Majority rule in the absence of a majority

Part II: Reinforcement, uniqueness, and continuity

New Developments in Judgment Aggregation and Voting Theory
Freudenstadt

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Review of notation and terminology

- Let $\mathcal{K}$ be a finite set of propositions (or ‘issues’, or ‘properties’).
- $\{\pm 1\}^{\mathcal{K}}$ is thus the set of all assignments of truth values to $\mathcal{K}$.
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- Let $\Delta(\mathcal{X})$ denote the set of all profiles.
- A **judgement aggregation rule** is a correspondence $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$.
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  (Here, $\widetilde{\mu}_k := \sum_{\mathbf{x} \in \mathcal{X}} \mu(\mathbf{x})\, x_k \ \in \ [-1,1]$, the ‘support’ for proposition k.)
- In particular, if $\phi(r) := r$ for all $r \in [-1,1]$, we get the **median rule**:

$$\mathrm{Median}(\mu) \quad := \quad \operatorname*{argmax}_{\mathbf{x} \in \mathcal{X}} \sum_{k \in \mathcal{K}} x_k \, \widetilde{\mu}_k \quad = \quad \operatorname*{argmax}_{\mathbf{x} \in \mathcal{X}} \mathbf{x} \bullet \widetilde{\mu}.$$

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- Let $\mathcal{A} := \{1, 2, 3, \ldots, A\}$ be a finite set of 'social alternatives'.
- Let $\mathcal{K} := \{(a, b);\ \ a, b \in \mathcal{A} \text{ and } a < b\}$.
- Let $\mathcal{P}_\mathcal{A}$ be the set of all strict preference orders over $\mathcal{A}$.
- For any $(\succ) \in \mathcal{P}_\mathcal{A}$ define $\mathbf{x}^\succ \in \{\pm 1\}^\mathcal{K}$ as follows:

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- Let $\mathcal{X}^{\mathrm{pr}}_\mathcal{A} := \{\mathbf{x}^\succ;\ (\succ) \in \mathcal{P}_\mathcal{A}\}$. This judgement space is called the **permutahedron**. Judgement aggregation over $\mathcal{X}^{\mathrm{pr}}_\mathcal{A}$ is equivalent to classic Arrovian preference aggregation.
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- Let $\mathcal{X}^{\mathrm{pr}}_\mathcal{A} := \{\mathbf{x}^\succ;\ \ (\succ) \in \mathcal{P}_\mathcal{A}\}$. This judgement space is called the **permutahedron**. Judgement aggregation over $\mathcal{X}^{\mathrm{pr}}_\mathcal{A}$ is equivalent to classic Arrovian preference aggregation.
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Example: the permutahedron

- Let $\mathcal{A} := \{1, 2, 3, \ldots, A\}$ be a finite set of 'social alternatives'.
- Let $\mathcal{K} := \{(a, b);\ \ a, b \in \mathcal{A} \text{ and } a < b\}$.
- Let $\mathcal{P}_{\mathcal{A}}$ be the set of all strict preference orders over $\mathcal{A}$.
- For any $(\succ) \in \mathcal{P}_{\mathcal{A}}$ define $\mathbf{x}^{\succ} \in \{\pm 1\}^{\mathcal{K}}$ as follows:

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A judgement aggregation rule $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ satisfies *reinforcement* if: for any profiles μ_0 and μ_1 in $\Delta(\mathcal{X})$ with $F(\mu_0) \cap F(\mu_1) \neq \emptyset$, we have

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Idea. If two subpopulations both select judgement $\mathbf{x}$ from $\mathcal{X}$, then the combined population should also select $\mathbf{x}$ (and *only* $\mathbf{x}$).

Proposition. *The median rule satisfies reinforcement on every judgement space.*

On the permutahedron, the median rule is the *Kemeny rule*. Young and Levenglick (1978) proved that the Kemeny rule is the *only* neutral, Condorcet-admissible preference aggregation rule which satisfies reinforcement. **Question:** Does the Young-Levenglick theorem extend to other judgement spaces?

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Example: If $\mathrm{st}\,(\phi)$ is finite and not constant in a neighbourhood of zero, then ϕ is regular. In particular, any real-valued ϕ is regular.

If $\mathcal{X} \subseteq \{\pm 1\}^{\mathcal{K}}$, then $\mathrm{conv}(\mathcal{X}) \subset \mathbb{R}^{\mathcal{K}}$. Say $\mathcal{X}$ is **thick** if $\dim[\mathrm{conv}(\mathcal{X})] = |\mathcal{K}|$.

The main result of Part II is the following:

Theorem 2A. *Let $\mathcal{X}$ be a thick judgement space and let $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ be an additive support rule. Then: F is regular, upper hemicontinuous and satisfies reinforcement on $\Delta(\mathcal{X})$ if and only if F is the median rule.*

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Example. Let $>$ be an arbitrary linear ordering on $\mathcal{X}$. Define $F_{M,>} : \Delta(\mathcal{X}) \longrightarrow \mathcal{X}$ by $F_{M,>}(\mu) := \max_{>} [\mathrm{Median}(\mathcal{X}, \mu)]$. (That is: first apply the median rule. Then break any ties using the ordering $>$.)
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Example. Let $>$ be an arbitrary linear ordering on $\mathcal{X}$. Define $F_{M,>} : \Delta(\mathcal{X}) \longrightarrow \mathcal{X}$ by $F_{M,>}(\mu) := \max_{>} [\mathrm{Median}(\mathcal{X}, \mu)]$. (That is: first apply the median rule. Then break any ties using the ordering $>$.)
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Ideally, we would like to eliminate the condition of regularity....

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Plan of talk:

1. From upper hemicontinuity to continuity (Theorems 2C and 2F).
2. Uniqueness of gain function (Theorem 2B).
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Theorem 2A. *Let $\mathcal{X}$ be a thick judgement space, and let $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ be a judgement aggregation rule. Then:*
F is regular, upper hemicontinuous and satisfies reinforcement on $\Delta(\mathcal{X})$ if and only if F is the median rule.

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- **Question 1.** How unique is this representation? That is: given two gain functions ψ and ϕ, how 'similar' must they be if $F_\phi = F_\psi$?
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The answer to these questions depends upon the structure of $\mathcal{X}$.

For example, if $\mathcal{X}$ is supermajoritarian determinate, then for *any* ϕ and ψ, we have $F_\phi(\mathcal{X},\mu) = F_\psi(\mathcal{X},\mu)$ for all $\mu \in \Delta(\mathcal{X})$.

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Finally, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, define ${}^o\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_\phi(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$.
(This set may be empty.)

The boundary set ${}^{\partial}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}$

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For any $\mathbf{x} \in \mathcal{X}$, define $\mathcal{C}^{\phi}_{\mathbf{x}} := \{\mathbf{c} \in \mathcal{C}\ ;\ \mathbf{x} \in F_\phi(\mathbf{c})\}$ (the 'preimage' of $\mathbf{x}$).

Let $\mathcal{A}$ be the affine subspace of $\mathbb{R}^{\mathcal{K}}$ spanned by $\mathcal{C}$, and let $\mathrm{int}\,(\mathcal{C})$ be the relative interior of $\mathcal{C}$ as a subset of $\mathcal{A}$. (If $\mathcal{X}$ is thick, then this is just the interior of $\mathcal{C}$ as a subset of $\mathbb{R}^{\mathcal{K}}$).

Finally, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, define $\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_\phi(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$.
(This set may be empty.)

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we define $\mathcal{K}_{\pm}(\mathbf{x}, \mathbf{y}) := \{k \in \mathcal{K};\ x_k \neq y_k\}$.

For example, suppose $\mathcal{K} = \{1, 2, 3, 4, \ldots, K\}$.
If $\mathbf{x} = (-1, -1, -1, u, v, w, \ldots)$ and $\mathbf{y} = (1, 1, 1, u, v, w, \ldots)$, then $\mathcal{K}_{\pm}(\mathbf{x}, \mathbf{y}) = \{1, 2, 3\}$.

Let $\mathcal{C} := \mathrm{conv}(\mathcal{X}) \subseteq \mathbb{R}^{\mathcal{K}}$. Then $\widetilde{\mu} \in \mathcal{C}$ for all $\mu \in \Delta(\mathcal{X})$.
Thus, for any odd gain function $\phi : [-1, 1] \longrightarrow {}^*\mathbb{R}$, the additive support rule F_ϕ can be reinterpreted as a function $F_\phi : \mathcal{C} \rightrightarrows \mathcal{X}$, defined by
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Finally, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, define $\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_\phi(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$.
(This set may be empty.)

The boundary set ${}^{\mathcal{O}}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}$

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we define $\mathcal{K}_{\pm}(\mathbf{x}, \mathbf{y}) := \{k \in \mathcal{K};\ x_k \neq y_k\}$.

For example, suppose $\mathcal{K} = \{1, 2, 3, 4, \ldots, K\}$.
If $\mathbf{x} = (-1, -1, -1, u, v, w, \ldots)$ and $\mathbf{y} = (1, 1, 1, u, v, w, \ldots)$, then $\mathcal{K}_{\pm}(\mathbf{x}, \mathbf{y}) = \{1, 2, 3\}$.

