# Universal strategies for diffusion markets and possibility of asymptotic arbitrage

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**Abstract**

The paper investigates the investment problem in a generic diffusion stochastic market model. Volatilities and appreciation rates are allowed to be random and unknown, with unknown prior distributions. We study "universal" strategies that use price observation only and do not require any knowledge on prior distributions of market parameters, i.e., where market parameters are not available. We define bounded risk strategies in this class that ensure a positive average gain for all random volatilities and appreciation rates from a wide class. Moreover, the strategies ensure a strengthened form of asymptotic arbitrage as the diversification of the portfolio increases: a given positive gain is ensured with probability arbitrarily close to 1.

*JEL classification*: D52, D81, D84, G11

*Subject classification*: IM10, IE10, IE13

*Keywords*: portfolio selection, universal strategies, asymptotic arbitrage

## 1 Introduction

The paper investigates the investment problem for a stochastic diffusion market model that consists of the risk free bond or bank account and of risky stocks. It is assumed that the dynamics of the stocks is given by continuous correlated random processes with some standard deviations of the stock returns (the volatility coefficients, or volatilities). The dynamics of bonds is deterministic and exponentially increasing with a given risk free rate. Empirical research shows that the real volatility is time-varying, random and correlated with stock prices (see Black and Scholes (1972)). A number of deterministic

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and stochastic equations for the volatility and the appreciation rates were proposed (see, e.g., the bibliography in Dokuchaev and Savkin (2002)).

Consider now the optimal investment problem given some optimality criterion (i.e., an utility function). Suppose that market parameters are allowed to be directly observed, and that their evolution law (or the prior distribution, or the probability measure) is fixed and known (i.e., the volatilities and the appreciation rates evolve according to known equations). Then the optimal strategy (i.e., the current vector of stocks portfolio) is a function of the current vector of the volatilities and the appreciation rates (see, e.g., Merton (1969) and survey in Karatzas and Shreve (1998)). In fact, this strategy is optimal for a given evolution law only. If this evolution law is changed, then the optimality property of the strategy may disappear. In more realistic setting, the parameters of the market are not supposed to be directly observed, but their evolution low is supposed to be known (see again the bibliography in Dokuchaev and Savkin (2002) and Dokuchaev (2002)). Again, the optimality of a strategy depends on the correctness of prior distributions. If one uses dynamic programming method, then the solution of the corresponding Bellmann equation depends on the future distributions of the random coefficients, because this equation must be solved backward starting form terminal time where some Cauchy condition is imposed. Even the myopic strategy that is optimal for log utility requires either direct observation of the appreciation rate or correct prior distribution hypothesis to calculate the conditional expectation of it. In fact, any "optimal" strategy is optimal only for a given model of price evolution, for a given probability measure (the prior distribution), and for a given utility function.

On the other hand, strategies based on "technical analysis" are model-free: they require only historical data. This is why they are so popular among traders (see, e.g., survey and discussion in Lo *et al.* (2000)). Our aim is to reduce the gap between model-free strategies and strategies based on stochastic models.

The paper studies strategies which do not employ any distribution assumptions on stocks evolution. Such a strategy was introduced first by Cover (1991) (the so-called *universal portfolio* strategy). The algorithm asymptotically outperforms the best stock in the market, under some conditions that in fact reflect the hypothesis that stock prices oscillate around some stable values. But it is not a bounded risk algorithm, because the wealth may tend to zero for some "bad" samples of stock prices. Some statistical analysis of performance of this strategy for real data has been done in Blædel *et al* (1999). It appears that the spectacular results of universal portfolios do not necessary materialize for a given historical market. A possible reason is that the required condition of price oscillations is too restrictive.

Dokuchaev and Savkin (2002) and Dokuchaev (2002), Chapter 2, proposed bounded risk strategies for a single stock discrete-time market that have some properties of Cover's "universal" strategy but ensure gain for a market with a trend rather than for a market with stable oscillations.

The aim of the current paper is to obtain a continuous time analog of "universal portfolio strategy" that (i) uses only stock price observations and does not require any knowledge about the appreciation rate, the volatility or other market parameters; (ii) bounds risk closely to the risk-free investment; (iii) gives some additional positive in average gain.

We consider a market model which consist of risky stocks and a risk-free bond (or bank account). We propose a strategy which differs from the strategy of Cover (1991) and has the desired properties (i)-(iii). The additional gain is positive in average for any case where the historical probability measure is not a risk-neutral measure, under some mild additional assumptions on probability distributions. These conditions are such that the market is still incomplete. The strategy itself does not use probability assumptions. Thus, we obtain a strategy for someone who basically prefers risk-free investments but accepts some bounded risk for the sake of an additional gain. The strategy is expressed as an explicit function of historical prices.

In addition, we found that our strategies ensure asymptotic arbitrage as the diversification of the portfolio increases.

A risk-free profitable strategy is said to be arbitrage. It is commonly recognized that any reasonable market model must be arbitrage free. Harrison and Pliska (1981) have shown that the arbitrage opportunity does not exist in the finite diffusion stochastic market model if there exists a risk-neutral probability measure. But some opportunity of arbitrage as a limit or asymptotic arbitrage does exist for some generic models. One definition of asymptotic arbitrage was introduced by Kabanov and Kramkov (1994). Another related definition is that of so-called "free lunch" (Harrison and Kreps (1979)). There are many results on existence or nonexistence of "free lunches" and asymptotic arbitrage opportunities. For example, it is known that "free lunches" do not exist in a diffusion market model with sequences of strategies that are piecewise constant with a bounded number of switching, and "free lunches" do exist in the case of an unlimited number of switching and unlimited borrowing (see e.g. Dalang *et al* (1990), Duffie and Huang (1986), Frittelli and Lakner (1992), Harrison and Kreps (1979), Jouini and Kallal (1995), Jouini (1996), Kreps (1981), Kabanov and Kramkov (1998), Klein and Schachermayer (1996)).

