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DISTRIBUTIONS OF ITÔ PROCESSES: ESTIMATES FOR THE DENSITY AND FOR CONDITIONAL EXPECTATIONS OF INTEGRAL FUNCTIONALS*

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(Translated by A. A. Gushchin)

Abstract. We obtain a priori estimates for the L_2-norms of solutions of parabolic Itô equations describing evolution of the distributions of solutions of ordinary Itô stochastic differential equations with random coefficients. In the case of nondegenerate equations, estimates for the L_2-norms of derivatives with respect to space variables are also obtained. As a consequence, we establish a generalization of Itô's formula for functions that have only square-summable derivatives of the first and the second order (or even of the first order).

Key words. distributions of Itô processes, parabolic Itô equations, Itô's formula

This paper is devoted to studying the probability distributions of Itô processes with the help of stochastic differential equations of parabolic type techniques [1]. The paper continues the work [2]. In the case of degenerating Itô equations we obtain integral estimates for functionals of Itô processes improving the well-known Krylov–Fichera estimates [1, § 5.2]. (In the limit case when there is no diffusion, similar estimates for ordinary differential equations were obtained in [3] and [4] and used to study control problems for ordinary differential equations.) In the nondegenerate case (different from that considered in [2]) a smoothness of functionals with respect to an initial value of the process is established. As a consequence, we establish a generalization of Itô's formula for functions that have only square-summable derivatives of the first and the second order (or even of the first order).

Close results have been announced in [5].

1. Introduction. Let us consider a N-dimensional Wiener process $w(t)$ with independent components on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, $\Omega = \{\omega\}$. The process generates the filtration of complete σ-algebras $\mathcal{F}_t = \overline{\sigma[w(s),\ s \leqq t]}$, $t \in [0, T]$, in the usual way, where $T > 0$. We assume that $w(0) = 0$.

Let us consider an n-vector Itô equation

$$dy(t,\omega) = f\Big(y(t,\omega), t, \omega\Big)\, dt + \beta\Big(y(t,\omega), t, \omega\Big)\, dw(t). \tag{1.1}$$

Denote $Q = \mathbf{R}^n \times [0, T]$. We assume that functions $f(x,t,\omega)$: $Q \times \Omega \to \mathbf{R}^n$ and $\beta(x,t,\omega)$: $Q \times \Omega \to \mathbf{R}^{n\times N}$ are measurable, progressively measurable for any $x \in \mathbf{R}^n$ and uniformly bounded with their partial derivatives in x up to the second order inclusive for f and up to the third order inclusive for β.

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Let a be a random vector that does not depend on $w(t)-w(s)$, $t \geqq s$. Denote by $y^{a,s}(t)$ a solution of (1.1) with the initial condition

$$y(s)=a. \tag{1.2}$$

We shall study the functionals

$$V(x,s,\omega)=\mathbf{E}\left\{\int_s^T \varphi\Big(y^{x,s}(t,\omega),t,\omega\Big)\,dt \mid \mathcal{F}_s\right\}, \tag{1.3}$$

where $(x,s) \in Q$ and $\varphi(x,t,\omega)$ are measurable $\mathcal{F}_t$-adapted (see a more precise definition below) functions.

Let us introduce some notation: $\mathcal{B}_m$ is the Borel σ-algebra in $\mathbf{R}^m$; λ_m is the Lebesgue measure in $\mathbf{R}^m$ and $\overline{\mathcal{B}}_m$ is the completion in the Lebesgue measure of $\mathcal{B}_m$; $\mathcal{P}$ is the σ-algebra of progressively measurable (with respect to $\mathcal{F}_t$) subsets of $[0,T]\times\Omega$; $\overline{\mathcal{P}}$ is the completion in the measure $\lambda_1\times\mathbf{P}$ of $\mathcal{P}$; $\overline{\mathcal{B}_m\otimes\mathcal{F}}$ is the completion in the measure $\lambda_1\times\mathbf{P}$ of the σ-algebra $\mathcal{B}_m\otimes\mathcal{F}$.

Introduce the Hilbert spaces H^k, $k=0,\pm1,\pm2$: for $k\geqq 0$ we put $H^k=W_2^k(\mathbf{R}^n)$, H^{-k} is the completion of H^0 in the norm $\|u\|_{H^{-k}}=((I-\Delta)^{-k}u,u)_{H_0}^{1/2}$ (these spaces are the same as in [2]).

Introduce the Hilbert spaces

$$X^k[s,T]=L^2\Big([s,T]\times\Omega,\overline{\mathcal{P}},\lambda_1\times\mathbf{P},H^k\Big),$$
$$Y^k[s,T]=L^2\Big([s,T],\overline{\mathcal{B}}_1,\lambda_1,H^k\Big),\qquad Z_s^k=L^2(\Omega,\mathcal{F}_s,\mathbf{P},H^k)$$

and the Banach spaces

$$C_k[s,T]=C\Big([s,T]\to Z_T^k\Big)$$

of functions $u(t)$: $[s,T]\to Z_T^k$ continuous in t. We assume that these spaces contain functions of (x,t,ω) (or (x,t), or (x,ω)), although, strictly speaking, they contain equivalence classes. We denote for brevity $X^k=X^k[0,T]$, $Y^k=Y^k[0,T]$, $C_0=C_0[0,T]$.

