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The pricing of options in a financial market model with transaction costs and uncertain volatility

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Abstract

The paper introduces a financial market model with transactions costs and uncertain volatility. This model is a modification of the well-known Black–Scholes model. The solution to the problem of the pricing of the European call option is obtained by solving a nonlinear parabolic partial differential equation. The presented option pricing formula relates the price of an option to the underlying asset price and the bounds of the volatility of the underlying asset. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Most practitioners have adapted the famous Black–Scholes model as the premier model for pricing and hedging of options. The Black–Scholes model of a financial market consists of two assets: the risk free bond or bank account and the risky stock. It is assumed that the dynamics of the stock is given by a random process with some standard deviation of the stock returns (the volatility coefficient, or volatility). The dynamics of bonds is deterministic and exponentially increasing with a given risk-free rate. In the classic Black–Scholes model, the volatility is assumed to be given and fixed and transaction costs are not taken into account. However, in any real financial market, transaction costs have to be taken into account. Furthermore, empirical research shows that the real volatility is timevarying, random and correlated with stock prices (see Black and Scholes, 1973).

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Because the volatility coefficient appears in the formulas defining the fair price and the structure of hedging strategies, the estimation of the volatility from usually incomplete statistical data of stock prices is of a special importance (see Day and Levis, 1992; Derman et al., 1996; Gannon, 1996; Johnson, 1996; Kupiec, 1996; Mayhew, 1995; Taylor and Xu, 1994; Wilkie, 1995). Many authors emphasise that the main difficulty in modifying the Black–Scholes model is taking into account the fact that the volatility does (as it is shown by statistics) depend on both time and stock prices. Christie (1982) has shown that the volatility is correlated with stock prices. Lauterbach and Schultz (1990) notice that the Black–Scholes option pricing model consistently misprices warrants (see also Hauser and Lauterbach, 1997), and one of possible explanations of this fact is the invalidity of the Black–Scholes assumption that the equity return variance is constant.

In modified Black–Scholes models, a number of formulas and equations for volatility were proposed (see e.g. Christie, 1982; Finucame, 1989; Johnson and Shanno, 1987; Hcube, 1996; Hull and White, 1987; Masi et al., 1994; Scott, 1987). The principal assumption of the current paper is related to the bounds of the volatility.

Another problem arises out of the desire to take into account transaction costs. Black and Scholes (1972) noticed that in real financial markets transaction costs are quite large. Many authors remark that the return volatility is correlated with the trade volume, transactions costs and stock prices (Grossman and Zhou, 1996; Kupiec, 1996). A number of mathematical models with transaction costs were proposed (see Davis and Norman, 1990; Edirisinghe et al., 1993; Leland, 1985; Taksar et al., 1998). In this paper, we introduce and investigate a financial market model where the costs of jumps and of the high frequency component of the portfolio are taken into account.

In the present paper, the Black–Scholes model of a financial market is modified and investigated under the assumption that the volatility coefficient may be time-varying, uncertain and random. Moreover, in our modified model, transaction costs are taken into account. We prove that there exists a hedging strategy for the European call option. The rational price of the European call option is obtained by solving a nonlinear parabolic partial differential equation. The formula for the rational price leads to some quantitative conclusions relating the implied volatility and the pricing of option.

2. Definitions

The diffusion Black–Scholes model of a financial market consists of two assets: the risk free bond or bank account $B=(B_t)_{t\geq 0}$ and the risky stock $S=(S_t)_{t\geq 0}$. In this model, it is assumed that the dynamics of the stock is described by the following stochastic differential equation

$$\mathrm{d}S_t = aS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}w_t, \quad t>0, \tag{1}$$

where a is the appreciation rate, σ is the volatility coefficient, $w(t)$ is the standard Wiener process. The initial price $S_0 > 0$ is a given non-random value. The dynamics of the bond is described by the equation

$$B_t = e^{rt} B_0, \tag{2}$$

where $r \geq 0$ and B_0 are given constants.