We show that there exist asymptotic arbitrage opportunity for a very generic diffusion market for the class of strategies that does not require observations of market parameters as well as prior hypothesis on its distributions (we emphasize that the future distributions cannot be estimated from current observations because there is no information available that distributions are stationary or evolve under some given law). We propose a strategy based on price observations only that ensures a strengthened form of the asymptotic arbitrage of the first kind introduced by Kabanov and Kramkov (1994): a fixed positive gain is ensured with probability $1 - \varepsilon$ for arbitrarily small $\varepsilon > 0$ for a wide class of volatilities and appreciation rates that includes all bounded random volatilities. In Section 6, we compare this result with asymptotic arbitrage opportunity for the Merton's strategies applied for a large market.

Dokuchaev and Savkin (1997) also proposed some "universal" strategies that ensure asymptotic arbitrage, but only for a market model where all stocks are driven by independent Brownian motions; in addition, these strategies are different from ones introduced below, and they include volatility.

In Section 2 we present notation and definitions, and describe the model. A single-stock universal strategy is presented in Section 3, and in Section 4 a multi-stock universal strategy for the diffusion market is presented. In Section 5 we demonstrate that the strategy ensures the asymptotic arbitrage opportunity. In Section 6 we compare our strategy with Merton's strategy. The proofs are given in the Appendix.

# 2 Definitions

Consider the diffusion model of a securities market consisting of the risk free bond or bank account with the price $B(t)$, $t \geq 0$, and the risky stocks with prices $S_i(t)$, $t \geq 0$, $i = 1, 2, \ldots, N$. We consider both cases of $N < +\infty$ and $N = +\infty$. The prices of the stocks evolve according to the following stochastic differential equations

$$dS_i(t) = S_i(t) \left[ a_i(t)dt + \sum_{j=1}^{N} \sigma_{ij}(t)dw_j(t) \right], \quad t > 0, \tag{1}$$

where $a_i(t)$ is the appreciation rate, $\sigma_{ij}(t)$ is the volatility coefficient, $w_i(t)$ are standard Wiener processes. The initial price $S_i(0) > 0$ is a given non-random value. The price of the bond evolves according to the following equation

$$B(t) = \exp\left( \int_0^t r(s)ds \right) B(0), \quad t \geq 0, \tag{2}$$

where $r(t) \geq 0$ is a random process, and $B(0)$ is a given constant.

We assume that $w_i(\cdot)$ are independent processes.

Set vector processes $a^{(n)}(t) \triangleq (a_1(t),\dots,a_n(t))$, $\sigma^{(n)}(t) \triangleq \{\sigma_{ij}(t)\}_{i,j=1}^N$, $w^{(n)}(t) \triangleq (w_1(t),\dots,w_n(t))$ and $S^{(n)}(t) \triangleq (S_1(t),\dots,S_n(t))$. Furthermore, set processes $a(t) \triangleq a^{(N)}(t)$, $\sigma(t) \triangleq \sigma^{(N)}(t)$, $w(t) \triangleq w^{(N)}(t)$ and $S(t) \triangleq S^{(N)}(t)$.

Let $\mathcal{F}_t^n$ be a right-continuous monotonically increasing filtration of complete $\sigma$-algebras of events such that the process $w^{(n)}(t)$ is progressively measurable with respect to $\mathcal{F}_t^n$ and $w^{(n)}(t+\tau) - w^{(n)}(t)$ does not depend on $\mathcal{F}_t^n$ for all $t \ge 0$ and $\tau > 0$. Let $\mathcal{F}_t \triangleq \mathcal{F}_t^N$.

Note that we do not exclude a special case where $\mathcal{F}_t^n = \overline{\sigma\{w_i(s),\ s \le t,\ i = 1,\dots,n\}}$, i.e., it is the filtration, generated by $\{w_i(t)\}_{i=1}^n$. We do not exclude also a case where $\mathcal{F}_t^n$ generated by $w^{(n)}(\cdot)$ and by processes independent of $w^{(n)}(\cdot)$.

We consider a case where market is not specified perfectly, i.e., where parameters $\sigma(\cdot)$ and $a(\cdot)$ are not fixed. However, it will be assumed that they are from a given set.

Let $\Sigma$ be the set of all $\sigma(\cdot)$ such that $\sigma_{ij}(t)$ are random processes that are progressively measurable with respect to $\mathcal{F}_t$ and $\sum_{j=1}^N |\sigma_{ij}(t)| \le \widetilde{C}_i$ a.s. for all $t$ and $i$, where $\widetilde{C}_i > 0$ is a given constant, $i = 1,\dots,N$.

Furthermore, let $\mathcal{A}$ be the set of all $(a(\cdot), r(\cdot))$ such that $a_i(t)$ and $r(t)$ are random processes that are progressively measurable with respect to $\mathcal{F}_t$ and $|a_i(t)| \le C_i$, $r(t) \in [0, C_0]$ a.s. for all $t$ and $i$, where $C_i > 0$ is a given constant, $i = 0, 1, \dots, N$.

