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LOCAL SOJOURN TIME OF DIFFUSION AND DEGENERATING PROCESSES ON A MOBILE SURFACE*

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Abstract. The existence of the local sojourn time on the surface is established for multi-dimensional Itô processes, and equations are derived for probability distributions. An explicit formula of the type of the Tanaka formula is obtained for local time. Local time continuity is established. The limiting properties of the local time are investigated for degenerating diffusion.

Key words. local time, diffusion processes, diffusion degeneration

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Introduction. A vast literature is dedicated to problems connected with the probability distribution of functionals of local time (see [1], [2], [3] and references therein). The local time of scalar Brownian motion has been studied most thoroughly. It seems natural to have a description of the local time distribution for more general processes than Brownian motion (for example, for diffusion processes with a control-dependent drift) in order to include in the stochastic theory of optimal control new problems in which it is required to minimize and maximize the local sojourn time in sets of zero measure. The existence of a local sojourn time on smooth hypersurfaces was established in [4] for general multi-dimensional semimartingales but the distribution of local time was not studied there. Using the tool of Kolmogorov equations, we obtain below a description of local sojourn time of a general multi-dimensional Markov diffusion process on a time-dependent "piecewise smooth" or even fractal hypersurface (Example 4.1); we derive special analogues of Kolmogorov equations and establish the solvability of these equations, as well as the convergence of random variables whose limit is usually called local time. An explicit expression of a local time in terms of a stochastic integral is found, i.e., an analogue of Tanaka's formula is obtained. The limiting properties of a local time are investigated when diffusion disappears and the Itô equation transforms into an ordinary differential equation.

1. Statement of the problem. We consider on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ an n-dimensional Wiener process $w(t)$ with independent components such that $w(0)=0$. Consider a random n-vector a which is independent of $w(t)$. Let a random process $y(t)$ of dimension n be a strong solution of the Itô stochastic equation

$$dy(t) = f\big(y(t),\, t\big)\, dt + \beta\big(y(t),\, t\big)\, dw(t), \tag{1.1}$$

$$y(0) = a. \tag{1.2}$$

The functions $f(x,t)\colon\ \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^n$, $\beta(x,t)\colon\ \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^{n\times n}$ are bounded and measurable. It is assumed that the function $\beta(x,t)$ is continuous, the derivative

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$\partial\beta(x,t)/\partial x$ is bounded, and $b(x,t) = \frac{1}{2}\,\beta(x,t)\,\beta(x,t)^{\mathrm{T}} \geqq \delta I > 0$ for some $\delta > 0$ for all x, t.

As is known [8, section II.6], under such assumptions equations (1.1), (1.2) have a weak solution (unique with respect to distribution). The collection $(\Omega, \mathcal{F}, \mathbf{P}, w(t), y(t))$ introduced here is thus one of such weak solutions.

Let $D \subset \mathbf{R}^n$ be some domain. It is assumed that either $D = \mathbf{R}^n$ or the domain D is bounded and has a C^2-smooth boundary, $a \in D$ a.s..

Consider the first random exit times $\tau_D = \inf\{t\colon\ y(t) \notin D\}$. Let a bounded hypersurface $\Gamma(t) \subset D$ of dimension $n-1$ be given for almost all $t \in [0,T]$ and let $\partial\Gamma(t)$ be its edge. We introduce the sets

$$\Gamma_*(t,\varepsilon) = \Big\{x \in D\colon \inf_{y\in\Gamma(t)} |x-y| < \frac{\varepsilon}{2}\Big\},$$

$$\Gamma'(t,\varepsilon) = \Big\{x \in D\colon \inf_{y\in\partial\Gamma(t)} |x-y| < \frac{\varepsilon}{2}\Big\}. \tag{1.3}$$

(It can happen that $\partial\Gamma(t) = \varnothing$ and then $\Gamma'(t,\varepsilon) = \varnothing$.) For all $t \in [0,T]$, $\varepsilon > 0$ we give arbitrarily Borel sets $\Gamma(t,\varepsilon)$ such that

$$\Gamma_*(t,\,\varepsilon)\backslash\Gamma'(t,\,\varepsilon) \subseteq \Gamma(t,\,\varepsilon) \subseteq \Gamma_*(t,\,\varepsilon). \tag{1.4}$$

We will be interested in a local time spent by the process $y(t)$ on $\Gamma(t)$ (until its exit from D), i.e., in the limits, as $\varepsilon \to +0$, of the variables

$$l_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^{\tau_D\wedge t} \mathrm{Ind}\big\{y(s) \in \Gamma(s,\,\varepsilon)\big\}\,ds. \tag{1.5}$$

For f, β, $\Gamma(t)$ of a general form it will be shown that, for any $t > 0$, these variables converge in the mean square to a random variable $\hat{\mathbf{t}}(t)$. The convergence in distribution is established in section 5 and the convergence in the mean square in section 6 (note that the limit does not depend on the choice of $\Gamma(t,\varepsilon)$ in (1.4)). An equation will be found for the characteristic function

$$\phi(z) = \mathbf{E}\,e^{z\hat{\mathbf{t}}(t)}, \quad z \in \mathbf{C}. \tag{1.6}$$

Moreover, equations which are analogues of the backward Kolmogorov equation will be derived (and their solvability shown in the respective classes of functions) for functionals of the form

$$\phi_1 = \mathbf{E}\,\hat{\mathbf{t}}(T), \qquad \phi_2 = \mathbf{E}\int_0^{\tau_D\wedge T} \varphi\,\big(y(t),\,t\big)\,e^{\hat{\mathbf{t}}(t)}\,dt \tag{1.7}$$

for a given number $T > 0$ and the function φ.

2. Spaces and classes of functions. Let some number $T > 0$ be given. Denote $Q = D \times (0,T)$. Below, $\|\cdot\|_X$ denotes a norm in a space X, $(\cdot,\cdot)_X$ is a scalar product in a Hilbert space X.

We introduce the spaces of (complex-valued) functions. For integers $k \geqq 0$, let $H^k = \overset{\circ}{W}{}_2^k(D)$ be Sobolev's Hilbert spaces, and let H^{-k} be the space dual to H^k with a norm $\|\cdot\|_{H^{-1}}$ such that, for $u \in H^0$, $\|u\|_{H^{-1}}$ is the supremum $(u,v)_{H^0}$ with respect to $v \in H^0$, $\|v\|_{H^k} \leqq 1$.

Let ℓ_m denote the Lebesgue measure in $\mathbf{R}^m$, and let $\overline{\mathcal{B}}_m$ be a σ-algebra of Lebesgue sets in $\mathbf{R}^m$. We introduce the spaces $\mathcal{C}^k = C([0,T] \to H^k)$, $X^k = L^2([0,T], \overline{\mathcal{B}}_1, \ell_1, H^k)$, $Y^k = X^k \cap \mathcal{C}^{k-1}$ with the norm $\|u\|_{Y^k} = \|u\|_{X^k} + \|u\|_{\mathcal{C}^{k-1}}$.

The norm $(u,v)_{H^0}$ is assumed to be well defined for $u \in H^{-k}$, $v \in H^k$ as well (extending it in a natural manner from $u \in H^0$, $v \in H^k$).

Below let $\mu \in (1,2)$ be an arbitrary number for the case $n=1$,

$$\mu \in \left(1, \frac{n}{n-1}\right) \quad \text{for the case } n > 1. \tag{2.1}$$

We introduce the space $\mathcal{W} = W_\mu^{(1)}(D)$ and its conjugate space $\mathcal{W}^*$, as well as the space $\mathcal{X} = L^\infty([0,T], \overline{\mathcal{B}}_1, \ell_1, \mathcal{W}^*)$.

The following proposition is standard.

PROPOSITION 2.1. (i) *If D is a bounded domain, then $\mathcal{W}^* \subset H^{-1}$, $\|g\|_{H^{-1}} \leqq c\|g\|_{\mathcal{W}^*}$, where $c = c(n, D, \mu)$ is a constant, and $\mathcal{X} \subset X^{-1}$.*

(ii) *Let $D = \mathbf{R}^n$, $g \in \mathcal{W}^*$ and assume there exists a bounded domain $D_1 \subset \mathbf{R}^n$ such that $\langle \xi, g\rangle = 0$ ($\forall \xi \in \mathcal{W}$: $\operatorname{supp}\xi \subset \mathbf{R}^n \backslash D_1$). Then $g \in H^{-1}$, $\|g\|_{H^{-1}} \leqq c\|g\|_{\mathcal{W}^*}$, where $c = c(n, D, \mu)$ is a constant.*

LEMMA 2.1. *For $\xi \in H^1$, $\eta \in H^1$, we have $\xi\eta \in \mathcal{W}$ and $\|\xi\eta\|_{\mathcal{W}} \leqq c\|\xi\|_{H^1}\|\eta\|_{H^1}$, where $c = c(n, D, \mu)$ is a constant.*

Proof. Let $r = 2/\mu$, $r' = r(r-1)^{-1}$, $p = \mu r'$, $\overline{\eta} \in H^0$. We have $p = \mu r' = 2\mu(2-\mu)$, $\mu(2-\mu)^{-1} < n(n-2)^{-1}$, $p < 2n(n-2)^{-1}$ for $n > 2$,

$$\begin{aligned} \|\xi\overline{\eta}\|_{L_\mu(D)} &= \left(\int_D \xi^\mu \overline{\eta}^\mu \, dx\right)^{1/\mu} \leqq \left(\left(\int_D \xi^{\mu r'} dx\right)^{1/r'} \left(\int_D \overline{\eta}^2 dx\right)^{1/r}\right)^{1/\mu} \\ &= \|\overline{\eta}\|_{H^0} \left(\int_D \xi^p dx\right)^{1/p} \leqq c\|\overline{\eta}\|_{H^0}\|\xi\|_{H^1}, \end{aligned} \tag{2.2}$$

with the constant $c = c(n, D, \mu)$. The latter estimate is obtained using the embedding theorems for Sobolev spaces (see [5, p. 78]). Applying (2.2) to $(\partial\xi/\partial x_i)\,\eta$, $(\partial\eta/\partial x_i)\,\xi$, we obtain the required result.

LEMMA 2.2. *For $\xi \in H^1$, $g \in \mathcal{W}^* \cap H^0$ we have*

$$\xi g \in H^{-1} \quad \text{and} \quad \|\xi g\|_{H^{-1}} \leqq c\|\xi\|_{H^1}\|g\|_{\mathcal{W}^*},$$

where $c = c(n, D, \mu)$ is a constant.