Let $X_0 > 0$ be the initial wealth at time $t = 0$ of the investor. The total wealth of the investor at time $t > 0$ is

$$X_t = \beta_t B_t + \gamma_t S_t. \tag{3}$$

Here β_t is the number of the bonds, γ_t is the number of shares or the stock. The pair (β_t, γ_t) describes the state of the securities portfolio at time t. We call such pairs strategies. Some constraints will be imposed later on operations in the market, or, in other words, on strategies. We will consider the problem of investment or choosing a strategy and the corresponding problem of hedging of the European call option.

In practice, the volatility coefficient can be estimated from the measurement, S_t, and the task is more difficult for the. appreciation rate a, which is harder to estimate than σ. In the classic Black–Scholes model, σ is supposed to be known and fixed, and a is arbitrary and unknown. Our aim is to take into account transaction costs and the fact that the volatility coefficient σ does depend on both time t and the stock price S_t. In our model, the main assumptions are related to upper and lower bounds of the volatility coefficient and the nature of transaction costs.

Consider a right-continuous monoton increasing filtration of complete σ- algebras of events $\mathcal{F}_t$, $t > 0$, such that $w(t)$ is $\mathcal{F}_t$-measurable and $\mathcal{F}_t$ does not depend on $w(t+h) - w(t)$ for $h > 0$. We assume that $a = a(t)$ and $\sigma = \sigma(t)$ are square integrable random processes which are progressively measurable with respect to the filtration $\mathcal{F}_t$.

Assumption 1. The volatility coefficient $\sigma = \sigma(t)$ satisfies the following condition: $\sigma_1 \leq \sigma(t) \leq \sigma_2$ for some constants σ_1, σ_2, where $0 < \sigma_1 < \sigma_2$.

The main constraint in choosing a strategy in the classical problem without transaction costs is the so-called condition of self-financing.

Definition 1. A pair (β_t, γ_t) is said to be self-financing in a financial market model without transaction costs, if

$$dX_t = \beta_t \, dB_t + \gamma_t \, dS_t \tag{4}$$

Our aim is to extend this definition and the corresponding results to the case of transaction costs and uncertain volatility.

Definition 2. A pair (β_t, γ_t) is said to be an admissible strategy if the following conditions hold:

(1) γ_t, β_t are square integrable $\mathcal{F}_t$-adapted random processes;

(2) the process $\gamma(t)$ is piecewise continuous a.s. (almost surely);

(3) there exists a set of open random time intervals $I_k \subset [0,T], I_k = (\tau_k^-, \tau_k^+)$, such that τ_k^-, τ_k^+ are Markov time moments, $I_k \cap I_m = 0$ for $k \neq m$ a.s., mes $\{ [0,T] \cup_{k=1}^N I_k\} = 0$ a.s.,[2] where $N \leq +\infty$ is a random number of intervals, and I$\gamma(t)$ has the differential

$$d\gamma_t = \tilde{\gamma}_t \, dt + \hat{\gamma}_t \, dw(t) \text{ for } t \in I_k,$$

where $\tilde{\gamma}t, \tilde{\gamma}t$ are square integrable random processes which are progressively measurable with respect to the filtration $\mathscr{F}_t$;

(4) there exists a function $G(x,t)$: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that

$$\gamma_t = G(S_t, t), \tag{5}$$

and $G(x, t)$ is bounded on any bounded domain;

(5) the processes $a(t)\gamma_t$ and $\gamma_t S_t$ are square integrable.

In this definition, $I_k = (\tau_k^-, \tau_k^+)$ are time intervals when γ_t evolves continuously, and τ_k^-, τ_k^+ are times of jumps. We do not require that $\tau_k^+ = \tau_{k+1}^-$, because in an important case of strategies described below, the set $[0,T] \setminus \cup_{k=1}^N I_k$ may be an a.s. continuous (or non-countable) Kantor type set with zero Lebesgue measure.