Let $\bar{\mathcal{B}}$ be the set of $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \Sigma \times \mathcal{A}$ such that the process $\sum_{i=1}^n (a_i(t) - r(t))$ does not depend on $(\sigma(\cdot), w(\cdot))$ for all $n < +\infty$, $n \le N$.

Let $X(0) > 0$ be the initial wealth at time $t = 0$, $X(t)$ be the wealth at time $t > 0$. Without loss of generality, we assume that $X(0) = 1$. Though the number $N$ of the available assets may be infinite, we assume that only a finite number $n \le N$ of them is traded by the agent, and the wealth $X(t)$ at time $t \ge 0$ is

$$X(t) = \beta(t)B(t) + \sum_{i=1}^n \gamma_i(t)S_i(t). \tag{3}$$

Here $n < +\infty$, $\beta(t)$ is the quantity of the bond portfolio, $\gamma_i(t)$ is the quantity of the $i$th stock portfolio, $\gamma(t) = (\gamma_1(t),\dots,\gamma_n(t))$, $t \ge 0$. The pair $(\beta(t), \gamma(t))$ describes the state of the bond-stocks securities portfolio at time $t$. We call these pairs strategies.

We consider the problem of investment or choosing a strategy.

**Definition 2.1** *The process* $\widetilde{X}(t) \triangleq \exp\left(-\int_0^t r(s)ds\right) X(t)$ *is called the normalized wealth.*

Clearly, $\widetilde{X}(0) = 1$.

Let $\mathcal{G}_t^n$ be the filtration generated by $(S^{(n)}(t), B(t))$.

**Definition 2.2** *Let* $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \Sigma \times \mathcal{A}$ *be fixed. A pair* $(\beta(t), \gamma(t)) = (\beta(t), \gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))$ *is said to be an admissible strategy if* $n < +\infty$, $n \le N$ *and* $\beta(t)$, $\gamma_i(t)$, $\gamma_i(t)S_i(t)$, $i = 1, \ldots, n$, *are random processes that are progressively measurable with respect the filtration* $\mathcal{G}_t^n$ *and such that*

$$\mathbf{E}\int_0^T |\beta(t)|^2 dt < +\infty, \quad \mathbf{E}\int_0^T \left(|\gamma(t)|^2 + \sum_{i=1}^n S_i(t)^2 \gamma_i(t)^2\right) dt < +\infty \quad \forall T > 0. \tag{4}$$

The main constraint in choosing a strategy is so-called condition of self-financing.

**Definition 2.3** *A pair* $(\beta(t), \gamma(t))$ *is said to be self-financing, if*

$$dX(t) = \beta(t) dB(t) + \sum_{i=1}^n \gamma_i(t) dS_i(t). \tag{5}$$

Set $\widetilde{S}(t) \triangleq \exp\left(-\int_0^t r(s)ds\right) S(0)$. As known, (5) can be rewritten as

$$d\widetilde{X}(t) = \sum_{i=1}^n \gamma_i(t) d\widetilde{S}_i(t). \tag{6}$$

It will be convenient to define a class of strategies as deterministic functions of historical prices.

**Definition 2.4** *A function* $\Gamma(t, \cdot) : C([0,t]; \mathbf{R}^{n+1}) \to \mathbf{R}^n$, $t \ge 0$, *is said to be an admissible CL-strategy (closed-loop strategy) if the corresponding pair* $(\beta(t), \gamma(t))$ *defined from the closed system*

$$\begin{array}{l} \gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)) = \Gamma\left(t, [S^{(n)}(\cdot), B(\cdot)]|_{[0,t]}\right), \\ \beta(t) = \frac{X(t) - \sum_{i=1}^n \gamma_i(t) S_i(t)}{B(t)}, \end{array} \tag{7}$$

*where*

$$X(t) = X(0) + \sum_{i=1}^n \int_0^t \gamma_i(s) dS_i(s) + \int_0^t \frac{X(t) - \sum_{i=1}^n \gamma_i(t) S_i(t)}{B(t)} dB(t), \tag{8}$$

*is an admissible self-financing strategy for any* $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \Sigma \times \mathcal{A}$.

For the market model with a set of possible $(\sigma(\cdot), a(\cdot), r(\cdot))$, it is more practical to use the CL-strategies $\Gamma(t, \cdot)$ rather then the random processes $(\beta(t), \gamma(t))$.

Note that for different $(\sigma(\cdot), a(\cdot), r(\cdot))$, the random processes $(\beta(t), \gamma(t))$ with same $\Gamma(t, \cdot)$ in (7) may be different, as well as $\Gamma(t, \cdot)$ in (7) may be different for same processes $\gamma(t)$. For instance, let $n = N = 1$, $S(0) = 1$, and $\Gamma\left(t, [S(\cdot), B(\cdot)]|_{[0,t]}\right) \triangleq \ln S(t)$, then

$\gamma(t) = t/2 + w(t)$ for a market with $a(t) \equiv 1$, $\sigma(t) \equiv 1$, and $\gamma(t) = t/4 + w(t)/2$ for a market with $a(t) \equiv 1/2$, $\sigma(t) \equiv 1/2$. Similarly, the process $\gamma(t) \triangleq t/2 + w(t)$ generates $\Gamma\left(t, [S(\cdot), B(\cdot)]|_{[0,t]}\right) = \ln S(t)$ for a market with $a(t) \equiv 1$, $\sigma(t) \equiv 1$, and it generates $\Gamma\left(t, [S(\cdot), B(\cdot)]|_{[0,t]}\right) = \ln S(t)^2$ for a market with $a(t) \equiv 1/2$, $\sigma(t) \equiv 1/2$.

