EULER'S

ELEMENTS OF ALGEBRA

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ELEMENTS OF ALGEBRA,

BY LEONARD EULER,

TRANSLATED FROM THE FRENCH;

WITH THE

ADDITIONS OF LA GRANGE,

AND

THE NOTES OF THE FRENCH TRANSLATOR;

TO WHICH IS ADDED

AN APPENDIX,

CONTAINING THE DEMONSTRATION OF SEVERAL CURIOUS AND IMPORTANT NUMERICAL PROPOSITIONS, ALLUDED TO, BUT NOT INVESTIGATED, IN THE BODY OF THE WORK,
&c. &c.

SECOND EDITION.

VOL. I.

LONDON:
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PREFACE.

In offering to the public an English translation of the present work, the Editor does not feel himself induced to make any apology, convinced that the uncommon merit of the original, and the illustrious reputation of its author, have rendered it a subject of wonder and regret, with the mathematicians of our own country, that a translation should not have been undertaken long since. To this may be added the very great scarceness of the original, particularly the French edition of it, the high price which has for many years been demanded for it, and the avidity with which it has been sought after.

Provided, therefore, the Translator has executed his task with tolerable fidelity and skill, the admirers of Euler will feel themselves gratified and obliged; for these Elements, we need not hesitate to say, will furnish the most beautiful examples of analysis that modern Europe can boast of. The mathematical student, whether he wishes to direct his attention to the properties of whole numbers, fractions, series, logarithms, the genesis of equations, or the invention of the higher and more complex formulæ, by which the Diophantine Algebra has been systematised and illustrated, will in these volumes find the profoundest researches and the most satisfactory information. He will be highly pleased also, if we mistake not, with the wonderful simplicity and clearness of this great Author's manner. He will discover no chasm in the reasoning, no link broken or deficient in the concatenation of his ideas, and nothing taken for granted, that has not been previously proved; defects which, in other writers, so often impede the progress of beginners, and discourage them from prosecuting their studies: but here, all is luminous, easy, and obvious. In giving the most difficult demonstrations, and in illustrating the most abstruse subjects, the different steps of the rationale are so many axioms; and it was EULER's great talent to render their order and dependence, in their progress through the mind, clear and evident to the meanest capacity.

But the reader will find a fuller estimate of his general character in the Memoirs of his Life.

It only remains to say a few words on the present undertaking. Though the public may be pleased to find the present work of EULER accessible to the English reader, yet some apology may be expected from the present translator, for attempting what others are much better qualified to perform. He only regrets that they have not done it; and spared him a task, from which it is easy to incur disgrace, and impossible to acquire fame.

With respect to the language, it has been the object of the Translator to render it clear and scientific, without sacrificing any of the ease and familiarity of the original; but if the reader should have the opportunity, and wishes to take the trouble, of comparing the English translation with the French, he will find that, for this very purpose, a needless multiplication of words, a redundancy of colloquial idiom, or a certain degree of unnecessary *verbiage*, has been silently dropped in almost every page.

That nothing might be omitted to gratify the reasonable curiosity of the English reader, the

Advertisement of the Editors of the original, and that of the French translator, are added; which will give some account of the history of the work and the different editions of it.

August 25, 1797.

ADVERTISEMENT

TO

THE SECOND EDITION.

NOTWITHSTANDING the care that was taken in the first edition of this work, to avoid errors and to correct those of the French translator; yet many of the latter still remained uncorrected, and many new ones were inadvertently admitted; all of which it has been the object of the present Editor to remove: and he flatters himself that he has accomplished his design to a considerable extent. Great pains have also been taken to arrange the matter in each page, so as to bring the algebraical formulæ under one distinct point of view, a circumstance which ought always to be attended to in elementary works.

In some few instances, the present Editor has ventured to deviate a little from the original form of the author. In the course of the work, particularly in the Multiplication and Division of Algebraic quantities, a small number of examples have been altered, for the sake of a more eligible arrangement. In one or two places, the symbols employed by the author, have also been changed for others which are more commonly used by the best modern writers. In chapters 9 and 10, Vol. I., a little alteration is likewise made in placing the terms of proportions; and throughout both volumes, x^2 has been substituted for xx, x^3 for xxx, &c. These form the principal instances of deviations from the original, and if it should be thought that the Editor has thus exceeded his proper limits, he trusts that the candour of the reader will do him the justice to attribute these apparent innovations, to his anxious desire of rendering the work unexceptionable to the taste of modern algebraists, and not to any want of truly appreciating the transcendent abilities of the justly celebrated Author.

Such of the former notes as are retained (among which are all those of the French translator), have been placed at the bottom of the pages to which they refer, and several others have been added where they were thought necessary, beside those subjoined to the second volume; in the latter of which are demonstrated all the numerical propositions that the author has referred to, but not investigated, in the body of the work. These notes, as far as could be done, are so arranged as to form a concise abstract of the Theory of Numbers, which, being a subject that has not much engaged the attention of the English mathematicians, it is presumed, that those who have not an opportunity of consulting foreign writers on this branch of analysis, may there find some useful information. A few of these notes are new, the others have been chiefly derived from the works of Waring, Gauss, and Legendre.

The Praxis which was given in the former edition has been cancelled, having been found inadequate to the purpose for which it was intended. The young student to whom, certainly, a great variety of examples is necessary to exercise him in the different rules, and to whom only the Praxis was useful, will find a much more valuable acquisition in Bonnycastle's Algebra; a work abounding with a choice of examples, judiciously selected and methodically arranged.