Let $\mathcal{C} := \mathrm{conv}(\mathcal{X}) \subseteq \mathbb{R}^{\mathcal{K}}$. Then $\widetilde{\mu} \in \mathcal{C}$ for all $\mu \in \Delta(\mathcal{X})$.
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Let $\mathcal{A}$ be the affine subspace of $\mathbb{R}^{\mathcal{K}}$ spanned by $\mathcal{C}$, and let $\mathrm{int}\,(\mathcal{C})$ be the relative interior of $\mathcal{C}$ as a subset of $\mathcal{A}$. (If $\mathcal{X}$ is thick, then this is just the interior of $\mathcal{C}$ as a subset of $\mathbb{R}^{\mathcal{K}}$).

Finally, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, define ${}^{\mathcal{O}}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} \ := \ \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_\phi(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$.
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Let $\mathcal{C} := \mathrm{conv}(\mathcal{X}) \subseteq \mathbb{R}^{\mathcal{K}}$. Then $\widetilde{\mu} \in \mathcal{C}$ for all $\mu \in \Delta(\mathcal{X})$.
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Let $\mathcal{A}$ be the affine subspace of $\mathbb{R}^{\mathcal{K}}$ spanned by $\mathcal{C}$, and let $\mathrm{int}\,(\mathcal{C})$ be the relative interior of $\mathcal{C}$ as a subset of $\mathcal{A}$. (If $\mathcal{X}$ is thick, then this is just the interior of $\mathcal{C}$ as a subset of $\mathbb{R}^{\mathcal{K}}$).

Finally, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, define $\mathscr{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_\phi(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$.
(This set may be empty.)

The boundary set ${}^{\partial}\!\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}$

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we define $\mathcal{K}_{\pm}(\mathbf{x}, \mathbf{y}) := \{k \in \mathcal{K};\ x_k \neq y_k\}$.

For example, suppose $\mathcal{K} = \{1, 2, 3, 4, \ldots, K\}$.
If $\mathbf{x} = (-1, -1, -1, u, v, w, \ldots)$ and $\mathbf{y} = (1, 1, 1, u, v, w, \ldots)$, then $\mathcal{K}_{\pm}(\mathbf{x}, \mathbf{y}) = \{1, 2, 3\}$.

Let $\mathcal{C} := \mathrm{conv}(\mathcal{X}) \subseteq \mathbb{R}^{\mathcal{K}}$. Then $\widetilde{\mu} \in \mathcal{C}$ for all $\mu \in \Delta(\mathcal{X})$.
Thus, for any odd gain function $\phi : [-1, 1] \longrightarrow {}^*\mathbb{R}$, the additive support rule F_ϕ can be reinterpreted as a function $F_\phi : \mathcal{C} \rightrightarrows \mathcal{X}$, defined by $F_\phi(\mathbf{c}) := \operatorname*{argmax}_{\mathbf{x} \in \mathcal{X}} (\mathbf{x} \bullet \phi(\mathbf{c}))$, for all $\mathbf{c} \in \mathcal{C}$.

For any $\mathbf{x} \in \mathcal{X}$, define $\mathcal{C}^{\phi}_{\mathbf{x}} := \{\mathbf{c} \in \mathcal{C}\ ;\ \mathbf{x} \in F_\phi(\mathbf{c})\}$ (the 'preimage' of $\mathbf{x}$).

Let $\mathcal{A}$ be the affine subspace of $\mathbb{R}^{\mathcal{K}}$ spanned by $\mathcal{C}$, and let $\mathrm{int}\,(\mathcal{C})$ be the relative interior of $\mathcal{C}$ as a subset of $\mathcal{A}$. (If $\mathcal{X}$ is thick, then this is just the interior of $\mathcal{C}$ as a subset of $\mathbb{R}^{\mathcal{K}}$).

Finally, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, define ${}^{\partial}\!\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_\phi(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$.
(This set may be empty.)

The boundary set ${}^o\mathcal{B}^\phi_{\mathbf{x},\mathbf{y}}$

For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we define $\mathcal{K}_\pm(\mathbf{x}, \mathbf{y}) := \{k \in \mathcal{K};\ x_k \neq y_k\}$.

For example, suppose $\mathcal{K} = \{1, 2, 3, 4, \ldots, K\}$.
If $\mathbf{x} = (-1, -1, -1, u, v, w, \ldots)$ and $\mathbf{y} = (1, 1, 1, u, v, w, \ldots)$, then
$\mathcal{K}_\pm(\mathbf{x}, \mathbf{y}) = \{1, 2, 3\}$.

Let $\mathcal{C} := \mathrm{conv}(\mathcal{X}) \subseteq \mathbb{R}^{\mathcal{K}}$. Then $\widetilde{\mu} \in \mathcal{C}$ for all $\mu \in \Delta(\mathcal{X})$.
Thus, for any odd gain function $\phi : [-1, 1] \longrightarrow {}^*\mathbb{R}$, the additive support rule F_ϕ can be reinterpreted as a function $F_\phi : \mathcal{C} \rightrightarrows \mathcal{X}$, defined by
$F_\phi(\mathbf{c}) := \operatorname*{argmax}_{\mathbf{x} \in \mathcal{X}} (\mathbf{x} \bullet \phi(\mathbf{c}))$, for all $\mathbf{c} \in \mathcal{C}$.

For any $\mathbf{x} \in \mathcal{X}$, define $\mathcal{C}^\phi_{\mathbf{x}} := \{\mathbf{c} \in \mathcal{C}\ ;\ \mathbf{x} \in F_\phi(\mathbf{c})\}$ (the 'preimage' of $\mathbf{x}$).

Let $\mathcal{A}$ be the affine subspace of $\mathbb{R}^{\mathcal{K}}$ spanned by $\mathcal{C}$, and let $\mathrm{int}\,(\mathcal{C})$ be the relative interior of $\mathcal{C}$ as a subset of $\mathcal{A}$. (If $\mathcal{X}$ is thick, then this is just the interior of $\mathcal{C}$ as a subset of $\mathbb{R}^{\mathcal{K}}$).

Finally, for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, define ${}^o\mathcal{B}^\phi_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_\phi(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$.
(This set may be empty.)

Recall: $\mathcal{C} := \operatorname{conv}(\mathcal{X})$ and ${}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \operatorname{int}(\mathcal{C}) \ ; \ F_{\phi}(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$

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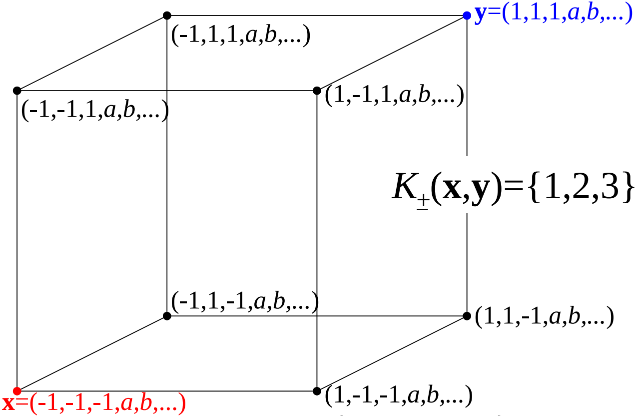
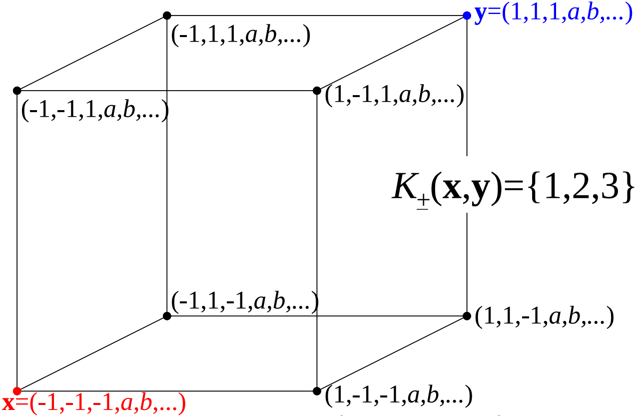


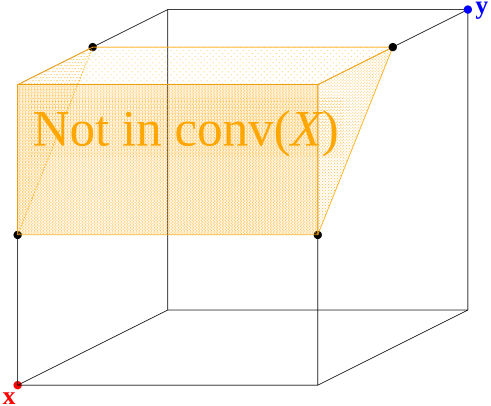
To visualize this, suppose that $\mathcal{X} = \{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots\}$.
Again, suppose $\mathcal{K} = \{1, 2, 3, 4, \ldots, K\}$, $\quad \mathbf{x} = (-1, -1, -1, u, v, w, \ldots)$, and $\mathbf{y} = (1, 1, 1, u, v, w, \ldots)$, so that $\mathcal{K}_\pm(\mathbf{x}, \mathbf{y}) = \{1, 2, 3\}$.