We give now constructive sufficient conditions of the admissibility of strategies. For this, we notice that a strategy (β_t, γ_t) is admissible, if β_t satisfies all the above assumptions, $\gamma_t = G(S_t, t)$, where $G(x, t)$: $\mathbf{R} \times [0,T] \to \mathbf{R}$ is a function bounded on any bounded domain and of a polynomial growth, and there exists a set of open domains D_k, $k = 1,2,\ldots$, with piecewise C^1-smooth boundaries ∂D_k, such that $\mathbf{R} \times [0,T] = \cup_k \geq 1 \, (D_k \cup \partial D_k)$, $D_k \cap D_m = 0$ if $k \neq m$, $G|_{D_k} \in W_2^{2,1}(D_k)$.[3] In this case, the corresponding intervals I_k are maximum connected open intervals $I_k = \{t: (S_t, t) \in D_m\}$, $[0,T] \setminus \cup_{k=1}^N I_k = \{t: (S_t, t) \in \cup_{k \geq 1} \partial D_k\}$.

For any admissible strategy, we introduce some transaction cost for the time interval $[0, t]$ as

$$\int_0^t \lambda_\tau \, d\tau + \sum_{k:\tau_k^- < t} C_k,$$

where λ_t is a given non-negative $\mathscr{F}_t$-adapted random function which depends on $(\beta_\tau, \gamma_\tau)$ and on S_τ, $\tau \leq t$, and C_k are the costs for jump of the stock portfolio value.

Definition 3. An admissible strategy (β_t, γ_t) is said to be self-financing in a financial market with transaction costs if

$$X_t = X_0 + \int_0^t \beta_\tau \, dB\tau + \int_0^t \gamma_\tau \, dS_\tau - \int_0^t \lambda_\tau \, d\tau - \sum_{k:\tau_k^- < t} C_k \;\; (\forall t > 0) \tag{6}$$

Assumption 2. We assume that $\lambda_t = c(t)|\hat{\gamma}(t) S_t|$, where $c(t)$ is a random $\mathscr{F}_t$-adapted function and $c(t) \in [0, \bar{c}]$ for all $t > 0$, where $\bar{c} \geq 0$ is a given constant. Furthermore,

[2] mes denotes the Lebesgue measure.

[3] We denote $W_2^{2,1}(D)$ the Sobolev space of functions $u = u(x, t)$ such that u, u'_t, u'_x, u''_{xx} belong $L_2(D)$ for a domain $D \subset \mathbf{R} \times \mathbf{R}$.

we assume that $C_k = \varphi(|\gamma_{\tau_k^-} - \gamma_{\tau_{k-1}^+}|)$, where $\varphi(x)$ is a given non-negative deterministic function.

In other words, the transaction cost over the time period $(0, t]$ is

$$\int_0^t c(\tau)|\hat{\gamma}_\tau S_\tau| \, \mathrm{d}\tau + \sum_{k:\tau_k^- < t} \varphi(|\gamma_{\tau_k^-} - \gamma_{\tau_{k-1}^+}|)$$

Notice that if $\gamma_t \equiv \text{const}$ then $\lambda_t \equiv 0$, $C_k \equiv 0$ and transactions costs are zero. Moreover, $\lambda_t \equiv 0$, $C_k \equiv 0$ if γ_t is a smooth enough process such that $\partial\gamma_t/\partial t$ exists. But the transaction costs are non-zero, if $\tilde{\gamma}t \neq 0$ or γ_t has jumps. In other words, in this assumption, the continuous 'slow' change of the value of stocks portfolio γ_t is not taken into account. A similar assumption was used by Leland (1985) in a hidden form for a limit of discrete time jump strategies as a number of jumps converges to infinity in a diffusion market model. A similar assumption has been used also by Grossman and Zhou (1996) for the analysis of the trade volume and the volatility in a financial market.

We have from Eq. (6) that the the class of admissible self-financing strategies does depend on the functions $c(t)$, φ. But we show below that the optimal option hedging strategy does not depend on $c(t)$, φ and does depend only on $\bar{c}$ (see Remark after Theorem 1 below). The case of $\bar{c}=0$, $\varphi \equiv 0$ corresponds to zero transaction cost.

