Explicit formulas for currents at branching long lines and for maximum of current amplitudes

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Abstract: A system of 'telegrapher's' equations for a number of long lines joined into a network is studied. Explicit formulas for Fourier transforms of current and voltage are derived. These formulas are very suitable for computer application as well as for the analytical study of processes on networks. As an example, the availability of formulas aids the derivation of explicit formulas for maxima of current amplitude over the given class of admissible external influences. These values may be used to indicate the characteristic of network robustness to excess voltage or electromagnetic impulse. The approach is based on the operational solution already proposed by the author for more general partial differential equations on graphs.

1 Introduction

The classical 'telegrapher's' equation describes the evolution of current in a long line [1]. There are a lot of technical and physical objects that are modelled by a system (network) of a number of jointed long lines (a network of underground pipes, a system of groundings, an ordinary electrical network of two-wire transmission lines with consumers and sources etc.). In this case the 'telegraph equations' are joined into a system with corresponding boundary value conditions. All the existing methods of calculating the currents in these systems are based on the finite-dimensional approximation of the continuous long lines (finite-elements methods, etc.), and so there is a loss of precision in this approximation. These methods usually need a prior manual analysis of the network topology. For some particular kinds of problem (when interaction between the processes at the different branches is realised only in a form of Kirchhoff's law at the nodes where branches are connected), we obtain a new method based on the approach [2] for the partial differential equations on graphs. We derive explicit formulas for the Fourier transforms of the currents and voltages in the branches of the branching network. These formulas give an exact solution of the system of 'telegrapher's equations'; the corresponding algorithm is suitable for computer applications. The input data are provided directly by the network topology, without the necessity for prior manual analysis to obtain additional equations for boundary-value descriptions. The availability of the formulas for the currents is used to derive the explicit formulas for current amplitude maximum being achieved over all the external influences (inputs) satisfying the constraints for input energy or input value at every point. This maximum may be used as a characteristic of network robustness to excess voltage.

Let us consider a system of n long lines with lengths $l_1, \ldots, l_n$ that are connected to a network (graph with branches and nodes). The equation for the current and voltage in each branch with number $k = 1, \ldots, n$ is [1]

$$\left.\begin{aligned} \frac{\partial U_k}{\partial y}(y, \omega) &= -(R_k + i\omega l_k) I_k(y, \omega) + e_k(y, \omega) \\ \frac{\partial I_k}{\partial y}(y, \omega) &= -(Y_k + i\omega C_k) U_k(y, \omega) + \delta(y - \bar{y}_k)\bar{I}_k(\omega) \end{aligned}\right\} \quad (1)$$

In eqn. 1, $I_k(y, \omega)$, $U_k(y, \omega)$ are the Fourier transforms of current and voltage in the kth branch, $y \in [0, l_k]$ is a longitudinal co-ordinate (we have chosen the orientation for every branch), ω is an angular frequency, $\omega = \mathbb{R}$. The constants R_k, l_k are the series resistance and inductance per unit length. The constants Y_k, C_k are the parallel conductance and capacitance per unit length. The action of the external electric field is characterised by the functions $e_k(y, \omega)$; $\bar{I}_k(\omega)$ are the external currents flowing into the given points $\bar{y}_k \in [0, l_k]$; and δ is a delta function.

We assume that interaction between the processes at the different branches is realised only at the nodes where branches are connected, and so all the eqns. 1 for $k = 1, \ldots, n$ are joined into the system with supplementary boundary-value conditions in the form of the Kirchhoff's current and voltage laws at joining nodes. Ohm's law holds at the terminal nodes that belong to only one branch, so that $I_k(0, \omega) = \bar{Y}_k U(0, \omega)$ or $I_k(l_k, \omega) = \bar{Y}_k U_k(l_k, \omega)$, where the constants $\bar{Y}_k$ are the terminating admittances.

We suppose that

(i) Nonzero R_k or Y_k exist for some k.