Introduce the parameters

$$\mu_f=\sup_{x,t,\omega}\Big|f(x,t,\omega)\Big|,\qquad \mu_\beta=\sup_{x,t,\omega}\Big|\beta(x,t,\omega)\Big|,$$
$$\mu_f'=\sup_{x,t,\omega}\left|\frac{\partial f}{\partial x}(x,t,\omega)\right|,\qquad \mu_\beta'=\sup_{x,t,\omega}\left|\frac{\partial\beta}{\partial x}(x,t,\omega)\right|,$$
$$\mu_\beta''=\sup_{x,t,\omega}\left|\frac{\partial^2\beta}{\partial x^2}(x,t,\omega)\right|,\qquad \mu_1=\{n,N,T,\mu_f,\mu_\beta,\mu_\beta'\},\quad \mu_2=\{\mu_f',\mu_\beta''\}.$$

(Here $|\partial\beta/\partial x|$ and $|\partial^2\beta/\partial x^2|$ are the usual Euclidean norms of the derivatives of mappings from the finite-dimensional space $\{x\}=\mathbf{R}^n$ into the corresponding finite-dimensional spaces.)

2. Degenerating equations. Let

$$b(x,t,\omega)=\tfrac12\,\beta(x,t,\omega)\,\beta(x,t,\omega)^T.$$

Let us consider the Cauchy problem for the parabolic [1] Itô equation

$$\begin{aligned} d_t\pi(x,t,\omega)&=\sum_{i,j=1}^n\frac{\partial^2}{\partial x_i\partial x_j}\Big\{b_{ij}(x,t,\omega)\,\pi(x,t,\omega)\Big\}\,dt\\ &+\sum_{i=1}^n\frac{\partial}{\partial x_i}\Big\{f_i(x,t,\omega)\,\pi(x,t,\omega)+\xi(x,t,\omega)\Big\}\,dt\\ &-\sum_{j=1}^N\sum_{i=1}^n\frac{\partial}{\partial x_i}\Big\{b_{ij}(x,t,\omega)\,\pi(x,t,\omega)\Big\}\,dw^{(j)}(t), \end{aligned} \tag{2.1}$$

$$\pi(x,s,\omega)=\rho_s(x,\omega). \tag{2.2}$$

Introduce the operators L_s^* and $\mathfrak{L}_s^*$: we assume that $\pi = L_s^*\xi + \mathfrak{L}_s^*\rho$. It is known [1] that the operators L_s^*: $X^1[s,T] \to X^1[s,T]$, $\mathfrak{L}_s^*$: $Z_s^1 \to X^1[s,T]$ and L_s^*: $X^1[s,T] \to C_0[s,T]$ are continuous. From [1, Lem. 4.2.3] it follows that the last two operators can be extended to continuous operators L_s^*: $X^0[s,T] \to C_0[s,T]$ and $\mathfrak{L}_s^*$: $Z_s^0 \to C_0[s,T]$ and there exists a constant $c_0 = c_0(\mu_1,\mu_2) > 0$ depending only on μ_1 and μ_2 such that the norms of the two last operators do not exceed c_0.

We shall denote by L_s and $\mathfrak{L}_s$ the operators dual to L_s^*: $X^0[s,T] \to X^0[s,T]$ and $\mathfrak{L}_s^*$: $Z_s^0 \to X^0[s,T]$, $\varphi|_{[s,T]}$ denotes the restriction of $\varphi \in X^0$ onto $\mathbf{R}^n \times [s,T] \times \Omega$.

THEOREM 2.1. *Let a function $\varphi(x,t,\omega)$: $Q\times\Omega \to \mathbf{R}$ be a $(\mathcal{B}_{n+1}\otimes\mathcal{F},\mathcal{B}_1)$-measurable* [6, p. 282] *function in X^0. Then the following holds for the function $V(x,s,\omega)$ defined by* (1.3) *and the function $v(x,s,\omega) = \mathfrak{L}_s(\varphi|_{[s,T]})$:*

a) *$v \in C_0$, $v = L_0\varphi$;*

b) *$V \in C_0$ and $v(x,s,\omega) = V(x,s,\omega)$ for any s for a. e.*

c) *$\|V\|_{C_0} \leqq c_0\|\varphi\|_{X_0}$, where $c_0 = c_0(\mu_1,\mu_2)$ depends only on μ_1 and μ_2 introduced in* §1.