Recall: $\mathcal{C} := \mathrm{conv}(\mathcal{X})$ and ${}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_{\phi}(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$



To visualize this, suppose that $\mathcal{X} = \{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots\}$.
Again, suppose $\mathcal{K} = \{1, 2, 3, 4, \ldots, K\}$, $\mathbf{x} = (-1, -1, -1, u, v, w, \ldots)$, and $\mathbf{y} = (1, 1, 1, u, v, w, \ldots)$, so that $\mathcal{K}_{\pm}(\mathbf{x}, \mathbf{y}) = \{1, 2, 3\}$.
Here we show a section through the cube $[-1, 1]^{\mathcal{K}}$, where the coordinates $\{1, 2, 3\}$ are allowed to vary, while coordinates $\{4, 5, 6, \ldots, K\}$ are held fixed at some values $a, b, c, \ldots$.

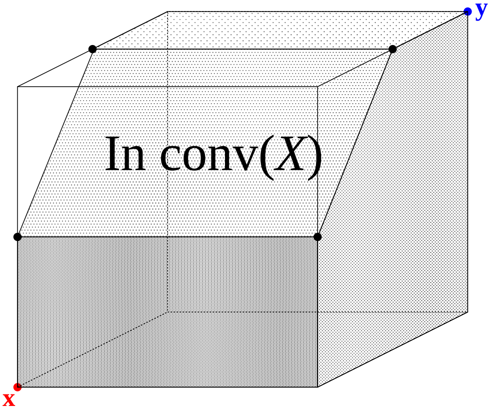
Recall: $\mathcal{C} := \mathrm{conv}(\mathcal{X})$ and ${}^{\circ}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_{\phi}(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$



Here we show a section through the cube $[-1, 1]^{\mathcal{K}}$, where the coordinates $\{1, 2, 3\}$ are allowed to vary, while coordinates $\{4, 5, 6, \ldots, K\}$ are held fixed at some values $a, b, c, \ldots$.

Suppose the orange region is the part of this section which is *not* in $\mathcal{C}$.

Recall: $\mathcal{C} := \mathrm{conv}(\mathcal{X})$ and ${}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_{\phi}(\mathbf{c}) = \{\mathbf{x},\mathbf{y}\}\}$

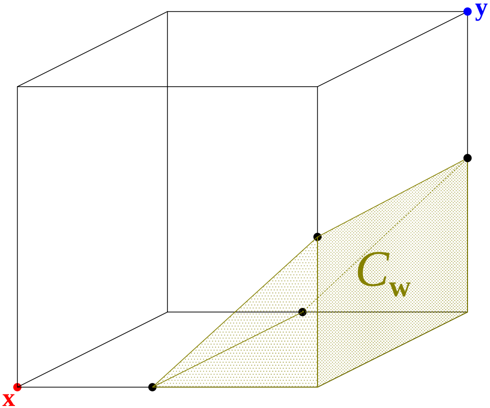


Here we show a section through the cube $[-1,1]^{\mathcal{K}}$, where the coordinates $\{1,2,3\}$ are allowed to vary, while coordinates $\{4,5,6,\ldots,K\}$ are held fixed at some values $a, b, c, \ldots$.

Suppose the orange region is the part of this section which is *not* in $\mathcal{C}$.

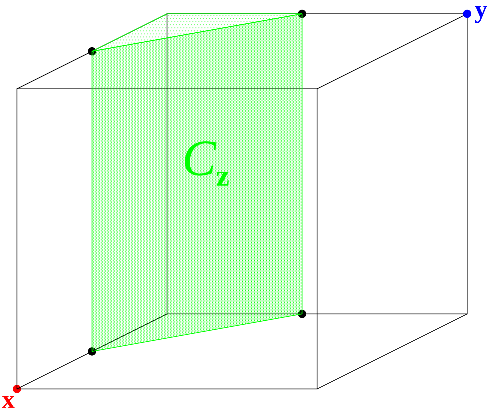
Thus, the grey region represents a section through $\mathcal{C}$.

Recall: $\mathcal{C} := \mathrm{conv}(\mathcal{X})$ and ${}^{\circ}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_{\phi}(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$



Suppose the brown region represents a section through $\mathcal{C}^{\phi}_{\mathbf{w}}$...

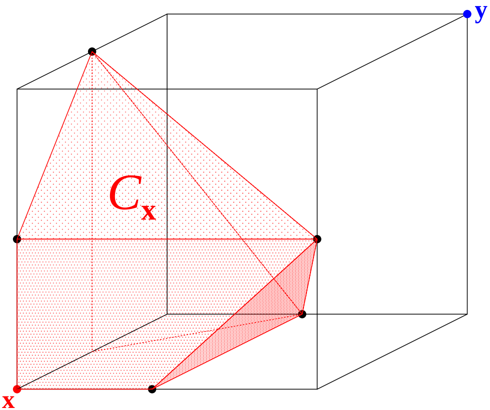
Recall: $\mathcal{C} := \mathrm{conv}(\mathcal{X})$ and ${}^{\circ}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_{\phi}(\mathbf{c}) = \{\mathbf{x},\mathbf{y}\}\}$



Suppose the brown region represents a section through $\mathcal{C}^{\phi}_{\mathbf{w}}$...

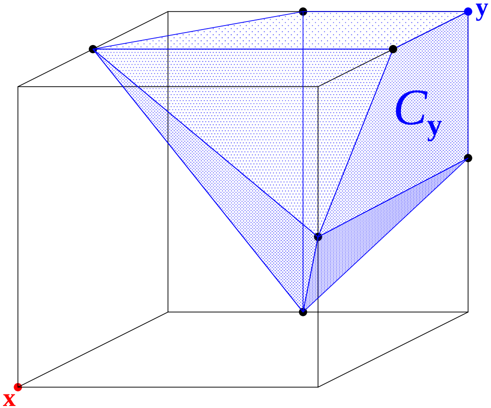
...and suppose the green region represents a section through $\mathcal{C}^{\phi}_{\mathbf{z}}$.

Recall: $\mathcal{C} := \mathrm{conv}(\mathcal{X})$ and $\mathscr{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_{\phi}(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$



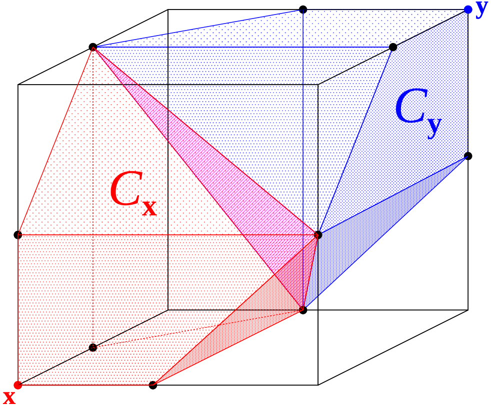
Suppose the red region represents a section through $\mathcal{C}^{\phi}_{\mathbf{x}}$...

Recall: $\mathcal{C} := \mathrm{conv}(\mathcal{X})$ and ${}^{\circ}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_{\phi}(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$



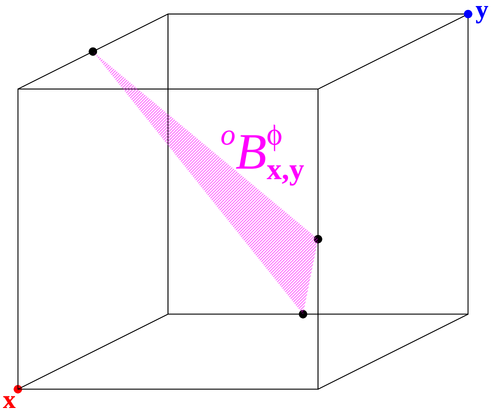
Suppose the red region represents a section through $\mathcal{C}^{\phi}_{\mathbf{x}}$...
...and suppose the blue region represents a section through $\mathcal{C}^{\phi}_{\mathbf{y}}$.

Recall: $\mathcal{C} := \mathrm{conv}(\mathcal{X})$ and ${}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_{\phi}(\mathbf{c}) = \{\mathbf{x},\mathbf{y}\}\}$



Then the purple triangle represents a section through $\mathcal{C}^{\phi}_{\mathbf{x}} \cap \mathcal{C}^{\phi}_{\mathbf{y}}$.

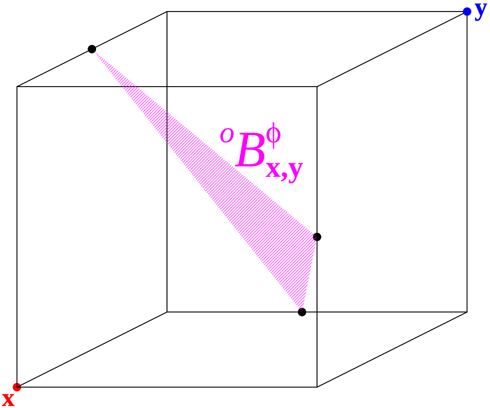
Recall: $\mathcal{C} := \mathrm{conv}(\mathcal{X})$ and ${}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C})\ ;\ F_{\phi}(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$



Then the purple triangle represents a section through $\mathcal{C}^{\phi}_{\mathbf{x}} \cap \mathcal{C}^{\phi}_{\mathbf{y}}$.