(ii) $|e_k(y, \omega)| < c$ for some constant $c > 0$ for every k, y, ω.

2 Formulas for currents and voltages

Our method is taken from Reference 2. The main idea is to describe the current and voltage distribution by the complex $2n$-vector

$$z(x, \omega) = \text{col}\,[U_1(l_1 x, \omega), I_1(l_1 x, \omega), \ldots, U_n(l_n x, \omega), I_n(l_n x, \omega)] \quad (2)$$

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defined for every $x \in [0, 1]$, $\omega \in \mathbb{R}$. This vector is convenient because we can write the boundary value problem for it:

$$\frac{\partial z}{\partial x}(x, \omega) = A(x, \omega)z(x, \omega) + E(x, \omega),$$

$$B_0 z(0, \omega) + B_1 z(1, \omega) = 0 \quad (3)$$

Here the external influence is

$$E(x, \omega) = \text{col}\,[l_1 e_1(l_1 x, \omega), \delta(l_1 x - \bar{y}_1)\bar{I}_1(\omega), \ldots, e_n(l_n x, \omega), \delta(l_n x - \bar{y}_n)\bar{I}_n(\omega)] \quad (4)$$

In eqn. 3, $A(\omega) = \text{diag}\,[A_1(\omega), \ldots, A_n(\omega)]$ is a complex $2n \times 2n$-matrix, where 2×2 matrices

$$A_k(\omega) = l_k \begin{pmatrix} 0 & -R_k - i\omega L_k \\ -Y_k - i\omega C_k & 0 \end{pmatrix}, \quad k = 1, \ldots, n$$

This means that the matrices A_k are placed on the main diagonal of A and all the remaining elements of A are zero.

The $2n \times 2n$-matrices B_0 and B_1 contain complete information about the topology of our network; we shall give the algorithm for their construction below:

The explicit final formula for the Fourier transform of the current and voltage is

$$z(x, \omega) = \int_0^1 G(x, \rho, \omega)E(\rho, \omega)\, d\rho \quad (5)$$

together with eqn. 2.

In eqn. 5, $G(x, \rho, \omega)$ is the Green's function for the problem of eqn. 3 defined [3] for $x \in [0, 1]$, $\rho \in [0, 1]$, $\omega \in \mathbb{R}$ as a complex $2n \times 2n$ matrix

$$G(x, \rho, \omega) = \begin{cases} -\Phi(x, \omega)[B_0 + B_1\Phi(1, \omega)]^{-1}B_1\Phi(1 - \rho, \omega) \\ \quad \text{for } 0 \leqslant x < \rho \\ \Phi(x, \omega)[B_0 + B_1\Phi(1, \omega)]^{-1}B_0\Phi(-\rho, \omega) \\ \quad \text{for } \rho < x \leqslant 1 \end{cases} \quad (6)$$

Note that the $2k$th component of eqn. 5 is undefined at the point $x = l_{2k}^{-1}\bar{y}_{2k}$, and has a jump discontinuity at this point for $k = 1, \ldots, n$ (see eqn. 1).

The matrix $\Phi(x, \omega) = \exp\,[A(\omega)x]$ is the fundamental matrix of the first eqn. 3: $\Phi(x, \omega) = \text{diag}\,[\Phi_1(x, \omega), \ldots, \Phi_n(x, \omega)]$, where

$$\Phi_k(x, \omega) = \begin{pmatrix} \cosh(\gamma_k x) & -\dfrac{\gamma_k}{\zeta_k}\sinh(\gamma_k x) \\ -\dfrac{\gamma_k}{\xi_k}\sinh(\gamma_k x) & \cosh(\gamma_k x) \end{pmatrix}$$

$\xi_k = l_k(R_k + i\omega L_k)$, $\zeta_k = l_k(Y_k + i\omega C_k)$, $\gamma_k = \sqrt{(\xi_k \zeta_k)}$ (we take the principal value for the root and assume that $\sinh 0/0 = 1$).