Proof. We introduce the set $BH^1 = \{\eta \in H^1 \cap L_\infty(D)\colon \|\eta\|_{H^1} \leqq 1\}$. Now

$$\begin{aligned} \|\xi g\|_{H^{-1}} &= \sup_{\eta \in BH^1} (\eta, \xi g)_{H^0} \leqq \sup_{\eta \in BH^1} (\eta\xi, g)_{H^0} \leqq \sup_{\eta \in BH^1} \|\eta\xi\|_{\mathcal{W}}\|g\|_{\mathcal{W}^*} \\ &\leqq c \sup_{\eta \in BH^1} \|\eta\|_{H^1}\|\xi\|_{H^1}\|g\|_{\mathcal{W}^*} \leqq c\|\xi\|_{H^1}\|g\|_{\mathcal{W}^*}, \end{aligned} \tag{2.3}$$

where c is the constant from Lemma 2.1. Hence we obtain the required result.

We introduce the parameter

$$\mathcal{P} = \left\{ n,\, T,\, D,\, \delta,\, \sup_{x,t} \left|f(x,t)\right|,\, \sup_{x,t}\left|\beta(x,t)\right|,\, \sup_{x,t,i}\left|\frac{\partial\beta(x,t)}{\partial x_i}\right| \right\}. \tag{2.4}$$

3. A parabolic equation with the coefficient from $\mathcal{X}$. Let us introduce and consider the parabolic equations which in what follows will be used to describe the evolution of functionals (1.6)–(1.7) and are analogues of the backward Kolmogorov equation.

We introduce the operator $\mathcal{A}$:

$$\mathcal{A}V = \sum_{i,j=1}^{n} b_{ij}(x,\, t)\, \frac{\partial^2 V}{\partial x_i \partial x_j}\,(x) + \sum_{i=1}^{n} f_i(x,\, t)\, \frac{\partial V}{\partial x_i}\,(x). \tag{3.1}$$

Here b_{ij}, f_i, x_i are the components of the matrix b and the vectors f, x, respectively. We shall consider the boundary value problem

$$\frac{\partial V}{\partial t} + \mathcal{A}V + gV = -\varphi, \quad V(x,\, t)\,|_{x\in\partial D} = 0,\ V(x,\, T) = R(x). \tag{3.2}$$

As is known (see [5]) for $g \in L_\infty(Q)$, $\varphi \in X^{-1}$, $R \in H^0 = L_2(D)$ this problem is uniquely solvable in the class Y^1.

Theorem 3.1. *Let $z \in \mathbf{C}$, $h \in \mathcal{X}$, $\varphi \in X^{-1}$, $R \in H^0$ be given. Let $g_\varepsilon \in L_\infty(Q)$, $g \in \mathcal{X}$, $h_\varepsilon \in L_\infty(Q)$, $\varphi_\varepsilon \in L_\infty(Q)$, $R_\varepsilon \in H^0$ be functions such that $g_\varepsilon = zh_\varepsilon$, $g = zh$, $\|h_\varepsilon - h\|_{\mathcal{X}} \to 0$, $\|\varphi_\varepsilon - \varphi\|_{X^{-1}} \to 0$, $\|R_\varepsilon - R\|_{H^0} \to 0$, $\varepsilon \to 0$. Let V_ε be the solutions of problem* (3.2), *corresponding to $g = g_\varepsilon$, $\varphi = \varphi_\varepsilon$. Then, for $\varepsilon \to +0$ the sequence V_ε has a limit V in Y^1, which is uniform with respect to z: $|z| \leqq 1$, uniquely defined, and*

$$\|V\|_{Y^1} \leqq c\left(\|\varphi\|_{X^{-1}} + \|R\|_{H^0}\right), \tag{3.3}$$

where $c > 0$ is the constant depending on the parameters $\mathcal{P}$, μ, $\|g\|_{\mathcal{X}}$.

Note that in the theorem, V is assumed to depend linearly on φ, R for any given g. Moreover, $V = 0$ if $\varphi = 0$, $R = 0$. Hence it follows that the operator assigning the solution V to the pair $(\varphi, R) \in X^{-1} \times H^0$ is also linear and homogeneous. We introduce operators $L(g)$, $\mathcal{L}(g)$ such that $V = L(g)\,\varphi + \mathcal{L}(g)\,R$ for the corresponding value of V from Theorem 3.1. According to the theorem, $L(g)$: $X^{-1} \to Y^1$, $\mathcal{L}(g)$: $H^0 \to Y^1$ are linear continuous operators; by Lemma 2.2, $gV \in X^{-1}$.

Definition 3.1. *We say that V is a solution in the class Y^1 of problem* (3.2) *with generalized $g \in \mathcal{X}$.*

Proof of Theorem 3.1. First we shall show that substituting $g = g_\varepsilon$ into (3.2) the constant in inequality (3.3), where $V = V_\varepsilon$, does not increase for $\varepsilon \to +0$.

Below we use the simple inequality

$$uv \leqq \frac{u^2}{2\gamma} + \frac{v^2\gamma}{2} \quad (\forall\, u, v, \gamma \in \mathbf{R},\ \gamma > 0).$$

Let $v \in H^1 \cap C^2(D)$. For any $t \in [0, T]$ and the operator $\mathcal{A} = \mathcal{A}(x, t)$ defined in (3.1) we have the estimates

$$\begin{aligned}(v,\, \mathcal{A}v)_{H^0} &= \left(v,\ \sum_{i,j=1}^{n} b_{ij}\, \frac{\partial^2 v}{\partial x_i \partial x_j}\right)_{H^0} + \left(v,\ \sum_{i=1}^{n} f_i\, \frac{\partial v}{\partial x_i}\right)_{H^0} \\ &= -\sum_{i,j=1}^{n} \left(\frac{\partial v}{\partial x_i},\ b_{ij}\, \frac{\partial v}{\partial x_j}\right)_{H^0} - \sum_{i,j=1}^{n} \left(v,\ \frac{\partial b_{ij}}{\partial x_i}\, \frac{\partial v}{\partial x_j}\right)_{H^0} \\ &\quad + \sum_{i=1}^{n} \left(v,\ f_i\, \frac{\partial v}{\partial x_i}\right)_{H^0} \leqq -\delta \sum_{i=1}^{n} \left\|\frac{\partial v}{\partial x_i}\right\|^2_{H^0}\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i,j=1}^{n} \|v\|_{H^0} \left\| \frac{\partial b_{ij}}{\partial x_i} \right\|_{L_\infty(Q)} \left\| \frac{\partial v}{\partial x_j} \right\|_{H^0} \\
&+ \sum_{i=1}^{n} \|v\|_{H^0} \|f_i\|_{L_\infty(Q)} \left\| \frac{\partial v}{\partial x_i} \right\|_{H^0} \leqq -\delta \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{H^0}^2 + \frac{\delta}{4} \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{H^0}^2 \\
&+ \frac{C}{\delta} \sum_{i,j=1}^{n} \|v\|_{H^0}^2 \left(\left\| \frac{\partial b_{ij}}{\partial x_i} \right\|_{L_\infty(Q)}^2 + \|f_i\|_{L_\infty(Q)}^2 \right), \qquad (3.4)
\end{aligned}
$$

with the constant $C = C(n)$. Hence for any $v \in H^1$ and for any $t \in [0,T]$ we have

$$
(v,\, \mathcal{A}v)_{H^0} \leqq -\frac{3\delta}{4} \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{H^0}^2 + C_1 \|v\|_{H^0}^2, \qquad (3.5)
$$

where the constant C_1 depends on the parameter $\mathcal{P}$.

For any $v \in H^1$ and for any $t \in [0,T]$, we have

$$
\begin{aligned}
\big(v,\, \varphi_\varepsilon(\cdot,\, t)\big)_{H^0} &\leqq \|v\|_{H^1}^2 \|\varphi_\varepsilon(\cdot,\, t)\,\|_{H^{-1}}^2 \leqq C_D \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{H^0} \big\|\varphi_\varepsilon(\cdot,\, t)\big\|_{H^{-1}} \\
&\leqq \frac{\delta}{4} \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{H^0}^2 + C_2 \big\|\varphi_\varepsilon(\cdot,\, t)\big\|_{H^{-1}}^2, \qquad (3.6)
\end{aligned}
$$

where the constant C_D depends on n, D and the constant C_2 depends on $\mathcal{P}$.

For any $v \in H^1$,

$$
\begin{aligned}
(v,\, g_\varepsilon v)_{H^0} &\leqq \|v^2\|_{\mathcal{W}} \|g_\varepsilon\|_{\mathcal{W}^*} \leqq C_3 \|v\|_{H^1} \|v\|_{H^0} \|g_\varepsilon\|_{\mathcal{W}^*} \\
&\leqq \frac{\delta}{4} \sum_{i=1}^{n} \|v\|_{H^1}^2 + \widehat{C}_3 \|v\|_{H_0}^2 \leqq \frac{\delta}{4} \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{H^0}^2 + C_3 \|v\|_{H_0}^2, \qquad (3.7)
\end{aligned}
$$

where the constants $\widehat{C}_3$, C_3 depend on $\sup_\varepsilon \|g_\varepsilon\|_{\mathcal{X}}$, δ, n, D.

By virtue of (3.5)–(3.7) we have, for the solution $V = V_\varepsilon$ of problem (3.2) with $g = g_\varepsilon$, $\varphi = \varphi_\varepsilon$, $\varepsilon \in (0, \varepsilon_1]$,

$$
\begin{aligned}
&\big\|V_\varepsilon(\cdot,\, t)\big\|_{H^0}^2 - \big\|V_\varepsilon(\cdot,\, T)\big\|_{H^0}^2 \\
&\quad = 2 \int_t^T \big(V_\varepsilon(\cdot,\, s),\ \mathcal{A}V_\varepsilon(\cdot,\, s) + g_\varepsilon V_\varepsilon(\cdot,\, s) + \varphi_\varepsilon(\cdot,\, s)\big)_{H^0}\, ds \\
&\quad \leqq \int_t^T \left\{ -\delta \sum_{i=1}^{n} \left\| \frac{\partial V_\varepsilon}{\partial x_i}(\cdot,\, s) \right\|_{H^0}^2 + C_4 \Big(\big\|V_\varepsilon(\cdot,\, s)\big\|_{H^0}^2 + \big\|\varphi_\varepsilon(\cdot,\, s)\big\|_{H^{-1}}^2 \Big) \right\} ds, \qquad (3.8)
\end{aligned}
$$

where the constant C_4 depends on $\mathcal{P}$, μ, $\|g\|_{\mathcal{X}}$.

Hence we immediately obtain

$$
\|V_\varepsilon\|_{Y^1} \leqq C_* \big(\|\varphi_\varepsilon\|_{X^{-1}} + \|R\|_{H^0} \big) \quad (\forall\, \varepsilon \in (0,\, \varepsilon_1]), \qquad (3.9)
$$

where the constant C_* depends on $\mathcal{P}$, μ, $\|g\|_{\mathcal{X}}$.