**Definition 2.5** *Let $T > 0$ be fixed, and let $C(t)$ be a random process such that $C(t) \in (0, 1]$ for all $t$ a.s. An admissible CL-strategy $\Gamma(t, \cdot)$ is said to be a bounded risk strategy with the bound $C(\cdot)$ if*

$$\widetilde{X}(t) \geq C(t) \quad a.s. \quad \forall t \in [0, T] \quad \forall (\sigma(\cdot), a(\cdot), r(\cdot)) \in \Sigma \times \mathcal{A}.$$

The following definition is a particular case of the classical definition of arbitrage (see Harrison and Pliska (1981)).

**Definition 2.6** *Let $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \Sigma \times \mathcal{A}$ be fixed, let $(\beta(t), \gamma(t))$ be an admissible self-financing strategy, let $\widetilde{X}(t)$ be the corresponding normalized wealth, and let $T > 0$ be a given non-random time. If*

$$\mathbf{P}(\widetilde{X}(T) \geq 1) = 1, \quad \mathbf{P}(\widetilde{X}(T) > 1) > 0,$$

*then this strategy is said to be arbitrage.*

The following definition is a particular case of the definition of asymptotic arbitrage from Kabanov and Kramkov (1994), (1998).

**Definition 2.7** *Let $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \Sigma \times \mathcal{A}$ and $T > 0$ be fixed, let $(\beta^{(m)}(t), \gamma^{(m)}(t))$, $m = 1, 2, \ldots$ be a sequence of admissible self-financing strategies, let $X^{(m)}(t)$ be the corresponding total wealth, $X^{(m)}(0) = 1$ $(\forall m)$. Let $\widetilde{X}^{(m)}(t)$ be the corresponding normalized wealth. Suppose that there exists real numbers $\kappa > 1$, $p_0 > 0$ such that for any $\varepsilon > 0$ there exists a number $\bar{m}$ such that*

$$\widetilde{X}^{(m)}(t) \geq 1 - \varepsilon \quad a.s. \ \ \forall t \in [0, T], \quad \mathbf{P}(\widetilde{X}^{(m)}(T) \geq \kappa) \geq p_0 \quad \forall m \geq \bar{m}.$$

*Then the sequence $(\beta^{(m)}(t), \gamma^{(m)}(t))$ is said to be asymptotic arbitrage of the first kind.*

The following definition strengthen the requirements of Definitions 2.6–2.7; it assumes a positive gain with probability arbitrarily close to 1.

**Definition 2.8** *Let $\mathcal{B} \subseteq \Sigma \times \mathcal{A}$ be a given subset of the set $\Sigma \times \mathcal{A}$. Let $\Gamma^{(m)}(t, \cdot)$, $m = 1, 2, \ldots$ be a sequence of admissible CL-strategies, and let $X^{(m)}(t)$ be the corresponding total wealth, $X^{(m)}(0) = 1$ $(\forall m)$. Let $\widetilde{X}^{(m)}(t)$ be the corresponding normalized wealth.*

*Let $T > 0$ be given. Suppose that there exists a real number $\kappa > 1$ such that for any $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \Sigma \times \mathcal{A}$, $\varepsilon > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exists a number $\widehat{m}$ such that*

$$\widetilde{X}^{(m)}(t) \geq 1 - \varepsilon \quad a.s. \ \ \forall t \in [0, T], \quad \mathbf{P}(\widetilde{X}^{(m)}(T) \geq \kappa - \varepsilon_1) \geq 1 - \varepsilon_2 \quad \forall m \geq \widehat{m}.$$

*Then the sequence $\Gamma^{(m)}(t, \cdot)$ is said to be asymptotic arbitrage that almost guarantees the gain $\kappa$ for the class $\mathcal{B}$.*

# 3 A strategy for a single stock market

In this Section, we assume that $N = n = 1$. Set

$$\psi(y) \triangleq e^y - y.$$

Clearly, $\psi(0) = 1$, $\psi(y) > 1$ ($\forall y \neq 0$). Set $v(t) \triangleq \frac{1}{2} \int_0^t \sigma(s)^2 ds$.

**Theorem 3.1** *Let*

$$\begin{array}{ll} \widetilde{X}(t) \triangleq \psi[\log \widetilde{S}(t)] - v(t), & X(t) \triangleq \exp\left(\int_0^t r(s)ds\right) \widetilde{X}(t), \\ \gamma(t) \triangleq \frac{1}{S(0)} - \widetilde{S}(t)^{-1}, & \beta(t) \triangleq \frac{X(t) - \gamma(t)S(t)}{B(t)}. \end{array} \tag{9}$$

*Then, for any $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \Sigma \times \mathcal{A}$, the pair $(\beta(t), \gamma(t))$ is an admissible and self-financing strategy with the corresponding wealth $X(t)$ and normalized wealth $\widetilde{X}(t)$. The process $(\beta(t), \gamma(t))$ does not depend on the distributions of the parameters. Furthermore, there exist a function $\Gamma(t, \cdot) : C([0, t]; \mathbf{R}) \to \mathbf{R}$ such that $\gamma(t) = \Gamma(t, \widetilde{S}(\cdot)|_{[0,t]})$. The function $\Gamma(t, \cdot)$ does not depend on $(\sigma(\cdot), a(\cdot), r(\cdot))$, and it is an admissible bounded risk CL-strategy with the bound $C(t) = 1 - v(t)$:*

$$\widetilde{X}(t) \geq 1 - v(t) \quad a.s. \quad \forall t \geq 0 \quad \forall (\sigma(\cdot), a(\cdot), r(\cdot)) \in \Sigma \times \mathcal{A}.$$