Royal Military Academy,
Woolwich,
March 14, 1810.

EULER.

LEONARD EULER was the son of a clergyman in the neighbourhood of Basil, and was born on the 15th of April, 1707. His natural turn for mathematics soon appeared from the eagerness and facility with which he became master of the elements under the instructions of his father, by whom he was sent to the university of Basil at an early age. There, his abilities and his application were so distinguished, that he attracted the particular notice of John Bernoulli. That excellent mathematician seemed to look forward to the youth's future achievements in science, while his own kind care strengthened the powers by which they were to be accomplished. In order to superintend his studies, which far outstripped the usual routine of the public lecture, he gave him a private lesson regularly once a week; when they conversed together on the acquisitions which the pupil had been making since their last interview, considered whatever difficulties might have occurred in his progress, and arranged the reading and exercises for the ensuing week. Under such eminent advantages, the capacity of Euler did not fail to make rapid improvements; and in his seventeenth year, the degree of Master of Arts was conferred on him. On this occasion, he received high applause for his probationary discourse, the subject of which was a comparison between the Cartesian and Newtonian systems.

His father, having all along intended him for his successor, enjoined him now to relinquish his mathematical studies, and to prepare himself by those of theology and general erudition for the ministerial functions. After some time, however, had been consumed, this plan was given up. The father, himself a man of learning and liberality, abandoned his own views for those to which the inclination and talents of his son were of themselves so powerfully directed; persuaded, that in thwarting the propensities of genius, there is a sort of impiety against nature, and that there would be real injustice to mankind in smothering those abilities which were evidently destined to extend the boundaries of science. LEONARD was permitted, therefore, to resume his favorite pursuits; and, at the age of nineteen, transmitting two dissertations to the Academy of Sciences at Paris, one on the masting of ships, and the other on the philosophy of sound, he commenced that splendid career which continued, for so long a period, the admiration and the glory of Europe.

About the same time he stood candidate for a vacant professorship in the university of Basil; but having lost the election, he resolved, in consequence of this disappointment, to leave his native country; and in 1727 he set out for Petersburg, where his friends, the young Bernoullis, had settled about two years before, and where he flattered himself with prospects of literary preferment under the patronage of Catherine I. Those prospects, however, were not immediately realised; nor was it till after he had been frequently and long disappointed, that he obtained any settlement. His first appears to have been the chair of natural philosophy; and when Daniel Bernoulli removed from Petersburg, EULER succeeded him as professor of the mathematics. In this situation he remained for several years, engaged in the most laborious researches, enriching the academical collections of the continent with papers of the highest value, and producing almost daily improvements in the various branches of physical, and more particularly analytical science. In 1741, he complied with a very pressing invitation from Frederic the Great, and resided at Berlin till 1766. Throughout this period, he continued the same literary labours, directed by the same wonderful sagacity and comprehension of intellect. As he advanced with his own discoveries and inventions, the field of knowledge seemed to widen before his view, and new subjects still multiplied on him for farther speculation. The toils of intense study, with him, seemed only to invigorate his future exertions. Nor did the energy of Euler's powers give way, even when the organs of the body were overpowered: for in the year 1735, having completed in three days certain astronomical calculations, which the academy called for in haste, but which several mathematicians of eminence had declared could not be performed within a shorter period than some months, the intense application threw him into a fever, in which he lost the sight of one eye. Shortly after his return to Petersburg, in 1766, he became totally blind. His passion for science, however, suffered no decline; the powers of his mind were not impaired, and he continued as indefatigable as ever. Though the distresses of age likewise were crowding fast upon him, for he had now passed his sixtieth year; yet it was in this latter period of his life, under infirmity, bodily pain, and loss of sight, that he produced some of the most valuable works; such as command our astonishment, independently of the situation of the author, from the labour and originality which they display. In fact, his habits of study and composition, his inventions and discoveries, closed only with his life. The very day on which he died, he had been engaged in calculating the orbit of Herschel's planet, and the motions of aërostatic machines. His death happened suddenly in September 1783, when he was in the seventy-sixth year of his age.

Such is the short history of this illustrious man. The incidents of his life, like that of most other laborious students, afford very scanty materials for biography; little more than a journal of studies and a catalogue of publications; but curiosity may find ample compensation in surveying the character of his mind. An object of such magnitude, so far elevated above the ordinary range of human intellect, cannot be approached without reverence, nor nearly inspected, perhaps, without some degree of presumption. Should an apology be necessary, therefore, for attempting the following estimate of

Euler's character, let it be considered, that we can neither feel that admiration, nor offer that homage, which is worthy of genius, unless, aiming at something more than the dazzled sensations of mere wonder, we subject it to actual examination, and compare it with the standards of human nature in general.