Let us show that the sequence $\{V_\varepsilon\}$ is a Cauchy sequence in Y^1. Let $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$. Denote $W = V_{\varepsilon_1} - V_{\varepsilon_2}$. We have

$$
\frac{\partial W}{\partial t} + \mathcal{A}W + g_{e_1} W = -\xi, \quad W(x,\, t)\,|_{x \in \partial D} = 0,\ W(x,\, T) = 0, \qquad (3.10)
$$

where $\xi = \varphi_{\varepsilon_1} - \varphi_{\varepsilon_2} + (g_{\varepsilon_1} - g_{\varepsilon_2})V_{\varepsilon_2}$. We obtain

$$\|\varphi_{\varepsilon_1} - \varphi_{\varepsilon_2}\|_{X^{-1}} \longrightarrow 0, \qquad \|g_{\varepsilon_1} - g_{\varepsilon_2}\|_{\mathcal{X}} \longrightarrow 0, \tag{3.11}$$

since $\{\varphi_{\varepsilon_i}\}$, $\{g_{\varepsilon_i}\}$ are the Cauchy sequences converging in the corresponding spaces. By Lemma 2.2,

$$\begin{aligned}\big\|(g_{\varepsilon_1} - g_{\varepsilon_2})\, V_{\varepsilon_2}\big\|_{X^{-1}} &= \int_0^T \big\|(g_{\varepsilon_1} - g_{\varepsilon_2})\, V_{\varepsilon_2}(\cdot,\, t)\big\|_{H^{-1}}\, dt \\ &\leqq \int_0^T \|g_{\varepsilon_1} - g_{\varepsilon_2}\|_{\mathcal{W}^*} \big\|V_{\varepsilon_2}(\cdot,\, t)\big\|_{H^1}\, dt \\ &\leqq \|g_{\varepsilon_1} - g_{\varepsilon_2}\|_{\mathcal{X}} \int_0^T \big\|V_{\varepsilon_2}(\cdot,\, t)\big\|_{H^{-1}}\, dt \\ &= \|g_{\varepsilon_1} - g_{\varepsilon_2}\|_{\mathcal{X}} \|V_{\varepsilon_2}\|_{X^1} \longrightarrow 0 \end{aligned} \tag{3.12}$$

for $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, by virtue of (3.9), (3.11). Therefore, $\|\xi\|_{X^{-1}} \to 0$. Applying estimate (3.9) to the solution W of the boundary value problem (3.10), we obtain

$$\|W\|_{Y^1} \leqq C_* \|\xi\|_{X^{-1}} \longrightarrow 0. \tag{3.13}$$

Hence it follows that the sequence $\{V_\varepsilon\}$, $\varepsilon = \varepsilon_i \to 0$, is Cauchy (and therefore converging) in the Banach space Y^1. Estimate (3.3) follows from (3.9), the latter estimate implying the uniqueness of V.

We will show that the sequence $\{V_\varepsilon\}$ converges in Y^1 uniformly with respect to $z \in \mathbf{C}$: $|z| \leqq 1$. Let $\varepsilon \to 0$. Denote $W = V_\varepsilon - V$. We have

$$\frac{\partial W}{\partial t} + \mathcal{A} W + g_\varepsilon W = -\xi, \quad W(x,t)\,|_{x \in \partial D} = 0,\ W(x,\, T) = R_\varepsilon - R, \tag{3.14}$$

where $\xi = \varphi_\varepsilon - \varphi + (g_\varepsilon - g)\, V$. We have

$$\|\varphi_\varepsilon - \varphi\|_{X^{-1}} \longrightarrow 0, \qquad \|g_\varepsilon - g\|_{\mathcal{X}} = z\|h_\varepsilon - h\|_{\mathcal{X}} \longrightarrow 0 \tag{3.15}$$

uniformly with respect to $|z| \leqq 1$. By Lemma 2.2,

$$\begin{aligned}\big\|(g_\varepsilon - g)\, V\big\|_{X^{-1}} &= \int_0^T \big\|(g_\varepsilon - g)\, V_{\varepsilon_2}(\cdot,\, t)\big\|_{H^{-1}}\, dt \leqq \int_0^T \|g_\varepsilon - g\|_{\mathcal{W}^*} \big\|V(\cdot,\, t)\big\|_{H^1}\, dt \\ &\leqq \|g_\varepsilon - g\|_{\mathcal{X}} \int_0^T \big\|V(\cdot,\, t)\big\|_{H^1}\, dt = \|g_\varepsilon - g\|_{\mathcal{X}} \|V\|_{X^1} \longrightarrow 0 \end{aligned} \tag{3.16}$$

for $\varepsilon \to 0$ and $\|\xi\|_{X^{-1}} \to 0$ uniformly with respect to $|z| \leqq 1$. Applying estimate (3.3) to the solution W of the boundary value problem (3.14), we obtain

$$\|W\|_{Y^1} \leqq C_* \big(\|\xi\|_{X^{-1}} + \|R_\varepsilon - R\|_{H^0}\big) \longrightarrow 0.$$

Hence it follows that the sequence $\{V_\varepsilon\}$, $\varepsilon \to 0$, converges uniformly with respect to $|z| \leqq 1$. Theorem 3.1 is proved.

4. On a class of hypersurfaces. It is natural to assume that to study functionals of type (1.6)–(1.7) it is helpful to use equations (3.2) with $g \in \mathcal{X}$, $\varphi \in X^{-1}$, such that as functions g_ε, φ_ε, appearing in Theorem 3.1, one can take the functions $\varepsilon^{-1}\mathrm{Ind}\{x \in \Gamma(t,\varepsilon)\}$ for the hypersurface $\Gamma(t) \subset D$ (the sets $\Gamma(t,\varepsilon)$ are defined by (1.3)–(1.4)). In what follows it will be shown that the limit of $\varepsilon^{-1}\mathrm{Ind}\{x \in \Gamma(t,\varepsilon)\}$

belongs to both X^{-1} and $\mathcal{X}$ for $\Gamma(t)$ from a sufficiently wide class of "piecewise C^1-smooth" hypersurfaces.

By $e^{(j)}$ we denote the jth unit vector in $\mathbf{R}^n$.

Let Γ be some surface. Denote by $N(x,j)$ the number of intersections of the hypersurface Γ by the ray from $x=(x_1,x_2,\dots,x_n)$ to $(x_1,\dots,x_{j-1},-\infty,x_{j+1},\dots,x_n)$, the points $\widehat{x}_k(x,j)$ being the corresponding intersection points; it is assumed that $N(x,j)=+\infty$ if the ray is tangential to Γ.

We introduce functions $\gamma_j\colon \Gamma\to\mathbf{R}$ such that $\gamma_j(x)=|\cos\alpha_j(x)|$, where $\alpha_j(x)$ is the angle between $e^{(j)}$ and the normal to Γ at the point $x\in\Gamma$ if this normal is defined; $\gamma_j(x)=0\,(\forall\, j)$ if x is the point of violation of smoothness at which the normal is not defined. Let us define the functions

$$G_j(x)=\sum_{k=1}^{N(x,j)}\gamma_j\bigl(\widehat{x}_k(x,\,j)\bigr). \tag{4.1}$$

Assume that $G_j(x)=+\infty$ if $N(x,j)=+\infty$.

Theorem 4.1. *Let a set $\widehat{\Gamma}\subset\mathbf{R}^n$ be given, which is the union of a finite number $\mathcal{N}$ of polyhedra $\widehat{\Gamma}_i$ of dimension $n-1$ with pairwise nonintersecting interiors, $\widehat{\Gamma}=\cup_{i=1}^{\mathcal{N}}\widehat{\Gamma}_i$. Let $\mathcal{B}\colon \mathbf{R}^n\to\mathbf{R}^n$ be some continuous bijective function, let the hypersurface $\Gamma\subset D$ be such that $\Gamma=\mathcal{M}(\widehat{\Gamma})$. It is assumed that the functions $\mathcal{B}\colon \widehat{\Gamma}_i\to\mathbf{R}^n$ are C^1-smooth bijections, $\mathcal{B}(x)=x$, if x is the vertex of some $\widehat{\Gamma}_i$, $|\mathbf{n}(x)-\mathbf{n}_i|\leqq\delta_0$ if x belongs to the interior part of Γ_i, $i=1,\dots,\mathcal{N}$. Here $\Gamma_i=\mathcal{B}(\widehat{\Gamma}_i)$, $\mathbf{n}(x)$ is the normal to Γ in x, $\mathbf{n}_i$ is the normal to $\widehat{\Gamma}_i$, $\delta_0\leqq n^{-2}/2$ (the direction of the normals is fixed, $|\mathbf{n}(x)|=1$, $|\mathbf{n}_i|=1$). Then $N(x,j)<+\infty$ for all j for almost all x. We define the generalized functions*

$$g=\sum_{j=1}^{n}\frac{\partial G_j}{\partial x_j}. \tag{4.2}$$

Then $g\in\mathcal{W}^*\cap H^{-1}$, the functions $g_\varepsilon(x)=\varepsilon^{-1}\mathrm{Ind}\{x\in\Gamma(\varepsilon)\}$ converge to g in the metric of $\mathcal{W}^*$ and H^{-1}, and assume

$$\|g\|_{\mathcal{W}^*}\leqq c\sum_{j=1}^{n}\|G_j\|_{L_\nu(D_1)},\qquad \|g\|_{H^{-1}}\leqq c\sum_{j=1}^{n}\|G_j\|_{L_2(D_1)},$$

where $\nu=\mu(\mu-1)^{-1}$, D_1 is a bounded domain in $\mathbf{R}^n$ such that $\Gamma\subset D_1\subseteq D$, $c=c(n,D_1)$ is a constant.

Remark 4.1. It is not difficult to note that the assumptions of Theorem 4.1 are fulfilled for disks, spheres, and many other piecewise C^1-smooth $(n-1)$-dimensional surfaces.

