Special problems for students[1]

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Remember that Mathematics is not just a set of tools and algorithms. The most important thing is solving difficult problems. There are several of them below. If you have difficulties with definitions, you may need additional reading and preliminary research in library.

1 Problems from Mathematical and Statistical Finance

Problem 1 *[10] Consider American put options for a Black-Scholes market with zero risk-free interest rate (i.e., with zero borrowing cost, or, in terms of bond-and-stock market, with bond $B(t) \equiv 1$). Prove that the fair price is the same for American and European put.*

Problem 2 *[10] Consider a diffusion market model with a single stock $S(t)$ with zero risk-free interest rate (i.e., with zero borrowing cost, and with zero interest bank account). Assume that $dS(t) = S(t)dw(t)$, where $w(t)$ is the scalar Wiener process (or, for simplicity, you can assume that $S(t) = S(0) + w(t)$). A (self-financing) strategy of an investor is the number of shares $\gamma(t)$, and the corresponding wealth is such that $X(t) = X(0) + \int_0^t \gamma(s)dS(s)$. John's initial wealth is $X(0) = S(0)$, and he uses the following strategy: $\gamma(t) = \mathbb{I}_{\{S(t) \geq S(0)\}}$, where $\mathbb{I}$ denotes the indicator function. (This means that John keeps one share of stock when $S(t) \geq S(0)$ and keeps zero amount of shares if $S(t) < S(0)$, i.e., in that case he keeps all money in risk-free cash account). John hopes to have the wealth $X_T = \max(S(0), S(T))$ at time T. Is this feasible?*

Problem 3 *[10] Consider a diffusion market model with a single stock such that volatility process $\sigma(t)$ is non-random and depends on time, and the risk-free interest rare r is a non-random constant. Prove that the initial wealth that can ensure replication of the option claim (i.e., the fair price) can be calculated for put and call using Black-Scholes formula with σ replaced for $\left(\frac{1}{T}\int_0^T \sigma(t)^2 dt\right)^{1/2}$.*

Problem 4 *[3] Find an analog of explicit Black-Scholes formula for European call option with the payoff $\max(0, S(T)^{-1} - K)$, where $S(t)$ is the stock price at time t, $K \in (0, +\infty)$.*

[1] A solution can ensure my personal support for any scholarship applications for all students. The number in [ ] for a particular problem is the number of marks that can be credited for students from my classes. The marks will be credited only for the first solutions submitted. This number represents my estimation of the value of problem.

Problem 5 *[20] An investor looks for opportunities to obtain annual profit $\%10^8$ (i.e, $\%100,000,000$), and asks a quantitative analyst to create a self-financing strategy for the Black-Scholes model with given $T = 1$ (year), $\sigma > 0$, $r \geq 0$, $S(0) = \$1$ such that the initial wealth $X(0) = \$1$ has to be raised to the wealth $X(T)$ such that*

$$X(T) \geq \$10^6 \quad \textit{if either} \quad S(T) \leq \$10^6 \quad \textit{or} \quad S(T) \geq \$(10^6 + 1).$$

(Note that it is allowed that $X(T) < \$10^6$ if $\$10^6 < S(T) < \$(10^6+1)$.) Does this strategy exist?

Problem 6 *[10-20]* Asuume that the stock prices $S(t)$ are observed at times $t = t_1, ..., t_n$. Suggest a method for estimation of the parameters $(a, \sigma_1, \sigma_2, p)$ under the hypothesis that $dS(t) = S(t)[adt + \sigma dw(t)]$, where $w(t)$ is a Wiener process, the appreciation rate $a \in \mathbf{R}$ is a non-random constant, and the volatility $\sigma = \sigma(\omega)$ is random, independent of $w(\cdot)$, time independent, and can take only two values, σ_1 and σ_2, with probabilities p and $1 - p$ correspondingly.

2 Related Mathematical Problems

Problem 7 *[20] Let $A > B > 0$, where A and B are symmetric real $n \times n$-dimensional matrices. Prove that $A^{-1} < B^{-1}$.*

Problem 8 *[10] Let $A \in \mathbf{R}^{n\times n}$. Prove that all eigenvalues of A are inside of the circle $\{z : |z| < 1\}$ if and only if the solution of the equation*

$$x_{k+1} = Ax_k + e_k, \quad k = 0, 1, 2, ...$$

for any $x_0 \in \mathbf{R}^n$ and for any bounded sequence $\{e_k\} \subset \mathbf{R}^n$ is bounded in $k > 0$.

Problem 9 *[3] Does it exist an example of a probability distribution on $\mathbf{R}$ such that its support coincides with the set of all rational numbers? If yes, give an example, if no, prove it.*

Problem 10 *Let ξ_k be random variables such that $\mathbf{E}|\xi_k|^n < +\infty$, $k = 1, ..., n$. Prove that $\mathbf{E}|\xi_1\xi_2 \cdots \xi_n| < +\infty$.*

Hint: use Hölder inequality.

3 General Mathematical Problems

Problem 11 *[3] Let* $\mathbf{Q}^2$ *be the set of all pairs* $(x, y) \in \mathbf{R}^2$ *such that* x *and* y *are rational numbers. We consider a random direct line* L *in* $\mathbf{R}^2$ *such that* $(0, 0) \in L$ *with probability 1, and that the angle between* L *and the vector* $(1, 0)$ *has the uniform distribution on* $[0, \pi)$. *Find the probability that the set* $L \cap \mathbf{Q}^2$ *is finite.*

Problem 12 *[8] Prove that the roots of polynomial of order* n *depends continuously on its coefficients.*

Problem 13 *[1] Let* $X = \{x_1, x_2, x_3\}$, *and let* $r(x_i, x_j) = 0$ *for* $i = j$, $r(x_i, x_j) = 3$ *for* $(i, j) = (1, 3)$ *or* $(j, i) = (1, 3)$; *otherwise,* $r(x_i, x_j) = 1$. *Is it a metric?*