Thus, the interior of this purple triangle represents a section through ${}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}$.

Recall: $\mathcal{C} := \mathrm{conv}(\mathcal{X})$ and ${}^o\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} := \{\mathbf{c} \in \mathrm{int}(\mathcal{C})\ ;\ F_\phi(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\}$



Then the purple triangle represents a section through $\mathcal{C}^{\phi}_{\mathbf{x}} \cap \mathcal{C}^{\phi}_{\mathbf{y}}$.
Thus, the interior of this purple triangle represents a section through ${}^o\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}$.

We shall see later that, if F_ϕ is upper hemicontinuous, then the sets ${}^o\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}$ alone completely determine the behaviour of F_ϕ (Proposition 2H).

$\mathcal{K}_\pm(\mathbf{x},\mathbf{y}) = \{k \in \mathcal{K};\ x_k \neq y_k\}$ and ${}^{\circ}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} = \{\mathbf{c} \in \operatorname{int}(\mathcal{C});\ F_\phi(\mathbf{c}) = \{\mathbf{x},\mathbf{y}\}\}$.

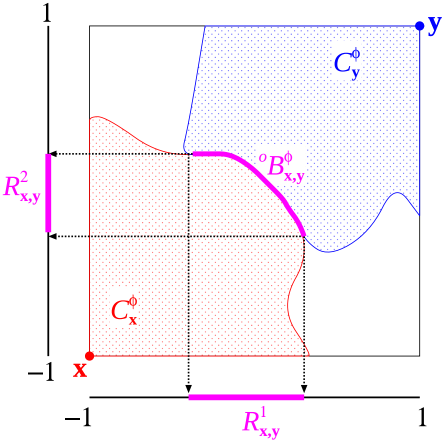
For all $k \in \mathcal{K}_\pm(\mathbf{x},\mathbf{y})$, let $\mathcal{R}^k_{\mathbf{x},\mathbf{y}}$:=projection of ${}^{\circ}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}$ onto the kth coordinate. Finally, define

$$\mathcal{R}^{\phi}_{\mathcal{X}} := \bigcup_{\substack{\mathbf{x},\mathbf{y}\in\mathcal{X} \\ d(\mathbf{x},\mathbf{y})\geq 3}} \ \bigcup_{k\in\mathcal{K}_\pm(\mathbf{x},\mathbf{y})} \mathcal{R}^k_{\mathbf{x},\mathbf{y}}$$

(a subset of $[-1,1]$).

Lemma. *Let $\mathcal{X}$ be any judgement space, and let $\phi : [-1,1] \longrightarrow {}^*\mathbb{R}$ be any gain function such that F_ϕ is upper hemicontinuous. If $\mathcal{X}$ is not supermajoritarian determinate, then $\mathcal{R}^{\phi}_{\mathcal{X}}$ is a nonempty open set.*
(In particular, this holds if $\mathcal{X}$ thick and non-proximal).

$\mathcal{K}_\pm(\mathbf{x},\mathbf{y}) = \{k \in \mathcal{K};\ x_k \neq y_k\}$ and ${}^o\mathcal{B}^\phi_{\mathbf{x},\mathbf{y}} = \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C});\ F_\phi(\mathbf{c}) = \{\mathbf{x},\mathbf{y}\}\}$.
For all $k \in \mathcal{K}_\pm(\mathbf{x},\mathbf{y})$, let $\mathcal{R}^k_{\mathbf{x},\mathbf{y}}$:=projection of ${}^o\mathcal{B}^\phi_{\mathbf{x},\mathbf{y}}$ onto the kth coordinate.



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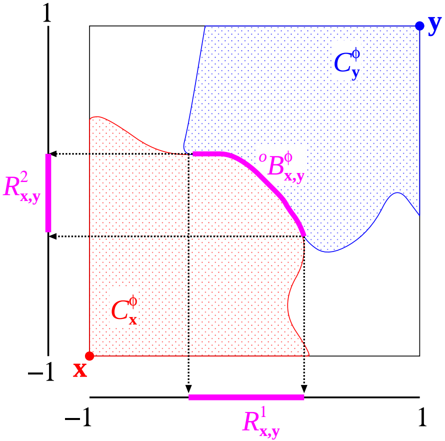

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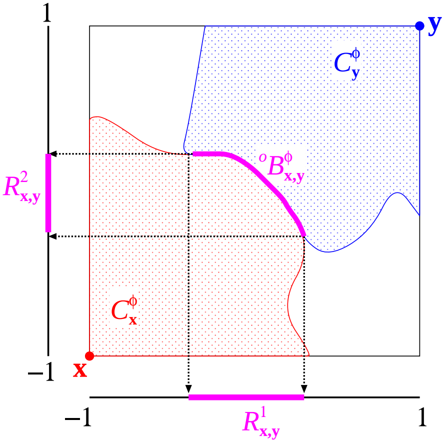

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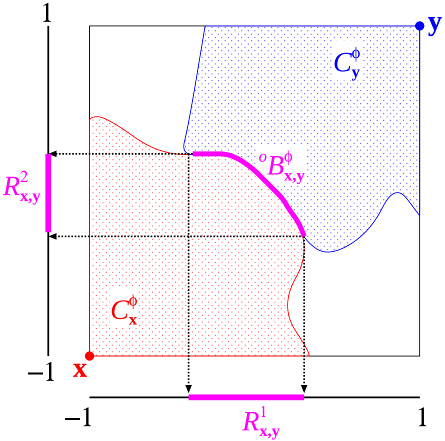
For all $k \in \mathcal{K}_\pm(\mathbf{x},\mathbf{y})$, let $\mathcal{R}^k_{\mathbf{x},\mathbf{y}}$:=projection of ${}^o\mathcal{B}^\phi_{\mathbf{x},\mathbf{y}}$ onto the kth coordinate.



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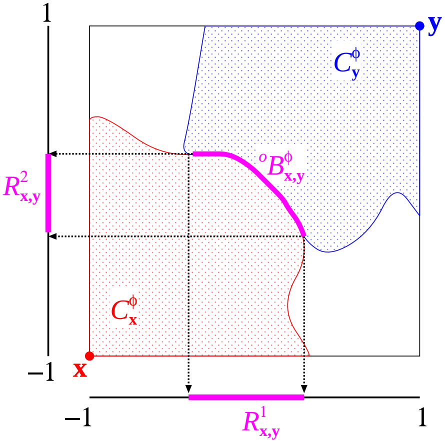

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Also, recall that $\mathcal{X}$ is **thick** if $\dim[\mathrm{conv}(\mathcal{X})] = |\mathcal{K}|$.

Theorem 2B. *Let $\phi : [-1,1] \longrightarrow \mathbb{R}$ and $\psi : [-1,1] \longrightarrow \mathbb{R}$ be odd, continuous, real-valued gain functions. Let $\mathcal{X}$ be a thick judgement space, such that $\mathcal{R}^{\phi}_{\mathcal{X}} \cup \{0\}$ is connected. Then: $F_{\phi}(\mathcal{X},\mu) = F_{\psi}(\mathcal{X},\mu)$ for all $\mu \in \Delta(\mathcal{X})$ if and only if there is some scalar $s > 0$ such that $\psi(r) = s\,\phi(r)$ for all $r \in \mathcal{R}^{\phi}_{\mathcal{X}}$.*

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Let $\phi : [-1,1] \longrightarrow \mathbb{R}$ be a gain function, and $\mathcal{X}$ a judgement space. Recall:

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Continuity implies upper hemicontinuity

Recall, a judgement aggregation rule $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ is **upper hemicontinuous** (UHC) if, for any sequence $\mu_n \xrightarrow[n\to\infty]{} \mu \in \Delta(\mathcal{X})$, if $\mathbf{x} \in F(\mu_n)$ for all $n \in \mathbb{N}$, then $\mathbf{x} \in F(\mu)$.

Theorem 2C. *If $\phi : [-1,1] \longrightarrow \mathbb{R}$ is continuous, then the additive support rule F_ϕ is upper hemicontinuous on $\Delta(\mathcal{X})$, for any judgement space $\mathcal{X}$.*

Question. Is this theorem still true for $\phi : [-1,1] \longrightarrow {}^*\mathbb{R}$?
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Proposition 2D *Let $\phi : [-1,1] \longrightarrow \mathbb{R}$ be any real-valued gain function. If $\mathcal{X}$ is a thick judgement space, and $F_\phi : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ is upper hemicontinuous, then ϕ must be continuous on $\mathcal{R}^{\phi}_{\mathcal{X}}$.*

Does upper hemicontinuity imply that ϕ must be continuous and/or real-valued on all of $[-1,1]$? In general, no.