Proof of Theorem 4.1. Denote $S_0=\cup\partial\Gamma_i$; we have $\ell_n(S_0)=0$. Consider a set of open domains $D_i\subseteq D$ such that $\overline{D}=\cup_{i=1}^{\mathcal{N}}\overline{D}_i$, $D_i\cap D_j=\varnothing$ if $i\neq j$, $\Gamma_i=\Gamma\cap\overline{D}_i$ ($\overline{D}_i$, $\overline{D}$ denote the closures of domains). Denote $\Gamma_i(\varepsilon)=\Gamma(\varepsilon)\cap\overline{D}_i$. Denote by $P_j(x)$ the straight line passing through x and parallel to $e^{(j)}$. We consider functions $\psi(\cdot,\varepsilon)\colon \Gamma(\varepsilon)\longrightarrow\Gamma\backslash S_0$ such that $\psi(x)\in\Gamma_i$ if $x\in\Gamma_i(\varepsilon)$, $|\psi(x,\varepsilon)-x|\leqq c\varepsilon$, with constants c such that these functions exist for all $\varepsilon\leqq\varepsilon_*$ for some small $\varepsilon_*>0$. We consider functions $G_j^\varepsilon(x)$, $x=(x_1,\dots,x_n)$. These functions are assumed to be continuous in x_j. The functions $G_j^\varepsilon(x)$ are assumed to be constant on each segment $P_j(x)$ that lies in $D\backslash\Gamma(\varepsilon)$ (each segment has its own constants), $G_j^\varepsilon(x)=0$ if

$(x_1, \dots, x_{j-1}, y_j, x_{j+1}, \dots, x_n) \notin \Gamma(\varepsilon)$ $(\forall\, y_j < x_j)$, and $\partial G_j^\varepsilon(x)/\partial x_j = \varepsilon^{-1}\gamma_j^2(\psi(x,\varepsilon))$, $x \in P_j(x) \cap \Gamma(\varepsilon)$. These conditions define $G_j^\varepsilon(x)$ uniquely.

We have $\sum_{j=1}^n \gamma_j^2(x) = 1$ $(\forall\, x \in \Gamma \backslash S_0)$. Then,

$$\sum_{j=1}^n \frac{\partial G_j^\varepsilon}{\partial x_j}(x) = \frac{1}{\varepsilon}\sum_{j=1}^n \gamma_j^2\big(\psi(x,\,\varepsilon)\big) = \frac{1}{\varepsilon}, \quad x \in \Gamma(\varepsilon),$$
$$\sum_{j=1}^n \frac{\partial G_j^\varepsilon}{\partial x_j}(x) = 0, \quad x \notin \Gamma(\varepsilon), \qquad \sum_{j=1}^n \frac{\partial G_j^\varepsilon}{\partial x_j}(x) = g_\varepsilon(x), \quad x \in D.$$

We have $\ell_n(x \in \mathbf{R}^n\colon\ P_j(x) \cap S_0 \neq \varnothing) = 0$, $j = 1, \dots, n$.

By Proposition 2.1 [7], $N(j,x) < +\infty$ is fulfilled for almost all x (we can assume the ray to be a solution of a trivial ordinary differential equation; then Proposition 2.1 [7] can be reformulated for this case). By the piecewise C^1-smoothness of Γ we have $N(x,j) \leqq \text{const.}$ for almost all x.

Let $\mathcal{N} = 1$, $\Gamma = \Gamma_1$ be such that $\mathbf{n}_1 = (\mathbf{n}_1^{(1)}, \dots, \mathbf{n}_1^{(n)})$, $|\mathbf{n}_1^{(j)}| = n^{-1/2}$, $j = 1, \dots, n$; then $\gamma_j(x) \geqq \delta_1$ $(\forall\, x \in \Gamma \backslash S_0)$ for some $\delta_1 > 0$. Denote by $\mu_j(x,\varepsilon)$ the length of a minimal segment that lies in $\Gamma(\varepsilon)$, with ends in $\partial\Gamma(\varepsilon)$, passes through x, and is parallel to $e^{(j)}$. We have $\varepsilon^{-1}\mu_j(x,\varepsilon) \longrightarrow \gamma_j^{-1}(x)$. By the continuity of the normal on the given C^1-smooth surface $\Gamma = \Gamma_1$ we conclude that $G_i^\varepsilon(x) \to G_i(x)$ for $\varepsilon \to +0$ for almost all x, $i = 1, \dots, n$,

$$\sup_x \big|G_j^\varepsilon(x)\big| \leqq \varepsilon^{-1}\mu_j(x,\,\varepsilon)\, \sup_x \gamma_j^2\big(\psi(x,\,\varepsilon)\big) \leqq c,$$

where $x \in P_j(x) \cap \Gamma(\varepsilon)$, the constant c depending on Γ. Therefore, $\|G_i^\varepsilon - G_i\|_{L_\nu(D)} \to 0$ as $\varepsilon \to +0$ $(\forall\, \nu > 1,\ i = 1, \dots, n)$, or $g_\varepsilon = \sum_{j=1}^n \partial G_j^\varepsilon/\partial x_j(x) \longrightarrow g$ in $\mathcal{W}^*$. Let $u \in C^2(D) \cap \mathcal{W}^* \cap H^1$. We have

$$\begin{aligned}\langle g_\varepsilon - g,\, u\rangle &= \sum_{j=1}^n \int_{\partial D} \big(G_j^\varepsilon(x) - G_j(x)\big)\, u(x)\, \cos\big(\mathbf{n}(x),\, e_j\big)\, dx \\ &\quad - \sum_{j=1}^n \int_D \big(G_j^\varepsilon(x) - G_j(x)\big)\ \frac{\partial u}{\partial x_j}(x)\, dx \\ &= -\sum_{j=1}^n \int_D \big(G_j^\varepsilon(x) - G_j(x)\big)\ \frac{\partial u}{\partial x_j}(x)\, dx,\end{aligned}$$

since $G_j^\varepsilon \equiv G_j$ on ∂D (when $D = \mathbf{R}^n$, the integrals on ∂D in the second expression vanish). Thus, for $\nu = \mu(\mu-1)^{-1}$, we have

$$\big|\langle g_\varepsilon - g,\, u\rangle\big| \leqq \sum_{j=1}^n \big(G_j^\varepsilon - G_j\big)_{L_\nu(D)}\, \|u\|_{\mathcal{W}}.$$

Hence $g_\varepsilon = \sum_{j=1}^n \partial G_j^\varepsilon/\partial x_j(x) \longrightarrow g$ in $\mathcal{W}^*$. Theorem 4.1 in the above particular case is proved.

Let $\mathcal{N} = 1$ and let the normal $\mathbf{n}_1$ have an arbitrary direction. We introduce new coordinates $\widetilde{x}_j = \sum_{k=1}^n \theta_{jk}x_k$, where $\theta \in \mathbf{R}^{n\times n}$ is a matrix such that $\theta^{\mathrm{T}} = \theta^{-1}$ and $|\widetilde{\mathbf{n}}_1^{(j)}| = n^{-1/2}$, $\mathbf{n}_1^{(j)}\widetilde{\mathbf{n}}_1^{(j)} \geqq 0$, $j = 1, \dots, n$, where $\widetilde{\mathbf{n}}_1 = (\widetilde{\mathbf{n}}_1^{(1)}, \dots, \widetilde{\mathbf{n}}_1^{(n)})$ is the corresponding normal to $\theta\widehat{\Gamma}$ in terms of the new coordinates. We denote by

$\widetilde{N}(x,j)$ the number of intersections of the hypersurface $\widetilde{\Gamma} = \theta\Gamma$ by the ray from $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \dots, \widetilde{x}_n)$ to $(\widetilde{x}_1, \dots, \widetilde{x}_{j-1}, -\infty, \widetilde{x}_{j+1}, \dots, \widetilde{x}_n)$, the points $\widetilde{x}'_k(x,j)$ being the corresponding intersection points; it is assumed that $\widetilde{N}(x,j) = +\infty$ if the ray is tangential to Γ. Then, $\widetilde{N}(x,j) < +\infty$ for all j for almost all x. We define the functions $\widetilde{G}_j(\widetilde{x}) = \sum_{k=1}^{\widetilde{N}(\widetilde{x},j)} \gamma_j(\widetilde{x}'_k(\widetilde{x}, j))$. Here $\widetilde{\gamma}_j(\widetilde{x})$ are the corresponding functions constructed by the rule indicated before the formulation of the theorem but in terms of the new coordinates. Denote $\widetilde{g} = \sum_{j=1}^n \partial\widetilde{G}_j/\partial\widetilde{x}_j$. As above, we obtain $\widetilde{g}_\varepsilon(\widetilde{x}) = \mathrm{Ind}\{x = \theta^{-1}\widetilde{x} \in \Gamma(\varepsilon)\} \longrightarrow \widetilde{g}$ in $\mathcal{W}^*$. Denote by $\Delta G_j(x)$ a jump of the function G_j in x on the straight line $P_j(x)$ with respect to the order of growth of x_j. We have $\Delta G_j(x) = \gamma_j(x)$. Denote by $\Delta\widetilde{G}_j(\widetilde{x})$ analogous jumps in terms of the new coordinates. The fact that the transformation of coordinates is actually a rotation implies that $\Delta\widetilde{G}_j(\widetilde{x}) = \sum_{k=1}^n \theta_{jk}\Delta G_k(\theta^{-1}\widetilde{x})$ if $N(x,j) < +\infty$, $x \in \Gamma\backslash S_0$. Thus $\langle\widetilde{\xi}, \widetilde{g}\rangle = \langle\xi, g\rangle$ for $\widetilde{\xi} \in \mathcal{W}$, $\xi(x) = \widetilde{\xi}(\theta x)$. Therefore, Theorem 4.1 is proved for this case.

Let $\mathcal{N} > 1$, $g_\varepsilon^{(i)}(x) = \mathrm{Ind}\{x \in \Gamma_i(\varepsilon)\}$. We have $g_\varepsilon = \sum_{i=1}^{\mathcal{N}} g_\varepsilon^{(i)}$, $g_\varepsilon^{(i)}(x) \longrightarrow g^{(i)} \in \mathcal{W}^*$ in $\mathcal{W}^*$, $g^{(i)} = \sum_{j=1}^n \partial G_j^{(i)}/\partial x_j$, where $G_j^{(i)}$ are constructed in the same way as in the formulation of Theorem 4.1 for the corresponding $\Gamma = \Gamma_i$. We have $G_j(x) = \sum_{i=1}^{\mathcal{N}} G^{(i)}(x)$. Hence we obtain the proof of Theorem 4.1.

We shall give an example of the surface $\Gamma = \Gamma(t)$ which changes in time, approaching a fractal, and $g = g(t) \in X^{-1}$ holds for $g(t)$ defined for each t in Theorem 4.1.

Example 4.1. Let $n = 2$, $T = 2$,

$$\Gamma(t) = \Big\{(x_1, x_2)\colon\ x_2 = \sin\Big(x_1(1-t)^{-1/3}\Big),\ x_1 \in [-1,\, 1]\Big\};$$

then $g = g(t) \in X^{-1}$.