*Moreover, if $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \bar{\mathcal{B}}$, then*

$$\mathbf{E}\widetilde{X}(t) = \mathbf{E}\psi\left(\int_0^t [a(s) - r(s)]ds\right) \qquad \forall t \geq 0,$$

*and*

$$E\widetilde{X}(t) > 1 \quad \textit{if and only if} \quad \mathbf{P}\left(\int_0^t [a(s) - r(s)]ds \neq 0\right) > 0.$$

Note that $\int_0^t [a(s) - r(s)]ds = 0$ ($\forall t > 0$) if and only if $a(s) \equiv r(s)$.

*Remark 3.1.* Let $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \Sigma \times \mathcal{A}$ and let $\sigma(\cdot)$ does not depend on $w(\cdot)$. If $\mathbf{P}\left(\int_0^t [a(s) - r(s)]ds \neq 0\right) > 0$ and $\sigma \equiv 0$, then $\mathbf{E}\widetilde{X}(t) > 1$ and $\widetilde{X}(t) \geq 1$ a.s., i.e. the strategy defined in Theorem 3.1 is arbitrage. Let $\mathbf{P}(v(t) > 0) > 0$, then $\mathbf{P}(S(t) \in I) > 0$ for any interval $I \subset \mathbf{R}_+$. In that case, it follows from (9) that $\mathbf{P}(\widetilde{X}(t) < 1) > 0$. Thus, the strategy defined in Theorem 3.1 is arbitrage if and only if $\mathbf{P}\left(a(\cdot) \neq r(\cdot)\right) > 0$ and $\sigma \equiv 0$.

Note that for the single-stock discrete-time market a strategy with similar features was studied in Dokuchaev and Savkin (2002).

# 4 A strategy for a multi-stock market

In this section, we assume that $1 \leq n < +\infty$, $n \leq N \leq +\infty$. Introduce the following functions

$$v_j(t) \triangleq \int_0^t \left(\sum_{i=1}^n \sigma_{ij}(s)^2\right) ds, \quad \nu(t) \triangleq \frac{1}{2n^2} \sum_{i=1}^N v_i(t). \tag{10}$$

Set

$$\alpha_i(t) \triangleq \int_0^t [a_i(s) - r(s)]ds, \qquad A_n(t) \triangleq \frac{1}{n} \sum_{i=1}^n \alpha_i(t), \tag{11}$$

$$\eta_i(t) \triangleq \sum_{j=1}^N \int_0^t \sigma_{ij}(s)dw_j(s), \quad \bar{\eta}(t) \triangleq \frac{1}{n} \sum_{i=1}^n \eta_i(t). \tag{12}$$

**Theorem 4.1** *Let the process $\widetilde{Y}(t)$ evolves as*

$$\begin{cases} d\widetilde{Y}(t) \triangleq \frac{1}{n}\widetilde{Y}(t) \sum_{i=1}^n \frac{d\widetilde{S}_i(t)}{\widetilde{S}_t(t)}, \\ \widetilde{Y}(0) = 1. \end{cases} \tag{13}$$

*Let*

$$\widetilde{X}(t) \triangleq \psi\left(A_n(t) + \bar{\eta}(t) - \nu(t)\right) - \nu(t), \quad X(t) \triangleq \exp\left(\int_0^t r(s)ds\right) \widetilde{X}(t), \tag{14}$$

$$\gamma_i(t) \triangleq \frac{\widetilde{Y}(t)}{n\widetilde{S}_i(t)} - \frac{1}{n\widetilde{S}_i(t)}, \tag{15}$$

$$\gamma(t) \triangleq (\gamma_1(t), \ldots, \gamma_n(t)), \quad \beta(t) \triangleq \frac{X(t) - \sum_{i=1}^n \gamma_i(t) S_i(t)}{B(t)}. \tag{16}$$

*Then, for any $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \Sigma \times \mathcal{A}$, the pair $(\beta(t), \gamma(t))$ is an admissible and self-financing strategy with the corresponding wealth $X(t)$ and the normalized wealth $\widetilde{X}(t)$. The process $(\beta(t), \gamma(t))$ does not depend on the future values of volatility. Furthermore, there exist a function $\Gamma(t, \cdot) : C([0,t]; \mathbf{R}^n) \to \mathbf{R}^n$ such that $\gamma(t) = \Gamma(t, \widetilde{S}(\cdot)|_{[0,t]})$. The* *function $\Gamma(t,\cdot)$ does not depend on $(\sigma(\cdot),a(\cdot),r(\cdot))$, and it is an admissible bounded risk CL-strategy with the bound $C(t)=1-\nu(t)$:*

$$\widetilde{X}(t)\ge 1-\nu(t)\quad a.s.\quad \forall t>0\quad \forall(\sigma(\cdot),a(\cdot),r(\cdot))\in\Sigma\times\mathcal{A}. \tag{17}$$

*Moreover, if $(\sigma(\cdot),a(\cdot),r(\cdot))\in\bar{\mathcal{B}}$, then*

$$\mathbf{E}\widetilde{X}(t)=\mathbf{E}\psi\left(A_n(t)\right)\quad \forall t>0, \tag{18}$$

*and*

$$\mathbf{E}\widetilde{X}(t)>1\quad \textit{if and only if}\quad \mathbf{P}(A_n(t)\neq 0)>0. \tag{19}$$

For $T>0$, $\bar{v}>0$, introduce the class $\Sigma(\bar{v},T)\subset\Sigma$ of all $\sigma(\cdot)$ such that

$$\frac{1}{n}\sum_{j=1}^{N}v_j(T)\le\bar{v}\quad \text{a.s.}\quad \forall n, \tag{20}$$

where $v_i(t)$ are defined in (10).

