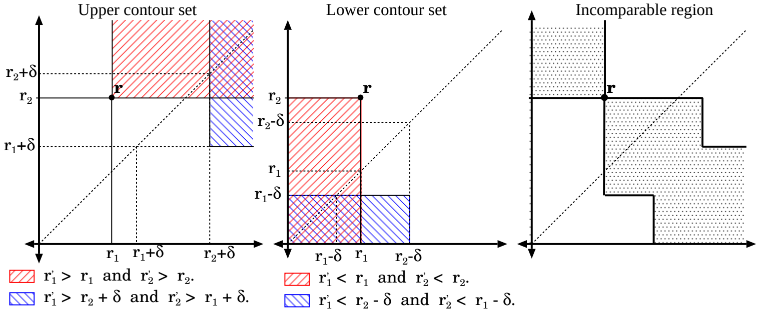
$$\Big(\mathbf{s} \underset{s,\boldsymbol{\psi}}{\blacktriangleleft} \mathbf{r}\Big) \iff \Big((\boldsymbol{\psi}, \mathbf{s}) \underset{s}{\trianglelefteq} (\boldsymbol{\psi}, \mathbf{r})\Big)$$

Two-person Suppes-Sen social preorder on $\mathbb{R}^2$



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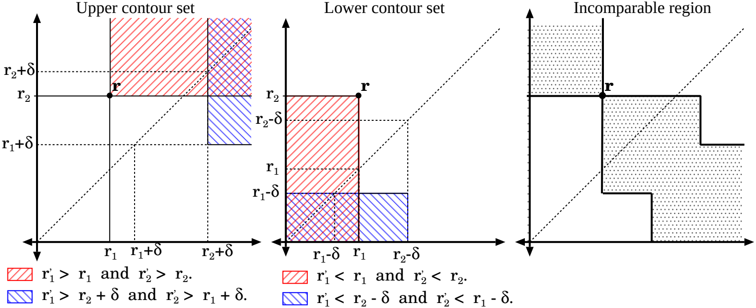
$$\iff \Big(\text{either } s_1 \leq r_1 \text{ and } s_2 \leq r_2, \quad \text{or} \quad s_2 < r_1 - \delta \text{ and } s_1 < r_2 - \delta\Big).$$

Two-person Suppes-Sen social preorder on $\mathbb{R}^2$



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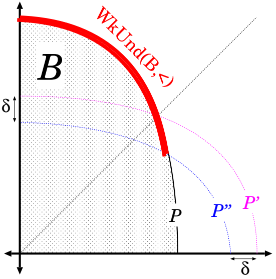

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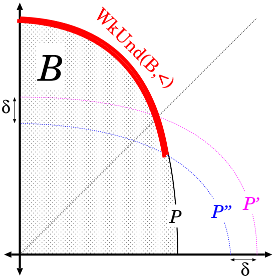

Again, let $\mathcal{I} := \{1,2\}$, and let $(\underset{s,\psi}{\blacktriangleright})$ be the preorder on $\mathbb{R}^2$ from previous slide.
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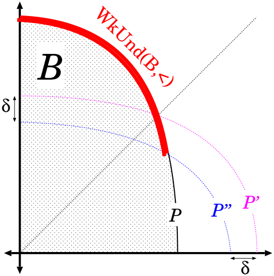

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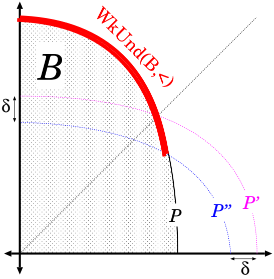

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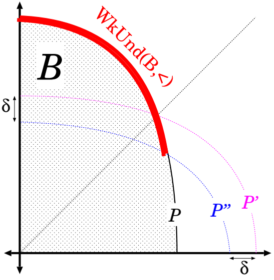
Bargaining solutions pick a small (usually singleton) subset of $\mathcal{P}$ (usually by maximizing some SWO on $\mathbb{R}^2$).

Suppes-Sen for bilateral bargaining problems



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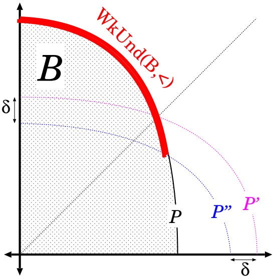
Let wkUnd $\left(\mathcal{B}, \underset{s,\psi}{\underline{\blacktriangleright}}\right) := \{\mathbf{b} \in \mathcal{B};\ \nexists\ \mathbf{b}' \in \mathcal{B}$ with $\mathbf{b} \underset{s,\psi}{\blacktriangleleft} \mathbf{b}'\}$ (the *weakly undominated set*) —the ‘bargaining solution’ defined by $(\trianglerighteq)$.