5. Probability interpretation of a solution of a parabolic equation for generalized g, φ. Below it is assumed that the assumptions of section 1 are fulfilled for the process $y(t)$ and equations (1.1)–(1.2). In addition, it is assumed that an initial random vector $a = y(0)$ has a distribution density $\rho \in H^0 = L_2(D)$. It is assumed that for almost all $t \in [0,T]$ the hypersurface $\Gamma(t)$ is given, for which the assumptions of Theorem 4.1 are fulfilled, $g(t)$ are the corresponding elements of $\mathcal{W}^* \cap H^{-1}$ defined by Theorem 4.1, and g is the corresponding element of X^{-1}.

Let $\Gamma(t,\varepsilon)$ be defined by (1.3)–(1.4), and let $l_\varepsilon(t)$ be the variables given by (1.5).

THEOREM 5.1. *Let $\rho \in L_2(D)$, $g \in \mathcal{X}$.*

(a) *For any $t > 0$ there exists a random variable $\hat{\mathbf{t}}(t)$ such that $l_\varepsilon(t) \to \hat{\mathbf{t}}(t)$ in distribution as $\varepsilon \to +0$.*

(b) *Let $z \in \mathbf{C}$. We introduce the generalized function $g_1 = zg$. Let $V_1 = L(g_1)g$, $V = zV_1$ (in other words, $V = zL(zg)g$). Then $V \in Y^1$,*

$$1 + \big(V(\cdot,\, 0),\, \rho\big)_{H^0} = \lim_{\varepsilon\to+0} \mathbf{E}\exp\big\{z\, l_\varepsilon(T)\big\} = \mathbf{E}\exp\big\{z\,\hat{\mathbf{t}}(T)\big\}, \tag{5.1}$$

where the limit exists uniformly with respect to $z \in \mathbf{C}$: $|z| \leqq 1$.

COROLLARY 5.1. *Under the assumptions of Theorem* 5.1,

$$\mathbf{E}\, e^{k\hat{\mathbf{t}}(t)} < +\infty, \qquad \mathbf{E}\,\hat{\mathbf{t}}(t)^k < +\infty \quad (\forall\, k > 0).$$

Remark 5.1. The existence of a local time of general multi-dimensional Itô processes on a smooth hypersurface was in principle established in [4]. More precisely, in this paper the existence was proved of continuous nondecreasing processes $\hat{\mathbf{t}}(t)$ increasing only on hypersurfaces and such that they can be obtained as a limit a.s. of the variables $\hat{l}_\varepsilon(t) = \int_0^t \widehat{g}_\varepsilon(y(t))\,dt$, where $\widehat{g}_\varepsilon$ are some nonnegative functions; however, the question of whether, in the general case, one can take $\widehat{g}_\varepsilon = g_\varepsilon$ (an indicator of the hypersurface neighborhood with a normalizing multiplier) was left open.

In what follows, $\hat{\mathbf{t}}(t)$ are the variables given in Theorem 5.1.

THEOREM 5.2. *Let the assumptions of Theorem* 5.1 *be fulfilled*, $z \in \mathbf{C}$, $\varphi \in X^{-1}$, *the function* $R \in L_2(D)$ *be measurable*, $\varphi_\varepsilon \in L_2(Q)$ *be measurable functions such as those of Theorem* 3.1 *for a given* φ, $V = L(zg)\,\varphi + \mathcal{L}(zg)\,R$. *Then*,

$$\big(V(\cdot,\,0),\,\rho\big)_{H^0} = \lim_{\varepsilon\to+0} \mathbf{E}\bigg\{ R\big(y(T)\big)\,\mathrm{Ind}\,\{\tau_D > T\}\exp\{\,z\,l_\varepsilon(T)\} + \int_0^{\tau_D\wedge T} \exp\big\{\,z\,l_\varepsilon(t)\big\}\,\varphi_\varepsilon\big(y(t),\,t\big)\,dt\bigg\}. \tag{5.2}$$

Proof of Theorem 5.2. We introduce functions $R_\varepsilon \in C^2(D)$ such that $\|R_\varepsilon - R\|_{H^0} \to 0$. We introduce the functions $g_\varepsilon(x,t) = \varepsilon^{-1}\mathrm{Ind}\,\{x \in \Gamma(t,\varepsilon)\}$, $V_\varepsilon = L(zg_\varepsilon)\,\varphi + \mathcal{L}(zg_\varepsilon)\,R_\varepsilon$. By virtue of [5, section IV.9] we have $V_\varepsilon \in W_q^{2,1}(Q)\ \ \forall\, q > 1$. We introduce the function

$$W(t) = V_\varepsilon\big(y(t),\,t\big)\exp\bigg\{z\int_0^t g_\varepsilon\big(y(r),\,r\big)\,dr\bigg\}.$$

We have

$$d_tW(t) = \exp\bigg\{z\int_0^t g_\varepsilon\big(y(r),\,r\big)\,dr\bigg\}\,d_tV_\varepsilon\big(y(t),\,t\big) + zg_\varepsilon\big(y(r),\,r\big)\,W(t)\,dt.$$

Since $V_\varepsilon \in W_q^{2,1}(Q)\ \ \forall\, q > 1$, we can apply the Itô formula [8, section II.10] to $d_tV_\varepsilon(y(t),t)$. Using this formula, we obtain

$$\begin{aligned}\mathbf{E}\,\big(W(\tau_D\wedge T) - W(0)\big) &= \mathbf{E}\,\big(W(\tau_D\wedge T) - V_\varepsilon(y(0),\,0)\big)\\ &= -\mathbf{E}\int_0^{\tau_D\wedge T}\exp\bigg\{z\int_0^t g_\varepsilon\big(y(r),\,r\big)\,dr\bigg\}\varphi_\varepsilon\big(y(t),\,t\big)\,dt.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{E}\,V_\varepsilon\big(y(0),\,0\big) &= \big(V_\varepsilon(\cdot,\,0),\,\rho\big)_{H^0}\\ &= \mathbf{E}\bigg\{R_\varepsilon\big(y(T)\big)\,\mathrm{Ind}\,\{\tau_D > T\}\exp\bigg(z\int_0^{\tau_D\wedge T} g_\varepsilon\big(y(t),\,t\big)\,dt\bigg)\\ &\qquad + \int_0^{\tau_D\wedge T}\exp\bigg\{z\int_0^t g_\varepsilon\big(y(r),\,r\big)\,dr\bigg\}\,\varphi_\varepsilon\big(y(t),\,t\big)\,dt\bigg\}.\end{aligned} \tag{5.3}$$

By Theorem 3.1, we have $V_\varepsilon \to V$ in Y^1. Hence we obtain the required result.

Proof of Theorem 5.1. Let us first prove part (b). Let

$$g_\varepsilon(x,t) = \varepsilon^{-1}\mathrm{Ind}\,\big\{x \in \Gamma(t,\varepsilon)\big\},\quad V_\varepsilon = zL(zg_\varepsilon)\,g_\varepsilon.$$

This means that V_ε is a solution of the problem

$$\frac{\partial V_\varepsilon}{\partial t} + \mathcal{A}V_\varepsilon + zg_\varepsilon V_\varepsilon = -zg_\varepsilon,\quad V_\varepsilon(x,\,t)\,|_{x\in\partial D} = 0,\ V_\varepsilon(x,\,T) = 0. \tag{5.4}$$

Similarly to (5.3), by the Itô formula [8, section II.10] we have

$$V_\varepsilon(a,0) = \mathbf{E}\int_0^{\tau_D\wedge T} g_\varepsilon\big(y(t),t\big)\exp\left(z\int_0^t g_\varepsilon\big(y(r),r\big)\,dr\right)dt$$

$$= \mathbf{E}\exp\left(z\int_0^{\tau_D\wedge T} g_\varepsilon\big(y(t),t\big)\,dt\right) - 1. \tag{5.5}$$

Therefore,

$$\phi_\varepsilon(z) = \big(V_\varepsilon(\cdot,0),\rho\big)_{H^0} + 1 = \mathbf{E}\exp\big\{z\,l_\varepsilon(T)\big\}. \tag{5.6}$$

Hence we obtain statement (b) of Theorem 5.1.

Let us prove statement (a). By Theorem 3.1, we have $V_\varepsilon \to V$ in Y^1 uniformly with respect to $z \in \mathbf{C}$: $|z| \leqq 1$. We introduce the function $\phi(z) = (V(\cdot,0),\rho)_{H^0} + 1$. As has been proved, $\phi_\varepsilon(z) \to \phi(z)$ uniformly with respect to $z \in \mathbf{C}$: $|z| \leqq 1$. By Levy's theorem this means that $\phi(z)$ is the characteristic function of some random variable $\hat{\mathbf{t}}(T)$, to which the variables $l_\varepsilon(T)$ converge in distribution. Theorem 5.1 is proved.

THEOREM 5.3. *Let $g \in X^{-1}$, $V = L(0)g$; then $(V(\cdot,0),\rho)_{H^0} = \mathbf{E}\,\hat{\mathbf{t}}(T)$.*

Proof. We have $\mathbf{E}\,l_\varepsilon = (\rho, V_\varepsilon(\cdot,0))_{H^0}$, where $V_\varepsilon = L(0)\,g_\varepsilon$, $g_\varepsilon(x,t) = \varepsilon^{-1}\mathrm{Ind}\,\{x \in \Gamma(t,\varepsilon)\}$. By Theorem 5.1, $\mathbf{E}\,l_\varepsilon(T) \to \mathbf{E}\,\hat{\mathbf{t}}(T)$ as $\varepsilon \to +0$, and, by Theorem 3.1, $V_\varepsilon \to V$ in Y^1. Hence we obtain the required result.

Remark 5.2. Using the approach of [9], [10], it is possible to obtain analogues of Theorems 5.1–5.3 for non-Markov Itô processes $y(t)$ under the assumptions of [9] (i.e., when $\beta dw(t) = \widetilde{\beta}d\widetilde{w}(t) + \widehat{\beta}d\widehat{w}(t)$, where $\widetilde{\beta}$, $\widehat{\beta}$ are random functions which are nonanticipative with respect to $\widehat{w}(t)$, $\widetilde{\beta}\,\widetilde{\beta}^T \geqq \delta I > 0$).

6. An explicit formula for local time and a strong convergence. Let the assumption of section 5 be fulfilled. We retain all the notation of section 5 (in particular, for g, $l_\varepsilon(T)$, g_ε).

Below let β_j, $j = 1,\dots,n$, be the columns of the matrix β in (1.1), $\mathcal{F}_t$ be the flow of σ-algebras of events generated by $\{a, w(s), s \leqq t\}$. Let $\mathcal{M}(g)$: $X^{-1} \to H^0$ be the operator assigning the value $V(x,t)\,|_{t=0} = \mathcal{M}(g)\,\varphi$ to the function $\varphi \in X^{-1}$.