Note that the condition (20) is not restrictive. For example, let $|\sigma_{ij}(t)|\le\text{const}$ and $K(i,t)\le\text{const}$, where $K(i,t)$ is the number of $j$ such that $\sigma_{ij}(t)\neq 0$. Then (20) is satisfied.

*Remark 4.1* By (18), the performance of the strategy is better for $A_n(t)>0$ than for $A_n(t)<0$, because the shape of $\psi$ is asymmetric. Using similar approach, we can find strategies to ensure (18) for $\psi$'s with other shape, for instance, $\psi(x)=e^{-x}+x$, or $\psi(x)=(e^x+e^{-x})/2$.

**Corollary 4.1** *For any $T>0$, $\bar{v}>0$,*

$$\widetilde{X}(t)\ge 1-\bar{v}/2n=1-\varepsilon_n\quad a.s.\quad \forall t\in[0,T]\quad \forall(\sigma(\cdot),a(\cdot),r(\cdot))\in\Sigma(\bar{v},T)\times\mathcal{A}, \tag{21}$$

*where $\varepsilon_n\triangleq\bar{v}/(2n)\to 0$ as $n\to+\infty$. In other words, the maximum loss for the strategies defined in Theorem 4.1 converges to zero as the number $n$ of the traded stocks increases.*

We cannot conclude yet that (19) and (21) ensure asymptotic arbitrage as it defined in Definitions 2.7–2.8 because a lower boundary of gain is not established. In the following section, we give some sufficient conditions that ensure asymptotic arbitrage.

# 5 Asymptotic arbitrage

In this section, we assume that $N=+\infty$. Let $T>0$ be a fixed time.

For $\theta > 0$ and $p \in (0,1]$, introduce the set $\mathcal{A}(\theta,p,T) \subset \mathcal{A}$ such that for any $(a(\cdot),r(\cdot)) \in \mathcal{A}(\theta,p,T)$ there exists a number $\widehat{n} = \widehat{n}(a(\cdot),r(\cdot))$ such that

$$\mathbf{P}\left(|A_n(T)| \geq \theta\right) \geq p \quad \forall n > \widehat{n}, \tag{22}$$

where $A_n(T) \triangleq \frac{1}{n}\sum_{i=1}^n \left(\int_0^T [a_i(s) - r(s)]ds\right)$.

Let $\chi$ denote the indicator function.

**Theorem 5.1** *Let $T > 0$, $\bar{v} > 0$, $\theta > 0$, and $p \in (0,1]$ be fixed. Consider the sequence of the strategies $(\beta(t),\gamma(t)) = (\beta^{(n)}(t),\gamma^{(n)}(t))$, defined in Theorem 4.1. Let $\widetilde{X}^{(n)}(t)$ be the corresponding normalized wealth. Then*

*(i) For any $(\sigma(\cdot),a(\cdot),r(\cdot)) \in (\Sigma(\bar{v},T) \times \mathcal{A}(\theta,p,T)) \cap \bar{\mathcal{B}}$, $\varepsilon > 0$, there exists a number $\bar{n}$ such that*

$$\begin{aligned}&\widetilde{X}^{(n)}(t) \geq 1-\varepsilon \quad a.s. \quad \forall t \in [0,T],\\ &\mathbf{E}\widetilde{X}^{(n)}(T) \geq p\psi(-\theta) + \mathbf{P}\left(|A_n(T)| < \theta\right)\mathbf{E}\{\psi(A_n(T))\,\big|\,|A_n(T)| < \theta\} \geq p\psi(-\theta)\end{aligned} \tag{23}$$

*for all $n \geq \bar{n}$.*

*(ii) For any $(\sigma(\cdot),a(\cdot),r(\cdot)) \in \Sigma(\bar{v},T) \times \mathcal{A}(\theta,p,T)$, $\varepsilon > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exists a number $\bar{n}$ such that*

$$\widetilde{X}^{(n)}(t) \geq 1-\varepsilon \quad a.s \quad \forall t \in [0,T], \quad \mathbf{P}\left(\widetilde{X}^{(n)}(T) \geq \psi(-\theta) - \varepsilon_1\right) \geq p - \varepsilon_2 \tag{24}$$

*for all $n \geq \bar{n}$.*

*(iii) Let $\mathcal{A}_u \subset \mathcal{A}(\theta,p,T)$ be a set such that there exists a number $\widehat{n}$ such that (22) holds for all $(a(\cdot),r(\cdot)) \in \mathcal{A}_u$. Then for any $\varepsilon > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exists a number $\bar{n}$ such that (24) holds for all $(\sigma(\cdot),a(\cdot),r(\cdot)) \in \Sigma(\bar{v},T) \times \mathcal{A}_u$ and $n \geq \bar{n}$.*

**Corollary 5.1** *(i) For any $p \in (0,1]$ and $(\sigma(\cdot),a(\cdot),r(\cdot)) \in \Sigma(\bar{v},T) \times \mathcal{A}(\theta,p,T)$, the sequence of strategies in Theorem 5.1 is asymptotic arbitrage of the first kind (Definition 2.7) if $p\psi(-\theta) > 1$.*