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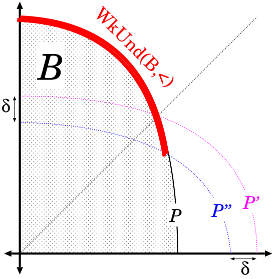
Equivalently: $\mathbf{b} \in \mathrm{wkUnd}\left(\mathcal{B}, \underset{s,\psi}{\blacktriangleright}\right)$ if and only if:

Suppes-Sen for bilateral bargaining problems



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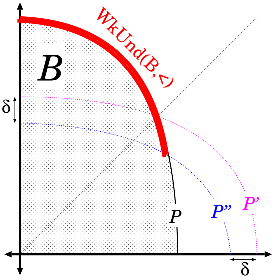

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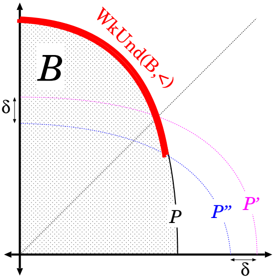

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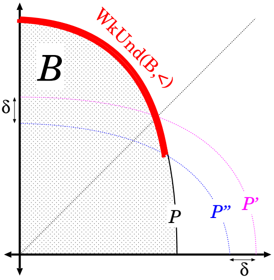

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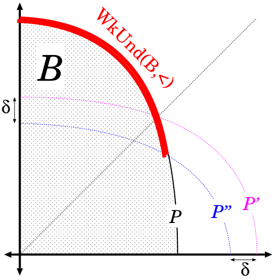

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Let $\mathcal{P}'$ be the reflection of $\mathcal{P}$ across the diagonal. Let $\mathcal{P}'' := \mathcal{P}' - (\delta, \delta)$.
Then $\mathbf{b} \in \mathrm{wkUnd}\left(\mathcal{B}, \underset{s,\psi}{\underline{\blacktriangleright}}\right)$ if and only if $\mathbf{b} \in \mathcal{P}$ *and* there is no $\mathbf{b}' \in \mathcal{P}''$ which Pareto-dominates $\mathbf{b}$.

The Approximate Maximin Social Preorder

Fix an interpersonal preorder $(\succeq)$ on $\mathcal{X}$.

The Approximate Maximin Social Preorder

Fix an interpersonal preorder $(\succeq)$ on $\mathcal{X}$. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we set

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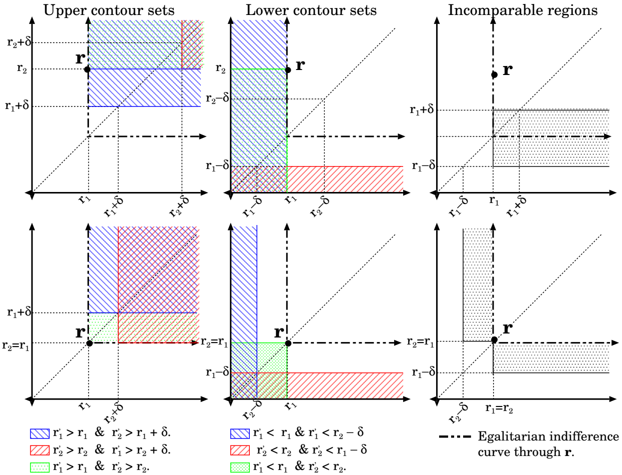
Two-person approximate maximin on $\mathbb{R}^2$



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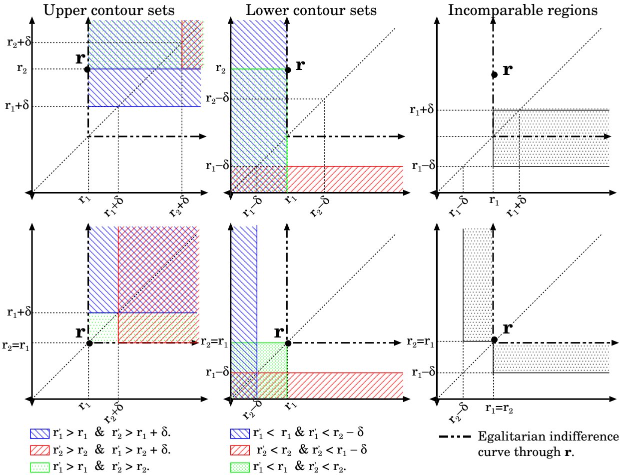