Remark. The estimate in statement c) of the theorem resembles the Krylov–Fichera inequality [1, §5.2], which allows one to estimate the norm of the function $\mathbf{E}V(x,t,\omega)$ in $L_p(Q)$ for nonrandom f,β,φ.

Proof. One may easily see that $v = L_0\varphi$ and, if $\xi \equiv 0$,

$$\Big(v(\cdot,s,\cdot),\,\rho_s(\cdot)\Big)_{H^0} = (\varphi,p)_{X^0[s,T]}$$

for $p = \mathfrak{L}_s^*\rho_s$. Since the choice of ρ is arbitrary, we obtain that

$$\Big(v(x,s,\omega),\,\rho_s(x,\omega)\Big)_{H^0} = \mathbf{E}\bigg\{\int_s^T \Big(\varphi(x,t,\omega),\,p(x,t,\omega)\Big)_{H^0} dt \mid \mathcal{F}_s\bigg\} \quad \text{a. s.} \tag{2.3}$$

If $\rho_s \equiv 0$,

$$(v,\xi)_{X^0[s,T]} = (\varphi,\pi)_{X^0[s,T]} \tag{2.4}$$

for $\pi = L_s^*\xi$ (here and below $(\cdot,\cdot)$ stands for the scalar product in the corresponding Hilbert space). From (2.3) one can also deduce the estimate

$$\sup_t\Big(\mathbf{E}\Big\|v(x,t,\omega)\Big\|_{H^0}^2\Big)^{1/2} \leqq c_0\|\varphi\|_{X^0}$$

with the constant $c_0 = c_0(\mu_1,\mu_2)$ introduced before the statement of the theorem.

Let us establish that $v \in C_0$.

It is enough to show that $L_0\varphi \in C_0$ ($\forall\,\varphi \in \mathfrak{X}$) for some set $\mathfrak{X} \subset X^0$ that is everywhere dense in X^0. Let Π: $C_0 \to C_0$ be the operator such that, for $u \in C_0$, $(\Pi u)(t)$ is the projection of $u(t) \in Z_T^0$ onto Z_t^0 for all t. Let $\overline{T}$: $X^0 \to C_0$ be the operator which maps a function g into a solution $u = \overline{T}g$ of the problem

$$\frac{\partial u}{\partial t} + Au = -g, \qquad u|_{t\in T} = 0.$$

Define the operator R: $X^0 \to X^0$ by assuming that $R^*\pi = h$ for the dual operator R^*, where $h = \pi - z$ and z is a solution of the problem

$$\begin{aligned} d_t z(x,t,\omega) &= A^*(x,t,\omega)\,z(x,t,\omega)\,dt \\ &\quad + \sum_{j=1}^N\sum_{i=1}^n \Big\{b_{ij}(x,t,\omega)\Big[\pi(x,t,\omega) - z(x,t,\omega)\Big]\Big\}\,dw_j(t), \qquad z(x,0,\omega) = 0, \end{aligned}$$

(the operators $\overline{\mathcal{T}}$ and R have been introduced and investigated in [2]). Similar to [2], L_0 may be represented as the superposition $L_0 = \mathcal{T}R$, where $\mathcal{T} = \Pi\overline{\mathcal{T}}$: $X^0 \to C_0$ is a continuous operator and $\mathcal{T}(X^0) \subset C_0$. From the form of R^* and [1, Thm. 4.2.1] we obtain that the set $\mathfrak{X} = \{\varphi \in X^0\colon\ R\varphi \in X^0\}$ is the one required: it is dense in X^0 and $L_0\varphi \in C_0$ $(\forall\, \varphi \in \mathfrak{X})$.

Let $\rho_s \in Z_s^1$ be the conditional (relative to $\mathcal{F}_s$) density of some vector a, which does not depend on $w(t) - w(s)$, $t \geqq s$; this means that ρ_s is the Radon–Nikodym derivative of the regular conditional (relative to $\mathcal{F}_s$) distribution of a with respect to the Lebesgue measure (see [1, §5.3]). It is known [1] that in this case $p(x,t,\omega) = \mathfrak{L}_s^*\rho_s$ is the conditional (relative to $\mathcal{F}_t$) density of the distribution of the solution of the equations (1.1)–(1.2) (in the case $\mathbf{E}|a|^2 = +\infty$ a solution of the equation is understood in the sense of [7, §1, p. 4]). Without loss of generality, we shall assume that a is a random vector on the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$, where $\widetilde{\Omega} = \Omega \times \Omega'$, $\Omega' = \mathbf{R}^n$, $\widetilde{\mathcal{F}} = \overline{\mathcal{F} \otimes \mathcal{B}_n}$ and

$$\widetilde{\mathbf{P}}(\Gamma_1 \times \Gamma_2) = \int_{\Gamma_1} \mathbf{P}(d\omega)\, \mathbf{P}'(\omega, \Gamma_2), \qquad \text{where} \quad \mathbf{P}'(\omega, \Gamma_2) = \int_{\Gamma_2} \rho_s(x,\omega)\, dx$$

for $\Gamma_1 \in \mathcal{F}$ and $\Gamma_z \in \mathcal{B}_n$. The symbol $\widetilde{\mathbf{E}}$ denotes expectation in $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}})$. We suppose that $\widetilde{\omega} = \|\omega, \omega'\|$, $\widetilde{\Omega} = \{\widetilde{\omega}\}$ and $a(\widetilde{\omega}) = \omega'$.

