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Stochastic Controls with Terminal Contingent Conditions

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Abstract

This paper considers a nonlinear stochastic control problem where the system dynamics is a controlled nonlinear backward stochastic differential equation and the state must coincide with a given random vector at the terminal time. A necessary condition of optimality in the form of a global maximum principle as well as a sufficient condition of optimality are presented. The general result is also applied to a backward linear-quadratic control problem and an optimal control is obtained explicitly as a feedback of the solution to a forward-backward equation. Finally, a nonlinear problem with additional integral constraints is discussed and it is shown that the duality gap is zero under the Slater condition.

Key words. backward stochastic differential equation, adjoint equation, maximum principle, linear-quadratic control, Lagrangian, duality gap.

AMS Subject Classifications. 93E, 49K

1 Introduction

Stochastic maximum principle has been extensively investigated since 1960s [7, 10, 2, 1, 8, 13, 16]. The research on the problem has mainly focused on the dynamic systems governed by the Itô (forward) stochastic differential equations with *initial* states

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specified. However, in studying (among others) the derivative securities (or contingent claims) which are now becoming increasingly popular financial tools for investment and risk hedging, one typically encounters stochastic systems where the *terminal* states are pre-determined, following the so-called *backward stochastic differential equations* (BSDEs). A good example is the option pricing problem where the replication of the option follows a backward equation. This calls for the research on evaluating and optimizing the performance of backward stochastic systems.

Linear BSDEs was initially introduced by Bismut [3] when he was studying adjoint equations associated with the stochastic maximum principle. The nonlinear extension was introduced by Pardoux and Peng [13]. Research on BSDE theory and applications has been very active in recent years. For a updated and systematic account of BSDE theory, see [15, Chapter 7].

In this paper, we study an optimal control problem where the dynamics follows a BSDE and therefore the terminal state must coincide with a prescribed random vector contingent on the terminal situation. This sort of problems come out naturally when we study a (forward) stochastic linear-quadratic control problem [4]. More interestingly, it can be used to model some optimal control problem of contingent claims. For example, part of the control may represent the rate of capital injection or withdrawal from a replication of a claim in order to achieve certain goal.

A control problem for BSDEs was considered in [9], where a necessary condition of optimality was obtained for a system with a state-linear drift. For a general controlled nonlinear BSDE, a stochastic maximum principle in a *local form* was derived by Peng [14]. In this paper, we attempt to prove the stochastic maximum principle in the *global form*. Note that the major difficulty in doing this is that the state of a backward system consists of *two* variables $y(t)$ and $z(t)$. The second one, $z(t)$, is hard to handle because there is no convenient pointwise (in t) estimation for it, as opposed to the first variable $y(t)$. This calls for a more delicate estimation of the variation of $z(t)$ in some Banach space when carrying out the spike variation approach that is typical for deriving a necessary condition. After the maximum principle is derived, the result is applied to a *backward* linear-quadratic (LQ) problem via a Riccati-like equation and an optimal control is presented in a closed form. Then we investigate when the derived stochastic maximum principle becomes sufficient. Finally, we study a problem with a finite number of additional integral constraints and show under the standard Slater condition that the duality gap is zero. As a consequence, necessary and sufficient conditions of optimality in a form of a duality equality are obtained.

The rest of the paper is organized as follows. In Section 2 the optimal control problem with BSDE dynamics is formulated. Section 3 is devoted to the necessary conditions of optimality (maximum principle). In Section 4 a linear-quadratic problem is studied as a special case. Section 5 deals with the sufficient conditions of optimality. In Section 6 a constrained problem is treated. Finally, Section 7 gives some concluding remarks.

2 Problem Formulation and Preliminaries

Let $T > 0$ be fixed. Consider a standard probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a standard d-dimensional Wiener process $w(t)$ (with $w(0) = 0$) which generates the filtration $\mathcal{F}_t = \sigma\{w(r) : 0 \le r \le t\}$ augmented by all the $\mathbf{P}$-null sets in $\mathcal{F}$.

Let ξ be an n-dimensional $\mathcal{F}_T$-measurable random vector. Consider the following control problem:

$$\text{Minimize } J(u(\cdot)) = \mathbf{E}g(y(0)) + \mathbf{E}\int_0^T \varphi(t, y(t), z(t), u(t))dt, \tag{2.1}$$

$$\text{Subject to } \begin{cases} dy(t) = f(t, y(t), z(t), u(t))dt + z(t)dw(t), \quad t < T, \\ y(T) = \xi. \end{cases} \tag{2.2}$$

Here $u(t) = u(t, \omega)$ is an m-dimensional control vector, $y(t) = y(t, \omega)$ an n-dimensional vector, and $z(t) = z(t, \omega)$ an $n \times d$ matrix, $\omega \in \Omega$. The pair $x(t) \equiv (y(t), z(t))$ is the state process.

In (2.1) and (2.2), $f(t, y, z, u, \omega) : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n\times d} \times \mathbf{R}^m \times \Omega \to \mathbf{R}^n$, $\varphi(t, y, z, u, \omega) : \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^{n\times d} \times \mathbf{R}^m \times \Omega \to \mathbf{R}$, and $g(y) : \mathbf{R}^n \to \mathbf{R}$ are given measurable functions.

Assumption 2.1 *The random functions $f(t, y, z, u, t)$ and $\varphi(t, y, z, u, \omega)$ are continuous for fixed $\omega \in \Omega$ and are progressively measurable with respect to $\mathcal{F}_t$ for fixed (y, z, u). The function $g(y) : \mathbf{R}^n \to \mathbf{R}^n$ is continuous.*

Assumption 2.2 *There exist continuous derivatives $\partial^k g(y)/\partial y^k$, $\partial^k f(t,y,z,u,\omega)/\partial y^k$, $\partial^k f(t,y,z,u,\omega)/\partial z^k$, $\partial^k \varphi(t,y,z,u,\omega)/\partial y^k$ and $\partial^k \varphi(t,y,z,u,\omega)/\partial z^k$, $k=1,2$. Moreover, the following estimates hold:*

$$|\varphi(t,y,z,u,\omega)|+|g(y)|\le C_0(|y|^2+|z|^2+1),$$
$$|f(t,y,z,u,\omega)|+\left|\frac{\partial\varphi}{\partial y}(t,y,z,u,\omega)\right|+\left|\frac{\partial\varphi}{\partial z}(t,y,z,u,\omega)\right|+\left|\frac{\partial g}{\partial y}(y)\right|\le C_1(|y|+|z|+1),$$
$$\left|\frac{\partial f}{\partial y}(t,y,z,u,\omega)\right|+\left|\frac{\partial f}{\partial z_i}(t,y,z,u,\omega)\right|+\left|\frac{\partial^2 f}{\partial y^2}(t,y,z,u,\omega)\right|+\left|\frac{\partial^2 f}{\partial z_i^2}(t,y,z,u,\omega)\right|$$
$$+\left|\frac{\partial^2\varphi}{\partial y^2}(t,y,z,u,\omega)\right|+\left|\frac{\partial^2\varphi}{\partial z_i^2}(t,y,z,u,\omega)\right|+\left|\frac{\partial^2 g}{\partial y^2}(y)\right|\le C_2,$$

where $C_k>0$ are constants, $k=0,1,2$, and z_i are the columns of the matrix z, $i=1,...,d$.