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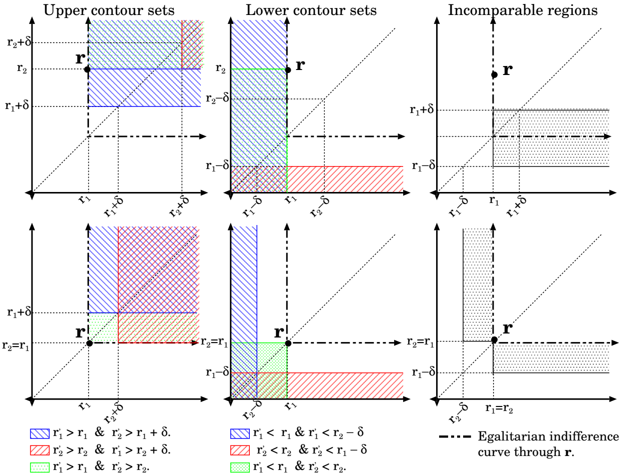

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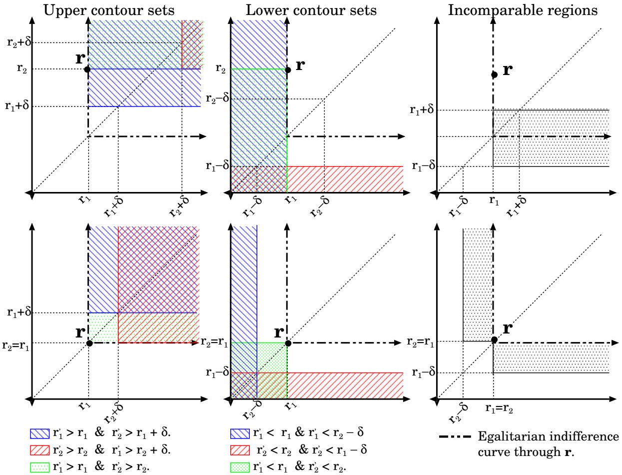

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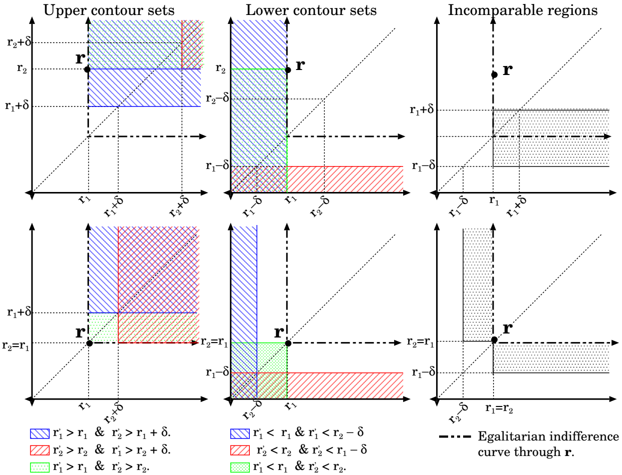

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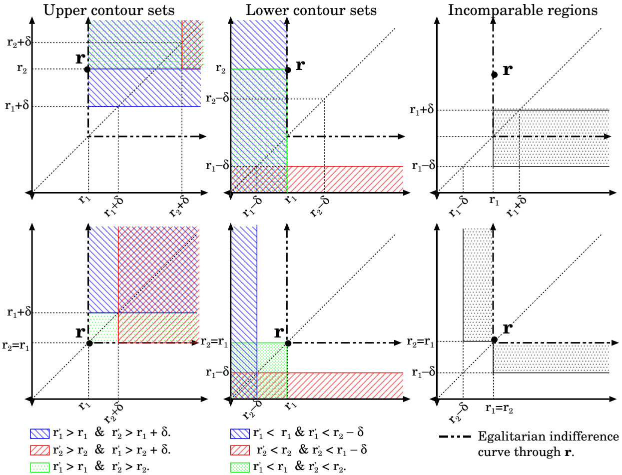

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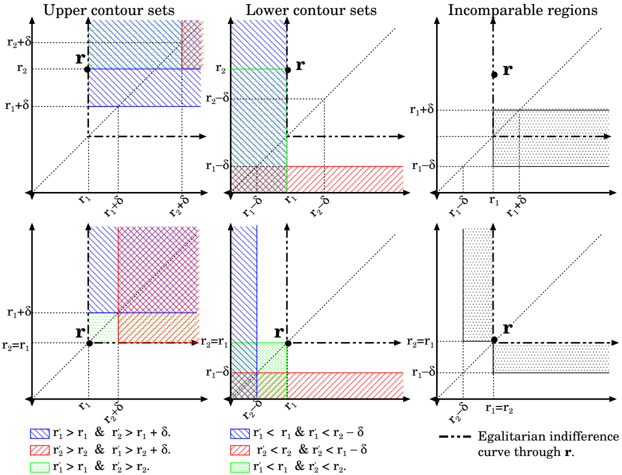