*(ii) This sequence is asymptotic arbitrage that almost guarantees the gain $\psi(\theta)$ for the class $\Sigma(\bar{v},T) \times \mathcal{A}(\theta,1,T)$ (Definition 2.8).*

*(iii) Property (24) is ensured for large $n$ uniformly in $(\sigma(\cdot),a(\cdot),r(\cdot)) \in \Sigma(\bar{v},T) \times \mathcal{A}_u$.*

*Remark 5.1.* The strategy $(\beta^{(n)}(t),\gamma^{(n)}(t))$ can be approximated by strategies $(\beta^{(n,m)}(t),\gamma^{(n,m)}(t))$, $m = 1,2,3,\ldots$, that are constant at the intervals $(lT/m,(l+1)T/m)$,

$l = 0, 1, \ldots, m-1$, and such that $\mathbf{E}|X^{(n,m)}(T) - X^{(n)}(T)|^2 \to 0$ as $m \to +\infty$ for the corresponding values of wealth. Hence the second inequalities in (23)–(24) may be ensured as a limit for these piecewise constant strategies, but the first inequalities there can be guaranteed only with probability that is close to 1, but not almost surely.

# 6 Comparison with Merton's strategies

Let us consider limit properties of the *Merton's strategy* for the market with a large number of stocks.

Let $n = N < +\infty$, let the matrix $\sigma(t)\sigma(t)^\top$ be invertible, and let $Q(t) \triangleq (\sigma(t)\sigma(t)^\top)^{-1}$. Consider the following closed-loop strategy:

$$\gamma_M(t)^\top \triangleq \widehat{a}(t)^\top \mathbf{S}(t)^{-1} Q(t) X_M(t). \tag{25}$$

This is a special case of the Merton's strategy, and it is optimal for the problem of maximizing $\mathbf{E} \ln \widetilde{X}(T)$ in the class of $\mathcal{F}_t$-adapted strategies.

Here $X_M(t)$ is the corresponding wealth, $\widehat{a}(t) \triangleq \mathbf{E}\{\widetilde{a}(t)|\mathcal{F}_t\}$, $\widetilde{a}(t) \triangleq a(t) - r(t)\mathbf{1}$, $\mathbf{1} = (1, 1, \ldots, 1)^\top \in \mathbf{R}^n$, $\mathbf{S}(t)$ is the diagonal matrix in $\mathbf{R}^{n\times n}$ with the diagonal $S_1(t), \ldots, S_n(t)$.

Let $X_0 = 1$ and $(\sigma(\cdot), a(\cdot), \sigma(\cdot)) \in \bar{\Sigma} \times \mathcal{A}(\theta, p, T)$ for some $\theta > 0$ and $p \in (0, 1]$. For strategy (25,

$$\mathbf{E} \ln \widetilde{X}_M(T) = \frac{1}{2} \mathbf{E} \int_0^T \widehat{a}(t)^\top Q(t) \widehat{a}(t) dt \to +\infty \quad \text{as} \quad n \to \infty$$

under some mild conditions, where $\widetilde{X}_M(t)$ is the corresponding normalized wealth. (For details regarding this strategy see, e.g., Dokuchaev (2002)). However, *the Merton's strategy (25) does require the prior distribution of* $a(t)$ *to calculate* $\widehat{a}(t)$, and if $\widehat{a}(t)$ in (25) is replaced for an estimation of $\widetilde{a}(t)$ based on a wrong hypothesis about the prior distribution of $a(t)$, then $\mathbf{E} \ln \widetilde{X}_M(T)$ can be negative. Moreover, it can be shown that for any given hypothesis on the prior distribution of $(\sigma(\cdot), a(\cdot), r(\cdot))$ and for any $\xi < 0$, there exists $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \bar{\Sigma} \times \mathcal{A}(\theta, p, T)$ such that this hypothesis is wrong and there exists $n > 0$ such that $\mathbf{E} \ln \widetilde{X}_M(T) \le \xi$ for the Merton's type strategy based on this hypothesis. On the other hand, our strategy (15) is a bounded risk strategy: $\widetilde{X}(T) \ge 1 - \nu$ and $\ln \widetilde{X}(T) \ge \ln(1 - \nu)$, and it does not use any hypothesis on the prior distributions.

Further, let us modify the strategy (25) such that

$$\gamma_M(t)^\top \triangleq \frac{1}{n} \widehat{a}(t)^\top \mathbf{S}(t)^{-1} Q(t) X_M(t). \tag{26}$$

Now $\mathbf{E} \ln \widetilde{X}_M(T)$ is bounded for large $n$ if $\widehat{a}$ is bounded. This strategy is not optimal for log utility anymore. It is optimal for the problem of minimizing $\mathbf{E}\widetilde{X}(T)^{\nu}$ with $\delta \triangleq 1-n$, but only in the case of for non-random and known $(Q, \widetilde{a})$, i.e., for the case that is out of our interests for now. However, we shall consider performance of the strategy for random coefficients without concern about optimality. Again, the resulting wealth for this strategy very depends on the correctness of the hypothesis on the prior distributions, and in that sense the performance of this strategy is worse than for the strategy (15). For instance, for any given hypothesis on the prior distribution of $(\sigma(\cdot), a(\cdot), r(\cdot))\bar{\Sigma} \times \mathcal{A}(\theta, 1, T) \cap \bar{\mathcal{B}}$ and for any $\varepsilon > 0$, there exist $n > 0$, and $(\sigma(\cdot), a(\cdot), r(\cdot)) \in \bar{\Sigma} \times \mathcal{A}(\theta, 1, T) \cap \bar{\mathcal{B}}$ such that $\mathbf{E} \ln \widetilde{X}_M(T) \le -\zeta + \varepsilon$, where $\zeta$ is the value of $\mathbf{E} \ln \widetilde{X}(T)$ calculated with the correct prior distribution.