THEOREM 6.1. *Let the initial vector a have a distribution density $\rho \in L_\infty(D)$, $g \in X^{-1}$, and let the matrix $\beta(x,t)$ be continuous. Let $V = L(0)g$. Then $V \in Y^1$ and $\mathbf{E}\,|l_\varepsilon(T) \to \hat{\mathbf{t}}(T)|^2 \to 0$ for $\varepsilon \to 0$, with the random variable*

$$\hat{\mathbf{t}}(T) = V(a,0) + \sum_{j=1}^n \int_0^{\tau_D\wedge T} \frac{\partial V}{\partial x}\big(y(t),t\big)\,\beta_j\big(y(t),t\big)\,dw_j(t). \tag{6.1}$$

Here, by $\partial V/\partial x$ we understand a Borel-measurable function which is a representative of the equivalence class $\partial V/\partial x \in L_2(Q)$. The functions

$$\xi_j(t) = \frac{\partial V}{\partial x}\big(y(t),t\big)\,\beta_j\big(y(t),t\big)$$

are such that $\mathbf{E}\int_0^T |\xi_j(t)|^2 dt < +\infty$ $(\forall j)$, and for some sequence random time functions $\xi_{\varepsilon,j}(t)$, $\varepsilon = \varepsilon_k \to 0$, that are progressively measurable with respect to the flow $\mathcal{F}_t$ we have

$$\mathbf{E}\int_0^T \big|\xi_{j,\varepsilon}(t) - \xi_j(t)\big|^2\,dt \longrightarrow 0, \qquad \sup_\varepsilon \mathbf{E}\int_0^T \big|\xi_{j,\varepsilon}(t)\big|^2\,dt < +\infty \quad (\forall j).$$

Corollary 6.1. *Let $p(x,t)$ be the distribution density of the process $y(t)$ that has a break on ∂D (i.e., $p = \mathcal{M}(0)^*\rho$). Then, using the assumptions and notation of Theorem* 6.1, *we have*

$$\mathbf{E}\,\hat{\mathbf{t}}(T)^2 = \int_D \left|V(x,\,0)\right|^2 \rho(x)\,dx + \sum_{j=1}^n \int_Q \left|\frac{\partial V}{\partial x}\,(x,\,t)\,\beta_j(x,\,t)\right|^2 p(x,\,t)\,dx\,dt.$$

Lemma 6.1. *Let, under the assumptions of Theorem* 6.1, $V_\varepsilon = L(0)g_\varepsilon$,

$$\xi_{j,\varepsilon}(t) = \frac{\partial V_\varepsilon}{\partial x}\,\bigl(y(t),\,t\bigr)\,\beta_j\bigl(y(t),\,t\bigr).$$

Then

$$\ell_\varepsilon(T) = V_\varepsilon(a,0) + \sum_{j=1}^n \int_0^{\tau_D\wedge T} \xi_{j,\varepsilon}(t)\,dw_j(t) \tag{6.2}$$

and $\mathbf{E}\left|l_\varepsilon(T) - \hat{\mathbf{t}}(T)\right|^2 \longrightarrow 0$ *for* $\varepsilon \to 0$.

Proof of Lemma 6.1. Let p be the same as in Corollary 6.1. Note that $p \in L_\infty(Q)$. Denote

$$\ell_\varepsilon^{x,0}(T) = \int_0^{\tau_D\wedge T} g_\varepsilon\bigl(y^{x,0}(t),\,t\bigr)\,dt.$$

For $V_\varepsilon = L(0)g_\varepsilon$ we have $g_\varepsilon = -\partial V_\varepsilon/\partial t - AV_\varepsilon$. By Theorem 9.1 of [5, section IV.9] $V_\varepsilon \in W_q^{2,1}(Q)$ $(\forall\, q > 1)$. For $x \in D$ the Itô formula [8, section II.10] implies that

$$\begin{aligned} -V_\varepsilon(a,\,0) &= V_\varepsilon\bigl(y(\tau_D\wedge T),\,\tau_D\wedge T\bigr) - V_\varepsilon(a,\,0)\\ &= -\int_0^{\tau_D\wedge T} g_\varepsilon\bigl(y(t),\,t\bigr)\,dt + \sum_{j=1}^n \int_0^{\tau_D\wedge T} \xi_j\bigl(y(t),\,t\bigr)\,dw_j(t). \end{aligned}$$

Therefore,

$$\ell_\varepsilon(T) = V_\varepsilon(a,\,0) + \sum_{j=1}^n \int_0^{\tau_D\wedge T} \frac{\partial V_\varepsilon}{\partial x}\,\bigl(y(t),\,t\bigr)\beta_j\bigl(y(t),\,t\bigr)\,dw_j(t).$$

Hence we obtain (6.2). The function $\partial V_\varepsilon(x,t)/\partial x$ is continuous and thus we conclude that the functions $\xi_{\varepsilon,j}(t)$ are progressively measurable with respect to the flow $\mathcal{F}_t$.

Denote $W_\varepsilon = V_\varepsilon - V$. We have

$$\begin{aligned} \mathbf{E}\int_0^{\tau_D\wedge T} \left|\xi_{j,\varepsilon}(t) - \xi_j(t)\right|^2 dt &\leqq c\mathbf{E}\int_0^{\tau_D\wedge T} \left|\frac{\partial W_\varepsilon}{\partial x}\,\bigl(y(t),\,t\bigr)\right|^2 dt\\ &\leqq c\int_Q \left|\frac{\partial W_\varepsilon}{\partial x}\,(x,\,t)\right|^2 p(x,\,t)\,dx\,dt\\ &\leqq c\|p\|_{L_\infty(Q)}\|W_\varepsilon\|_{Y^1} \longrightarrow 0, \end{aligned} \tag{6.3}$$

$$\begin{aligned} \mathbf{E}\left|V_\varepsilon(a,\,0) - V(a,\,0)\right|^2 &\leqq c\int_D \left\|W_\varepsilon(x,\,0)\right\|^2 \rho(x)\,dx\\ &\leqq c\|\rho\|_{L_\infty(D)}\|W_\varepsilon\|_{Y^1} \longrightarrow 0 \end{aligned} \tag{6.4}$$

by Theorem 3.1 ($c > 0$ is some constant). Lemma 6.1 is proved.

The proof of Theorem 6.1 immediately follows from Lemma 6.1 and (6.2)–(6.4).

We shall derive an analogue of Tanaka's formula for the case with stationary coefficients (see [1], [2], [11], [12]).

THEOREM 6.2. *Let $\rho \in L_\infty(D)$, $\beta(x,t) \equiv \beta(x)$, $f(x,t) \equiv f(x)$, $\Gamma(t) \equiv \Gamma$ be time independent, and let the function $\beta(x)$ be continuous. Let $F(x)$ be a solution of the problem $\mathcal{A}F = g$, $F|_{\partial D} = 0$ in the class H^1. Then, $F \in C(D)$ and (a.s.)*

$$\hat{\mathbf{t}}(T) = F\left(y(\tau_D \wedge T)\right) - F(a) - \sum_{j=1}^{n} \int_0^{\tau_D \wedge T} \frac{\partial F}{\partial x}\big(y(t)\big)\,\beta_j\big(y(t)\big)\,dw_j(t). \tag{6.5}$$

By $\partial F/\partial x$ we understand here a Borel measurable function which is a representative of the equivalence class $\partial F/\partial x \in L_2(D)$. The functions

$$\widehat{\xi}_j(t) = \frac{\partial F}{\partial x}\big(y(t)\big)\,\beta_j\big(y(t)\big)$$

are such that $\mathbf{E}\int_0^T |\widehat{\xi}_j(t)|^2\,dt < +\infty$ $(\forall\, j)$ and $\mathbf{E}\int_0^T |\widehat{\xi}_{j,\varepsilon}(t) - \widehat{\xi}_j(t)|^2\,dt \longrightarrow 0$ for some sequence of functions $\widehat{\xi}_{j,\varepsilon}(t)$, $\varepsilon = \varepsilon_k \to 0$, progressively measurable with respect to the flow $\mathcal{F}_t$ and such that

$$\sup_{\varepsilon} \mathbf{E} \int_0^T \big|\widehat{\xi}_{j,\varepsilon}(t)\big|^2\,dt < +\infty \quad (\forall\, j).$$

Proof. Let F_ε be a solution of the problem $\mathcal{A}F_\varepsilon = g_\varepsilon$, $F_\varepsilon|_{\partial D} = 0$ in the class H^1. We assume that (4.1), (4.2) hold for g. By Theorem III.14.1 of [6], $F \in C(D)$, $F_\varepsilon \in C(D)$, and

$$\|F_\varepsilon - F\|_{C(D)} \longrightarrow 0, \qquad \|F_\varepsilon - F\|_{H^1} \longrightarrow 0. \tag{6.6}$$

Let $V_\varepsilon = L(0)g$, $U_\varepsilon = V_\varepsilon + F_\varepsilon$. We have

$$\frac{\partial U_\varepsilon}{\partial t}(x,\,t) + \mathcal{A}U_\varepsilon(x,\,t) = 0, \quad U_\varepsilon(x,\,t)\,|_{x \in \partial D} = 0,\ U_\varepsilon(T,\,x) = F(x). \tag{6.7}$$

Denote

$$\widehat{\xi}_{j,\varepsilon}(t) = \frac{\partial V_\varepsilon}{\partial x}\big(y(t),\,t\big)\,\beta_j\,\big(y(t)\big), \qquad \widehat{\xi}_{j,\varepsilon}(t) = \frac{\partial F_\varepsilon}{\partial x}\big(y(t)\big)\,\beta_j\,\big(y(t)\big). \tag{6.8}$$

By Lemma 6.1,

$$\ell_\varepsilon(T) = U_\varepsilon(a,\,0) - F_\varepsilon(a) + \sum_{j=1}^{n} \int_0^{\tau_D \wedge T} \xi_{j,\varepsilon}(t)\,dw_j(t). \tag{6.9}$$

We have

$$\xi_{j,\varepsilon}(t) = \frac{\partial U_\varepsilon}{\partial x}\big(y(t),\,t\big)\,\beta_j\big(y(t)\big) - \widehat{\xi}_{j,\varepsilon}(t), \tag{6.10}$$

$$F_\varepsilon\big(y(\tau_D \wedge T)\big) = U_\varepsilon\big(y(\tau_D \wedge T),\ \tau_D \wedge T\big). \tag{6.11}$$

By the Itô formula [8, section II.10] and (6.7),

$$U_\varepsilon\big(y(\tau_D \wedge T),\,\tau_D \wedge T\big) = U_\varepsilon(a,\,0) + \sum_{j=1}^{n} \int_0^{\tau_D \wedge T} \frac{\partial U_\varepsilon}{\partial x}\big(y(t),\,t\big)\,\beta_j\big(y(t)\big)\,dw_j(t). \tag{6.12}$$

Relations (6.6)–(6.12) yield the proof of Theorem 6.2.