We can now rewrite Definition 3.

Definition 4. An admissible strategy (β_t, γ_t) is said to be self-financing in a financial market with transaction costs if

$$X_t = X_0 + \int_0^t \beta_\tau \, \mathrm{d}B_\tau + \int_0^t \gamma_\tau \, \mathrm{d}s_\tau - \int_o^t c(\tau)|\hat{\gamma}_\tau S_\tau| \, \mathrm{d}\tau - \sum_{k:\tau_k^- < t} \varphi(|\gamma_{\tau_k^-} - \gamma_{\tau_{k-1}^+}|).$$

Consider the problem of finding the price of options. Let $F(x){:}\mathbf{R} \to \mathbf{R}$ be a given non-negative function and $T > 0$ be a given time. Consider a call option of European type with the option writer obligation $F(S_t)$.

In the case of the standard call option of European type, the function $F(x) = (x-K)_+ = \max(0, x-K)$, where K is the option striking price. We consider more general $F(x)$ which may describe exotic options.

The approach of Black and Scholes is based on the idea that the option price dynamics can be determined by the dynamics of a risk free (hedging) strategy in the investment problem (see Black and Scholes, 1973).

Definition 5. A strategy (β_t, γ_t) is said to be a hedge in a financial market with transaction costs and uncertain volatility if the following conditions holds:
(1) (β_t, γ_t) is admissible and self-financing, and the function G in Eq. (5) depends on parameters σ_1, σ_1, $\bar{c}$, $\varphi(\cdot)$, T, $F(\cdot)$;
(2)

$$X_t \geq 0 (\forall t \in [0, T]) a.s. \tag{7}$$

(3)

$$X_T \geq F(S_T) a.s. \tag{8}$$

for all admissible $c(t)$, $\sigma(t)$.

In the approach of Black and Scholes, the option price is the initial wealth which may be raised to the option writer obligation by some investment transactions. Following this approach, we define the fair (rational) price of options.

Definition 6. Let $\prod$ be the set of all values of the initial wealth X_0 such that there exists an admissible strategy which is a hedge. Then, the fair (rational) price $\hat{C}$ for the option in this class of admissible strategies is defined as

$$\hat{C} = \inf_{X_0 \in \prod} X_0.$$

We will extend the Black and Scholes results to the case of the uncert volatility coefficient and transactions costs.

3. The main results

In this section, we assume for the sake of simplicity, that $r=0$ in Eq. (2) (It is not essential because of the deterministic character of B_t). We assume that $F(x)$ is piecewise smooth and $|F(x)| + |dF(x)/dx| \leq const(|x|+1)$. Furthermore, we assume that one of the following conditions holds:

(1) The function $F(x)$ is a convex function and there are non-zero transaction costs (in other words, $\bar{c} \neq 0$, $\varphi \neq 0$).

(2) The function $F(x)$ may be non-convex, but the transaction costs are absent (in other words, $\bar{c}=0$, $c(t) \equiv 0$, $\varphi(x) \equiv 0$).

Notice that the function $F(x) = (x-K)_+$ from the standard European call option is convex.

Suppose $H(x, t)$ is a solution of the boundary value problem for the following nonlinear parabolic equation

$$\frac{\partial H}{\partial t}(x, t) + \frac{1}{2} \max_{\sigma \in [\sigma_1, \sigma_2]} \left\{ \sigma^2 x^2 \frac{\partial^2 H}{\partial x^2}(x, t) \right\} + \bar{c}\sigma_2 \left| \frac{\partial^2 H}{\partial x^2}(x, t) \right| x^2 = 0, \tag{9}$$

$$H(x, T) = F(x), \tag{10}$$

in the domain $x>0$, $t \in [0, T]$. It is known, that this equation has an unique solution with locally square integrable derivatives (see Krylov, 1987).