The topology of the network is defined by the ordered set $\{s_m, p_m, r_m, (i_1, \ldots, i_{p_m}), (j_1, \ldots, j_{r_m})\}_{m=1}^q$. Here q is the number of nodes, p_m is the number of branches with numbers $i_1, \ldots, i_{p_m}$ exiting from the mth nodes, r_m is the number of the branches with numbers $j_1, \ldots, j_{p_m}$ entering into the mth nodes, $s_m = p_m + r_m$.

We construct the matrices B_0 and B_1 by joining all the strings of $s_m \times n$-matrices $Q_m = Q_m(i, j)$, $P_m = P(i, j)$, $m = 1, \ldots, q$, $i = 1, \ldots, s_m$, $j = 1, \ldots, n$:

$$B_0 = \begin{vmatrix} P_1 \\ \vdots \\ P_q \end{vmatrix} \quad B_1 = \begin{vmatrix} Q_1 \\ \vdots \\ Q_q \end{vmatrix}$$

where Q_m and P_m are constructed by the following algorithm.

(i) Let $s_m = 1$. If $p_m = 1$, then $P_m(1, 2i_1 - 1) = \bar{Y}_k$ and $P_m(1, 2i_1) = -1$; if $r_m = 1$ then $Q_m(1, 2j_1 - 1) = \bar{Y}_k$ and $Q_m(1, 2j_1) = -1$. All the remaining components of P_m and Q_m must be zero.

(ii) Let $S_m > 1$. If $r_m > 0$, then $Q_m(p_m + d, 2j_d - 1) = -Q_m(p_m + d, 2j_{d+1} - 1) = 1$ for $d = 1, \ldots, r_m - 1$, and $Q_m(s_m, 2j_1 - 1) = Q_m(s_m, 2j_2) = \cdots = Q_m(s_m, 2j_{r_m}) = 1$. If $p_m > 0$, then $P(s, 2i_s - 1) = -P_m(s, 2i_{s+1} - 1) = 1$ for $s = 1, \ldots, p_m - 1$ and $P_m(s_m, 2i_1) = P_m(s_m, 2i_2) = \cdots = P_m(s_m, 2i_{p_m}) = 1$. If $r_m > 1$ and $p_m > 1$, then $P_m(p_m, 2i_{p_m} - 1) = -Q_m(p_m, 2j_1 - 1) = 1$. All the remaining components of P_m and Q_m must be zero.

Let us discuss the correctness of our formulas. The existence of nonzero R_k or Y_k is sufficient for matrix $[B_0 + B_1\Phi(1, \omega)]$ to be invertible, and for the problem of eqn. 3 to be well posed [3]. Let us show this. Let $z_0 \in \mathbb{C}^n$ and $[B_0 + B_1\Phi(1, \omega)]z_0 = 0$, then $z(x, \omega) = \Phi(x, \omega)z_0$ satisfies eqn. 3 with $E(x, \omega) \equiv 0$. There is an energy dissipation in eqn. 1 for k with nonzero R_k or Y_k, and so the corresponding evolution system is stable in the temporal domain. Thus $z_0 = 0$, and the problem in eqn. 3 is well posed.

Integral eqn. 5 exists because of supposition (ii).

3 Maximum of current amplitude

Having obtained explicit formulas for currents, we can derive the explicit formulas for the maximum current amplitude at a branching network. This problem is important for applications, because we can use this value for the characterisation of the robustness to excess voltage or electromagnetic impulse. We think that this problem has no explicit solution other than our explicit formulas for currents.

In eqn. 1, the external influences are characteristised for every $\omega \in \mathbb{R}$ by the ordered set $F(\omega) = \{e_k(., \omega), \bar{I}_k(\omega), \bar{y}_k\}_{k=1}^n$.