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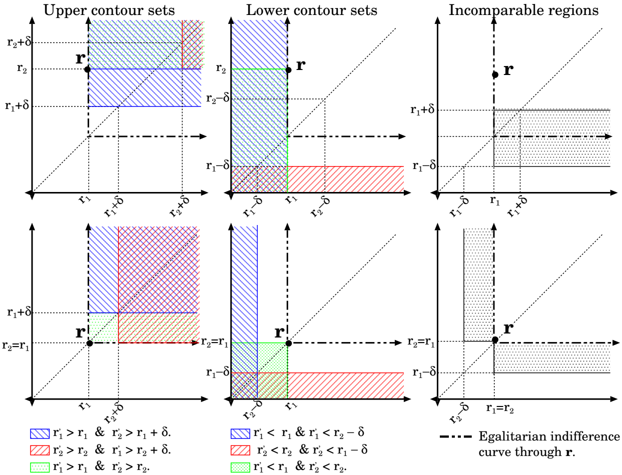

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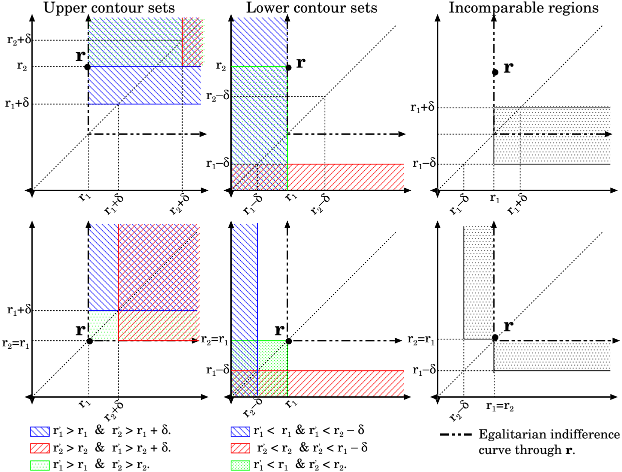

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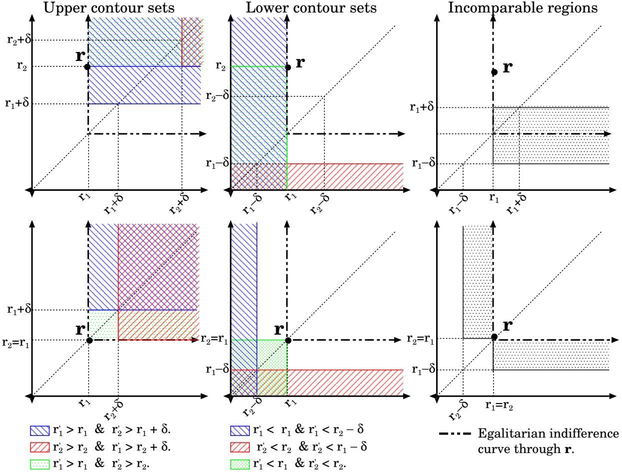

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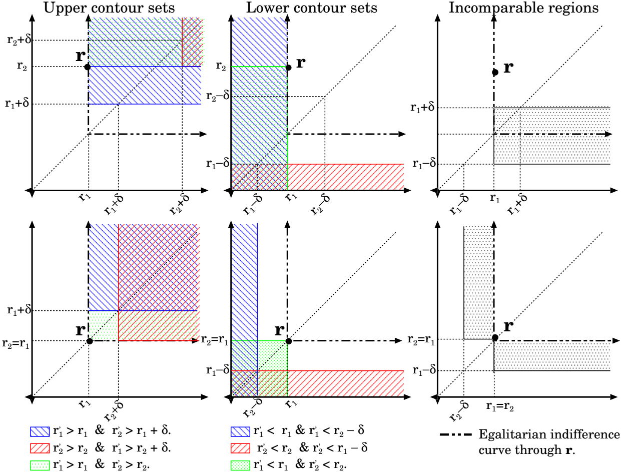

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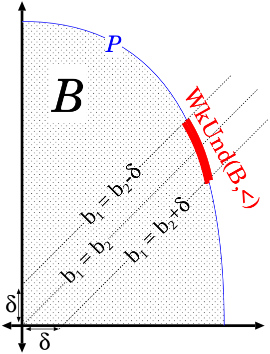