Furthermore, let

$$X_t = H(S_t, t) + \int_0^t \alpha(t)\, dt, \tag{11}$$

where

$$\alpha(t) = \max_{\sigma \in [\sigma_1, \sigma_2]} \left\{ \frac{\sigma^2 - \sigma(t)^2}{2} S_t^2 \frac{\partial^2 H}{\partial x^2}(S_t, t) \right\} + [\bar{c}\sigma_2 - c(t)\sigma(t)] \left| \frac{\partial^2 H}{\partial x^2}(S_t, t) \right| S_t^2. \tag{12}$$

Let

$$\gamma_t = \frac{\partial H}{\partial x}(S_t, t), \; \beta_t = \frac{X_t - \gamma_t S_t}{B_t}. \tag{13}$$

Now we are in a position to present the main results of this paper.

Theorem 1. *The strategy Eq. (13) is a hedge, and the corresponding total wealth X_t is defined in Eq. (11).*

Theorem 2. *The rational price of the option is*

$$\hat{C} = H(S_0, 0). \tag{14}$$

Theorem 3. *Let $F(x)$ be a convex function. Then*

$$H(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F\left(x \exp\left\{ \hat{\sigma} y \sqrt{t} - \frac{t\hat{\sigma}^2}{2} \right\} \right) \exp\left(-\frac{y^2}{2} \right) \mathrm{d}y, \tag{15}$$

where

$$\hat{\sigma} = \sqrt{\sigma_2^2 + 2\bar{c}\sigma_2}. \tag{16}$$

Moreover, if $F(x) = (x - K)_+$, where $K > 0$ is a constant, then the rational price of the option is

$$\hat{C} = H(S_0, 0) = S_0 N(d_+) - KN(d_-), \tag{17}$$

where

$$d_\pm = (\hat{\sigma}\sqrt{T})^{-1} \left(\ln \frac{S_0}{K} \pm \frac{T\hat{\sigma}^2}{2} \right).$$

$N(d_\pm)$ is the cumulative standard normal distribution evaluated at $d_\pm$,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-\frac{y^2}{2}} \mathrm{d}y$$

Remark. The process γ_t has no jumps for the strategy which is optimal in the problem of the option pricing. Hence we have proved that we can not improve hedge by using discontinuous strategies either in a case of $C_k \equiv 0$ or $C_k \neq 0$.

4. Proof of results

Proposition. *Let $F(x)$ be a convex function. Then the solution of the Eqs. (9) and (10) coincides with the solution of the equations*

$$\frac{\partial H}{\partial t}(x, t) + \frac{1}{2}\hat{\sigma}^2 x^2 \frac{\partial^2 H}{\partial x^2}(x, t) = 0 \tag{18}$$

$$H(x, T) = F(x), \tag{19}$$

where $\hat{\sigma} = \sqrt{\sigma_2^2 + 2\bar{c}\sigma_2}$.

Proof of Proposition. Let $H(x, t)$ be a solution of Eqs. (18) and (19). Suppose that there exists $t_0 \in [0, T)$ such that the function $H(\cdot, t_0)$ is not convex. Since T is arbitrary and the coefficients of the equations are constants, it is enough to consider only $t_0 = 0$. Suppose, the function $H(\cdot, 0)$ is not convex. Then there exist $x_1 > 0$, $x_2 > 0$ such that $H(x_1, 0) + H(x_1, 0) < 2H((x_1 + x_2)/2, 0)$. Consider the classical problem of the option pricing with the volatility coefficient or and without transaction costs. Let

$$S_0 = \frac{x_1 + x_2}{2},\ S_t^{(i)} = \frac{2x_i S_t}{x_1 + x_2},\ \gamma_t^{(i)} = \frac{\partial H}{\partial x}(S_t^{(i)}, t),\ i = 1,2.$$

Then $\bar{\gamma}_t = [\gamma_t^{(1)} + \gamma_t^{(2)}]/2$ is admissible. Let $\bar{\beta}_t$ be such that$(\bar{\beta}_t, \bar{\gamma}_t)$ is a self-financing strategy, $\bar{x}_t$ be the corresponding wealth. It is obvious that $(\bar{\beta}_t, \bar{\gamma}_t)$is a hedge, and $\bar{X}_0 < H(S_0, 0)$. However, it contradicts the Black and Scholes formula for the rational price. Hence, $H(\cdot, 0)$ is a convex function and $H(\cdot, t)$ is convex for any time t, and $H''_{xx}(x, t) \geq 0$. Hence, Eq. (9) holds. This completes the proof of Proposition.