Assumption 2.3 *There exists a number $p>2$ such that $\xi\in L^p(\Omega,\mathcal{F}_T,\mathbf{P};\mathbf{R}^n)$.*

Let Δ be a given non-empty subset of $\mathbf{R}^m$. Consider a right-continuous and non-decreasing filtration $\mathcal{A}_t$ consisting of complete σ-algebras such that $\mathcal{A}_t\subseteq\mathcal{F}_t$.

Introduce a set U of admissible controls consisting of all functions $u(t,\omega):[0,T]\times\Omega\to\mathbf{R}^m$ which are progressively measurable with respect to $\mathcal{A}_t$, $u(t,\omega)\in\Delta$ a.e. a.s., and $\mathbf{E}\int_0^T|u(t,\omega)|^p<+\infty$, where the number p is defined in Assumption 2.3.

Notice that $\mathcal{A}_t\neq\mathcal{F}_t$ allows one to consider some smaller but important classes of controls. For example, the case when $\mathcal{A}_t$ is the completion of $\{\Omega,\emptyset\}$, corresponds to a problem when only deterministic controls are admissible. The case $\mathcal{A}_t=\mathcal{F}_{(t-h)\vee 0}$ corresponds to the problem with a time delay $h>0$ in control.

Let E be an Euclidean space and $r>1$ be a number. Introduce the set $\widetilde{\mathcal{M}}_r(E)$ of all functions $\zeta(t,\omega):[0,T]\times\Omega\to E$ which are progressively measurable with respect to $\mathcal{F}_t$ and such that $\|\zeta\|_{\mathcal{M}_r(E)}=\left\{\mathbf{E}\left(\int_0^T\|\zeta\|_E^2dt\right)^{r/2}\right\}^{1/r}<+\infty$. Let the Banach space $\mathcal{M}_r(E)$ be the completion of $\widetilde{\mathcal{M}}_r(E)$. We also introduce the following space

$$Y_r=\mathcal{M}_r(\mathbf{R}^n),\ Z_r=\mathcal{M}_r(\mathbf{R}^{n\times d}),\ X=Y_2\times Z_2,\mathcal{C}_r=L^r(\Omega,\mathcal{F}_T,\mathbf{P};C([0,T]\to\mathbf{R}^n)).$$

The following concerns the existence and uniqueness of solutions to the BSDE (2.2).

Theorem 2.1 *([9, p. 54]) For any $u(\cdot)\in U$, there exist an unique pair $(y(\cdot),z(\cdot))\in(Y_p\cap\mathcal{C}_p)\times Z_p$ such that (2.2) holds.*

3 Necessary Condition of Optimality

Assume that the process $(x^0(\cdot), u^0(\cdot)) \equiv (y^0(\cdot), z^0(\cdot), u^0(\cdot))$ is an optimal solution of the control problem (2.1) and (2.2). Consider the following forward Ito equation:

$$\begin{cases} d\psi(t) = \Big\{ -\left(\frac{\partial f}{\partial y}\right)_0^*(t)\psi(t) + \left(\frac{\partial \varphi}{\partial y}\right)_0^*(t) \Big\} dt + \sum_{i=1}^d \Big\{ -\left(\frac{\partial f}{\partial z_i}\right)_0^*(t)\psi(t) + \left(\frac{\partial \varphi}{\partial z_i}\right)_0^*(t) \Big\} dw_i(t), \\ \psi(0) = \frac{\partial g}{\partial y}(y^0(0))^*. \end{cases} \tag{3.1}$$

Here and after we use the notation that $(\phi)_0(t) \equiv \phi(x^0(t), u^0(t))$ for any function $\phi(\cdot)$. Furthermore, $w_i(t)$ are the components of the vector $w(t)$, and $z_i(t)$ are the columns of the matrix $z(t)$.

Introduce the function $\widehat{H} : [0,T] \times \Delta \times \Omega \to \mathbf{R}$:

$$\widehat{H}(t,u) \triangleq \psi(t)^* f(t, y^0(t), z^0(t), u) - \varphi(t, y^0(t), z^0(t), u).$$

Proposition 3.1 *[1]. The conditional expectation $\tilde{H}(t,u) = \mathbf{E}\Big\{\widehat{H}(t,u)|\mathcal{A}_t\Big\}$ exists, and there exists a variant $H(t,u) : [0,T] \times \Delta \times \Omega \to \mathbf{R}$ of $\tilde{H}(t,u)$ (i.e., $\mathbf{P}\Big(H(t,u) = \tilde{H}(t,u),\ \forall (t,u)\Big) = 1$) such that the process $H(t,u(t))$ is $\mathcal{A}_t$-adapted for any $u(\cdot) \in U$.*

The function $H(t,u,\omega)$ is the so-called *regular conditional expectation* [6, 1]. We assume from now on that $H(t,u) = H(t,u,\omega)$ is such as determined by Proposition 3.1.

Theorem 3.1 (Maximum Principle) *The following inequality holds:*

$$H(t, u^0(t,\omega), \omega) = \max_{v \in \Delta} H(t, v, \omega), \quad a.e.t \in [0,T],\ \mathbf{P}\text{-}a.s. \tag{3.2}$$

The rest of this section is devoted to the proof of Theorem 3.1. Let μ denotes an arbitrary pair $(t', v') \in (0,t] \times L^\infty(\Omega, \mathcal{A}_{t'}, \mathbf{P}; \mathbf{R}^m)$ such that $v'(\omega) \in \Delta$ $\mathbf{P}$-a.s.. For each μ and $\varepsilon \geq 0$, introduce the set $Q(\varepsilon) = \{t \in [0,T] :\ |t - t'| \leq \varepsilon/2\}$. Specify a number ε_μ so small that $Q(\varepsilon) \subset [0,T]$. Construct a variation $u(\cdot, \varepsilon|\mu)$ of $u^0(\cdot)$ in the following way. Let $u(\cdot, 0|\mu) \equiv u^0(\cdot)$. For $\varepsilon \in (0, \varepsilon_\mu]$, let

$$u(t,\varepsilon|\mu) = \begin{cases} u^0(t) & \text{if } t \notin Q(\varepsilon) \\ v' & \text{if } t \in Q(\varepsilon). \end{cases} \tag{3.3}$$

The resulting set of curves in the space U is called a *variation bundle*. The parameter μ enumerates the curves, and the bundle vertex is at $u^0(\cdot)$.

Let μ be fixed. Denote $u^\varepsilon(\cdot) = u(\cdot, \varepsilon|\mu)$ and $x^\varepsilon(\cdot) \equiv (y^\varepsilon(\cdot), z^\varepsilon(\cdot))$ the corresponding solution of (2.2). Furthermore, we shall employ the notation $(\phi)_\varepsilon(t) \equiv \phi(x^\varepsilon(t), u^\varepsilon(t))$ for any function $\phi(\cdot)$.