Proposition 2E. *Let $M \in \mathbb{N}$, and let $\mathcal{X}^{\mathrm{pr}}_{M}$ be the permutahedron on M alternatives. Let $\phi : [-1,1] \longrightarrow {}^{*}\mathbb{R}$ be a gain function such that ϕ is continuous, real-valued, and unbounded on $\left(-1+\frac{2}{M},\ 1-\frac{2}{M}\right)$, and ϕ is infinite on $\left[-1,\ -1+\frac{2}{M}\right] \sqcup \left[1-\frac{2}{M},\ 1\right]$. Then F_ϕ is upper hemicontinuous on $\Delta(\mathcal{X}^{\mathrm{pr}}_{M})$.*

Thus, the strict converse of Theorem 2C is false. Instead, we have...

Upper hemicontinuity implies continuity?

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(a) *Let $\mathcal{R} \subseteq \mathcal{R}^\phi_\mathcal{X}$ be a connected component of $\mathcal{R}^\phi_\mathcal{X}$, and fix $r_1, r_2 \in \mathcal{R}$ with $0 < r_1 < r_2$. Define $\overline{\phi} : \mathcal{R} \longrightarrow \mathbb{R}$ by*

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(*Note:* ϕ_d is well-defined in ${}^*\mathbb{R}$ even if d is infinite or infinitesimal.)

Then define $H^d(\mathcal{X}, \mu) := F_{\phi_d}(\mathcal{X}, \mu)$. (a 'homogeneous' rule)

Example: $H^1(\mathcal{X}, \mu) = \mathrm{Median}(\mathcal{X}, \mu)$.

Proposition: *Let $\mathcal{X}$ be any judgement space, and let $\mu \in \Delta(\mathcal{X})$.*

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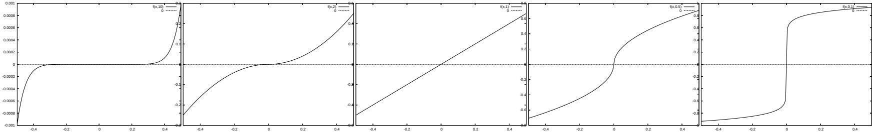

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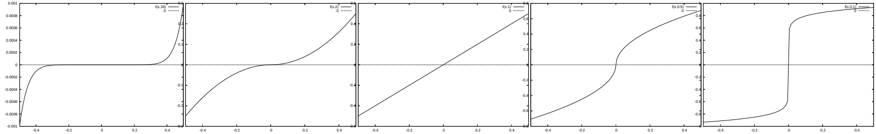

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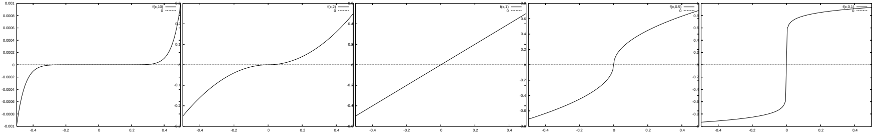

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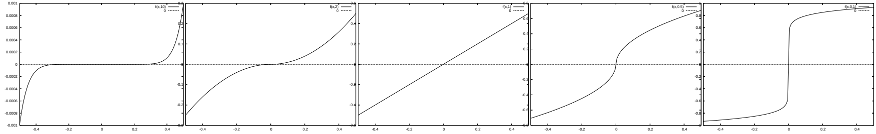

ϕ_{10} ϕ_2 ϕ_1 $\phi_{1/2}$ $\phi_{1/10}$

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Example: $H^1(\mathcal{X},\mu) = \mathrm{Median}\,(\mathcal{X},\mu)$.

Proposition: *Let $\mathcal{X}$ be any judgement space, and let $\mu \in \Delta(\mathcal{X})$.*

(a) $\lim_{d\to\infty} H^d(\mathcal{X},\mu) = \mathrm{LexiMin}\,(\mathcal{X},\mu)$.

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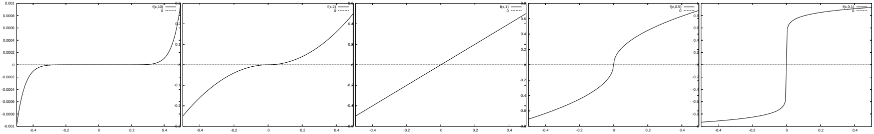

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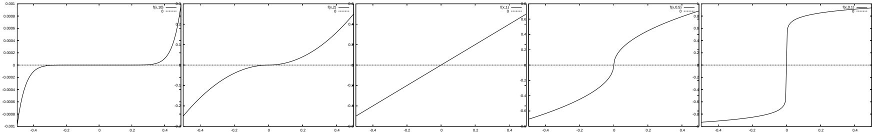

ϕ_{10} ϕ_2 ϕ_1 $\phi_{1/2}$ $\phi_{1/10}$

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Homogeneous Rules

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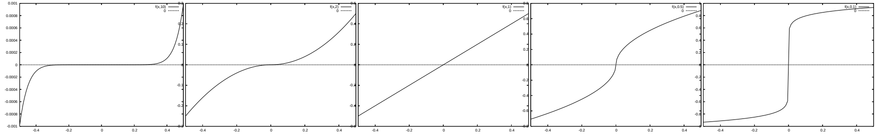

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Neutral Reinforcement & Homogeneous rules

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Proof sketches

Let $\mathcal{X}$ be a judgement space, and let $\mathcal{C} := \mathrm{conv}(\mathcal{X})$.

Let $\phi : [-1,1] \longrightarrow {}^*\mathbb{R}$ be any gain function. For any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, recall that

$$\mathcal{C}_{\mathbf{x}}^{\phi} \;:=\; \{\mathbf{c} \in \mathcal{C} \; ; \; \mathbf{x} \in F_\phi(\mathbf{c})\}, \qquad \mathcal{B}_{\mathbf{x},\mathbf{y}}^{\phi} \;:=\; \mathcal{C}_{\mathbf{x}}^{\phi} \cap \mathcal{C}_{\mathbf{y}}^{\phi},$$

$$\text{and} \quad {}^{o}\mathcal{B}_{\mathbf{x},\mathbf{y}}^{\phi} \;:=\; \{\mathbf{c} \in \mathrm{int}\,(\mathcal{C}) \; ; \; F_\phi(\mathbf{c}) = \{\mathbf{x}, \mathbf{y}\}\} \;\subseteq\; \mathcal{B}_{\mathbf{x},\mathbf{y}}^{\phi}.$$

The proofs of Theorems 2A and 2E depend on the following result:

Proposition 2H. Let $\phi, \psi : [-1,1] \longrightarrow {}^*\mathbb{R}$ be gain functions. Suppose the rules $F_\phi : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ and $F_\psi : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ are UHC. Then
$\Big(F_\psi(\mu) = F_\phi(\mu)$ for all $\mu \in \Delta(\mathcal{X}) \Big) \iff$

$$\Big({}^{o}\mathcal{B}_{\mathbf{x},\mathbf{y}}^{\phi} \subseteq \mathcal{B}_{\mathbf{x},\mathbf{y}}^{\psi} \cup (\mathcal{C} \setminus \mathcal{C}_{\mathbf{x}}^{\psi}) \text{ for every } \mathbf{x}, \mathbf{y} \in \mathcal{X} \text{ with } d(\mathbf{x},\mathbf{y}) \geq 3 \Big).$$

Proof sketch. "$\Longrightarrow$" is obvious: if $F_\psi = F_\phi$, then ${}^{o}\mathcal{B}_{\mathbf{x},\mathbf{y}}^{\phi} = {}^{o}\mathcal{B}_{\mathbf{x},\mathbf{y}}^{\psi}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, and hence, ${}^{o}\mathcal{B}_{\mathbf{x},\mathbf{y}}^{\phi} \subseteq \mathcal{B}_{\mathbf{x},\mathbf{y}}^{\psi}$.
"$\Longleftarrow$" First, why only require the RHS for $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ with $d(\mathbf{x},\mathbf{y}) \geq 3$?
Reason: If $d(\mathbf{x},\mathbf{y}) \leq 2$, then supermajoritarian efficiency alone dictates that F_ϕ and F_ψ must behave identically when choosing between $\mathbf{x}$ and $\mathbf{y}$.
(If $d(\mathbf{x},\mathbf{y}) = 1$ and $x_k = 1$ while $y_k = -1$, then any SME rule must choose $\mathbf{x}$ over $\mathbf{y}$ if $\widetilde{\mu}_k > 0$. If $d(\mathbf{x},\mathbf{y}) = 2$ and $x_j = x_k = 1$ while $y_j = y_k = -1$, then any SME rule must choose $\mathbf{x}$ over $\mathbf{y}$ if $\widetilde{\mu}_j + \widetilde{\mu}_k > 0$.)

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Proof sketch "$\Longleftarrow$" (continued). For any $\mathbf{x} \in \mathcal{X}$ it can be shown that:

(a) $\mathcal{C}^\phi_{\mathbf{x}}$ and $\mathcal{C}^\psi_{\mathbf{x}}$ are connected, and are the closures of their interiors.