The function $\varphi(y^{\omega',s}(t,\widetilde{\omega}), t, \omega)$ is $(\overline{\mathcal{B}}_{n+1}, \mathcal{B}_1)$-measurable as a function of ω', t for a. e. ω and $(\widetilde{\mathcal{F}}, \mathcal{B}_1)$–measurable as a function of $\widetilde{\omega} = \|\omega, \omega'\|$ for a. e. t. This follows from the fact that the vector $y^{a,s}(t)$ has a density and a $\mathcal{F}_t$-conditional density and, hence, the preimages of Borel sets of zero measure have zero measure for the function $y^{\omega',s}(t,\widetilde{\omega})$ considered as a function of $t, \widetilde{\omega}$ for a. e. ω and a function of $\widetilde{\omega}$ for any $t \geqq s$ (it can be easily deduced from this that the mapping $\operatorname{col}[y^{\omega',s}(t,\omega), t]$: $\Omega' \times [s,T] \longrightarrow \mathbf{R}^{n+1}$ is $(\overline{\mathcal{B}}_{n+1}, \overline{\mathcal{B}}_{n+1})$-measurable for a. e. ω).

From (2.3) and Fubini's theorem we obtain

$$\begin{aligned}
\mathbf{E}\Big(\rho_s(x,\omega),\, V(x,s,\omega)\Big)_{H^0} &= \mathbf{E}\int_{\mathbf{R}^n} \rho_s(x,\omega)\, \mathbf{E}\Big\{\int_s^T \varphi\Big[y^{x,s}(t,\omega), t, \omega\Big]\, dt \mid \mathcal{F}_s\Big\}\, dx \\
&= \mathbf{E}\Big\{\int_{\mathbf{R}^n} dx\, \rho_s(x,\omega) \int_s^T \mathbf{E}\Big\{\varphi\Big[y^{x,s}(t,\omega), t, \omega\Big] \mid \mathcal{F}_t\Big\}\, dt\Big\} \\
&= \int_s^T \mathbf{E}\int_{\mathbf{R}^n} \rho_s(x,\omega)\, \varphi\Big[y^{x,s}(t,\omega), t, \omega\Big]\, dx\, dt = \int_s^T \widetilde{\mathbf{E}}\varphi\Big[y^{a,s}(t,\widetilde{\omega}), t, \omega\Big]\, dt \\
&= \int_s^T \mathbf{E}\Big(p(x,t,\omega), \varphi(x,t,\omega)\Big)_{H^0} dt = \mathbf{E}\Big(\rho_s(x,\omega),\, v(x,s,\omega)\Big)_{H^0}
\end{aligned} \tag{2.5}$$

(summability of the functions is seen from (2.3) and the fact that $p \in X^0[s,T]$, $\varphi \in X^0$). Since the choice of ρ_s and a is arbitrary, it follows that $V(x,s,\omega) = v(x,s,\omega)$ on C_0. The theorem has been proved.

THEOREM 2.2. *Let $\rho_s(x,\omega) \in L^2(\Omega, \mathcal{F}_s, \mathbf{P}, H^0)$ be the conditional (relative to $\mathcal{F}_s$) density of the distribution of a random n-vector a, which does not depend on $w(t) - w(s)$, $t \geqq s$. Then the function $p(x,t,\omega) = \mathfrak{L}_s^*\rho_s$ is the conditional (relative to $\mathcal{F}_t$) density of the distribution of the process $y^{a,s}(t,\omega)$. The following estimate holds*:

$$\|p\|_{C^0[s,T]} \leqq c_0 \|\rho\|_{Z_s^0},$$

where $c_0 = c_0(\mu_1, \mu_2)$ depends only on μ_1 and μ_2.

Proof. It has been already noted that the operator $\mathfrak{L}_s^*$: $Z_s^0 \to C_0[s,T]$ is continuous and there exists a constant $c_0 = c_0(\mu_1, \mu_2)$ such that the norm of the operator does not exceed c_0.

One may see from (2.3) and the chain of equalities (2.5) that

$$\int_s^T \widetilde{\mathbf{E}}\varphi\Big[y^{a,s}(t,\widetilde{\omega}), t, \omega\Big]\, dt = \int_s^T dt \int_{\mathbf{R}^n} \rho_s(x,\omega)\, \varphi\Big[y^{x,s}(t,\omega), t, \omega\Big]\, dx$$

$$= \mathbf{E}\int_s^T dt \int_{\mathbf{R}^n} p(x,t,\omega)\,\varphi(x,t,\omega)\,dx.$$

(Here $\widetilde{\mathbf{E}}$ and $\widetilde{\omega}$ are the same as in the proof above.) Since the choice of φ is arbitrary, from this one can obtain routinely the statement of the theorem.