Introduce the set $\widehat{X} \subset X$ such that any $\widehat{x}(\cdot) \equiv (\widehat{y}(\cdot), \widehat{z}(\cdot)) \in \widehat{X}$ is the solution of (2.2) for some $\widehat{u}(\cdot) \in U$. By definition, for any $x(\cdot) \equiv (y(\cdot), z(\cdot)) \in \widehat{X}$, there exists a process $\overline{y}(\cdot)$ such that $(\overline{y}(\cdot), z(\cdot)) \in X$ and $dy(t) = \overline{y}(t)dt + z(t)dw(t)$. Introduce the following functional $L : \widehat{X} \times U \to \mathbf{R}$:

$$\begin{aligned} L(x(\cdot), u(\cdot)) \triangleq\ & \mathbf{E} \textstyle\int_0^T \psi(t)^* \Big(f(t, y(t), z(t), u(t)) - \overline{y}(t) \Big) dt \\ & - \mathbf{E} g(y(0)) - \mathbf{E} \textstyle\int_0^T \varphi(t, y(t), z(t), u(t)) dt. \end{aligned} \tag{3.4}$$

Note that in the above definition $x(\cdot)$ is not necessarily the state corresponding the control $u(\cdot)$. Then

$$\begin{aligned} J(x^0(\cdot), u^0(\cdot)) - J(x^\varepsilon(\cdot), u^\varepsilon(\cdot)) = \ & L(x^\varepsilon(\cdot), u^\varepsilon(\cdot)) - L(x^0(\cdot), u^0(\cdot)) \\ = \ & L(x^\varepsilon(\cdot), u^\varepsilon(\cdot)) - L(x^0(\cdot), u^\varepsilon(\cdot)) \\ & + L(x^0(\cdot), u^\varepsilon(\cdot)) - L(x^0(\cdot), u^0(\cdot)). \end{aligned}$$

Lemma 3.1 *We have*

$$\frac{1}{\varepsilon}(L(x^\varepsilon(\cdot), u^\varepsilon(\cdot)) - L(x^0(\cdot), u^\varepsilon(\cdot))) \to 0 \quad as \quad \varepsilon \to 0+.$$

Proof. Introduce the processes

$$\widehat{\psi}_i(t) \triangleq \left(\frac{\partial f}{\partial z_i}\right)_0^* (t)\psi(t) + \left(\frac{\partial \varphi}{\partial z_i}\right)_0^* (t), \quad i = 1, ..., d. \tag{3.5}$$

It can be easily seen from Ito's formula that

$$\begin{aligned} \int_0^T \psi(t)^* \overline{y}(t) dt = \psi(T)^* y(T) - \psi(0)^* y(0) - \int_0^T d\psi(t)^* y(t) - \int_0^T \psi(t)^* z(t) dw(t) \\ - \sum_{i=1}^d \int_0^T \widehat{\psi}_i(t)^* z_i(t) dt. \end{aligned}$$

Hence (3.4) can be rewritten as

$$\begin{aligned} L(x(\cdot), u(\cdot)) = \ & \mathbf{E} \Big\{ -\psi(T)^* y(T) + \psi(0)^* y(0) + \textstyle\int_0^T d\psi(t)^* y(t) + \int_0^T \psi(t)^* z(t) dw(t) \\ & + \textstyle\sum_{i=1}^d \int_0^T \widehat{\psi}_i(t)^* z_i(t) dt + \int_0^T \psi(t)^* f(t, y(t), z(t), u(t)) dt \Big\} \\ & - \mathbf{E} g(y(0)) - \mathbf{E} \textstyle\int_0^T \varphi(t, y(t), z(t), u(t)) dt. \end{aligned} \tag{3.6}$$

Let

$$h(t) \triangleq y^\varepsilon(t) - y^0(t), \quad \widehat{h}(t) \triangleq z^\varepsilon(t) - z^0(t),$$

and denote by $\widehat{h}_i$ the columns of the matrix $\widehat{h}$. Then $h(T) = 0$, and

$$\begin{aligned}
L(x^\varepsilon, u^\varepsilon) - L(x^0, u^\varepsilon) = E\Big\{ &\psi(0)^* h(0) - \psi(T)^* h(T) + \int_0^T d\psi(t)^* h(t) + \int_0^T \psi(t)^* \widehat{h}(t) dw(t) \\
&+ \sum_{i=1}^d \int_0^T \widehat{\psi}_i(t)^* \widehat{h}_i(t) dt - g(y^\varepsilon(0)) + g(y^0(0)) \\
&+ \int_0^T \psi(t)^* \Big(f(t, y^\varepsilon(t), z^\varepsilon(t), u^\varepsilon(t)) - f(t, y^0(t), z^0(t), u^\varepsilon(t)) \Big) dt \\
&- \int_0^T \Big(\varphi(t, y^\varepsilon(t), z^\varepsilon(t), u^\varepsilon(t)) - \varphi(t, y^0(t), z^0(t), u^\varepsilon(t)) \Big) dt \Big\} \\
= E\Big\{ &\psi(0)^* h(0) + \int_0^T d\psi(t)^* h(t) + \int_0^T \psi(t)^* \widehat{h}(t) dw(t) \\
+ \sum_{i=1}^d \int_0^T \widehat{\psi}_i(t)^* \widehat{h}_i(t) dt &- \frac{\partial g}{\partial y}(y(0)) h(0) + \int_0^T \Big\{ \psi(t)^* \Big(\Big(\frac{\partial f}{\partial y}\Big)_0 (t) h(t) + \sum_{i=1}^d \Big(\frac{\partial f}{\partial z_i}\Big)_0 (t) \widehat{h}_i(t) \Big), \\
&- \Big(\frac{\partial \varphi}{\partial y}\Big)_0 (t) h(t) - \sum_{i=1}^d \Big(\frac{\partial \varphi}{\partial z_i}\Big)_0 (t) \widehat{h}_i(t) \Big\} dt - \alpha_0 + \alpha_1 - \alpha_2 = -\alpha_0 + \alpha_1 - \alpha_2,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_0 &\triangleq \mathbf{E}\Big(g(y^\varepsilon(0)) + g(y^0(0)) - \frac{\partial g}{\partial y}(y(0)) h(0) \Big), \\
\alpha_1 &\triangleq \mathbf{E} \int_0^T a_1(t) dt, \quad \alpha_2 \triangleq \mathbf{E} \int_0^T a_1(t) dt, \\
a_1(t) &\triangleq \psi(t)^* \Big(f(t, y^\varepsilon(t), z^\varepsilon(t), u^\varepsilon(t)) - f(t, y^0(t), z^0(t), u^\varepsilon(t)) \\
&\quad - \Big(\frac{\partial f}{\partial y}\Big)_0 (t) h(t) - \sum_{i=1}^d \Big(\frac{\partial f}{\partial z_i}\Big)_0 (t) \widehat{h}_i(t) \Big), \\
a_2(t) &\triangleq \varphi(t, y^\varepsilon(t), z^\varepsilon(t), u^\varepsilon(t)) - \varphi(t, y^0(t), z^0(t), u^\varepsilon(t)) - \Big(\frac{\partial \varphi}{\partial y}\Big)_0 (t) h(t) - \sum_{i=1}^d \Big(\frac{\partial \varphi}{\partial z_i}\Big)_0 (t) \widehat{h}_i(t).
\end{aligned}$$

Let us now introduce the following proposition.