Whoever is acquainted with the memoirs of those great men, to whom the human race is indebted for the progress of knowledge, must have perceived that while mathematical genius is distinct from the other departments of intellectual excellence, it likewise admits in itself of much diversity. The subjects of its speculation are become so extensive and so various, especially in modern times, and present so many interesting aspects, that it is natural for a person whose talents are of this cast, to devote his principal curiosity and attention to particular views of the science. When this happens, the faculties of the mind acquire a superior facility of operation with respect to the objects towards which they are most frequently directed, and the invention becomes habitually most active and most acute in that channel of inquiry. The truth of these observations is strikingly illustrated by the character of Euler. His studies and discoveries lay not among the lines and figures of geometry, those characters, to use an expression of Galileo, in which the great book of the universe is written; nor does he appear to have had a turn for philosophising by experiment, and advancing to discovery through the rules of inductive investigation. The region, in which he delighted to speculate, was that of pure intellect. He surveyed the properties and affections of quantity under their most abstracted form. With the same rapidity of perception, as a geometrician ascertains the relative position of portions of extension, EULER ranges among those of abstract quantity, unfolding their most involved combinations, and tracing their most intricate proportions. That admirable system of mathematical logic and language, which at once teaches the rules of just inference, and furnishes an instrument for prosecuting deductions, free from the defects which obscure and often falsify our reasonings on other subjects; the different species of quantity, whether formed in the understanding by its own abstractions, or derived from modifications of the representative system of signs; the investigation of the various properties of these, their laws of genesis, the limits of comparison among the different species, and the method of applying all this to the solution of physical problems: these were the researches on which the mind of EULER delighted to dwell, and in which he never engaged without finding new objects of curiosity, detecting sources of inquiry that had passed unobserved, and exploring fields of speculation before unknown.

The subjects, which we have here slightly enumerated, form, when taken together, what is called the Modern Analysis; a science eminent for the profound discoveries which it has revealed, for the refined artifices that have been devised in order to bring the most abstruse parts of mathematics within the compass of our reasoning powers, and in order to apply them in solving actual phænomena, as well as for the remarkable degree of systematic simplicity with which the various methods of investigation that it employs may be combined so as to confirm and throw light on one another. The materials, indeed, had been collecting for years, from about the middle of the seventeenth century; the foundations had been laid by Newton, Leibnitz, the elder Bernoullis, and a few others; but EULER raised the superstructure; it was reserved for him to work upon the materials, and to arrange this noble monument of human industry and genius in its present symmetry. Through the whole course of his scientific labours, the ultimate and the constant aim on which he set his mind, was the perfection of Calculus and Analysis. Whatever physical inquiry he began with, this always came in view, and very frequently received more of his attention than that which was professedly the main subject. His ideas ran so naturally in this train, that even in the perusal of Virgil's poetry he met with images that would recall the associations of his more familiar studies, and lead him back, from the fairy scenes of fiction, to the element, more congenial to his nature, of mathematical abstraction. That the sources of analysis might be ascertained in their full extent, as well as the various modifications of form and restrictions of rule that become necessary in applying it to different views of nature; he appears to have nearly gone through a complete course of philosophy. The theory of rational mechanics, the whole range of physical astronomy, the vibrations of elastic fluids, as well as the movements of those which are incompressible, naval architecture and tactics, the doctrine of chances, probabilities, and political arithmetic, were successively subjected to the analytical method; and all these sciences received fresh confirmation and farther improvement.

It cannot be denied that, in general, his attention is more occupied with the analysis itself, than with the subject to which he is applying it; he seems more taken up with his instruments and tools, than with the work which they are to assist him in executing. But this can hardly be made a ground of censure or regret, since it is the very circumstance to which we owe the present perfection of those instruments; a perfection to which he could never have brought them but by the unremitted attention and enthusiastic preference which he paid his favorite object. If he now and then exercised his ingenuity on a physical, or perhaps metaphysical, hypothesis, he must have been aware as well as any one, that his conclusions would perish of course with that from which they were derived. What he regarded, was the means of arriving at those conclusions, the new views of analysis which the investigation might open, and the new expedients of calculus to which it might give birth. This was his uniform pursuit, all other inquiries were prosecuted with reference to it; in which, consisted the peculiar character of his mathematical genius.

The faculties that are subservient to invention he possessed in a very remarkable degree. His memory was at once so retentive and so ready, that he had perfectly at command all those numerous and complex formulæ which enunciate the rules and more important theorems of analysis. As is reported of Leibnitz, he could also repeat the Æneid from beginning to end; and could trust his recollection for the first and last lines in every page of the edition which he had been accustomed to use. These are instances of a kind of memory, more frequently to be found where the capacity is inferior to the ordinary standard, than accompanying original, scientific genius. But in EULER, they seem not to have been not so much the result of natural constitution, as of his most wonderful attention; a faculty, which, if we consider the testimony of Newton* sufficient evidence, is the great constituent of inventive power. It is that complete retirement of the mind within itself, during which the senses are locked up; that intense meditation, on which no extraneous idea can intrude; that firm straightforward progress of thought, deviating into no irregular sally; which can alone place mathematical objects in a light sufficiently strong to illuminate them fully, and preserve the perceptions of the mind's eye in the same order that it moves along. "Two of EULER's pupils (we are told by M. Fuss, "a pupil himself) had calculated a converging "series as far as the seventeenth term, but found, "on comparing the written results, that they dif-"fered one unit at the fiftieth figure; they com-"municated this difference to their master, who "went over the whole calculation by head, and his "decision was found to be the true one.—For the "purpose of exercising his little grandson in the

* This opinion of Sir Isaac Newton is recorded by Dr. Pemberton.

"extraction of roots, he has been known to form "to himself the table of the six first powers of all "numbers, from 1 to 100, and to have preserved "it actually in his memory."