Proof of Theorem 1. Let $F(\cdot)$ be a convex function. From Proposition, the Eqs. (15), (18) and (19) hold for H defined by Eqs. (9) and (10). Let

$$G(x, t) = \frac{\partial H}{\partial x}(x, t)$$

The fundamental solution for Eqs. (18) and (19) is known (see Shiryaev et al., 1994). Using this solution, we can easily obtain the formula for G and make the conclusion that G has continuous derivatives G'_t, G'_x, G''_{xx} in Q for any domain $Q = D \times (0, T_*)$, where $D \subset \mathbf{R}^+$, $T_* \in (0, T)$ (or $G \in C_2^{2,1}(Q)$).

It is obvious that this strategy is admissible with

$$\hat{\gamma}_t = \frac{\partial G}{\partial x}(S_t, t)\sigma(t)S_t = \frac{\partial^2 H}{\partial x^2}(S_t, t)\sigma(t)S_t, \tag{20}$$

$$\lambda_t = c(t)\left|\frac{\partial^2 H}{\partial x^2}(S_t, t)\sigma(t)\right| S_t^2. \tag{21}$$

¿From the Ito's formula and Eqs. (11) and (12), we have that

$$dX_t = \mathrm{d}_t H(S_t, t) + \alpha(t)\,\mathrm{d}t = G(S_t, t)\,\mathrm{d}S_t + \left[\frac{\partial H}{\partial t}(S_t, t) + \frac{1}{2}\sigma(t)^2 S_t^2 \frac{\partial^2 H}{\partial x^2}(S_t, t)\right]$$

$$\times\,\mathrm{d}t + \alpha(t)\,\mathrm{d}t = G(S_t, t)\,\mathrm{d}S_t - \lambda_t\,\mathrm{d}t.$$

Hence the strategy is self-financing. Furthermore, it is obvious that $\alpha(t) \geq 0$ and Eqs. (7) and (8) hold.

In the case of zero transaction cost, we do not need the existence of derivatives G'_{t}, G'_{x}, G''_{xx} and the proof is similar. This completes the proof of Theorem 1

Proof of Theorem 2. In the classic case of zero transaction costs and a known constant volatility (when $\bar{c}=0$, $\varphi \equiv 0$, $\sigma_1 = \sigma_2$), we have $X_T = F(S_T)$ for a hedge, and fair price is $\hat{C} = \mathbf{E}^* F(S_T)$, where $\mathbf{E}^*$ is the expectation by such probability measure that S_t is martingale, and, hence, $\hat{C}$ is the rational (fair) price. We cannot use this method in our general case because we have only inequality $X_T \geq F(S_T)$ and the values $X_T - F(S_T)$ depend on strategies. However, we can use another approach which does not use martingale properties of hedge wealth. Note, that a different non-martingale approach was proposed by Wilmott and Atkinson (1993).

Let $(\tilde{b}_t, \tilde{\gamma}_t)$ be some other hedge, $\tilde{\gamma}_t = \tilde{G}(S_t, t), \tilde{X}t$ be the corresponding wealth, $\hat{C} = \tilde{X}_{\mathrm{O}} < \hat{C}$. Suppose that $\sigma(t) \equiv \sigma_2$, $c(t) \equiv \bar{c}$. Introduce the following function

$$\tilde{H}(x, t) = \int_0^x \tilde{G}(y, t)\,\mathrm{d}y.$$

Let I_k be the random time intervals introduced in Section 2 for admissible strategies, $k = 1,\ldots,N$. We have from the Ito's formula that