Proposition 3.2 *For any $r \in (1, p]$, $\nu \in (0, 1)$, there exists a constant $C' = C'(r, \nu) > 0$ such that*

$$\|h\|_{\mathcal{C}_r} + \|\widehat{h}\|_{Z_r} \le C' \varepsilon^\nu.$$

Proof. As in the proof of [9, Theorem 5.1], for a small enough $T>0$ it can be shown that

$$\|h\|^p_{\mathcal{C}_r}+\|\widehat{h}\|^p_{Z_r}\le \mathrm{const}\left(\int_0^T\Big|f(t,y^0(t),z^0(t),u^\varepsilon(t))-f(t,y^0(t),z^0(t),u^0(t))\Big|^p\,dt\right)^{p\nu}.$$

(To get this one only needs to slightly modify the proof in [9] by replacing the Cauchy-Schwartz inequality by the Holder inequality.) By (3.3),

$$\|h\|^p_{\mathcal{C}_r}+\|\widehat{h}\|^p_{Z_r}\le \mathrm{const}\,\varepsilon^{p\nu},\quad \forall\varepsilon\in[0,\varepsilon_\mu).$$

The general case of an arbitrary $T>0$ can be obtained by subdividing the interval $[0,T]$ into a finite number of small intervals, using the flow property of the backward equation (see [9, Proposition 2.5]). This completes the proof of Proposition 3.2. □.

Lemma 4.1 then follows immediately from the following result.

Proposition 3.3 *For* $i=0,1,2$, $\varepsilon^{-1}|\alpha_i|\to 0\quad as\quad \varepsilon\to 0+.$

Proof. For a scalar random process $\theta(t)$, introduce the process

$$x^\theta(t)\stackrel{\Delta}{=}(1-\theta(t))x^0(t)+\theta(t)x^\varepsilon(t).$$

Denote $(\phi)_\theta(t)\equiv\phi(x^\theta(t),u^\varepsilon(t))$ and $(\phi)_\gamma(t)\equiv\phi(x^\theta(t),u^0(t))$ for any function $\phi(\cdot)$. It can be easily obtained from the Mean-Value Theorem that there exists a process $\theta(t)$ such that

$$|a_1(t)|\le\left|\psi(t)^*\Big\{\Big(\Big(\frac{\partial f}{\partial y}\Big)_\theta(t)-\Big(\frac{\partial f}{\partial y}\Big)_0(t)\Big)h(t)+\sum_{i=1}^d\Big(\Big(\frac{\partial f}{\partial z_i}\Big)_\theta(t)-\Big(\frac{\partial f}{\partial z_i}\Big)_0(t)\Big)\widehat{h}_i(t)\Big\}\right|.$$

Hence $|a_1(t)|\le|\overline{a}_1(t)|+|\overline{a}_2(t)|$, where

$$\overline{a}_1(t)\stackrel{\Delta}{=}\psi(t)^*\Big\{\Big(\Big(\frac{\partial f}{\partial y}\Big)_\theta(t)-\Big(\frac{\partial f}{\partial y}\Big)_\gamma(t)\Big)h(t)+\sum_{i=1}^d\Big(\Big(\frac{\partial f}{\partial z_i}\Big)_\theta(t)-\Big(\frac{\partial f}{\partial z_i}\Big)_\gamma(t)\Big)\widehat{h}_i(t)\Big\},$$

$$\overline{a}_2(t)\stackrel{\Delta}{=}\psi(t)^*\Big\{\Big(\Big(\frac{\partial f}{\partial y}\Big)_\gamma(t)-\Big(\frac{\partial f}{\partial y}\Big)_0(t)\Big)h(t)+\sum_{i=1}^d\Big(\Big(\frac{\partial f}{\partial z_i}\Big)_\gamma(t)-\Big(\frac{\partial f}{\partial z_i}\Big)_0(t)\Big)\widehat{h}_i(t)\Big\}.$$

Furthermore, let $r'\in(2,p)$, $\nu\in(1/2,1)$ be arbitrary, $r=r'(r'-1)^{-1}$, and $R=2r$. It can be easily seen that $\psi\in\mathcal{C}_{\overline{p}}$ for any $\overline{p}<p$, hence $\|\psi\|_{\mathcal{C}_{r'}}<+\infty$. By Proposition 3.2, we have

$$\mathbf{E}\int_0^T|\overline{a}_2(t)|dt\le C_2\|\psi\|_{\mathcal{C}_{r'}}\Big\{\mathbf{E}\Big(\int_0^T(|h(t)|+|\widehat{h}(t)|)^2dt\Big)^r\Big\}^{1/r}$$

$$\le \text{const}\, \|\psi\|_{\mathcal{C}_{r'}} \left(\|h\|_{\mathcal{C}_R} + \|\widehat{h}\|_{Z_R}\right)^2 \le \text{const}\, \|\psi\|_{\mathcal{C}_{r'}} C'(R,\nu)\varepsilon^{2\nu}.$$

Hence

$$\frac{1}{\varepsilon}\mathbf{E}\int_0^T |\overline{a}_2(t)|dt \to 0 \quad \text{as} \quad \varepsilon \to 0+.$$

Furthermore, one has

$$\mathbf{E}\int_0^T |\overline{a}_1(t)|dt \le 2d\|\psi\|_{\mathcal{C}_{r'}} \left\{\mathbf{E}\left(\int_{Q(\varepsilon)} (|h(t)| + |\widehat{h}(t)|) d \sup_{t,\omega}\left(\left|\frac{\partial f}{\partial y}\right| + \left|\frac{\partial f}{\partial z_i}\right|\right)dt\right)^r\right\}^{1/r}$$

$$\le 2dC_2\|\psi\|_{\mathcal{C}_{r'}} \left\{\mathbf{E}\left(\int_{Q(\varepsilon)} (|h(t)| + |\widehat{h}(t)|)dt\right)^r\right\}^{1/r}$$

$$\le \text{const}\, \|\psi\|_{\mathcal{C}_{r'}} \left\{\mathbf{E}\left(\int_{Q(\varepsilon)} (|h(t)| + |\widehat{h}(t)|)^2 dt\right)^{r/2} \left(\int_{Q(\varepsilon)} dt\right)^{r/2}\right\}^{1/r}$$

$$\le \text{const}\; \|\psi\|_{\mathcal{C}_{r'}} \left(\|h\|_{Y_r} + \|\widehat{h}\|_{Z_r}\right)\sqrt{\varepsilon} \le \text{const}\; \varepsilon^{\nu+1/2} = o(\varepsilon).$$

Hence $\alpha_1 = o(\varepsilon)$. This completes the proof of Proposition 3.3 for $i = 1$. The proof for $i = 0$ and $i = 2$ is similar. This completes the proof of Proposition 3.3 and hence that of Lemma 3.1. □

Lemma 3.2 *Let $v \in \Delta$, $t' \in [0,T)$, and $\Omega_\mu \in \mathcal{A}_{t'}$ be fixed. Let $u^\varepsilon(\cdot) \triangleq u(\cdot,\varepsilon|\mu)$, where $\mu = (t, v')$ with*

$$v'(\omega) \triangleq \begin{cases} u^0(t',\omega) & \text{if } \omega \notin \Omega_\mu \\ v & \text{if } \omega \in \Omega_\mu. \end{cases}$$

Then

$$\frac{1}{\varepsilon}(L(x^0(\cdot), u^\varepsilon(\cdot)) - L(x^0(\cdot), u^0(\cdot))) \to H(t', u^0(t',\omega),\omega) - H(t', v, \omega) \quad \text{as} \quad \varepsilon \to 0+, \ a.e.t'.$$

Proof. By (3.4),

$$\frac{L(x^0(\cdot), u^\varepsilon(\cdot)) - L(x^0(\cdot), u^0(\cdot))}{\varepsilon} = \frac{1}{\varepsilon}\mathbf{E}\int_0^T \Big\{\psi(t)(f(t,y^0(t),z^0(t),u^\varepsilon(t)) - f(t,y^0(t),z^0(t),u^0(t)))$$

$$-\varphi(t,y^0(t),z^0(t),u^\varepsilon(t)) + \varphi(t,y^0(t),z^0(t),u^0(t))\Big\}dt$$

$$= \frac{1}{\varepsilon}\int_{(t'-\varepsilon/2)\vee 0}^{(t'+\varepsilon/2)\wedge T} \int_{\Omega_\mu} \mathbf{P}(d\omega)(H(u^0(t,\omega),t,\omega) - H(v,t,\omega)) \to$$

$$\int_{\Omega_\mu} \mathbf{P}(d\omega)(H(u^0(t',\omega),t',\omega) - H(v,t',\omega)) \quad \text{as} \quad \varepsilon \to 0+$$

for a.e. t', $\mathbf{P}$-a.s. This convergence holds for any $\Omega_\mu \in \mathcal{A}_{t'}$. This completes the proof of Lemma 3.2. □

Theorem 3.1 then follows from Lemma 3.1 and Lemma 3.2.