The dexterity he had attained in analysis and calculation, is remarkably exemplified by the manner in which he manages formulæ of the greatest length and intricacy. He perceives, almost at a glance, the factors from which they may have been composed; the particular system of factors belonging to the question under present consideration; the various artifices by which that system may be simplified and reduced; and the relation of the several factors to the conditions of the hypothesis. His expertness in this particular probably resulted, in a great measure, from the ease with which he performed mathematical investigations by head. He had always accustomed himself to that exercise; and having practised it with assiduity, even before the loss of sight, which afterwards rendered it a matter of necessity, he is an instance to what an astonishing degree it may be acquired, and how much it improves the intellectual powers. No other discipline is so effectual in strengthening the faculty of attention; it gives a facility of apprehension, an accuracy and steadiness to the conceptions; and what is a still more valuable acquisition, it habituates the mind to arrangement in its reasonings and reflections. If the reader wants a farther commentary on its advantages, let him proceed to the work of EULER, of which we here offer a translation; and if he has any taste for the beauties of method, and of what is properly called composition, we venture to promise him the highest satisfaction and pleasure. The subject is so aptly divided, the order so luminous, the connected parts seem so truly to grow one out of the other, and are disposed altogether in a manner so suitable to their relative importance, and so conducive to their mutual illustration, that, when added to the precision as well as clearness with which every thing is explained, and the judicious selection of examples, we do not hesitate to consider it, next to Euclid's Geometry, the most perfect model of elementary writing of which the literary world is in possession.

When our reader shall have studied so much of these volumes as to relish their admirable style, he will be the better qualified to reflect on the circumstances under which they were composed. They were drawn up soon after our author was deprived of sight, and were dictated to his servant, who had originally been a tailor's apprentice, and, without being distinguished for more than ordinary parts, was completely ignorant of mathematics. But Euler, blind as he was, had a mind to teach his amanuensis, as he went on with the subject. Perhaps he undertook this task by way of exercise, with the view of conforming the operation of his faculties to the change which the loss of sight had produced. Whatever was the motive, his treatise had the advantage of being composed under an immediate experience of the method best adapted to the natural progress of a learner's ideas: from the want of which, men of the most profound knowledge are often awkward and unsatisfactory when they attempt elementary instruction. It is not improbable, that we may be farther indebted to the circumstance of our Author's blindness; for the loss of this sense is generally succeeded by the improvement of other faculties. As the surviving organs, in particular, acquire a degree of sensibility which they did not previously possess; so the most charming visions of poetical fancy have been the offspring of minds, on whom external scenes had long been closed. And perhaps a philosopher, familiarly acquainted with Euler's writings, might trace some improvement in perspicuity of method, and in the flowing progress of his deductions, after this calamity had befallen him: which, leaving "an universal blank of nature's works," favours that entire seclusion of the mind which concentrates the attention, and gives liveliness and vigour to the conceptions.

In men devoted to study, we are not to look for those strong complicated passions, which are contracted amidst the vicissitudes and tumult of public life. To delineate the character of EULER, requires no contrasts of colouring. Sweetness of disposition, moderation in the passions, simplicity of manners, were his leading features. Susceptible of the domestic affections, he was open to all their amiable impressions, and was remarkably fond of children. His manners were simple, without being singular, and seemed to flow naturally from a heart that could dispense with those habits by which many must be trained to artificial mildness, and with the forms that are often necessary for concealment. Nor did the equability and calmness of his temper indicate a defect of energy, but the serenity of a soul that overlooked the frivolous provocations, the petulant caprices, and jarring humours of ordinary mortals.

Possessing a mind of such wonderful comprehension, and dispositions so admirably formed to virtue and to happiness, EULER found no difficulty in being a Christian: accordingly "his faith was unfeigned," and his love "was that of a pure and undefiled heart." The advocates for the truth of revealed religion, therefore, may rejoice to add to the bright catalogue which already claims a Bacon, a Newton, a Locke, and a Hale, the illustrious name of EULER. But on this subject we shall permit one of his learned and grateful pupils* to sum up the character of his venerable master. "His piety was rational and "sincere; his devotion was fervent: he was fully "persuaded of the truth of Christianity—felt its im-"portance to the dignity and happiness of human "nature—and looked upon its detractors and op-"posers as the most pernicious enemies of man."

The length to which this account has been extended may require some apology; but the character of EULER is an object so interesting, that it is difficult to prescribe a limit to reflections, when they are once indulged. One is attracted by a sentiment of admiration, that almost arises to the emotion of sublimity; and curiosity becomes eager to examine what talents and qualities and habits belonged to a mind of such superior power.

* Mr. Fuss, Eulogy of M. L. Euler.

We hope, therefore, the student will not deem this an improper introduction to the work which he is about to peruse; as we trust he is prepared to enter on it with that temper and disposition which will open his mind both to the perception of excellence, and to the ambition of emulating what he cannot but admire.

ADVERTISEMENT

BY THE

EDITORS OF THE ORIGINAL.

WE present to the lovers of Algebra a work, of which a Russian translation appeared two years ago. The object of the celebrated Author was to compose an Elementary Treatise, by which a beginner, without any other assistance, might make himself complete master of Algebra. The loss of sight had suggested this idea to him, and his activity of mind did not suffer him to defer the execution of it. For this purpose M. EULER pitched on a young man whom he had engaged as a servant on his departure from Berlin, sufficiently master of arithmetic, but in other respects without the least knowledge of mathematics. He had learned the trade of a tailor, and with regard to his capacity was not above mediocrity. This young man, however, has not only retained what his illustrious master taught and dictated to him, but in a short time was able to perform the most difficult algebraic calculations, and to resolve with readiness whatever analytical questions were proposed to him.