Remark 6.1. One can obtain an analogue of Theorem 6.2 for $D = \mathbf{R}^n$. In that case, (6.5) is fulfilled for the function F which is a solution of the equation $\mathcal{A}F = g$ in the class of functions of polynomial order of growth. The well-known Tanaka formula for Brownian local time can be written in the form (6.5), where $n = 1$, $D = \mathbf{R}$, $y(t) = a + w(t)$, $\Gamma = \{0\}$, $F(x) = 2x^+$.

7. Local time continuity.

THEOREM 7.1. *Let the assumptions of Theorem* 6.1 *be fulfilled and let the notation of this theorem be retained. Let there exist a collection of hypersurfaces* $\{\Gamma(t)\} = \{\Gamma_h(t)\}$, $\Gamma_h(t) = \Gamma_0(t) + h$, $h \in \Delta$, *where* $\Delta \subset \mathbf{R}^n$ *is some open set*, $\Gamma_h(t) \subset D$ $(\forall\, h)$, $0 \in \Delta$. *Let* $\hat{\mathbf{t}}(T) = \hat{\mathbf{t}}(T, h)$ *be the corresponding variables for* $\Gamma(t) = \Gamma_h(t)$, $h \in \Delta$. *Then* $\mathbf{E}\,|\hat{\mathbf{t}}(T,h) - \hat{\mathbf{t}}(T,0)|^2 \longrightarrow 0$ *for* $h \to 0$.

Proof. Let $g(t) = g^h(t)$, $G_j^h(x,t)$ be the corresponding functions (4.2), $V^h = L(0)g^h$, $W^h = V^h - V^0$. We have

$$g^h(t) = \sum_{j=1}^n \frac{\partial G_j^h}{\partial x_j}\,(x,\,t).$$

Obviously, $G_j^h(x,t) \equiv G_j^0(x-h,t)$, $\|G_j^h - G_j^0\|_{L_2(Q)} \longrightarrow 0$ as $h \to 0$, $j = 1,\dots,n$, then $\|g^h - g^0\|_{X^{-1}} \to 0$. By Theorems 3.1 and 6.1,

$$\begin{aligned}\mathbf{E}\left|\hat{\mathbf{t}}(T,\,h) - \hat{\mathbf{t}}(T,\,0)\right|^2 &\leqq c\mathbf{E}\left|V^h(a,\,0) - V^0(a,\,0)\right|^2 + c\,\mathbf{E}\int_0^{\tau_D\wedge T}\left|\frac{\partial W^h}{\partial x}\,\big(y(t),\,t\big)\right|^2 dt\\ &\leqq c\int_D \left\|W^h(x,\,0)\right\|^2\rho(x)\,dx + c\int_Q\left|\frac{\partial W^h}{\partial x}\,(x,\,t)\right|^2 p(x,\,t)\,dx\,dt\\ &\leqq c\,\|p\|_{L_\infty(Q)}\|W^h\|_{Y^1} \leqq c\|p\|_{L_\infty(Q)}\|g^h - g^0\|_{X^{-1}} \longrightarrow 0\end{aligned}$$

for some constant $c > 0$. Theorem 7.1 is proved.

THEOREM 7.2. *Let the assumptions and notation of Theorem* 6.1 *be retained. Let* $T_0 \in [0,T]$, $T_1 \in [0,T]$. *Then* $\mathbf{E}\,|\hat{\mathbf{t}}(T_1) - \hat{\mathbf{t}}(T_0)|^2 \longrightarrow 0$ *for* $T_1 \to T_0$.

Proof. Let $\varphi_0 \in X^{-1}$, $\varphi_1 \in X^{-1}$, $\varphi_i(t) = g(t)$ for $t \leqq T_i$, $\varphi_i(t) = 0$ for $t > T_i$, $i = 0,1$. Let $V_i = L(0)\,\varphi_i$, $W = V_1 - V_0$. Obviously, $\|\varphi_1 - \varphi_0\|_{X^{-1}} \longrightarrow 0$. By Theorem 3.1, $\|W\|_{Y^1} \to 0$. From here on, the proof of Theorem 7.2 is similar to that of Theorem 7.1.

THEOREM 7.3. *Let the assumptions of Theorem* 6.2 *be fulfilled. Then there exists a version of* $\hat{\mathbf{t}}(T)$ *a.s. continuous with respect to* T.

The proof is an immediate corollary of (6.5).

8. Diffusion degeneration. The author's papers [7], [13] deal with a number of questions connected with the limiting properties of processes under diffusion degeneration (including distributions of functionals of nonsmooth functions [13] and moments of the first exit from the domain [7]). Here the approach of [7], [13] is applied to the limiting properties of a local time in the case of diffusion tending to zero. It is established that the local time of random paths tends in a certain sense toward the local time of the limiting smooth function after averaging over the initial values (although the paths always remain stochastically nonsmooth and no averaging is performed with respect to the Wiener process).

Let us consider the collection of diffusion coefficients

$$\beta = \beta_\delta, \qquad \beta_\delta(x,\,t) \equiv \sqrt{\delta}B(x,\,t), \tag{8.1}$$

where the number $\delta \geqq 0$, $B(x,t)$: $\mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}^{n\times n}$ is a matrix function such that $B(x,t)\,B(x,t)^{\mathrm{T}} \geqq I$. We assume that, for $\delta > 0$, all the assumptions of section 1 are fulfilled, $D = \mathbf{R}^n$, the derivatives $\partial^m f(x,t)/\partial x^m$, $\partial^m \beta(x,t)/\partial x^m$ are bounded, $m = 1, 2, 3, 4$, a random initial vector a has a distribution density $\rho \in L_\infty(\mathbf{R}^n)$, and $B(x,t)$ is a continuous function. It is assumed that the hypersurface $\Gamma(t) \subset \mathbf{R}^n$ depends on time and the assumptions of Theorem 5.1 are fulfilled for it, $g(t)$ is the same element $\mathcal{W}^*$ as in Theorem 4.1, and $g \in X^{-1}$.

For $\delta > 0$ we denote by $\hat{\mathbf{t}}_\delta(T)$ the variable introduced in Theorem 6.1 and use the notation $y_\delta(t)$, $L_\delta(0)$, $\mathcal{M}_\delta(0)$ for the corresponding processes $y(t)$ and the operators $L(0)$, $\mathcal{M}(0)$ introduced in sections 3 and 6.

In particular, the operator $\mathcal{M}_\delta(0)$: $X^0 \longrightarrow H^0$ assigns the function $V(\cdot,0) \in H^1$, where $V = L_\delta(0)\,\varphi$, $\delta > 0$, to the function φ. Passing from parabolic equations to first order equations, we introduce similar operators for $\delta = 0$. By Theorem 4.2.1 of [14] (referring to more general equations), the operator $\mathcal{M}_0^*(0)$: $H^1 \to X^1$ is continuous. We define the operator $\mathcal{M}_0(0)$: $X^{-1} \to H^{-1}$ as its conjugate.

Similarly, by Theorem 4.2.1 of [14] the operators $L_\delta^*(0)$: $X^k \to X^k$, $\mathcal{M}_\delta^*(0)$: $H^k \to X^k$ are continuous and their norms are bounded with respect to $\delta \in [0,1]$, $k = 0,1,2$. For the conjugate operators this implies the following.

Proposition 8.1. *The operators $L_\delta(0)$: $X^{-k} \to X^{-k}$, $\mathcal{M}_\delta(0)$: $X^{-k} \to H^{-k}$ are continuous and their norms are bounded with respect to $\delta \in [0,1]$, $k = 0,1,2$.*

Let $g_\varepsilon(x,t) = \mathrm{Ind}\{x \in \Gamma(t,\varepsilon)\}$ as above. We denote

$$u_{\delta,\varepsilon} = \mathcal{M}_\delta(0)\,g_\varepsilon, \qquad u_{\delta,0} = \mathcal{M}_\delta(0)\,g. \tag{8.2}$$

Lemma 8.1. (a) *The norms $\|u_{\delta,\varepsilon}\|_{H^{-1}}$ are bounded uniformly with respect to $\varepsilon \in (0,1]$, $\delta \in [0,1]$.*

(b) *For $V_{\delta,\varepsilon} = L_\delta(0)\,g_\varepsilon$, $V_\delta = L_\delta(0)\,g$ the norms $\|V_{\delta,\varepsilon}(\cdot,t)\|_{H^{-1}}$, $\|V_\delta(\cdot,t)\|_{H^{-1}}$ are bounded uniformly in $t \in [0,T]$, $\varepsilon \in (0,1]$, $\delta \in [0,1]$.*

Proof. For fixed $t = 0$ statements (a) and (b) follow immediately from Proposition 8.1; statement (b) for all t is obtained by a change of the initial time.

Denote by $y_0^{x,0}(t)$ a solution of equations (1.1), (1.2) with $\beta \equiv 0$ and the initial condition $y(0) = x$, $x \in \mathbf{R}^n$.

Proposition 8.2. *The equality*

$$\frac{1}{\varepsilon}\int_0^T \mathrm{Ind}\big\{y_0^{x,0}(t) \in \Gamma(\varepsilon,\,t)\big\}\,dt = u_{0,\varepsilon}(x) = V_{0,\varepsilon}(x,\,0)$$

holds in $H^0 = L_2(\mathbf{R}^n)$. Moreover, for $\delta > 0$ we have

$$\frac{1}{\varepsilon}\,\mathbf{E}\int_0^T \mathrm{Ind}\big\{y_\delta(t) \in \Gamma(\varepsilon,\,t)\big\}\,dt = (\rho,\,u_{\delta,\varepsilon})_{H^0}.$$

(Here the first equality follows from Theorem 1.1 of [13], the second one from our Theorem 5.3.)

Theorem 8.1. *$u_{\delta,\varepsilon} \to u_{0,0}$ weakly in H^{-1} as $\delta \to 0$, $\varepsilon \to 0$.*

By Proposition 8.1 and Theorem 8.1 it is evident that the following definition is natural.

Definition 8.1. *For the collection of paths $\{y_0^{x,0}(t)\}_{x\in\mathbf{R}^n}$, we call $u_{0,0} \in H^{-1}$ a* local sojourn time *on Γ up to time T.*

Note that if $u_{0,0} \in H^0$ is fulfilled, then $u_{0,0} = u_{0,0}(x)$ is a Lebesgue measurable function of x. We shall give an example in which $u_{0,0} \in H^0$; the local time is given for any $x = y_0^{x,0}(0)$ and is not a generalized but a bounded measurable function of x.