4 Application: A Linear-Quadratic Problem

In this section, we apply Theorem 3.1 to a linear-quadratic problem as a particular case of the control problem (2.1)-(2.2).

Let $T > 0$, the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and the d-dimensional Wiener process $w(t)$ be such as defined in Section 2. Let $p > 2$ be a given number, and $\xi \in L^p(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbf{R}^n)$ be a given random vector.

Consider the following control problem:

$$\text{Minimize } J(y(\cdot), z(\cdot), u(\cdot)) = \frac{1}{2}\mathbf{E}\Big(y(0)^* G y(0) + \int_0^T u(t)^* \Gamma(t) u(t) dt\Big), \tag{4.1}$$

$$\text{Subject to } \begin{cases} dy(t) = \Big(A(t)y(t) + B(t)u(t) + \sum_{i=1}^d C_i(t) z_i(t)\Big)\, dt + z(t) dw(t), \\ y(T) = \xi. \end{cases} \tag{4.2}$$

Here $u(t) = u(t,\omega)$ is an m-dimensional control vector, $y(t) = y(t,\omega)$ is an n-dimensional vector, $z(t) = z(t,\omega)$ is an $n \times d$ dimensional matrix with $z_i(t)$ being the columns of $z(t)$. The pair $x(t) \equiv (y(t), z(t))$ is the state process.

In (4.1) and (4.2), $A(t) = A(t,\omega) : [0,T] \times \Omega \to \mathbf{R}^{n\times n}$, $B(t) = B(t,\omega) : [0,T] \times \Omega \to \mathbf{R}^{n\times m}$, $C_i(t) = C_i(t,\omega) : [0,T] \times \Omega \to \mathbf{R}^{n\times n}$, $\Gamma(t) = \Gamma(t,\omega) : [0,T] \times \Omega \to \mathbf{R}^{m\times m}$ are bounded matrix processes which are progressively measurable with respect to $\mathcal{F}_t$, and $G \in \mathbf{R}^{n\times n}$ is a given (deterministic) matrix.

We assume that $G = G^* \geq 0$, $\Gamma(t,\omega) = \Gamma(t,\omega)^* \geq \delta I_m$ for all t, ω, where $\delta > 0$, and I_m is the unit matrix in $\mathbf{R}^m$.

Introduce a set U_0 of admissible controls consisting of all functions $u(t,\omega) : [0,T] \times \Omega \to \mathbf{R}^m$ which are progressively measurable with respect to $\mathcal{F}_t$ and such that $\mathbf{E}\int_0^T |u(t,\omega)|^p dt < +\infty$.

Let $u^0(\cdot)$ be an optimal control, and $(y^0(\cdot), z^0(\cdot))$ be the corresponding state process. Then the adjoint process $\psi(\cdot)$ is the solution of the following equations:

$$\begin{cases} -d\psi(t) = A(t)^*\psi(t)dt + \sum_{i=1}^d C_i(t)^*\psi(t) dw_i(t), \\ \psi(0) = G y^0(0). \end{cases} \tag{4.3}$$

Theorem 4.1 *There is a unique optimal control $u^0(\cdot)$ for the problem (4.1)-(4.2) in the class U_0. Moreover, $u^0(\cdot)$ has the following representation:*

$$u^0(t) = \Gamma(t)^{-1} B(t)^* \psi(t). \tag{4.4}$$

Proof. The cost function (4.1) is a positive quadratic functional of controls because of the assumptions on G and $\Gamma(t)$. Hence by standard convex optimization theory an optimal control exists. Moreover, in the present case, the adjoint equation (3.1) has the form (4.3). By Theorem 3.1,

$$\psi(t)^* B(t) u^0(t) - u^0(t)^* \Gamma(t) u^0(t) \geq \psi(t)^* B(t) v - v^* \Gamma(t) v, \quad \forall v \in \mathbf{R}^m.$$

This implies (4.4). Hence (4.4) is the only control which satisfies the necessary conditions of optimality. It then must be the optimal control. This completes the proof of Theorem 4.1. □

Assume now that there exist a random $n \times n$ matrix process $P(t) = P(t,\omega)$ with the following properties:

(i) $P(t)$ is progressively measurable with respect to $\mathcal{F}_t$;

(ii) $P \in L^2([0,T] \times \Omega)$;

(iii) the following equation holds:

$$\begin{cases} dP(t) = -\left[P(t)A(t) + A(t)^* P(t) + P(t)B(t)\Gamma(t)^{-1}B(t)^* P(t)\right] dt - \sum_{i=1}^d C_i(t)^* P(t) dw_i(t), \\ P(0) = G. \end{cases} \tag{4.5}$$

This equation will play a role similar to the Riccati equation in forward LQ control theory. Note however that this equation does not have the symmetric property and the solvability of it in general remains open. However, the following gives a sufficient condition for the existence of its solutions.

Lemma 4.1 *Let $C_i(t,\omega) = c_i(t,\omega) I_n$, where I_n is the unit matrix in $\mathbf{R}^{n\times n}$, and $c_i(t) = c_i(t,\omega) : [0,T]\times\Omega \to \mathbf{R}$, are bounded scalar processes which are progressively measurable with respect to $\mathcal{F}_t$. Then there exists $P(t)$ satisfying the conditions* (i)-(iii) *above.*

Proof. Introduce the processes

$$q(t) = q(t,\omega) = \exp\left\{-\sum_{i=1}^d \int_0^t c_i(s,\omega) dw_i(s)\right\}, \quad \Gamma_q(t) = \Gamma_q(t,\omega) = q(t,\omega)^{-1}\Gamma(t,\omega).$$

For fixed ω, let $Q(t) = Q(t,\omega)$ be the solution of the following conventional Riccati equation

$$\begin{cases} \frac{dQ}{dt} = -Q(t)A(t) - A(t)^* Q(t) - Q(t)B(t)\Gamma_q(t)^{-1}B(t)^* Q(t) - \frac{1}{2}\sum_{i=1}^d c_i(t) Q(t), \\ Q(0) = G. \end{cases} \tag{4.6}$$

This equation has a solution $Q(t) = Q(t,\omega) > 0$. Furthermore, it can be easily seen that

$$0 \le Q(t) \le G - \int_0^t [Q(s)A(s) + A(s)^* Q(s) - \frac{1}{2}\sum_{i=1}^d c_i(s)Q(s)]ds.$$