This fact must be a strong recommendation of the manner in which this work is composed, as the young man who wrote it down, who performed the calculations, and whose proficiency was so striking, received no instructions whatever but from this master, a superior one indeed, but deprived of sight.

Independently of so great an advantage, men of science will perceive, with pleasure and admiration, the manner in which the doctrine of logarithms is explained, and its connexion with other branches of calculus pointed out, as well as the methods which are given for resolving equations of the third and fourth degrees.

Lastly, those who are fond of *Diophantine* problems, will be pleased to find, in the last Section of the Second Part, all these problems reduced to a system, and all the processes of calculation, which are necessary for the solution of them, fully explained.

ADVERTISEMENT

BY THE

FRENCH TRANSLATOR.

The Treatise of Algebra, which I have undertaken to translate, was published in German, 1770, by the Royal Academy of Sciences at Petersburg. To praise its merits, would almost be injurious to the celebrated name of its author; it is sufficient to read a few pages, to perceive, by the perspicuity with which every thing is explained, what advantage beginners may derive from it. Other subjects are the purpose of this advertisement.

I have departed from the division which is followed in the original, by introducing, in the first volume of the French translation, the first Section of the Second Volume of the original, because it completes the analysis of determinate quantities. The reason for this change is obvious: it not only favours the natural division of Algebra into determinate and indeterminate analysis, but it was necessary to preserve some equality in the size of the two volumes, on account of the additions which are subjoined to the Second Part.

The reader will easily perceive that those additions come from the pen of M. DE LA GRANGE; indeed, they formed one of the principal reasons that engaged me in this translation; I am happy in being the first to show more generally to mathematicians, to what a pitch of perfection two of our most illustrious mathematicians have lately carried a branch of analysis but little known, the researches of which are attended with many difficulties, and, on the confession even of these great men, present the most difficult problems that they have ever resolved.

I have endeavoured to translate this algebra in the style best suited to works of the kind; my chief anxiety was to enter into the sense of the original, and render it with the greatest perspicuity: perhaps I may presume to give my translation some superiority over the original, because that work having been dictated and admitting of no revision from the author himself, it is easy to conceive that in many passages it would stand in need of correction. If I have not submitted to translate literally, I have not failed to follow my author step by step; I have preserved the same divisions in the articles, and it is only in so few places that I have taken the liberty of suppressing some details of calculation, and inserting one or two lines of illustration in the text, that I believe it unnecessary to enter into an explanation of the reasons by which I was justified in doing so.

Nor shall I take any more notice of the notes which I have added to the first part; they are not so numerous as to make me fear the reproach of having unnecessarily increased the volume; and they may throw light on several points of mathematical history, as well as make known a great number of tables that are of subsidiary utility.

With respect to the correctness of the press, I believe it will not yield to that of the original; I have carefully compared all the calculations, and having repeated a great number of them myself, have by those means been enabled to correct several faults beside those which were indicated in the *Errata*.

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FIRST VOLUME.

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ERRATA IN VOL. I.

Page	Line	
26,	20,	in the note, for form, read forms.
34,	7,	in the note, for $a=1$, read $a^0=1$.
34,	8,	in the note, for $^{0+1}$, read a^{0+1}.
84,	12,	for $\sqrt{a^5}$, read $\sqrt{a}$.
243,	14,	for $\frac{b}{c}=s\frac{ab}{c}$, read $\frac{b}{c}s=\frac{ab}{c}$.

ELEMENTS

OF

ALGEBRA.

PART I.

Containing the Analysis of Determinate Quantities.

SECTION I.

Of the different Methods of calculating Simple Quantities.

CHAP. I.

Of Mathematics in general.

ARTICLE I.

WHATEVER is capable of increase or diminution, is called *magnitude*, or *quantity*.

A sum of money therefore is a quantity, since we may increase it and diminish it. It is the same with a weight, and other things of this nature.

2. From this definition, it is evident, that the different kinds of magnitude must be so various as to render it difficult to enumerate them: and this is the origin of the different branches of the Mathe-

matics, each being employed on a particular kind of magnitude. Mathematics, in general, is the *science of quantity;* or, the science which investigates the means of measuring quantity.

3. Now we cannot measure or determine any quantity, except by considering some other quantity of the same kind as known, and pointing out their mutual relation. If it were proposed, for example, to determine the quantity of a sum of money, we should take some known piece of money, as a louis, a crown, a ducat, or some other coin, and show how many of these pieces are contained in the given sum. In the same manner, if it were proposed to determine the quantity of a weight, we should take a certain known weight; for example, a pound, an ounce, &c. and then show how many times one of these weights is contained in that which we are endeavouring to ascertain. If we wished to measure any length or extension, we should make use of some known length, such as a foot.

4. So that the determination, or the measure of magnitude of all kinds, is reduced to this: fix at pleasure upon any one known magnitude of the same species with that which is to be determined, and consider it as the *measure* or *unit;* then, determine the proportion of the proposed magnitude to this known measure. This proportion is always expressed by numbers; so that a number is nothing but the proportion of one magnitude to another arbitrarily assumed as the unit.

5. From this it appears, that all magnitudes may be expressed by numbers; and that the foundation of all the Mathematical Sciences must be laid in a

complete treatise on the science of Numbers, and in an accurate examination of the different possible methods of calculation.