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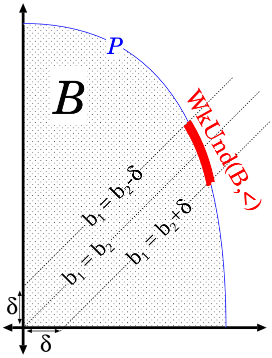

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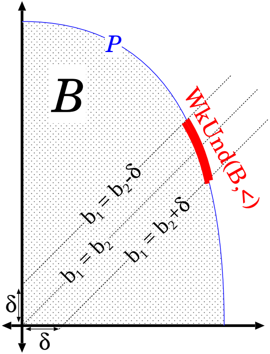

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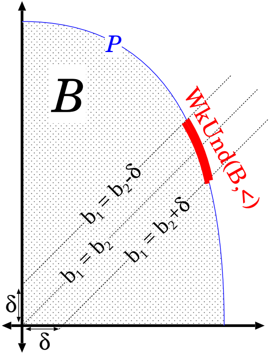
Approximate maximin bilateral bargaining



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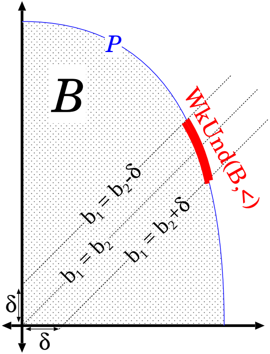

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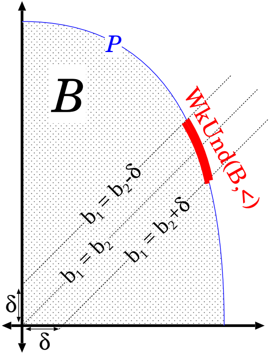
Approximate maximin bilateral bargaining [Skip]



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(Anonomity) If $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is a permutation, and $\mathbf{r} \in \mathbb{R}^{\mathcal{I}}$, then $\mathbf{r} \overset{\blacktriangle}{\approx} \sigma(\mathbf{r})$.

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Metric social preorders

A *social welfare order* (SWO) is a complete preorder ($\underline{\blacktriangleright}$) on $\mathbb{R}^{\mathcal{I}}$ satisfying:

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A $(\succeq)$-social preorder $(\trianglerighteq)$ is *metric* if there is some SWO ($\underline{\blacktriangleright}$) such that, for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$, we have

$$\left(\mathbf{x} \trianglerighteq \mathbf{y}\right) \quad \Longleftrightarrow \quad \left(\mathbf{u}(\mathbf{x}) \;\underline{\blacktriangleright}\; \mathbf{u}(\mathbf{y}), \text{ for all } u \in \mathcal{U}(\succeq)\right).$$

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Example: The approximate maximin preorder $(\underset{\text{am}}{\trianglerighteq})$ is metric: let $(\trianglerighteq)$ be the maximin SWO on $\mathbb{R}^{\mathcal{I}}$ (by Theorem 3).

Characterization of approximate maximin: setup

- Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$. We say $\mathbf{x}$ and $\mathbf{y}$ are *fully $(\succeq)$-comparable* if the set $\{x_i\}_{i \in \mathcal{I}} \cup \{y_i\}_{i \in \mathcal{I}}$ is totally ordered by $(\succeq)$. (i.e. *complete* interpersonal comparability between everyone in these two social states).

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- Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$. We say $\mathbf{x}$ and $\mathbf{y}$ are *fully* ($\succeq$)*-comparable* if the set $\{x_i\}_{i \in \mathcal{I}} \cup \{y_i\}_{i \in \mathcal{I}}$ is totally ordered by ($\succeq$). (i.e. *complete* interpersonal comparability between everyone in these two social states).
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- Suppose $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathcal{X}^{\mathcal{I}}$ are fully ($\succeq$)-comparable. Their *rank structure* is the complete order ($\gtrdot$) on $\{1, 2, 3\} \times \mathcal{I}$ defined:

$$\forall\ n, m \in \{1, 2\},\ \ \forall\ i, j \in \mathcal{I}, \quad \Big((n, i) \gtrdot (m, j)\Big) \quad \iff \quad \left(x_i^n \succeq x_j^m\right).$$