Hence any solution $Q(t) = Q(t,\omega)$ of (4.6) is uniformly bounded, and it can be easily seen that $Q(t)$ is progressively measurable with respect to $\mathcal{F}_t$. Let $P(t) = q(t)Q(t)$. It can be verified directly that this matrix process satisfies (i)-(iii). This completes the proof of Lemma 4.1. □

Theorem 4.2 *Assume that there exists $P(t)$ such that the conditions* (i)-(iii) *above hold. Then the optimal control $u^0(\cdot)$ for the problem (4.1)-(4.2) can be represented as*

$$u^0(t) = \Gamma(t)^{-1}B(t)^*P(t)\widetilde{y}(t), \tag{4.7}$$

where $\widetilde{y}(t)$ is the solution of the equation

$$\begin{cases} \frac{d}{dt}\widetilde{y}(t) = \left[A(t) + B(t)\Gamma^{-1}B(t)^*P(t)\right]\widetilde{y}(t), \\ \widetilde{y}(0) = y^0(0). \end{cases} \tag{4.8}$$

Proof. Let $\widetilde{\psi}(t) \triangleq P(t)\widetilde{y}(t)$. We have

$$d\widetilde{\psi}(t) = dP(t)\widetilde{y}(t) + P(t)d\widetilde{y}(t)$$

$$= \Big\{ -\Big(P(t)A(t) + A(t)^*P(t) + P(t)B(t)\Gamma(t)^{-1}B(t)^*P(t)\Big)dt - \sum_{i=1}^d C_i(t)^*P(t)dw_i(t)\Big\}\widetilde{y}(t)$$

$$+P(t)\Big(A(t) + B(t)\Gamma^{-1}B(t)^*P(t)\Big)\widetilde{y}(t)$$

$$= -\Big\{A(t)^*P(t) - \sum_{i=1}^d C_i(t)^*P(t)dw_i(t)\Big\}\widetilde{y}(t) = -A(t)^*\widetilde{\psi}(t)dt - \sum_{i=1}^d C_i(t)^*\widetilde{\psi}(t)dw_i(t).$$

So $\widetilde{\psi}(t)$ satisfies the same equations as $\psi(t)$. Hence $\widetilde{\psi}(\cdot) = \psi(\cdot)$ by the uniqueness. This completes the proof of Theorem 4.2. □

Note that $(\widetilde{y}(\cdot), y^0(\cdot), z^0(\cdot))$ satisfies the following so-called *forward-backward stochastic differential equation* (FBSDE):

$$\begin{cases} \frac{d}{dt}\widetilde{y}(t) = \left[A(t) + B(t)\Gamma^{-1}B(t)^*P(t)\right]\widetilde{y}(t), \\ dy^0(t) = \Big(A(t)y^0(t) + B(t)\Gamma(t)^{-1}B(t)^*P(t)\widetilde{y}(t) + \sum_{i=1}^d C_i(t)z_i(t)\Big)dt + z(t)dw(t), \\ \widetilde{y}(0) = y^0(0), \ \ y^0(T) = \xi. \end{cases} \tag{4.9}$$

Therefore, the optimal control (4.7) is a "feedback" of the solution to the equation (4.9).

The following result is straightforward.

Corollary 4.1 *If $C_i \equiv 0$ $(\forall i)$ and $A(t)$, $B(t)$ are deterministic, then the matrix $P(t)$ is deterministic and $\widetilde{y}(t) = \mathbf{E}y^0(t)$.*

5 Sufficient Condition of Optimality

In this section, we examine when the necessary condition of optimality (3.2) becomes sufficient. We assume that $\mathcal{A}_t \equiv \mathcal{F}_t$. Let $u^0(\cdot)$ be an admissible control and $x^0(\cdot) \equiv (y^0(\cdot), z^0(\cdot))$ be the corresponding state process. Introduce the function $\mathcal{H} : [0,T] \times \mathbf{R}^n \times \mathbf{R}^{n\times d} \times \mathbf{R}^m \times \Omega \to \mathbf{R}$:

$$\mathcal{H}(t,y,z,u,\omega) \triangleq \psi(t)^* f(t,y,z,u,\omega) - \varphi(t,y,z,u,\omega).$$

(Note the natural relationship between $\mathcal{H}$ and $\widehat{H}$ that appears in the maximum principle.)

Before stating the main result, we need to have some more notation. Let $v : \mathcal{X} \to \mathbf{R}$ be a locally Lipschitz continuous function, where $\mathcal{X}$ is a convex set in $\mathbf{R}^n$. The Clarke *generalized gradient* of v at $\widehat{x} \in \mathcal{X}$, denoted by $\partial_x v(\widehat{x})$, is a set defined by

$$\partial_x v(\widehat{x}) \triangleq \{p \in \mathbf{R}^n : \ p^*\xi \le v^0(\widehat{x};\xi), \quad \forall \xi \in \mathbf{R}^n\},$$

where

$$v^0(\widehat{x};\xi) \triangleq \limsup_{x\in\mathcal{X},\ x+h\xi\in\mathcal{X}, x\to\widehat{x}, h\to 0} \frac{v(x+h\xi)-v(x)}{h}.$$

Theorem 5.1 *Let Δ be either an open set or a convex set in $\mathbf{R}^m$. Assume that the function $g(\cdot)$ is convex, and the function $\mathcal{H}(t,y,z,u,\omega)$ is concave and Lipschitz continuous in (y,z,u) for fixed (t,ω). Then $u^0(\cdot)$ is an optimal control of the problem (2.2)–(2.1) if it satisfies (3.2).*

Proof. Let $u(\cdot)$ be an arbitrary admissible control, $x(\cdot) \equiv (y(\cdot), z(\cdot))$ be the corresponding state process, and

$$h(t) = y(t) - y^0(t), \quad \widehat{h}(t) = z(t) - z^0(t), \quad \Delta f(t) = f(y(t), z(t), u(t), t) - f(y^0(t), z^0(t), u^0(t), t).$$

We have $h(T) = 0$, and

$$\psi(T)^*h(T) - \psi(0)^*h(0) = \int_0^T (d\psi(t)^*h(t) + \psi(t)^*dh(t) + \sum_{i=1}^d \widehat{\psi}_i(t)^*\widehat{h}_i(t)dt),$$

where $\widehat{h}_i(t)$ is the columns of the matrix $\widehat{h}(t)$, and the processes $\widehat{\psi}_i(t)$ is defined in (3.5). Hence

$$-\mathbf{E}\psi(0)^*h(0) = \mathbf{E}\int_0^T \Big\{ -\psi(t)^*\Big(\frac{\partial f}{\partial y}\Big)_0(t)h(t) + \Big(\frac{\partial \varphi}{\partial y}\Big)_0(t)h(t)$$
$$+\psi(t)^*\Delta f(t) - \sum_{i=1}^d \Big(\psi(t)^*\Big(\frac{\partial f}{\partial z_i}\Big)_0(t) - \Big(\frac{\partial \varphi}{\partial z_i}\Big)_0(t)\Big)\widehat{h}_i(t)\Big\}dt,$$

or

$$-\mathbf{E}\psi(0)^*h(0) = \mathbf{E}\int_0^T \Big\{ -\Big(\frac{\partial \mathcal{H}}{\partial y}\Big)_0(t)h(t) - \sum_{i=1}^d \Big(\frac{\partial \mathcal{H}}{\partial z_i}\Big)_0(t)\widehat{h}_i(t) + \psi(t)^*\Delta f(t)\Big\}dt.$$