This fundamental part of mathematics is called Analysis, or Algebra*.

6. In Algebra then we consider only numbers which represent quantities, without regarding the different kinds of quantity. These are the subjects of other branches of the mathematics.

7. Arithmetic treats of numbers in particular, and is the *science of numbers properly so called;* but this science extends only to certain methods of calculation which occur in common practice: Algebra, on the contrary, comprehends in general all the cases which can exist in the doctrine and calculation of numbers.

CHAP. II.

Explanation of the Signs + Plus *and* − Minus.

8. When we have to add one given number to another, this is indicated by the sign + which is placed before the second number, and is read *plus.*

* Several mathematical writers make a distinction between *Analysis* and *Algebra*. By the term *Analysis*, they understand the method of determining those general rules which assist the understanding in all mathematical investigations; and by *Algebra*, the instrument which this method employs for accomplishing that end. This is the definition given by M. Bezout in the preface to his Algebra. F. T.

Thus $5+3$ signifies that we must add 3 to the number 5, in which case, every one knows that the result is 8; in the same manner $12+7$ make 19; $25+16$ make 41; the sum of $25+41$ is 66, &c.

9. We also make use of the same sign + *plus*, to connect several numbers together; for example, $7+5+9$ signifies that to the number 7 we must add 5 and also 9, which make 21. The reader will therefore understand what is meant by

$$8+5+13+11+1+3+10,$$

viz. the sum of all these numbers, which is 51.

10. All this is evident; and we have only to mention, that, in Algebra, in order to generalize numbers, we represent them by letters, as a, b, c, d, &c. Thus, the expression $a+b$, signifies the sum of two numbers, which we express by a and b, and these numbers may be either very great or very small. In the same manner, $f+m+b+x$, signifies the sum of the numbers represented by these four letters.

If we know therefore the numbers that are represented by letters, we shall at all times be able to find, by arithmetic, the sum or value of such expressions.

11. When it is required, on the contrary, to subtract one given number from another, this operation is denoted by the sign —, which signifies *minus*, and is placed before the number to be subtracted: thus $8-5$ signifies that the number 5 is to be taken from the number 8; which being done, there remain 3. In like manner $12-7$ is the same as 5; and $20-14$ is the same as 6, &c.

12. Sometimes also we may have several numbers to subtract from a single one; as, for instance,

50—1—3—5—7—9. This signifies, first, take 1 from 50, and there remain 49; take 3 from that remainder, and there will remain 46; take away 5, and 41 remain; take away 7, and 34 remain; lastly, from that take 9, and there remain 25: this last remainder is the value of the expression. But as the numbers 1, 3, 5, 7, 9, are all to be subtracted, it is the same thing if we subtract their sum, which is 25, at once from 50, and the remainder will be 25 as before.

13. It is also easy to determine the value of similar expressions, in which both the signs + *plus* and — *minus* are found: for example;

12—3—5+2—1 is the same as 5.

We have only to collect separately the sum of the numbers that have the sign + before them, and subtract from it the sum of those that have the sign —. Thus, the sum of 12 and 2 is 14; and that of 3, 5, and 1, is 9; hence 9 being taken from 14, there remain 5.

14. It will be perceived from these examples that the order in which we write the numbers is perfectly indifferent and arbitrary, provided the proper sign of each be preserved. We might with equal propriety have arranged the expression in the preceding article thus; 12+2—5—3—1, or 2—1—3—5+12, or 2+12—3—1—5, or in still different orders; where it must be observed, that in the expression proposed, the sign + is supposed to be placed before the number 12.

15. It will not be attended with any more difficulty if, in order to generalize these operations, we make use of letters instead of real numbers. It is evident, for example, that

$$a-b-c+d-e,$$

signifies that we have numbers expressed by *a* and *d*, and that from these numbers, or from their sum, we must subtract the numbers expressed by the letters *b*, *c*, *e*, which have before them the sign —.

16. Hence it is absolutely necessary to consider what sign is prefixed to each number: for in Algebra, simple quantities are numbers considered with regard to the signs which precede, or affect them. Farther, we call those *positive quantities*, before which the sign + is found; and those are called *negative quantities*, which are affected by the sign —.

17. The manner in which we generally calculate a person's property, is an apt illustration of what has just been said. For we denote what a man really possesses by positive numbers, using, or understanding the sign +; whereas his debts are represented by negative numbers, or by using the sign —. Thus, when it is said of any one that he has 100 crowns, but owes 50, this means that his real possession amounts to 100—50; or, which is the same thing, +100—50, that is to say 50.

18. As negative numbers, in like manner, may be considered as debts, because positive numbers represent real possessions, we may say that negative numbers are less than nothing. Thus, when a man has nothing of his own, and owes 50 crowns, it is certain that he has 50 crowns less than nothing; for if any one were to make him a present of 50 crowns to pay his debts, he would still be only at the point nothing, though really richer than before.

19. In the same manner therefore as positive numbers are incontestably greater than nothing,

negative numbers are less than nothing. Now we obtain positive numbers by adding 1 to 0, that is to say, 1 to nothing; 1 to 1, 1 to 2, 1 to 3, &c. This is the origin of the series of numbers called *natural numbers*; the following being the leading terms of this series:

$0, +1, +2, +3, +4, +5, +6, +7, +8, +9, +10,$

and so on to infinity.