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- The social preorder $(\trianglerighteq)$ is *minimally decisive* (MinDec) if $\mathbf{x}$ and $\mathbf{y}$ are $(\trianglerighteq)$-comparable whenever they are fully $(\succeq)$-comparable. (e.g: the approximate maximin preorder $(\underset{\text{am}}{\trianglerighteq})$ is minimally decisive. But Suppes-Sen is not.)
- Suppose $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathcal{X}^{\mathcal{I}}$ are fully $(\succeq)$-comparable. Their *rank structure* is the complete order $(\geq)$ on $\{1, 2, 3\} \times \mathcal{I}$ defined:

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- We say $(\succeq)$ satisfies *minimally richness* (MR) if, for any complete order $(\geq)$ on $\{1, 2, 3\} \times \mathcal{I}$, there exist fully $(\succeq)$-comparable $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3 \in \mathcal{X}^{\mathcal{I}}$ whose rank structure is $(\geq)$. (An almost trivial assumption.)

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(Example: any two complete preorders have the same scope).

Characterization of approximate maximin

We say $(\trianglerighteq)$ satisfies *minimal charity* (MinCh) if there exist $\mathbf{x}, \mathbf{y} \in \mathcal{X}^{\mathcal{I}}$ and $i \in \mathcal{I}$ such that $x_i \prec y_i \preceq y_j \prec x_j$ for all $j \in \mathcal{I} \setminus \{i\}$; yet $\mathbf{x} \trianglelefteq \mathbf{y}$.

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Example: Let $(\underset{\text{lex}}{\blacktriangleright})$ be 'leximin' SWO on $\mathbb{R}^{\mathcal{I}}$.

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(a) *If* $(\trianglerighteq)$ *satisfies* (MinCh) *and* (MinDec), *then* $(\underset{\text{am}}{\trianglerighteq})$ *extends* $(\trianglerighteq)$.

(b) $(\trianglerighteq)$ *satisfies* (MinCh) *and has same scope as* $(\underset{\text{am}}{\trianglerighteq})$ *iff* $(\trianglerighteq)$ *refines* $(\underset{\text{am}}{\trianglerighteq})$.

(c) $(\trianglerighteq)$ *extends* $(\underset{\text{am}}{\trianglerighteq})$ *if and only if* $(\trianglerighteq)$ *is* $(\underset{\text{am}}{\trianglerighteq})$.

Interpretation: $(\underset{\text{am}}{\trianglerighteq})$ has 'maximal scope' within the set of minimally decisive, minimal charitable, metric social preorders.

Example: Let $(\underset{\text{lex}}{\blacktriangleright})$ be 'leximin' SWO on $\mathbb{R}^{\mathcal{I}}$. Define metric preorder $(\underset{\text{lex}}{\trianglerighteq})$ by

$$\left(\mathbf{x} \underset{\text{lex}}{\trianglerighteq} \mathbf{y}\right) \iff \left(\mathbf{u}(\mathbf{x}) \underset{\text{lex}}{\blacktriangleright} \mathbf{u}(\mathbf{y}), \text{ for all } u \in \mathcal{U}(\succeq)\right).$$

Then $(\underset{\text{lex}}{\trianglerighteq})$ satisfies (MinDec) and (MinCh); thus $(\underset{\text{am}}{\trianglerighteq})$ extends $(\underset{\text{lex}}{\trianglerighteq})$.

Part III:
Risky social choice

Now we turn to individual/social choice under risk.

von Neumann-Morgenstern interpersonal preorders

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- (*Linearity*) For all $\rho, \rho'_1, \rho'_2 \in \mathfrak{P}$ and $s, s' \in (0,1)$ with $s + s' = 1$, $\left(\rho'_1 \succeq \rho'_2\right) \implies \left((s\rho + s'\rho'_1) \succeq (s\rho + s'\rho'_2)\right)$.

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Example: For any $u : \mathcal{X} \longrightarrow \mathbb{R}$ and $\rho \in \mathfrak{P}$, let $u^*(\rho) := \int_{\mathcal{X}} u \, \mathrm{d}\rho \quad = \quad$ (the expected u-payoff of lottery ρ.)

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Also, if $\mathcal{X}$ is a compact metric space, and $\mathfrak{P} =$ space of all Borel measures with weak* topology, then any continuous vNMIP has a multiutility representation (Dubra, Maccheroni and Ok, 2004 or Evren, 2008).

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- Let $\mathcal{U}(\succeq)$ be the set of all vNM utility functions for $(\succeq)$.

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- (Anonymity) If $\sigma : \mathcal{I} \longrightarrow \mathcal{I}$ is any permutation, then for all $\boldsymbol{\rho} \in \mathfrak{P}^{\otimes\mathcal{I}}$, $\boldsymbol{\rho} \triangleq \sigma(\boldsymbol{\rho})$.