3. Nondegenerate case. Let

$$w_1(t) = \left\|w^{(1)}(t),\ldots,w^{(d)}(t)\right\|, \qquad w_2(t) = \left\|w^{(d+1)}(t),\ldots,w^{(N)}(t)\right\|$$

be two parts of $w(t)$, $0 < d < N$. As to the initial probability space $(\Omega,\mathcal{F},\mathbf{P})$ we shall assume that $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$, $\mathbf{P} = \mathbf{P}_1 \times \mathbf{P}_2$ and the processes $w_i(t)$, $i = 1,2$, are given on the probability spaces $(\Omega_i,\mathcal{F}_i,\mathbf{P}_i)$; for $\omega \in \Omega$ we have $\omega = \|\omega_1,\omega_2\|$, $w_i(t,\omega) = w_i(t,\omega_i)$, $i = 1,2$.

Let $\mathcal{F}_t^{(i)} = \overline{\sigma[w_i(s),\ s \leqq t]}$ be the completion in Ω of the σ-algebras generated by $w_i(\cdot)$. Denote by $\overline{\mathcal{P}}_1$ the completion of the σ-algebra in $[0,T]\times\Omega$ generated by processes progressively measurable relative to $\mathcal{F}_t^{(1)}$.

Introduce the spaces $\widehat{X}^k = L^2([0,T]\times\Omega,\ \overline{\mathcal{P}}_1,\lambda_1\times\mathbf{P},H^k)$.

Denote by $\mathcal{X}$ the class of functions $u(x,t,\omega) = u(x,t,\omega_1,\omega_2) \in X^0$ such that

$$\|u\|_{\mathcal{X}} = \left(\int_{\Omega_1}\mathbf{P}_1(d\omega_1)\int_Q \operatorname*{ess\,sup}_{\omega_2}\left|u(x,t,\omega_1,\omega_2)\right|^2 dx\,dt\right)^{1/2} < +\infty.$$

Introduce the $(N-d)\times n$ matrix $\widehat{\beta} = \|\beta_{d+1},\ldots,\beta_N\|$ (here β_j are columns of β).

THEOREM 3.1. *Let the function $\widehat{\beta}(x,t,\omega_1)$ does not depend on $w_2(\cdot)$ and*

$$\det\left\{\widehat{\beta}(x,t,\omega_1)\,\widehat{\beta}(x,t,\omega_1)^T\right\} \geqq \delta > 0 \qquad (\forall\, x,t,\omega_1)$$

for some $\sigma > 0$. Let $\Phi_0 \in \mathcal{X}$, $\Phi_i(x,t,\omega) = \Phi_i(x,t,\omega_1) \in \widehat{X}^0$, $i = 1,\ldots,n$, and

$$\varphi(x,t,\omega) = \Phi_0(x,t,\omega) + \sum_{i=1}^n \frac{\partial\Phi_i}{\partial X_i}(x,t,\omega_1).$$

Then the following estimate holds for the functional (1.3):

$$\|V\|_{\mathcal{X}} \leqq c\left(\|\Phi_0\|_{\mathcal{X}} + \sum_{i=1}^n \|\Phi_i\|_{\widehat{X}^0}\right), \tag{3.1}$$

where $c = c(\mu_1,\delta) > 0$ depends only on δ and the parameter μ_1 introduced in § 1.

Proof. It is enough to consider non-negative functions $\varphi(x,t,\omega)$.

Introduce the set

$$B_+ = \Bigg\{\xi(x,t,\omega) \in X^1\colon\ \xi(x,t,\omega) \text{ — is bounded and finite}$$
$$\xi(x,t,\omega) \geqq 0,\ \mathbf{E}\int_Q\left(\mathbf{E}\left\{\xi(x,t,\omega)\mid\mathcal{F}_t^{(1)}\right\}\right)^2 dx\,dt \leqq 1\Bigg\}.$$

Let $\xi \in B_+$. Put $\pi = L^*\xi$, $\overline{\pi}(x,t,\omega) = \mathbf{E}\{\pi(x,t,\omega)\mid\mathcal{F}_t^{(1)}\}$, $\bar{\xi}(x,t,\omega) = \mathbf{E}\{\xi(x,t,\omega)\mid\mathcal{F}_t^{(1)}\}$,

$$\overline{\Phi}_0(x,t,\omega) = \frac{\mathbf{E}\{\Phi_0(x,t,\omega)\,\pi(x,t,\omega)\mid\mathcal{F}_t^{(1)}\}}{\mathbf{E}\{\pi(x,t,\omega)\mid\mathcal{F}_t^{(1)}\}},$$

$$\bar{f}(x,t,\omega) = \frac{\mathbf{E}\{f(x,t,\omega)\,\pi(x,t,\omega)\mid\mathcal{F}_t^{(1)}\}}{\mathbf{E}\{\pi(x,t,\omega)\mid\mathcal{F}_t^{(1)}\}}.$$