Fix $a_k \geqslant 0$, $b_k \geqslant 0$, $0 \leqslant \alpha_k \leqslant \beta_k \leqslant l_k$ for $k = 1, \ldots, n$. Let us introduce the sets $F_p(\omega) = \{F(\omega)\}$ being the classes of admissible external influences:

(i) For $1 \leqslant p < +\infty$, class $F_p(\omega)$ is the set of such $F(\omega) = \{e_k(., \omega), \bar{I}_k(\omega), \bar{y}_k\}_{k=1}^n$ that

$$\left(\int_0^{l_k} |e_k(y, \omega)|^p\, dy\right)^{1/p} \leqslant a_k, \quad |\bar{I}_k(\omega)| \leqslant b_k, \quad \bar{y}_k \in [\alpha_k, \beta_k]$$

$$\text{for every } k = 1, \ldots, n$$

(ii) Class $F_\infty(\omega)$ is the set of such $F(\omega) = \{e_k(., \omega), \bar{I}_k(\omega), \bar{y}_k\}_{k=1}^n$ that

$$\sup_{y \in [0, l_k]} |e_k(y, \omega)| \leqslant a_k, \quad |\bar{I}_k(\omega)| \leqslant b_k, \quad \bar{y}_k \in [\alpha_k, \beta_k]$$

$$\text{for every } k = 1, \ldots, n$$

Our aim is to obtain the value

$$\sup_{\mathfrak{F}(\omega) \in F_p(\omega)} |I_k(y, \omega)|$$

for every $k = 1, \ldots, n$, $y \in [0, l_k]$, $1 \leqslant p \leqslant +\infty$.

The result is that, for $p \in (1, +\infty]$, we have

$$\sup_{\mathfrak{F}(\omega) \in F_p(\omega)} |I_k(y, \omega)| = \sum_{m=1}^{n} \left\{ a_k \left[\int_0^1 |G_{2k, 2m-1}(y/l_k, \rho, \omega)|^q\, d\rho \right]^{1/q} + b_k \sup_{\rho \in [\alpha_k/l_k, \beta_k/l_k]} |G_{2k, 2m}(y/l_k, \rho, \omega)| \right\} \quad (7)$$

where $q = p(p-1)^{-1}$ for every $p \in (1, \infty)$, $q = 1$ for $p = +\infty$. For $p = 1$, we have

$$\sup_{\mathfrak{F}(\omega) \in F_p(\omega)} |I_k(y, \omega)| = \sum_{m=1}^{n} \left\{ a_k \sup_{\rho \in [0, 1]} |G_{2k, 2m-1}(y/l_k, \rho, \omega)| + b_k \sup_{\rho \in [\alpha_k/l_k, \beta_k/l_k]} |G_{2k, 2m}(y/l_k, \rho, \omega)| \right\} \quad (8)$$

Here, G_{km} are the components of matrix G defined by eqn. 6.

Eqns. 7 and 8 are very suitable for computer application. Analogous results may be obtained for

$$\sup_{\mathfrak{F}(\omega) \in F_p(\omega)} |U_k(y, \omega)|$$

(the corresponding changes in eqns. 7 and 8 are obvious).

To prove eqns. 7 and 8, we have to remark that the right-hand part of eqn. 7 or eqn. 8 is the functional norm of the $2k$th string of $G(y/l_k, ., \omega)$ being presented as an element of Banach space [4], which is dual to such space of function $e_k(., \omega)$ that $\mathfrak{F}(\omega) \varepsilon F_p(\omega)$ defines a ball there.

4 References

1 WEBER, E.: 'Linear transient analysis. Vol. 2' (Wiley and Sons, New York, 1956)

2 DOKUCHAEV, N.G.: 'Operation method for the boundary value problems on graphs', *Differential Equations* [*Differentzial'nye uravnenia*], 1990, **26**, (11), pp. 2006–2008 (in Russian)

3 KRENER, A.J.: 'A causal realization theory. Part 1: Linear deterministic systems', *SIAM J. Control*, 1987, **25**, (3), pp. 499–525

4 RUDIN, W.: 'Functional analysis' (McGraw-Hill, New York, 1973)