(b) $\partial\mathcal{C}^\phi_{\mathbf{x}} = \bigcup_{\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}} \mathcal{B}^\phi_{\mathbf{x},\mathbf{y}} = \bigcup_{\mathbf{y} \in \mathcal{X} \setminus \{\mathbf{x}\}} \mathrm{cl}\left({}^o\mathcal{B}^\phi_{\mathbf{x},\mathbf{y}}\right)$ (and likewise for $\mathcal{C}^\psi_{\mathbf{x}}$.)

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Proof sketch "$\Longleftarrow$" (continued). For any $\mathbf{x} \in \mathcal{X}$ it can be shown that:

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Theorem 2B. *Let $\phi, \psi : [-1,1] \longrightarrow \mathbb{R}$ be odd, continuous gain functions. Let $\mathcal{X}$ be a thick judgement space, such that $\mathcal{R}^{\phi}_{\mathcal{X}} \cup \{0\}$ is connected. Then:*

$$\left(F_{\phi}(\mathcal{X},\mu) = F_{\psi}(\mathcal{X},\mu) \text{ for all } \mu \in \Delta(\mathcal{X})\right) \iff$$

$$\left(\text{There exists } s > 0 \text{ such that } \psi(r) = s\,\phi(r) \text{ for all } r \in \mathcal{R}^{\phi}_{\mathcal{X}}\right).$$

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(There exists $s > 0$ such that $\psi(r) = s\,\phi(r)$ for all $r \in \mathcal{R}^{\phi}_{\mathcal{X}}$).

Proof sketch. "$\Longleftarrow$" Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, with $d(\mathbf{x}, \mathbf{y}) \geq 3$. We claim that ${}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{B}^{\psi}_{\mathbf{x},\mathbf{y}} \cup (\mathcal{C} \setminus \mathcal{C}^{\psi}_{\mathbf{x}})$. Let $\mathbf{b} \in {}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}$. Then

$$(\mathbf{x} - \mathbf{y}) \bullet \psi(\mathbf{b}) = \sum_{k \in \mathcal{K}_{\pm}(\mathbf{x},\mathbf{y})} (x_k - y_k)\,\psi(b_k) \underset{(\dagger)}{=} \sum_{k \in \mathcal{K}_{\pm}(\mathbf{x},\mathbf{y})} (x_k - y_k)\,s\,\phi(b_k)$$

$$= s \sum_{k \in \mathcal{K}_{\pm}(\mathbf{x},\mathbf{y})} (x_k - y_k)\,\phi(b_k) = s\,(\mathbf{x} - \mathbf{y}) \bullet \phi(\mathbf{b}) \underset{(*)}{=} 0. \qquad (\diamond)$$

Here, $(\dagger)$ is because $\psi(r) = s\,\phi(r)$ for all $r \in \mathcal{R}^{\phi}_{\mathcal{X}}$, while $b_k \in \mathcal{R}^{\phi}_{\mathcal{X}}$ for all $k \in \mathcal{K}_{\pm}(\mathbf{x}, \mathbf{y})$, because $d(\mathbf{x}, \mathbf{y}) \geq 3$. Next, $(*)$ is because $\mathbf{b} \in {}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}$.
Thus, $\mathbf{x} \bullet \psi(\mathbf{b}) = \mathbf{y} \bullet \psi(\mathbf{b})$. Now, if $\mathbf{x} \bullet \psi(\mathbf{b}) \geq \mathbf{z} \bullet \psi(\mathbf{b})$ for all $\mathbf{z} \in \mathcal{X}$, then statement $(\diamond)$ implies that $F_\psi(\mathbf{b}) \supseteq \{\mathbf{x}, \mathbf{y}\}$, so $\mathbf{b} \in \mathcal{B}^{\psi}_{\mathbf{x},\mathbf{y}}$. Otherwise, if $\mathbf{x} \bullet \psi(\mathbf{b}) < \mathbf{z} \bullet \psi(\mathbf{b})$ for some $\mathbf{z} \in \mathcal{X}$, then $\mathbf{x} \notin F_\psi(\mathbf{b})$, so $\mathbf{b} \in \mathcal{C} \setminus \mathcal{C}^{\psi}_{\mathbf{x}}$. Thus, ${}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{B}^{\psi}_{\mathbf{x},\mathbf{y}} \cup (\mathcal{C} \setminus \mathcal{C}^{\psi}_{\mathbf{x}})$, for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ with $d(\mathbf{x}, \mathbf{y}) \geq 3$. Thus, Proposition 2H says that $F_\phi(\mathcal{X}, \mu) = F_\psi(\mathcal{X}, \mu)$ for all $\mu \in \Delta(\mathcal{X})$.

Theorem 2B. *Let $\phi, \psi : [-1,1] \longrightarrow \mathbb{R}$ be odd, continuous gain functions. Let $\mathcal{X}$ be a thick judgement space, such that $\mathcal{R}^{\phi}_{\mathcal{X}} \cup \{0\}$ is connected. Then:* $(F_{\phi}(\mathcal{X},\mu) = F_{\psi}(\mathcal{X},\mu)$ *for all* $\mu \in \Delta(\mathcal{X}))$ $\iff$

$$(\textit{There exists } s > 0 \textit{ such that } \psi(r) = s\,\phi(r) \textit{ for all } r \in \mathcal{R}^{\phi}_{\mathcal{X}}).$$

Proof sketch. "$\Longleftarrow$" Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, with $d(\mathbf{x},\mathbf{y}) \geq 3$. We claim that ${}^{\circ}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{B}^{\psi}_{\mathbf{x},\mathbf{y}} \cup (\mathcal{C} \setminus \mathcal{C}^{\psi}_{\mathbf{x}})$. Let $\mathbf{b} \in {}^{\circ}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}$. Then

$$\begin{aligned}(\mathbf{x}-\mathbf{y}) \bullet \psi(\mathbf{b}) &= \sum_{k \in \mathcal{K}_{\pm}(\mathbf{x},\mathbf{y})} (x_k - y_k)\,\psi(b_k) \underset{(\dagger)}{=} \sum_{k \in \mathcal{K}_{\pm}(\mathbf{x},\mathbf{y})} (x_k - y_k)\,s\,\phi(b_k) \\ &= s \sum_{k \in \mathcal{K}_{\pm}(\mathbf{x},\mathbf{y})} (x_k - y_k)\,\phi(b_k) \quad = \quad s\,(\mathbf{x}-\mathbf{y}) \bullet \phi(\mathbf{b}) \underset{(*)}{=} 0. \qquad (\diamond)\end{aligned}$$

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(There exists $s > 0$ such that $\psi(r) = s\,\phi(r)$ for all $r \in \mathcal{R}^{\phi}_{\mathcal{X}}$).

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$$(\text{There exists } s > 0 \text{ such that } \psi(r) = s\,\phi(r) \text{ for all } r \in \mathcal{R}^{\phi}_{\mathcal{X}}).$$

Proof sketch. "$\Longleftarrow$" Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, with $d(\mathbf{x}, \mathbf{y}) \geq 3$. We claim that $\mathcal{O\!B}^{\phi}_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{B}^{\psi}_{\mathbf{x},\mathbf{y}} \cup (\mathcal{C} \setminus \mathcal{C}^{\psi}_{\mathbf{x}})$. Let $\mathbf{b} \in \mathcal{O\!B}^{\phi}_{\mathbf{x},\mathbf{y}}$. Then

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Thus, $\mathbf{x} \bullet \psi(\mathbf{b}) = \mathbf{y} \bullet \psi(\mathbf{b})$. Now, if $\mathbf{x} \bullet \psi(\mathbf{b}) \geq \mathbf{z} \bullet \psi(\mathbf{b})$ for all $\mathbf{z} \in \mathcal{X}$, then statement $(\diamond)$ implies that $F_\psi(\mathbf{b}) \supseteq \{\mathbf{x}, \mathbf{y}\}$, so $\mathbf{b} \in \mathcal{B}^{\psi}_{\mathbf{x},\mathbf{y}}$. Otherwise, if $\mathbf{x} \bullet \psi(\mathbf{b}) < \mathbf{z} \bullet \psi(\mathbf{b})$ for some $\mathbf{z} \in \mathcal{X}$, then $\mathbf{x} \notin F_\psi(\mathbf{b})$, so $\mathbf{b} \in \mathcal{C} \setminus \mathcal{C}^{\psi}_{\mathbf{x}}$. Thus, $\mathcal{O\!B}^{\phi}_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{B}^{\psi}_{\mathbf{x},\mathbf{y}} \cup (\mathcal{C} \setminus \mathcal{C}^{\psi}_{\mathbf{x}})$, for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ with $d(\mathbf{x}, \mathbf{y}) \geq 3$. Thus, Proposition 2H says that $F_\phi(\mathcal{X}, \mu) = F_\psi(\mathcal{X}, \mu)$ for all $\mu \in \Delta(\mathcal{X})$.