PROPOSITION 3.1. *For* $\xi \in B_+$,

a) $\overline{\pi}(x,t,\omega) \geqq 0$ *for all* t *for a. e.* x, ω;

b) $|\overline{\Phi}_0(x,t,\omega)| \leqq \operatorname{ess\,sup}_{\omega} |\Phi_0(x,t,\omega)|$ *for a. e.* x, t;

c) $|\bar{f}(x,t,\omega)| \leqq \sup_{x,t,\omega} |f(x,t,\omega)|$.

Statement a) of this proposition follows from Theorem 2.1 and the remaining statements follow from a).

One can easily see that the function $\overline{\pi}$ satisfies the equation

$$\begin{aligned} d_t\overline{\pi}(x,t,\omega) = & \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \Big\{ b_{ij}(x,t,\omega)\,\overline{\pi}(x,t,\omega) \Big\}\, dt \\ & + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Big\{ \bar{f}_i(x,t,\omega)\,\overline{\pi}(x,t,\omega) + \bar{\xi}(x,t,\omega) \Big\}\, dt \\ & - \sum_{j=1}^{N}\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Big\{ \beta_{ij}(x,t,\omega)\,\overline{\pi}(x,t,\omega) \Big\}\, dw^{(j)}(t), \overline{\pi}(x,0,\omega) = 0. \end{aligned} \tag{3.2}$$

Equation (3.2) is a superparabolic [1] Itô equation for any bounded measurable function $\bar{f}$ (it would be a parabolic [1] Itô equation if $d = N$). As is seen from the proof of Theorem 4.1.1 in [1], one may take $\sum (b^i u)_i$ instead of $\sum b^i u_i$ in the right-hand side of (4.1.1) in [1]. Applying Theorem 4.1.1 in [1], we get the estimate

$$\|\overline{\pi}\|_{\widetilde{X}^1} \leqq c_1 \|\bar{\xi}\|_{\widetilde{X}^0} \leqq c_1, \tag{3.3}$$

where $c_1 = c_1(\mu_1, \delta)$ does not depend on the choice of $\xi \in B_+$.

We have

$$\begin{aligned} \|V\|_{X^0} = & \sup_{\xi \in B_+} (V, \xi)_{X^0} \leqq \sup_{\xi \in B_+} (\varphi, \pi)_{X^0} \\ & \leqq \sup_{\xi \in B_+} \left| (\Phi_0, \pi)_{X^0} + \sum_{i=1}^{n} \left(\Phi_i, \frac{\partial \pi}{\partial x_i} \right)_{X^0} \right| \end{aligned} \tag{3.4}$$

for non-negative $\varphi(x,t,\omega)$. Evidently,

$$\begin{aligned} \left| (\Phi_0, \pi)_{X^0} \right| = \left| (\overline{\Phi}_0, \overline{\pi})_{\widetilde{X}^0} \right| & \leqq \|\overline{\Phi}_0\|_{\widetilde{X}^0} \|\overline{\pi}\|_{\widetilde{X}^0}, \\ \left| \left(\Phi_i, \frac{\partial \pi}{\partial x_i} \right)_{X^0} \right| & \leqq \|\Phi_i\|_{\widetilde{X}^0} \|\overline{\pi}\|_{X^1}. \end{aligned}$$

From this and (3.3)–(3.4), we obtain the estimate (3.1).

THEOREM 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied. Let* ρ *and* p *be the same as in Theorem* 2.2,

$$\overline{p}(x,t,\omega) = \mathbf{E}\Big\{ p(x,t,\omega) \mid \mathcal{F}_t^{(1)} \Big\}, \qquad \overline{\rho}(x,\omega) = \mathbf{E}\Big\{ \rho(x,\omega) \mid \mathcal{F}_s^{(1)} \Big\}.$$

Then

$$\|p\|_{\widetilde{X}^1[s,T]} \leqq c \|\overline{\rho}\|_{Z_s^0},$$

where $c = c(\mu_1, \delta)$ *depends only on* μ_1 *and* δ.

Theorem 3.2 is proved similarly to the proof of Theorem 3.1.

THEOREM 3.3. *Let the function* $\beta(x,t,\omega) = \beta(t)$ *in* (1.1) *does not depend on* x *and* ω,

$$\det \Big\{ \beta(t)\, \beta(t)^T \Big\} \geqq \delta > 0 \qquad (\forall t),$$

and the function $\varphi = \varphi(x,t) \in L^2(Q)$ *in* (1.3) *is nonrandom. Then*

$$\sum_{i=1}^{n} \left\| \frac{\partial V}{\partial x_i}(x,t,\omega) \right\|_{X^0} \leqq c\|\varphi\|_{L_2(Q)}, \tag{3.5}$$

where $c = c(\mu_1, \delta)$.