In addition, our strategy does not require that the matrix $\sigma(t)\sigma(t)^\top$ is invertible and does not include $Q(t)$.

# 7 Appendix: Proofs

*Proof of Theorem 4.1.* Let

$$\bar{Y}(t) \triangleq \exp\left(A_n(t) + \bar{\eta}(t) - \nu(t)\right).$$

By Itô's formula,

$$d\bar{Y}(t) = \frac{1}{n}\bar{Y}(t)\sum_{i=1}^n \frac{d\widetilde{S}_i(t)}{\widetilde{S}_i(t)}.$$

Hence $\bar{Y}(t) = \widetilde{Y}(t)$, and $\widetilde{X}(t) = \widetilde{Y}(t) - A_n(t) - \bar{\eta}(t)$. Thus, $\widetilde{X}(t)$ is the normalized wealth defined by strategy (15)–(16). Further, (14) implies (17)-(18), and (18) implies (19). This completes the proof. □

Theorem 3.1 is a special case of Theorem 4.1.

*Proof of Theorem 5.1.* The first inequalities in (23)-(24) for large $n$ are ensured by Corollary 4.1. The function $\psi(y)$ is increasing in $y > 0$ and decreasing in $y < 0$. Then

$$\begin{aligned}
\mathbf{E}\widetilde{X}^{(n)}(T) &= \mathbf{E}\psi(A_n(T)) \\
&= \mathbf{P}\left(|A_n(T)| \ge \theta\right) \mathbf{E}\psi(A_n(T)) \,\big|\, |A_n(T)| \ge \theta\} \\
&\qquad\qquad + \mathbf{P}\left(|A_n(T)| < \theta\right) \mathbf{E}\{\psi(A_n(T)) \,\big|\, |A_n(T)| < \theta\} \\
&\ge p\psi(-\theta) + \mathbf{P}\left(|A_n(T)| < \theta\right) \mathbf{E}\{\psi(A_n(T)) \,\big|\, |A(T)| < \theta\} \\
&\ge p\psi(-\theta)
\end{aligned}$$

for $n > \bar{n}$. This completes the proof of Theorem 5.1 (i).

Furthermore,

$$\widetilde{X}^{(n)}(T) = \psi(A_n(T) + \bar{\eta}(T) - \nu(T)) - \nu(T).$$

By (20),

$$\nu(T) \to 0 \quad \text{as} \quad n \to +\infty, \tag{A.27}$$

and

$$\begin{aligned} \mathbf{E}|\bar{\eta}(T)|^2 &= \frac{1}{n^2}\mathbf{E}\left|\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^T \sigma_{ij}(t)dw_j(t)\right|^2 \\ &= \frac{1}{n^2}\mathbf{E}\left|\sum_{j=1}^{N}\int_0^T \sum_{i=1}^{n}\sigma_{ij}(t)dw_j(t)\right|^2 \\ &= \frac{1}{n^2}\sum_{j=1}^{N} v_j(t) \le \frac{\bar{v}}{n} \to 0 \quad \text{as} \quad n \to +\infty. \end{aligned} \tag{A.28}$$

Given $\varepsilon_1$ and $\varepsilon_2$ from statements (ii) and (iii) of Theorem 5.1, let $\bar{n} > 0$ and $\varepsilon_3 \in (0, \theta)$ be such that

$$\begin{aligned} &\psi(-\theta + \delta) \ge \psi(-\theta) - \tfrac{\varepsilon_1}{2} \quad \forall \delta \in (-\varepsilon_3, \varepsilon_3), \\ &\nu(T) \le \min\left(\tfrac{\varepsilon_3}{2}, \tfrac{\varepsilon_1}{2}\right), \quad \mathbf{P}\left(|\bar{\eta}(T)| \ge \tfrac{\varepsilon_3}{2}\right) < \varepsilon_2, \quad \mathbf{P}(|A_n(T)| \ge \theta) \ge p \qquad \forall n > \bar{n}. \end{aligned}$$

These $\bar{n}$ and $\varepsilon_3$ do exist by (A.27)-(A.28). For Theorem 5.1 (ii), $\bar{n}$ depends on $a(\cdot)$. For Theorem 5.1 (iii), $\bar{n}$ depends on $\mathcal{A}_u$. We have that

$$\begin{aligned} \mathbf{P}\left(\widetilde{X}^{(n)}(T) \ge \psi(-\theta) - \varepsilon_1\right) &= \mathbf{P}\left(\ \psi[A_n(T) + \bar{\eta}(T) - \nu(T)] - \nu(T) \ge \psi(-\theta) - \varepsilon_1\right) \\ &\ge \mathbf{P}\left(|\bar{\eta}(T)| < \tfrac{\varepsilon_3}{2}, \quad |A_n(T)| \ge \theta\right) \\ &\ge p - \varepsilon_2 \qquad \forall n > \bar{n}. \end{aligned}$$

This completes the proof of Theorem 5.1. $\square$

## Acknowledgment

This work was supported by the Australian Research Council.

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