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Thus, $\mathbf{x} \bullet \psi(\mathbf{b}) = \mathbf{y} \bullet \psi(\mathbf{b})$. Now, if $\mathbf{x} \bullet \psi(\mathbf{b}) \geq \mathbf{z} \bullet \psi(\mathbf{b})$ for all $\mathbf{z} \in \mathcal{X}$, then statement $(\diamond)$ implies that $F_\psi(\mathbf{b}) \supseteq \{\mathbf{x},\mathbf{y}\}$, so $\mathbf{b} \in \mathcal{B}^{\psi}_{\mathbf{x},\mathbf{y}}$. Otherwise, if $\mathbf{x} \bullet \psi(\mathbf{b}) < \mathbf{z} \bullet \psi(\mathbf{b})$ for some $\mathbf{z} \in \mathcal{X}$, then $\mathbf{x} \notin F_\psi(\mathbf{b})$, so $\mathbf{b} \in \mathcal{C} \setminus \mathcal{C}^{\psi}_{\mathbf{x}}$. Thus, ${}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{B}^{\psi}_{\mathbf{x},\mathbf{y}} \cup (\mathcal{C} \setminus \mathcal{C}^{\psi}_{\mathbf{x}})$, for all $\mathbf{x},\mathbf{y} \in \mathcal{X}$ with $d(\mathbf{x},\mathbf{y}) \geq 3$. Thus, Proposition 2H says that $F_\phi(\mathcal{X},\mu) = F_\psi(\mathcal{X},\mu)$ for all $\mu \in \Delta(\mathcal{X})$.

Theorem 2B. *Let $\phi, \psi : [-1,1] \longrightarrow \mathbb{R}$ be odd, continuous gain functions. Let $\mathcal{X}$ be a thick judgement space, such that $\mathcal{R}^{\phi}_{\mathcal{X}} \cup \{0\}$ is connected. Then:* $(F_\phi(\mathcal{X},\mu) = F_\psi(\mathcal{X},\mu)$ *for all* $\mu \in \Delta(\mathcal{X}))$ $\iff$

$$(\text{There exists } s > 0 \text{ such that } \psi(r) = s\,\phi(r) \text{ for all } r \in \mathcal{R}^{\phi}_{\mathcal{X}}).$$

Proof sketch. "$\Longleftarrow$" Let $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, with $d(\mathbf{x},\mathbf{y}) \geq 3$. We claim that ${}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}} \subseteq \mathcal{B}^{\psi}_{\mathbf{x},\mathbf{y}} \cup (\mathcal{C} \setminus \mathcal{C}^{\psi}_{\mathbf{x}})$. Let $\mathbf{b} \in {}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}$. Then

$$(\mathbf{x}-\mathbf{y}) \bullet \psi(\mathbf{b}) \;=\; \sum_{k \in \mathcal{K}_\pm(\mathbf{x},\mathbf{y})} (x_k - y_k)\,\psi(b_k) \underset{(\dagger)}{=} \sum_{k \in \mathcal{K}_\pm(\mathbf{x},\mathbf{y})} (x_k - y_k)\, s\,\phi(b_k)$$

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$$(\mathbf{x}-\mathbf{y}) \bullet \phi(\mathbf{b}) \quad = \quad 0 \quad = \quad (\mathbf{x}-\mathbf{y}) \bullet \psi(\mathbf{b}). \tag{1}$$

Without loss of generality, suppose $\mathcal{K}_{\pm}(\mathbf{x},\mathbf{y}) = [1 \ldots J]$ for some $J \geq 3$, while $\mathbf{x}_{[1\ldots J]} = (1,-1,-1,\ldots,-1) = -\mathbf{y}_{[1\ldots J]}$. Then equation (1) becomes:

$$\phi(b_1) \;=\; \sum_{j=2}^{J} \phi(b_j) \quad \text{and} \quad \psi(b_1) \;=\; \sum_{j=2}^{J} \psi(b_j). \tag{2}$$

Finally, define $\widetilde{\mathbf{b}} := (\phi(b_j))_{j=1}^{J} \in \mathbb{R}^J$, and let $\widetilde{\mathcal{B}}_{\mathbf{x},\mathbf{y}} := \{\widetilde{\mathbf{b}};\ \mathbf{b} \in {}^{o}\mathcal{B}^{\phi}_{\mathbf{x},\mathbf{y}}\}$. Define $\tau := \psi \circ \phi^{-1}$. Then for all $\widetilde{\mathbf{b}} \in \widetilde{\mathcal{B}}_{\mathbf{x},\mathbf{y}}$, equation (2) becomes:

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Using the fact that $\mathcal{R}^{\phi}_{\mathcal{X}} \cup \{0\}$ is connected, while ϕ and ψ are continuous, we can 'stitch together' these local affine transformations, to obtain a single $s > 0$ and $t \in \mathbb{R}$ such that $\psi(r) = s\,\phi(r) + t$ for all $r \in \mathcal{R}^{\phi}_{\mathcal{X}}$.

But $\psi(0) = 0 = \phi(0)$ (because ψ and ϕ are odd); thus, continuity forces $t = 0$.

Thus, $\psi(r) = s\,\phi(r)$ for all $r \in \mathcal{R}^{\phi}_{\mathcal{X}}$. $\square$

Theorem 2G. *Let $\mathcal{X}$ be a thick judgement space, and let $F : \Delta(\mathcal{X}) \rightrightarrows \mathcal{X}$ be a judgement aggregation rule. Then:* (F *is regular, upper hemicontinuous and satisfies neutral reinforcement on* $\Delta(\mathcal{X})$) $\iff$ ($F = H^d$ *for some* $d \in (0, \infty)$).

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But then Theorem 2B ('uniqueness') implies that $F_\phi = H^d$. $\square$

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For a suitable $\mathbf{b}_0$ and $\mathbf{b}_1$, it can be shown that this forces $d = 1$. $\square$

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Thus, at most one of $\mathbf{x}, \mathbf{y}$ is in $F_\psi(\mathcal{X}, \mu)$, contradicting $(*)$. □

Separability + SME $\implies$ additive (finite populations)

For any $N \in \mathbb{N}$, let $\mathcal{I}_N := \{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\}$.

Let $\Delta_N(\mathcal{X})$ be the set of all $\mu \in \Delta(\mathcal{X})$ such that $\mu(\mathbf{x}) \in \mathcal{I}_N$ for all $\mathbf{x} \in \mathcal{X}$. (All profiles generated by a population of N voters with uniform weights).

Define $\mathcal{X}^* \; := \; \bigcup_{m=1}^{\infty} \mathcal{X}^m \quad$ and $\quad \Delta_N(\mathcal{X}^*) \; := \; \bigcup_{m=1}^{\infty} \Delta_N(\mathcal{X}^m)$.

Given any $M \in \mathbb{N}$ and any $\mu \in \Delta_N(\mathcal{X}^M)$, let $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(M)} \in \Delta_N(\mathcal{X})$ be the projections of μ onto the M coordinates.

Given a rule $F : \Delta_N(\mathcal{X}) \rightrightarrows \mathcal{X}$, we define $F^M : \Delta_N(\mathcal{X}^M) \rightrightarrows \mathcal{X}^M$ by

$$F^M(\mu) \quad := \quad F(\mu^{(1)}) \times F(\mu^{(2)}) \times \cdots \times F(\mu^{(M)}), \quad \text{for all } \mu \in \Delta_N(\mathcal{X}^M).$$

This yields a function $F^* : \Delta_N(\mathcal{X}^*) \rightrightarrows \mathcal{X}^*$, the **separable extension** of F.
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Proof sketch. "$\Longleftarrow$" is a straightforward computation.

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Now let $\mathcal{P}$ be the closure of $\mathcal{D}$, where we define

$$\mathcal{D} \ := \ \left\{ \mathbf{g}^{\mathbf{y},\mu} - \mathbf{g}^{\mathbf{x},\mu}\ ;\ M \in \mathbb{N},\ \mu \in \Delta_N(\mathcal{X}^M),\ \mathbf{x} \in F(\mathcal{X}^M, \mu),\ \text{and } \mathbf{y} \in \mathcal{X}^M \right\}.$$

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Claim 1. *$\mathcal{P}$ is a compact, convex polyhedron in $\mathbb{R}^{\mathcal{Q}_N^+}$.*

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Thus, $\mathcal{P}$ is a compact, convex polyhedron. $\diamondsuit_{\text{Claim 1}}$

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$$\iff \Big((\mathbf{g}(\mathbf{y}, \mu) - \mathbf{g}(\mathbf{x}, \mu)) \notin \mathbb{R}_+^{\mathcal{Q}} \setminus \{\mathbf{0}\} \text{ for all } \mathbf{y} \in \mathcal{X}^M\Big)$$

$$\iff \Big(\mathcal{D}_{M,\mu,\mathbf{x}} \cap \mathbb{R}_+^{\mathcal{Q}} = \{\mathbf{0}\}\Big), \text{ where } \mathcal{D}_{M,\mu,\mathbf{x}} := \{\mathbf{g}^{\mathbf{y},\mu} - \mathbf{g}^{\mathbf{x},\mu}\ ;\ \mathbf{y} \in \mathcal{X}^M\}.$$

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$\Big(\nexists \ \mathbf{y} \in \mathcal{X}^M$ with $g_q(\mathbf{y}, \mu) \geq g_q(\mathbf{x}, \mu)$ for all $q \in \mathcal{Q}$, and $\mathbf{g}(\mathbf{y}, \mu) \neq \mathbf{g}(\mathbf{x}, \mu)\Big)$
$\iff \Big((\mathbf{g}(\mathbf{y}, \mu) - \mathbf{g}(\mathbf{x}, \mu)) \notin \mathbb{R}_+^{\mathcal{Q}} \setminus \{\mathbf{0}\}$ for all $\mathbf{y} \in \mathcal{X}^M\Big)$
$\iff \Big(\mathcal{D}_{M,\mu,\mathbf{x}} \cap \mathbb{R}_+^{\mathcal{Q}} = \{\mathbf{0}\}\Big)$, where $\mathcal{D}_{M,\mu,\mathbf{x}} := \{\mathbf{g}^{\mathbf{y},\mu} - \mathbf{g}^{\mathbf{x},\mu} \ ; \ \mathbf{y} \in \mathcal{X}^M\}$.