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The approximate utilitarian vNMSP

Let $\boldsymbol{\rho} \in \mathfrak{P}^{\otimes \mathcal{I}}$.

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Theorem 5.

The approximate utilitarian vNMSP

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Theorem 5. (a) *Every* $(\succeq)$*-vNMSP on* $\mathfrak{P}^{\otimes \mathcal{I}}$ *extends and refines* $(\underset{u}{\trianglerighteq})$*. That is, for all* $\boldsymbol{\rho}$ *and* $\boldsymbol{\rho}'$ *in* $\mathfrak{P}^{\otimes \mathcal{I}}$*:*

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(b) *Also, if* $\boldsymbol{\rho} \underset{u}{\trianglerighteq} \boldsymbol{\rho}'$*, then* $\sum_{i\in\mathcal{I}} u^*(\rho_i) \geq \sum_{i\in\mathcal{I}} u^*(\rho_i')$*, for all* $u \in \mathcal{U}(\succeq)$*.*

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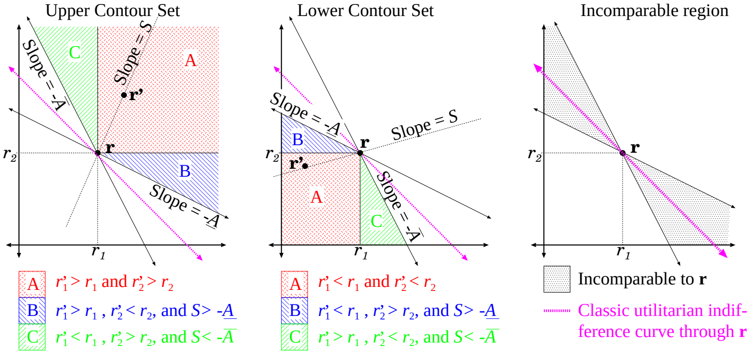
(c) *Suppose $\mathcal{U}(\succeq)$ provides a multiutility representation for $(\succeq)$. Then*

$$\forall\, \boldsymbol{\rho}, \boldsymbol{\rho}' \in \mathfrak{P}^{\otimes \mathcal{I}}, \quad \left(\boldsymbol{\rho} \underset{u}{\trianglerighteq} \boldsymbol{\rho}'\right) \iff \left(\sum_{i \in \mathcal{I}} u^*(\rho_i) \geq \sum_{i \in \mathcal{I}} u^*(\rho_i'), \text{ for all } u \in \mathcal{U}\right).$$



Suppose $(\succeq)$ has a multiutility representation, so that

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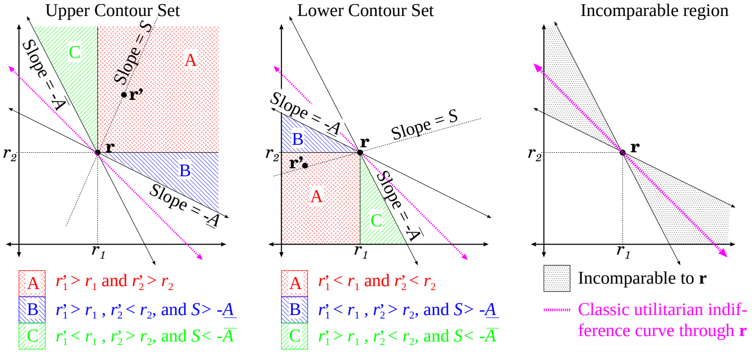

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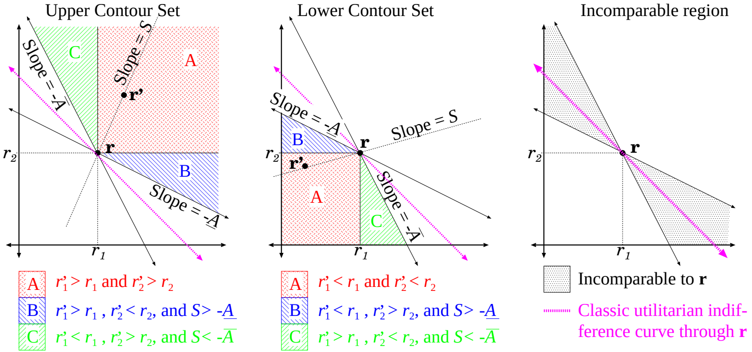