But if instead of continuing this series by successive additions, we continued it in the opposite direction, by perpetually subtracting unity, we should have the series of negative numbers:

$0, -1, -2, -3, -4, -5, -6, -7, -8, -9, -10,$

and so on to infinity.

20. All these numbers, whether positive or negative, have the known appellation of whole numbers, or *integers*, which consequently are either greater or less than nothing. We call them *integers*, to distinguish them from fractions, and from several other kinds of numbers of which we shall hereafter speak. For instance, 50 being greater by an entire unit than 49, it is easy to comprehend that there may be between 49 and 50 an infinity of intermediate numbers, all greater than 49, and yet all less than 50. We need only imagine two lines, one 50 feet, the other 49 feet long, and it is evident there may be drawn an infinite number of lines all longer than 49 feet, and yet shorter than 50.

21. It is of the utmost importance through the whole of Algebra, that a precise idea be formed of those negative quantities about which we have been speaking. I shall, however, content myself with remarking here, that all such expressions as

$+1-1$, $+2-2$, $+3-3$, $+4-4$, &c.

are equal to 0, or nothing. And that

$+2-5$ is equal to -3:

for if a person has 2 crowns, and owes 5, he has not only nothing, but still owes 3 crowns. In the same manner

$7-12$ is equal to -5, and $25-40$ is equal to -15.

22. The same observations hold true, when, to make the expression more general, letters are used instead of numbers; thus 0, or nothing, will always be the value of $+a-a$; but if we wish to know the value of $+a-b$, two cases are to be considered.

The first is when a is greater than b; b must then be subtracted from a, and the remainder (before which is placed, or understood to be placed, the sign $+$) shows the value sought.

The second case is that in which a is less than b: here a is to be subtracted from b, and the remainder being made negative, by placing before it the sign $-$, will be the value sought.

CHAP. III.

Of the Multiplication of Simple Quantities.

23. When there are two or more equal numbers to be added together, the expression of their sum may be abridged: for example,

$a+a$ is the same with $2\times a$,

$a+a+a$ - - - - - - $3\times a$,

$a+a+a+a$ - - - - $4\times a$, and so on; where $\times$ is the sign of multiplication. In this manner we may form an idea of multiplication; and it is to be observed that,

$2\times a$ signifies 2 times, or twice a
$3\times a$ - - - - 3 times, or thrice a
$4\times a$ - - - - 4 times a, &c.

24. If therefore a number expressed by a letter is to be multiplied by any other number, we simply put that number before the letter; thus,

a multiplied by 20 is expressed by $20a$, and
b multiplied by 30 gives $30b$, &c.

It is evident also that c taken once, or $1c$, is the same as c.

25. Farther, it is extremely easy to multiply such products again by other numbers; for example:

2 times, or twice $3a$ makes $6a$.
3 times, or thrice $4b$ makes $12b$.
5 times $7x$ makes $35x$.

and these products may be still multiplied by other numbers at pleasure.

26. When the number by which we are to multiply is also represented by a letter, we place it immediately before the other letter; thus, in multiplying b by a, the product is written ab; and pq will be the product of the multiplication of the number q by p. Also, if we multiply this pq again by a, we shall obtain apq.

27. It may be farther remarked here, that the order in which the letters are joined together is indifferent; thus ab is the same thing as ba; for b multiplied by a is the same as a multiplied by b. To understand this, we have only to substitute for a and b known numbers, as 3 and 4; and the truth will be self-evident; for 3 times 4 is the same as 4 times 3.

28. It will not be difficult to perceive, that when we have to put numbers, in the place of letters joined together, in the manner we have described, they cannot be written in the same way by putting them one after the other. For if we were to write 34 for 3 times 4, we should have 34 and not 12. When therefore it is required to multiply common numbers, we must separate them by the sign ×, or by a point: thus, 3×4, or 3.4, signifies 3 times 4, that is 12. So, 1×2 is equal to 2; and 1×2×3 makes 6. In like manner 1×2×3×4×56 makes 1344; and 1×2×3×4×5×6×7×8×9×10 is equal to 3628800, &c.

29. In the same manner we may discover the value of an expression of this form, $5.7.8.abcd$. It shows that 5 must be multiplied by 7, and that this product is to be again multiplied by 8; that we are then to multiply this product of the three numbers by a, next by b, then by c, and lastly by d. It may be observed also, that instead of 5.7.8 we may write its value, 280; for we obtain this number when we multiply the product of 5 by 7, or 35, by 8.

30. The results which arise from the multiplication of two or more numbers are called *products*; and the numbers, or individual letters, are called *factors*.

31. Hitherto we have considered only positive numbers, and there can be no doubt, but that the products which we have seen arise are positive also: viz. $+a$ by $+b$ must necessarily give $+ab$. But we must separately examine what the multiplication of $+a$ by $-b$, and of $-a$ by $-b$, will produce.

32. Let us begin by multiplying $-a$ by 3 or $+3$;

now since $-a$ may be considered as a debt, it is evident that if we take that debt three times, it must thus become three times greater, and consequently the required product is $-3a$. So if we multiply $-a$ by $+b$, we shall obtain $-ba$, or, which is the same thing, $-ab$. Hence we conclude, that if a positive quantity be multiplied by a negative quantity, the product will be negative; and it may be laid down as a rule, that $+$ by $+$ makes $+$ or *plus*; and that, on the contrary, $+$ by $-$, or $-$ by $+$, gives $-$ or *minus*.