Thus, (F is SME on $\Delta_N(\mathcal{X}^M)$) $\iff$
$\Big(\mathcal{D}_{M,\mu,\mathbf{x}} \cap \mathbb{R}_+^{\mathcal{Q}} = \{\mathbf{0}\}$, for all $\mathbf{x} \in F(\mathcal{X}^M, \mu)$ and $\mu \in \Delta_N(\mathcal{X}^M)\Big)$.

Theorem 1B. *Let $N \in \mathbb{N}$, let $\mathcal{X}$ be a judgement space, and let $F : \Delta_N(\mathcal{X}) \rightrightarrows \mathcal{X}$. The extension $F^* : \Delta_N(\mathcal{X}^*) \rightrightarrows \mathcal{X}^*$ is supermajoritarian efficient $\iff$ there is odd, increasing function $\phi_N : \mathcal{Q}_N \longrightarrow \mathbb{R}$ such that $F^M(\mu) \subseteq F_{\phi_N}(\mathcal{X}^M, \mu)$, for all $M \in \mathbb{N}$ and $\mu \in \Delta_N(\mathcal{X}^M)$.*

Proof sketch. " $\Longrightarrow$ " For any $M \in \mathbb{N}$, $\mu \in \Delta_N(\mathcal{X}^M)$, and $\mathbf{x} \in \mathcal{X}$, we define the vector $\mathbf{g}^{\mathbf{x},\mu}$ in $\mathbb{R}^{\mathcal{Q}_N^+}$, by setting $g_q^{\mathbf{x},\mu} := \gamma_{\mu,\mathbf{x}}(q)/|\mathcal{K}|$, for all $q \in \mathcal{Q}_N^+$. Now let $\mathcal{P}$ be the closure of $\mathcal{D}$, where we define

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Claim 1. *$\mathcal{P}$ is a compact, convex polyhedron in $\mathbb{R}^{\mathcal{Q}_N^+}$.*

Claim 2. *If F is SME, then $\mathcal{D} \cap \mathbb{R}_+^{\mathcal{Q}_N^+} = \{\mathbf{0}\}$.*

Proof idea: $\forall\ M \in \mathbb{N}$, $\mu \in \Delta(\mathcal{X}^M)$, and $\mathbf{x} \in \mathcal{X}^M$, $\Big(\mathbf{x} \in \mathrm{SME}\,(\mathcal{X}, \mu)\Big)$

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Now take the union over all $M \in \mathbb{N}$, $\mu \in \Delta(\mathcal{X}^M)$, and $\mathbf{x} \in \mathcal{X}^M$. $\Diamond_{\text{Claim2}}$

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Given a slight strengthening of the Separating Hyperplane Theorem, Claims 1 and 2 yield a strictly positive vector $\mathbf{v} \in \mathbb{R}_+^N$ which separates $\mathcal{P}$ from $\mathbb{R}_+^N$.

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Formal definition of ${}^*\mathbb{R}$: ultrafilters

Let $\mathcal{I}$ be an infinite set.

Let $\mathfrak{P} :=$ the power set of $\mathcal{I}$.

A *free filter* is a subset $\mathfrak{U} \subset \mathfrak{P}$ (i.e. a collections of subsets of $\mathcal{I}$) with the following properties:

- **(F0)** No finite subset of $\mathcal{I}$ is an element of $\mathfrak{U}$. (Hence, $\emptyset \notin \mathfrak{U}$.)
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Example: The set of all co-finite subsets of $\mathcal{I}$ is a free filter.

A free filter $\mathfrak{U}$ is a *free ultrafilter* if it also satisfies:

- **(UF)** For any $\mathcal{P} \in \mathfrak{P}$, either $\mathcal{P} \in \mathfrak{U}$ or $\mathcal{P}^{\complement} \in \mathfrak{U}$ (but not both).

Idea: Elements of $\mathfrak{U}$ are 'large' subsets of $\mathcal{I}$; if $\mathcal{U} \in \mathfrak{U}$ and a certain statement holds for all $i \in \mathcal{U}$, then this statement holds for 'almost all' $i \in \mathcal{I}$. (In particular, axioms (F0) and (UF) imply that $\mathcal{I} \in \mathfrak{U}$.)

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- **(UF)** For any $\mathcal{P} \in \mathfrak{P}$, either $\mathcal{P} \in \mathfrak{U}$ or $\mathcal{P}^{\complement} \in \mathfrak{U}$ (but not both).

Idea: Elements of $\mathfrak{U}$ are 'large' subsets of $\mathcal{I}$; if $\mathcal{U} \in \mathfrak{U}$ and a certain statement holds for all $i \in \mathcal{U}$, then this statement holds for 'almost all' $i \in \mathcal{I}$. (In particular, axioms (F0) and (UF) imply that $\mathcal{I} \in \mathfrak{U}$.)

Ultrafilter lemma. *Any free filter can be extended to a free ultrafilter.*

Proof. Use Zorn's Lemma.

Let $\mathcal{I}$ be an infinite set.

Let $\mathfrak{P} :=$ the power set of $\mathcal{I}$.

A *free filter* is a subset $\mathfrak{U} \subset \mathfrak{P}$ (i.e. a collections of subsets of $\mathcal{I}$) with the following properties:

- **(F0)** No finite subset of $\mathcal{I}$ is an element of $\mathfrak{U}$. (Hence, $\emptyset \notin \mathfrak{U}$.)
- **(F1)** If $\mathcal{U}, \mathcal{V} \in \mathfrak{U}$, then $\mathcal{U} \cap \mathcal{V} \in \mathfrak{U}$.
- **(F2)** For any $\mathcal{U} \in \mathfrak{U}$ and $\mathcal{P} \in \mathfrak{P}$, if $\mathcal{U} \subseteq \mathfrak{P}$, then $\mathcal{P} \in \mathfrak{U}$.

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Furthermore, $\mathbb{R}$ can be embedded as an ordered subfield of ${}^*\mathbb{R}$ by mapping any $r \in \mathbb{R}$ to the element ${}^*\overline{r}$ in ${}^*\mathbb{R}$, where $\overline{r} := (r, r, r, \ldots) \in \mathbb{R}^{\mathcal{I}}$.

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Formal definition of ${}^*\mathbb{R}$ as an ultraproduct

For any $r, s \in \mathbb{R}^{\mathcal{I}}$, define $r \underset{\mathfrak{U}}{\succsim} s$ if and only if $\{i \in \mathcal{I};\ \ r_i \geq s_i\} \in \mathfrak{U}$.
This yields a complete preorder $(\underset{\mathfrak{U}}{\succsim})$ on $\mathbb{R}^{\mathcal{I}}$.

Let $(\underset{\mathfrak{U}}{\approx})$ be the symmetric part of $(\underset{\mathfrak{U}}{\succsim})$ (an equivalence relation on $\mathbb{R}^{\mathcal{I}}$).
Thus, $r \underset{\mathfrak{U}}{\approx} s$ if they agree 'almost everywhere'. Define ${}^*\mathbb{R} := \mathbb{R}^{\mathcal{I}}/(\underset{\mathfrak{U}}{\approx})$.
For any $r \in \mathbb{R}^{\mathcal{I}}$, let ${}^*r$ denote the equivalence class of r in ${}^*\mathbb{R}$.

Define linear order $(>)$ on ${}^*\mathbb{R}$, by $({}^*r > {}^*s) \Leftrightarrow (r \underset{\mathfrak{U}}{\succ} s)$, for all ${}^*r, {}^*s \in {}^*\mathbb{R}$.

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Proof. Adapt the proof of Theorem 1B. $\Diamond_{\text{Claim1}}$

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(b) follows from (a) because ϕ is strictly increasing, so F_ϕ is monotone: for any $\mu \in \Delta(\mathcal{X})$ and any $\mathbf{x} \in F_\phi(\mu)$, the slightest increase in the support for $\mathbf{x}$ breaks the tie and makes $\mathbf{x}$ the unique winner.

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Thank you.

These presentation slides are available at

<http://euclid.trentu.ca/pivato/Research/SMEslides.pdf>