Denote $(\partial_{(x,u)}\mathcal{H})_0(t)$, etc., be the Clarke generalized gradients of $\mathcal{H}$ evaluated at $(x^0(t), u^0(t)) \equiv (y^0(t), z^0(t), u^0(t))$. The maximum principle (3.2) yields $0 \in (\partial_u\mathcal{H})_0(t)$, a.e.$t$, $\mathbf{P}$-a.s.. By [17, Lemma 2.3], $((\partial_x\mathcal{H})_0(t), 0) \in (\partial_{(x,u)}\mathcal{H})_0(t)$ for a.e. t, $\mathbf{P}$-a.s. It then follows from [17, Lemma 2.2(4)] that

$$\mathcal{H}(y(t), z(t), u(t), t) - \mathcal{H}(y^0(t), z^0(t), u^0(t), t) \le (\partial_y\mathcal{H})_0(t)h(t) + \sum_{i=1}^d (\partial_{z_i}\mathcal{H})_0(t)\widehat{h}_i(t).$$

Hence

$$-\mathbf{E}\psi(0)h(0) \le \mathbf{E}\int_0^T (-\mathcal{H}(y(t), z(t), u(t), t) + \mathcal{H}(y^0(t), z^0(t), u^0(t), t) + \psi(t)^*\Delta f(t))dt$$
$$= \mathbf{E}\int_0^T (\varphi(t, y(t), z(t), u(t)) - \varphi(t, y^0(t), z^0(t), u^0(t)))dt.$$

Furthermore,

$$g(y(0)) - g(y^0(0)) \ge \Big(\frac{\partial g}{\partial y}\Big)_0 h(0) = \psi(0)^*h(0).$$

due to the convexity assumption on $g(\cdot)$. Therefore,

$$\mathbf{E}(g(y(0)) - g(y^0(0))) + \mathbf{E}\int_0^T (\varphi(t, y(t), z(t), u(t)) - \varphi(t, y^0(t), z^0(t), u^0(t)))dt \ge 0.$$

This completes the proof of Theorem 5.1. □

6 Problem with Integral Constraints

In this section, we assume that $\sigma\{w_1(s),\ s \le t\} \subseteq \mathcal{A}_t$ $(\forall t)$, where $w_1(t)$ is the first component of the Wiener process $w(t)$ in (2.2). In particular, this assumption excludes the case of only deterministic controls.

Consider the following functionals

$$\Phi_i(u(\cdot)) = \mathbf{E} g_i(y(0)) + \mathbf{E}\int_0^T \varphi_i(t, y(t), z(t), u(t))dt, \quad i = 0, 1, 2, ..., N, \tag{6.1}$$

where $u(\cdot) \in U$ is a control, and the pair $x(t) \equiv (y(t), z(t))$ is the state process which evolves correspondingly to the equation (2.2). We assume that the functions g_i and φ_i have similar properties as specified in Assumptions 2.1 and 2.2.

Consider the following problem:

$$\begin{cases} \text{Minimize } \Phi_0(u(\cdot)) \text{ over } u(\cdot) \in U, \\ \text{Subject to } \Phi_1(u(\cdot)) \le 0, ..., \Phi_N(u(\cdot)) \le 0. \end{cases} \tag{6.2}$$

We assume the following *Slater condition*:

$$\exists u(\cdot) \in U : \quad \Phi_1(u(\cdot)) < 0, ..., \Phi_N(u(\cdot)) < 0. \tag{6.3}$$

Set

$$U_1 = \{u(\cdot) \in U : \quad \Phi_1(u(\cdot)) \le 0, ..., \Phi_N(u(\cdot)) \le 0\},$$

and introduce the Lagrangian

$$\L(u(\cdot), \mu) = \Phi_0(u(\cdot)) + \sum_{i=1}^N \mu_i \Phi_i(u(\cdot)),$$

where $\mu = (\mu_1, ..., \mu_n) \in \mathbf{R}^n$. We write $\mu \ge 0$ if $\mu_i \ge 0$, $\forall i = 1, \cdots, n$.

Theorem 6.1 (i) *The following relation holds:*

$$\inf_{u(\cdot)\in U_1} \Phi_0(u(\cdot)) = \inf_{u(\cdot)\in U} \sup_{\mu\ge 0} \L(u(\cdot), \mu) = \sup_{\mu\ge 0} \inf_{u(\cdot)\in U} \L(u(\cdot), \mu). \tag{6.4}$$

(ii) *The supremum on the right-hand side of (6.4) is achievable for a finite μ.*

(iii) *Each pair $(\mu, u(\cdot))$ achieving* sup inf *in (6.4) with*

$$u(\cdot) \in U_1, \quad \sum_{i=1}^N \mu_i \Phi_i(\cdot) = 0 \tag{6.5}$$

is a saddle point of the problem, i.e., it is the solution of the problem inf sup *as well as the solution of the minimization problems with constraints (6.2).*

(iv) *For each optimal control $u(\cdot)$ of the problem (6.2), there exists a finite $\mu \geq 0$ so that $(\mu, u(\cdot))$ is the solution of the problem*

$$\sup_{\mu \geq 0} \inf_{u(\cdot) \in U} L(u(\cdot), \mu), \tag{6.6}$$

and (6.5) holds.

The proof of this theorem will be given later in this section. Notice that the second equality in (6.4) shows that the so-called "duality gap" is zero for the constrained problem (6.2). Theorem 6.1 not only establishes the existence of Lagrange multipliers, but also shows how to calculate them. Moreover, Theorem 6.1 gives necessary and sufficient conditions of optimality as well as a sufficient condition of optimality (items (iv) and (iii) respectively).

Corollary 6.1 *Let $\widehat{u}(\cdot)$ be an optimal control for the problem (6.2). Then there exists $\mu \geq 0$ such that the following hold:*

(i) *μ is a solution of the problem (6.6).*

(ii) *(6.5) holds with $u(\cdot) = \hat{u}(\cdot)$.*

(iii) *The maximum principle (3.2) holds with $H(\cdot)$, $\psi(\cdot)$ defined for the following $g(\cdot)$, $\varphi(\cdot)$:*

$$g(y) = g_0(y) + \sum_{i=1}^{N} \mu_i g_i(y), \quad \varphi(t, y, z, u, \omega) = \varphi_0(t, y, z, u, \omega) + \sum_{i=1}^{N} \mu_i \varphi_i(t, y, z, u, \omega).$$

To prove Theorem 6.1, we employ the method which was originally proposed in [5] for optimal stopping with constraints. To start with , let us introduce the vector function $\Phi(u(\cdot)) \triangleq (\Phi_0(u(\cdot)), \Phi_1(u(\cdot)), ..., \Phi_N(u(\cdot))$.