Proof. Introduce the set

$$B = \left\{ \xi \in X^1 \colon \|\xi\|_{X^0} \leqq 1,\ \xi(x,t,\omega) \text{ is smooth and finite for all } \omega \right\}.$$

Denote $\mathcal{D} = \partial/\partial x_i$. We have

$$\left\| \frac{\partial V}{\partial x_i} \right\|_{X^0} = \sup_{\xi \in B} (V, \mathcal{D}\xi)_{X^0} = \sup_{\xi \in B} (\varphi, L^* \mathcal{D}\xi)_{X^0}. \tag{3.6}$$

Let $\pi = L^*\mathcal{D}\xi$, $\overline{\pi}(x,t) = \mathbf{E}\pi(x,t,\omega)$ for $\xi \in B$. Then

$$\frac{\partial \overline{\pi}}{\partial t}(x,t) = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\Big(b_{ij}(t)\overline{\pi}(x,t)\Big) + J(x,t) + \mathbf{E}\mathcal{D}\xi(x,t,\omega), \tag{3.7}$$

$$\overline{\pi}(x,0) = 0, \tag{3.8}$$

where

$$b = \|b_{ij}\| = \beta\beta^T, \qquad J(x,t) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \mathbf{E}\Big\{ f_i(x,t,\omega)\pi(x,t,\omega) \Big\}.$$

Evidently,

$$\left\| \frac{\partial V}{\partial x_i} \right\|_{X^0} = \sup_{\xi \in B} (\varphi, \overline{\pi})_{X^0} \leqq \|\varphi\|_{L_2(Q)} \|\overline{\pi}\|_{L_2(Q)}, \tag{3.9}$$

$$\|\overline{\pi}\|_{L_2(Q)} \leqq c_1 \Big\{ \|J\|_{Y^{-2}} + \|\mathbf{E}\mathcal{D}\xi\|_{Y^{-2}} \Big\}, \tag{3.10}$$

where $c_1 = c_1(\mu_1, \delta)$ is some constant. The last inequality follows from known properties of the problem dual to (3.7)–(3.8).

One can note that

$$\|J\|_{Y^{-2}} \leqq c\|\pi\|_{X^{-1}}, \tag{3.11}$$

where $c = c(\mu'_f)$. It is seen from (3.9)–(3.11) that the estimate (3.1) will be proved if we shall show that the norm of the operator $L^*\colon X^{-1} \to X^{-1}$ or the operator $L\colon X^1 \to X^1$ is bounded by a constant depending only on μ_1 and μ'_f. Let us prove this.

We have

$$\frac{\partial V}{\partial x_i}(x,s,\omega) = \mathbf{E}\left\{ \int_s^T \frac{\partial \varphi}{\partial x}\Big(y^{x,s}(t,\omega),t\Big)\, h^{x,s}(t,\omega)\, dt \mid \mathcal{F}_s \right\}, \tag{3.12}$$

for a smooth finite $\varphi \in X^1$, where $h^{x,s}(t,\omega)$ are solutions of the equation

$$\frac{dh^{x,s}}{dt}(t,\omega) = \frac{\partial f}{\partial x}\Big(y^{x,s}(t,\omega),t,\omega\Big)\, h^{x,s}(t,\omega),$$

$h^{x,s}(s) = \operatorname{col}\|e_i^{(j)}\|_{j=1}^n$, where $e_i^{(j)} = 1$ if $i = j$ and $e_i^{(j)} = 0$ if $i \neq j$ (one may use the results in [8, Chap. 8]).

Evidently, $|h^{x,s}(t,\omega)| \leqq c \quad (\forall t, \omega)$, where $c = c(T, \mu'_f, \mu'_\beta)$. Applying Theorem 2.1 to (3.12), we get the estimate

$$\left\| \frac{\partial V}{\partial x_i} \right\|_{X^0} \leqq c \left\| \frac{\partial \Phi}{\partial x} \right\|_{X^1},$$

where $c = c(\mu_1, \mu'_f)$. Theorem 3.3 has been proved.

4. Corollaries: Generalizations of Itô's formula. Let us introduce the classes of functions:

$$\mathcal{V}^{k,l} = \left\{ u(x,t)\colon\ u \in Y^k,\ \frac{\partial^r u}{\partial t^r} \in Y^0,\ r = 0, 1, \dots, l \right\},$$
$$\mathcal{W}^{k,l} = V^{k,l} \cap C\Big([0,T] \to H^0\Big).$$

THEOREM 4.1. *Under the assumptions of* § 2,

$$\mathbf{E}u\Big(y^{x,0}(T,\omega), T\Big) - u(x,0) = \mathbf{E}\int_0^T \left(\frac{\partial u}{\partial t} + Au\right)\Big(y^{x,0}(t,\omega), t, \omega\Big)\,dt$$

for any function $u = u(x,t) \in \mathcal{W}^{2,1}$ *for a. e.* $x \in \mathbf{R}^n$.