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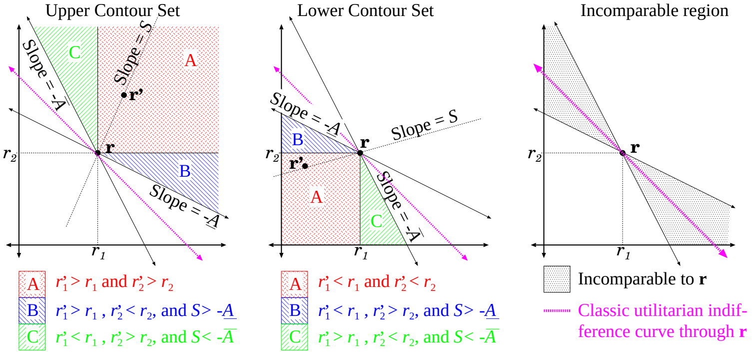

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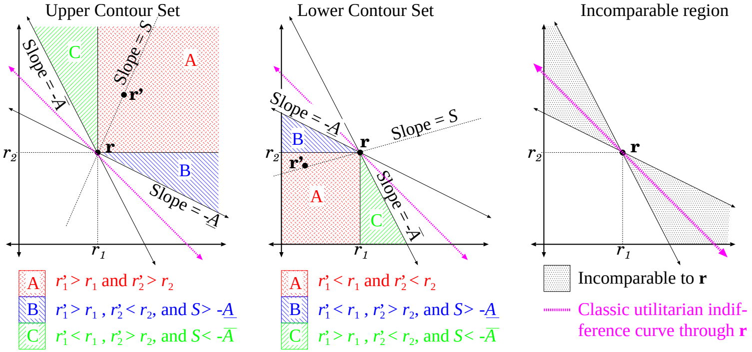

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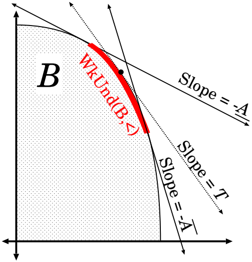
If we project $(\underset{u}{\trianglerighteq})$ through $\mathbf{u}^0_{\boldsymbol{\psi}}$, we get a preorder $(\underset{u,\boldsymbol{\psi}}{\blacktriangleright})$ on $\mathbb{R}^2$ shown above.

(Here, $\underline{A}$ and $\overline{A}$ are the minimum and maximum 'u^0-utility conversion ratios' between ψ_1 and ψ_2 which are induced by the vNMIP $(\succeq)$.)



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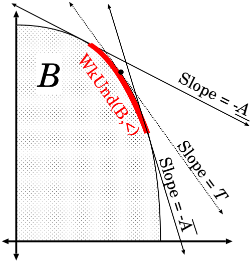
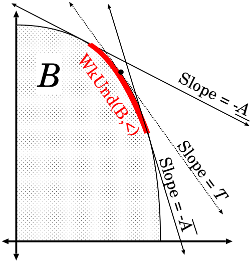
Approximate utilitarian bargaining solution



Let $\mathcal{B} \subset \mathbb{R}^2$ be a bargaining problem. The approximate utilitarian bargaining solution wkUnd $\left(\mathcal{B}, \underset{u,\psi}{\trianglerighteq}\right)$ is shown above. It is the set of all points on $\mathcal{P}$ whose *tangent slope* is between $-\overline{A}$ and $-\underline{A}$.



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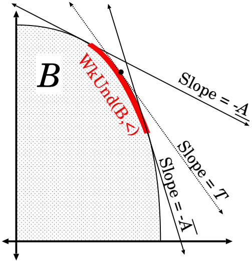

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Remark. If $(\succeq)$ is any vNMIP, and $(\trianglerighteq)$ is any $(\succeq)$-vNMSP, and $\mathcal{B}$ is a bargaining problem, then Theorem 5(a) implies that

$$\operatorname{wkUnd}\left(\mathcal{B}, \trianglerighteq\right) \quad \subseteq \quad \operatorname{wkUnd}\left(\mathcal{B}, \underset{u}{\trianglerighteq}\right).$$

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- More 'psychologically realistic' constructions of interpersonal preorders.

Thank you.

These presentation slides are available at

<http://euclid.trentu.ca/pivato/Research/approx.pdf>

Preprints:

- *Approximate interpersonal comparisons of well-being.* <http://mpra.ub.uni-muenchen.de/25224>
- *Aggregation of incomplete ordinal preferences with approximate interpersonal comparisons.* <http://mpra.ub.uni-muenchen.de/25271>
- *Risky social choice with approximate interpersonal comparisons of well-being.* <http://mpra.ub.uni-muenchen.de/25222>
- *Social choice with approximate interpersonal comparisons of welfare gains.* (preprint available upon request)