Lemma 6.1 *For any $u_1(\cdot) \in U$, $u_2(\cdot) \in U$ and $\delta > 0$, there exists $u(\cdot) \in U$ such that*

$$|2\Phi(u(\cdot)) - \Phi(u_1(\cdot)) - \Phi(u_2(\cdot))| \leq \delta. \tag{6.7}$$

Proof. For $\varepsilon > 0$, let

$$u_{1,\varepsilon}(\cdot) \triangleq u_1(\cdot), \quad u_{2,\varepsilon}(t, \omega) \triangleq \begin{cases} u_2(t, \omega) & \text{if } t \geq \varepsilon \\ u_1(t, \omega) & \text{if } t < \varepsilon. \end{cases}$$

It can be easily seen that

$$\Phi(u_{2,\varepsilon}(\cdot)) \to \Phi(u_2(\cdot)) \quad \text{as } \varepsilon \to 0+\,.$$

Hence it suffices to prove that for any $\varepsilon > 0$, $\delta > 0$ there exists $u(\cdot) \in U$ such that

$$|2\Phi(u(\cdot)) - \Phi(u_{1,\varepsilon}(\cdot)) - \Phi(u_{2,\varepsilon}(\cdot))| \leq \delta. \tag{6.8}$$

Now fix $\varepsilon > 0$. Consider the random number $\xi \triangleq w_1(\varepsilon)$, where $w_1(t)$ is the first component of the process $w(t)$. Set $\overline{\varphi}(\cdot) \triangleq (\varphi_0(\cdot), \varphi_1(\cdot), ..., \varphi_N(\cdot))$, $\overline{g}(\cdot) \triangleq (g_0(\cdot), g_1(\cdot), ..., g_N(\cdot))$. Let

$$\Gamma(u(\cdot)) \triangleq \overline{g}(y(0)) + \int_0^T \overline{\varphi}_i(t, y(t), z(t), u(t))dt, \tag{6.9}$$

where $u(\cdot) \in U$, and the pair $x(t) \equiv (y(t), z(t))$ is the state process corresponding to $u(\cdot)$.

Let $Z_i(x) : \mathbf{R} \to \mathbf{R}^{N+1}$ be defined as

$$Z_i(x) \triangleq \mathbf{E}\left\{\Gamma(u_{i,\varepsilon}(\cdot)) | \xi = x\right\} \rho(x), \quad i = 1, 2, \tag{6.10}$$

where $\rho(x)$ is the probability density function of $w_1(\varepsilon)$. In view of our assumptions, $Z_i \in L_1(\mathbf{R})$.

Consider the $(2N+2)$-dimensional function $Z(x) \triangleq [Z_1(x), Z_2(x)]$. By [5, Lemma 5.1], there exists a set $D \subset \mathbf{R}$ such that

$$\left|2\int_D Z_1(x)dx - \int_{\mathbf{R}} Z_1(x)dx\right| \leq \frac{\delta}{2}, \qquad \left|2\int_{\mathbf{R}\setminus D} Z_2(x)dx - \int_{\mathbf{R}} Z_2(x)dx\right| \leq \frac{\delta}{2}. \tag{6.11}$$

Set

$$u(t,\omega) \triangleq \begin{cases} u_{1,\varepsilon}(t,\omega) & \text{if } \xi \in D, \\ u_{2,\varepsilon}(t,\omega) & \text{if } \xi \notin D, \end{cases} \quad \text{for } t \in [0,T].$$

Then $u(t,\omega) = u_{1,\varepsilon}(t,\omega)$ for $t < \varepsilon$, hence $u(t,\omega)$ is progressively measurable with respect to $\mathcal{A}_t$, and $u(\cdot) \in U$. Furthermore,

$$\Phi(u(\cdot)) = \int_{\mathbf{R}} \mathbf{E}\left\{\Gamma(u(\cdot)) | \xi = x\right\} \rho(x)dx = \int_D \mathbf{E}\left\{\Gamma(u(\cdot)) | \xi = x\right\} \rho(x)dx$$

$$+ \int_{\mathbf{R}\setminus D} \mathbf{E}\left\{\Gamma(u(\cdot)) | \xi = x\right\} \rho(x)dx = \int_D Z_1(x)dx + \int_{\mathbf{R}\setminus D} Z_2(x)dx, \tag{6.12}$$

and

$$\Phi(u_{i,\varepsilon}(\cdot)) = \int_{\mathbf{R}} Z_i(x)dx. \tag{6.13}$$

Then (6.11)-(6.13) yields (6.8). This completes the proof of Lemma 6.1. $\square$

Let us denote by $\overline{\Phi(U)}$ the closure of the set $\Phi(U) \equiv \{\Phi(u(\cdot)) : u(\cdot) \in U\} \subset \mathbf{R}^{N+1}$.

Lemma 6.2 *The set $\overline{\Phi(U)}$ is convex.*

Proof. Let $z_1,\, z_2 \in \overline{\Phi(U)}$, $\alpha \in (0,1)$ be arbitrary. It is suffices to prove that for any $\delta > 0$ there exists $u(\cdot) \in U$ such that

$$|\alpha z_1 + (1-\alpha)z_2 - \Phi(u(\cdot))| \le \delta. \tag{6.14}$$

By definition of a closure, there exist $u_i(\cdot) \in U$ such that

$$|\Phi(u_i(\cdot)) - z_i| \le \frac{\delta}{4}, \quad i = 1, 2. \tag{6.15}$$

Let $\alpha = \sum_{i=1}^{+\infty} c_i 2^{-i}$, where $c_i \in \{0,1\}$. Introduce the numbers $\alpha_q \triangleq \sum_{i=1}^{q} c_i 2^i$, $q = 1, 2, \cdots$. We have $\alpha_q \to \alpha$ as $q \to +\infty$. Let k be such that

$$|\alpha_k - \alpha| \left(|\Phi(u_1(\cdot))| + |\Phi(u_2(\cdot))|\right) \le \frac{\delta}{4}. \tag{6.16}$$

Introduce the sets

$$\mathcal{A}_m \triangleq \left\{ \widehat{\alpha} \in (0,1) : \widehat{a} = \sum_{i=1}^{m} \widehat{c}_i 2^{-i}, \quad c_i \in \{0,1\} \right\},$$

$$\widehat{Z}_m \triangleq \left\{ z \in \mathbf{R}^{N+1} : \quad z = \widehat{\alpha}\Phi(u_1(\cdot)) + (1-\widehat{\alpha})\Phi(u_2(\cdot)), \quad \widehat{\alpha} \in \mathcal{A}_m \right\}, \quad m = 1, 2, ..., k.$$

Any element of $\widehat{Z}_m$ can be presented as either the middle or one of the edges of an interval connecting points which belong to $\widehat{Z}_{m-1}$. The number of elements in $\widehat{Z}_k$ is finite. It can be easily seen from Lemma 6.1 that for any $z \in \widehat{Z}_k$ there exists $u(\cdot) \equiv u(\cdot, z) \in U$ such that $|\Phi(u(\cdot, z)) - z| \le \delta/4$. Hence there exists $u(\cdot) \in U$ such that

$$|\alpha_k \Phi(u_1(\cdot)) + (1-\alpha_k)\Phi(u_2(\cdot)) - \Phi(u(\cdot))| \le \frac{\delta}{4} \tag{6.17}$$

By (6.15)-(6.17), the inequality (6.14) holds for this $u(\cdot)$. This completes the proof of Lemma 6.2. $\square$

Consequently, Theorem 6.1 follows from Lemma 6.2 and [5, Theorem 1.1] (see also [11, Theorem 0.1]).

Remark. It can be seen that the proof of Theorem 6.1 does not really depend on the specific structure of the equation (2.2). Hence this approach can be easily extended for a wide class of stochastic optimization problems with constraints.

7 Concluding Remarks

Study on controls of BSDE systems remains a relatively new endeavor and many research problems are open. For example, for the backward LQ problem we derived a Riccati-like equation which however lacks the symmetry property. What is a more appropriate Riccati equation? Can we have an optimal state feedback control in the conventional sense (i.e., the control is a function of the state (y, z))? Also, possible applications to contingent claims in finance promise great potential of the BSDE control problems.

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