Note that, in comparison with the theorem in [8, § 10.2], nondegeneracy of the matrix $\beta\beta^T$ is not required, and L_2-summability of derivatives of u instead of L_q-summability with $q = n+1$ is required (but the obtained statement is less strong since it holds only for almost all x). Below we shall escape the restriction on $u(x,t)$ that the derivative $\partial^2 u/\partial x^2$ is L_2-summable.

THEOREM 4.2. *Let the assumptions of Theorem* 3.1 *be satisfied,* $u = u(x,t) \in \mathcal{W}^{1,0}$ *and let the representation* $u(\cdot,t) = u_0 + \int_0^t \widehat{v}(s)\,ds$ *take place, where and* $u_0 \in H^0$, $\widehat{v} \in Y^{-1}$ *(the equality holds in* H^0 *for all* t*). Let a sequence* $\{\Phi_k\}_{k=1}^{\infty} \subset X^0$ *be such that*

$$\Phi_k \longrightarrow \widehat{v} + \sum_{i,j=1}^{n} b_{ij}(x,t,\omega_1)\,\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) \in X^{-1}$$

as $k \to +\infty$ *in the metric of* X^{-1}. *Then*

$$\mathbf{E}u\Big(y^{x,0}(T,\omega), T\Big) - u(x,0) = \mathbf{E}\int_0^T \frac{\partial u}{\partial x}\Big(y^{x,0}(t,\omega), t\Big)\, f\Big(y^{x,0}(t,\omega), t, \omega\Big)dt$$
$$+ \lim_{k \to +\infty} \mathbf{E}\int_0^T \Phi_k\Big(y^{x,0}(t,\omega), t, \omega\Big)\,dt,$$

moreover, the limit exists in $H^0 = L_2(\mathbf{R}^n)$, *the both sides of the equality, as functions of* $x \in \mathbf{R}^n$, *belong to* H^0 *and the equality holds in* $H^0 = L_2(\mathbf{R}^n)$ *(for a. e.* x*).*

The proof of Theorems 4.1–4.2 is based on an approximation of u by more smooth functions (as well as the proof of the theorem in [8, § 10.1]) and makes use of Theorems 2.1–2.2 and 3.1.

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INTEGRAL TRANSFORMS WITH INFINITELY DIVISIBLE KERNELS*

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Abstract. Given r characteristic functions $f_1(u), \ldots, f_r(u)$, none of which is identically equal to one, it is shown that the integral transform

$$\int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^r f_j(u_j)^{s_j} \right) dF(s_1, \ldots, s_r)$$

of the joint distribution function F of r non-negative random variables can be defined over a nonempty domain of natural numbers and it uniquely determines F. This result is used to obtain the converse of a multivariate version of a transfer theorem due to Gnedenko and Fahim, thus extending a result of Szasz and Frajeris in the univariate case. An application is also made to Lévy processes.

Key words. intergral transform, infinitely divisible, vector of random sums, the Lévy process

1. Introduction and summary. Given r probability characteristic functions $f_1(u) \leqq f_r(u)$, none of which is identically equal to one, we show that the integral transform

$$Q_F(\mathbf{u}) := \int_0^\infty \cdots \int_0^\infty \left(\prod_{j=1}^r f_j(u_j)^{s_j} \right) dF(s_1, \ldots, s_r)$$

can be defined over a domain of natural numbers, where F is any joint distribution function of r non-negative random variables. Such a transform has arisen earlier in the case $r = 1$ in papers by Feller [1], Gnedenko and Fahim [4], and Szasz and Frajeris [6], and in the monograph by Kruglov and Korolev [5]. This transform is shown here to determine F uniquely. This result is then used to answer a number of questions in the theory of independent random sums of independent random variables. In § 3 a multivariate extension of the transfer theorem by Gnedenko and Fahim [4] is obtained, and our uniqueness theorem is applied to give necessary and sufficient conditions that the joint limit distribution have independent coordinates. In connection with this, the converse to this transfer theorem obtained by Szasz and Frajeris in [6] is extended to the multidimensional case by means of our uniqueness theorem. In § 4 the results of § 3 are applied to Lévy processes.

2. The integral transform. Let $f(u)$ be an arbitrary characteristic function, and let $F(x)$ be the distribution function of a non-negative random variable. If $f(u) \neq 0$ for all u, let $J(f) = \mathbf{R}^1$; otherwise, let $J(f) = (-a, a)$, where

$$a = \sup \left\{ u \in \mathbf{R}^1 \colon f(u) \neq 0 \right\} = -\inf \left\{ u \in \mathbf{R}^1 \colon f(u) \neq 0 \right\}.$$

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