$$\tilde{H}(S_T, T) - \tilde{H}(S_0, 0) = \int_0^T \tilde{G}(S_t, t)\,\mathrm{d}S_t + \sum_{k=1}^{N}\Bigg\{\tilde{H}(S_{\tau_{k+1}^-}, \tau_{k+1}^-) - \tilde{H}(S_{\tau_{k^+}}, \tau_k^+)$$

$$+ \int_{I_k}\left[\frac{\partial \tilde{H}}{\partial t}(S_t, t) + \frac{1}{2}\sigma_2^2 S_t^2 \frac{\partial^2 \tilde{H}}{\partial x^2}(S_t, t)\right]\mathrm{d}t\Bigg\}.$$

Here we use some version of the Ito's formula for a function with non-smooth derivatives (see Krylov, 1980; Dokuchaev, 1994). The condition of self-financing and Eq. (8) give us that

$$\int_0^T \tilde{G}(S_t, t)\,\mathrm{d}S_t = \tilde{X}_T - \tilde{X}_0 + \int_0^T \lambda_t\,\mathrm{d}t + \sum_k C_k = F(S_T) + \xi + \int_0^T \lambda_t\,\mathrm{d}t - \tilde{X}_0.$$

Here $\xi \geq 0$ is some random value. Denote

$$\mathscr{L}\tilde{H} = \frac{\partial \tilde{H}}{\partial t} + \frac{1}{2}\sigma_2^2 x^2 \frac{\partial^2 \tilde{H}}{\partial x^2} + \bar{c}\sigma_2 \left| \frac{\partial^2 \tilde{H}}{\partial x^2} \right| x^2.$$

Then

$$\sum_{k=1}^{N} \left\{ \int_{I_k} \mathscr{L}\tilde{H}(S_t, t)\,\mathrm{d}t + \tilde{H}(S_{\tau_{k+1}^-}, \tau_{k+1}^-) - \tilde{H}(S_{\tau_k} + \tau_k^+) \right\}$$
$$= \tilde{H}(S_T, T) - \tilde{H}(S_0, 0) - F(S_T) - \tilde{\xi} + \tilde{X}_0,$$

where $\hat{\xi} \geq 0$ is some random value. Denote by $\mathscr{X}$ the space $W_2^{2,1}(Q)^*$ which is dual to the Sobolev space $W_2^{2,1}(Q)$, $Q = D \times [0, T]$, where $D \subset (0, +\infty)$ is an arbitrary interval. The element $\xi \in \mathscr{X}$ is said to be non-negative if $\langle \xi, f \rangle \geq 0$ for every $f \in W_2^{2,1}(Q)$ such that $f(x, t) \geq 0$. In this sense, $\mathscr{L}\tilde{\mathrm{H}} \leq 0$ as an element of $\mathscr{X}$. Then $H(x, 0) \leq \hat{H}(x, 0)$ because of Eq. (9). This completes the proof of Theorem 2.

Proof of Theorem 3. The fundamental solution for Eqs. (18) and (19) is known and Eqs. (18) and (19) hold for H defined by Eq. (15) (see Shiryaev et al., 1994). From Proposition, the Eqs. (9) and (10) hold for this H. For $F(x) = (x - K)_+$, the formula for $\hat{C}$ is a consequence of the Black–Scholes result. This completes the proof of Theorem 3.

5. Conclusions

In the classic Black–Scholes model, the volatility is assumed to be known and fixed. Moreover, in this model, transaction costs are not taken into account. This paper introduces a modification of the Black–Scholes model which includes time-varying, uncertain and random volatility, and takes transaction costs into account. The rational price of the European call option is obtained for this model. The formula for the rational price may have an interesting economic interpretation. According to this formula, the presence of transaction costs is analogous to the increase of the implied volatility. This can be interpreted as a mathematically rigorous confirmation of the empirical results of Kupiec (1996) and Derman et al. (1996).

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