Example 8.1. Let $n = 1$, $D = \mathbf{R}$, $\Gamma(t) \equiv \{0\}$. It is assumed that the functions $f(x,t)$, $B(x,t)$ are measurable and bounded together with all derivatives with respect to x, $|B(x,t)| \geqq 1$, the number $\delta \geqq 0$, the function $f(\cdot,t)$: $[0,T] \to C^1(\mathbf{R})$ is piecewise-continuous, $|f(0,t)| \geqq c_1$ $(\forall\, t)$, where $c_1 > 0$ is a constant. Then $u_{0,0}(x) = \sum_{k=1}^N s(\theta_k)\,|f(0,\theta_k)|^{-1}$, where N is the number of visits of zero by the process $y_0^{x,0}(t)$ for $t \in [0,T]$, θ_k are visit times, $k = 1,\dots,N$, $s(t) = 1$ for $t \in (0,T)$, $s(0) = s(T) = \frac{1}{2}$.

Proof of Theorem 8.1.

Lemma 8.2. *The sequence $u_{\delta,0}$ converges weakly to $u_{0,0}$ as $\delta \to 0$ in the space H^{-1}.*

Proof. We introduce the set $B^+ = \{\xi \in H^1\colon \xi(x) \geqq 0,\ \|\xi\|_{L_1(\mathbf{R}^n)} = 1\}$. For $\xi \in B^+$ denote $p_\delta = \mathcal{M}_\delta^*(0)\xi$. Note that the function $p_\delta(x,t)$ is the distribution density of the solution $y(t) = y_\delta(t)$ of equations (1.1), (1.2) provided that the vector $y(0)$ has the distribution density $\xi(x)$ with (8.1) taken into account. As is known, $\mathbf{E}\,\sup_t |y_\delta(t) - y_0(t)|^2 \longrightarrow 0$. Hence it follows that $p_\delta \to p_0$ weakly in X^0 as $\delta \to 0$. By Proposition 8.1, $\|p_\delta\|_{X^1} \leqq \text{const.}$ $(\forall\,\delta \in [0,1])$. Therefore,

$$p_\delta \to p_0 \quad \text{weakly in } X^1 \text{ as } \delta \to 0. \tag{8.3}$$

Hence it follows that

$$(u_{\delta,0} - u_{0,0},\, \xi)_{H^0} = (g,\, p_\delta - p_0)_{X^0} \longrightarrow 0 \tag{8.4}$$

holds for any $\xi \in B^+$. The linear span of B^+ is dense in H^1; hence, by Lemma 8.1 and relation (8.4), we obtain Lemma 8.2.

Let us continue the proof of Theorem 8.1. By virtue of Lemma 8.1 it is sufficient to prove that $u_{\delta,\varepsilon} \to u_{0,0}$ weakly in H^{-2} as $\delta \to 0$, $\varepsilon \to 0$. Let $\xi \in H^2$, $u_{\delta,1}$, $u_{\delta,0}$ be given by (8.2). We have

$$(u_{\delta,\varepsilon} - u_{0,0}, \xi)_{H^0} = R_1(\delta) + R_2(\delta,\, \varepsilon),$$

where

$$R_1(\delta) = (u_{\delta,0} - u_{0,0},\, \xi)_{H^0}, \qquad R_2(\delta,\, \varepsilon) = (u_{\delta,\varepsilon} - u_{\delta,0},\, \xi)_{H^0}.$$

We have $R_1(\delta) \to 0$ for $\delta \to 0$ by Lemma 8.1,

$$\begin{aligned} R_2(\delta,\, \varepsilon) &= \big(\mathcal{M}_\delta(0)\,(g_\varepsilon - g),\, \xi\big)_{H^0} = \big(g_\varepsilon - g,\, \mathcal{M}_\delta^*(0)\,\xi\big)_{H^0} \\ &\leqq \|g_\varepsilon - g\|_{X^{-1}} \big\|\mathcal{M}_\delta^*(0)\,\xi\big\|_{X^1} \leqq \text{const.}\, \|g_\varepsilon - g\|_{X^{-1}} \|\xi\|_{H^1}. \end{aligned}$$

Theorem 8.1 is proved.

Let $\mathcal{F}_a$, $\mathcal{F}_W$ be the σ-algebras of events generated by the initial vector a in (1.1) and the Wiener process $w(s)$, $s \in [0,T]$, respectively; let $(\Omega, \mathcal{F}, \mathbf{P})$ be the initial probability space, $\Omega = \{\omega\}$.

Let $V_\delta = L_\delta(0)g$, $\beta_\delta^{(j)}$ be the columns of the matrix β_δ. By Theorem 6.1 we have

$$\hat{\mathbf{t}}_\delta(T) = V_\delta(a,0) + \sum_{j=1}^n \int_0^T \frac{\partial V_\delta}{\partial x}\,\big(y_\delta(t),\, t\big)\, \beta_\delta^{(j)}\big(y_\delta(t),\, t\big)\, dw_j(t). \tag{8.5}$$

Denote

$$\eta_\delta = \hat{\mathbf{t}}_\delta(T) - \mathbf{E}\,\big\{\hat{\mathbf{t}}_\delta(T) \mid \mathcal{F}_a\big\}, \qquad \overline{\eta}_\delta = \mathbf{E}\,\big\{\eta_\delta \mid \mathcal{F}_W\big\}. \tag{8.6}$$

From (8.5) we have

$$\eta_\delta = \sum_{j=1}^{n} \int_0^T \frac{\partial V_\delta}{\partial x} \big(y_\delta(t),\, t\big)\, \beta_\delta^{(j)}\big(y_\delta(t),\, t\big)\, dw_j(t). \tag{8.7}$$

THEOREM 8.2. (a) *For any* $\rho \in H^1$ *we have*

$$\mathbf{E}\,\hat{\mathbf{t}}_\delta(T) \longrightarrow (u_{0,0},\, \rho)_{H^0} \quad \text{as } \delta \to 0. \tag{8.8}$$

(b) *Let* $\rho \in H^2$. *Then,*

$$\mathbf{E}\,|\overline{\eta}_\delta|^2 \leqq \delta \cdot \text{const.}. \tag{8.9}$$

Proof. Relation (8.8) follows from (8.5) and Lemma 8.2. Let us prove (8.9). Let $p(x,t,\omega)$ be a density of conditional (given $\mathcal{F}_W$) distribution of the process $y_\delta(t)$, i.e., for any domain $G \subset \mathbf{R}^n$,

$$\int_G p(x,\, t,\, \omega)\, dx = \mathbf{P}\big\{y_\delta(t) \in G\big\}.$$

Recall that $\mathcal{C}^k = C([0,T] \to H^k)$.

PROPOSITION 8.3. *Let the vector* $a = y_\delta(0)$ *be independent of* $w(t)$ *and have a distribution density* $\rho \in H^0$. *Then there exists a density of conditional* (*given* $\mathcal{F}_W$) *distribution of the process* $y_\delta(t)$: $p = p(x,t,\omega) \in L^2(\Omega, \mathcal{F}, \mathbf{P}, \mathcal{C}^0)$.

If $\rho \in H^k$, *then* $p(x,t,\omega) \in L^2(\Omega, \mathcal{F}, \mathbf{P}, \mathcal{C}^k)$, *and*

$$\mathbf{E}\,\big\|p(\cdot,\,\omega)\big\|_{\mathcal{C}^2}^2 \leqq \text{const.}\,\mathbf{E}\,\|\rho\|_{H^k}^2, \quad k = 0,\,1,\,2.$$

This proposition follows from Theorem 2.2 of [10]; the latter estimate holds by Theorem 4.2.1 of [14] due to the properties of the parabolic equation for p derived in Theorem 5.3.1 of [14].

Let us continue the proof of Theorem 8.2. We have

$$\overline{\eta}_\delta = \sum_{j=1}^{n} \int_0^T dw_j(t) \int_{\mathbf{R}^n} \frac{\partial V_\delta}{\partial x}\,(x,\, t)\, \beta_\delta^{(j)}(x,\, t)\, p(x,\, t,\, \omega)\, dx,$$

$$\begin{aligned}
\mathbf{E}\,|\overline{\eta}_\delta|^2 &= \mathbf{E} \sum_{j=1}^{n} \int_0^T dt \int_{\mathbf{R}^n} \left| \frac{\partial V_\delta}{\partial x}\,(x,\, t)\, \beta_\delta^{(j)}(x,\, t)\, p(x,\, t,\, \omega) \right|^2 dx \\
&\leqq \text{const.}\ \mathbf{E} \sum_{j=1}^{n} \int_0^T \big\|V_\delta(\cdot,\, t)\big\|_{H^{-1}}^2 \big\|\beta_\delta^{(j)}(x,\, t)\, p(x,\, t,\, \omega)\big\|_{H^2}^2\, dt \\
&\leqq \delta\, \text{const.}\ \mathbf{E}\,\big\|p(\cdot,\, \omega)\big\|_{X^2}^2 \leqq \delta\, \text{const.}\ \mathbf{E}\,\|\rho\|_{H^2}^2.
\end{aligned}$$

The inequality preceding the last one is fulfilled by virtue of (8.1) and Lemma 8.1 (a). Theorem 8.2 is proved.

Remark 8.1. Let $\rho \in W_2^k(\mathbf{R}^n)$, $k > n/2 + 2$, $\delta > 0$ be fixed, the derivatives $\partial^m f/\partial x^m$, $\partial^m \beta/\partial x^m$ be bounded, $m = 1, \dots, k+2$. By the approach used above for Theorem 8.2 one can prove that for $\varepsilon \to +0$ there exists in the space $L^2(\Omega, \mathcal{F}, \mathbf{P})$ a limit of the variables $\bar{l}_\varepsilon(x,t)$, where $x \in \mathbf{R}^n$, $t > 0$,

$$\bar{l}_\varepsilon(x,\, t) = \mathbf{E}\,\big\{l_\varepsilon(x,\, t) \mid \mathcal{F}_W\big\}, \qquad l_\varepsilon(x,\, t) = \frac{1}{\varepsilon^n} \int_0^t \text{Ind}\Big\{\big|y(r) - x\big| < \varepsilon\Big\}\, dr.$$

(Note that for $n > 1$ no limit of the variables $l_\varepsilon(x,t)$ exists without conditional averaging, since, as is known, there does not exist a local time of a multi-dimensional process at this point.)

To conclude, we would like to note that, based on Theorem 3.1, control problems with functionals depending on a local time were solved in [15], [16]. Moreover, Theorems 4.1 and 5.1 enable one to interpret the problems solved in [17] as local time control problems.

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