33. It remains to resolve the case in which $-$ is multiplied by $-$; or, for example, $-a$ by $-b$. It is evident, at first sight, with regard to the letters, that the product will be ab; but it is doubtful whether the sign $+$, or the sign $-$, is to be placed before the product; all we know is, that it must be one or the other of these signs. Now I say that it cannot be the sign $-$: for $-a$ by $+b$ gives $-ab$, and $-a$ by $-b$ cannot produce the same result as $-a$ by $+b$; but must produce a contrary result, that is to say, $+ab$; consequently we have the following rule: $-$ multiplied by $-$ produces $+$, that is, the same as $+$ multiplied by $+$ *.

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* As this conclusion is not so satisfactory as could be wished, we will endeavour to give another demonstration, founded upon principles which may be thought less objectionable.

First, we know that $+a$ multiplied by $+b$ gives the product $+ab$; and if $+a$ be multiplied by a quantity less than b, as $b-c$, the product must necessarily be less than ab; in short, from ab we must subtract the product of a, multiplied by c; hence $a\times\overline{b-c}$ must be expressed by $ab-ac$; therefore it follows that

34. The rules which we have explained are expressed more briefly as follows:

Like signs, multiplied together, give +; unlike or contrary signs give −. Thus, when it is required to multiply the following numbers; $+a$, $-b$, $-c$;

$a \times -c$ gives the product $-ac$; that is + *plus* into − *minus* gives − *minus*.

If now we consider the product arising from the multiplication of the two quantities $\overline{a-b}$, and $\overline{c-d}$, we know that it is less than that of $\overline{a-b} \times c$, or of $ac-bc$; in short, from this product we must subtract that of $\overline{a-b} \times d$; but the product $\overline{a-b} \times \overline{c-d}$ becomes $ac-bc-ad$, to which is to be annexed the product of $-b \times -d$, and the question is only what sign we must employ for this purpose, whether + or −. Now we have seen that from the product $ac-bc$ we must subtract the product of $\overline{a-b} \times d$, that is, we must subtract a quantity less than ad; we have therefore subtracted already too much by the quantity bd; this product must therefore be added; that is, it must have the sign + prefixed; hence we see that $-b \times -d$ gives $+bd$ for a product; or − *minus* multiplied by − *minus* gives + *plus*.

There are some other circumstances in this chapter which it may not be amiss to apprize the reader of, that he may be guarded against receiving, as self-evident facts, things which require demonstration. In article 27 it is assumed that a multiplied by b is the same as b multiplied by a; but this should be considered rather as a proposition than as an axiom, for it is not one of those truths that carries its own evidence along with it, which is what properly constitutes an axiom, such as, *Equals added to equals, the wholes are equal.* Le Gendre, in his "Essai sur la Theorie des Nombres," has given it an elaborate demonstration, but it finally rests upon a truth which seems to require to be proved nearly as much as the proposition he has endeavoured to establish, the difficulty in the demonstration naturally arising from the simplicity of the truth to be demonstrated. The principle upon which M. Le Gendre has founded his demonstration is this: that $a \times b = b \times a$, if $a \times 1 = 1 \times a$; the latter, he conceives, may be

$+d$; we have first $+a$ multiplied by $-b$, which makes $-ab$; this by $-c$, gives $+abc$; and this by $+d$, gives $+abcd$.

35. The difficulties with respect to the signs being removed, we have only to show how to multiply numbers that are themselves products. If we were, for instance, to multiply the number ab by the number cd, the product would be $abcd$, and it is obtained by multiplying first ab by c, and then the result of that multiplication by d. Or if we had to multiply 36 by 12; since 12 is equal to 3 times 4, we should only multiply 36 first by 3, and then the product 108 by 4, in order to have the whole product of the multiplication of 12 by 36, which is consequently 432.

36. But if we wished to multiply $5ab$ by $3cd$, we might write $3cd \times 5ab$; however, as in the present instance the order of the numbers to be multiplied is indifferent, it will be better, as is also the custom, to place the common numbers before the letters, and to express the product thus; $5 \times 3abcd$, or $15abcd$; since 5 times 3 is 15.

So if we had to multiply $12pqr$ by $7xy$, we should obtain $12 \times 7pqrxy$, or $84pqrxy$.

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properly considered as an axiom; and it is perhaps more admissible as such than the former. Upon the whole, his demonstration, though not so clear as could be wished, is probably as satisfactory as any that can be given. ED.

CHAP. IV.

Of the Nature of whole Numbers, or Integers, with respect to their Factors.

37. We have observed that a product is generated by the multiplication of two or more numbers together, and that these numbers are called *factors*. Thus the numbers *a*, *b*, *c*, *d*, are the factors of the product *abcd*.

38. If, now, we consider whole numbers only, we shall soon find that some of them cannot result from any multiplication, and consequently have no factors; while others may be the products of two or more multiplied together, and may consequently have two or more factors. Thus 4 is produced by 2×2; 6 by 2×3; 8 by $2\times2\times2$; or 27 by $3\times3\times3$; and 10 by 2×5, &c.

39. But, on the other hand, the numbers 2, 3, 5, 7, 11, 13, 17, &c. cannot be represented in the same manner by factors, unless for that purpose we make use of unity, and represent 2, for instance, by 1×2. But the numbers which are multiplied by 1 remaining the same, it is not proper to reckon unity as a factor.

All numbers, therefore, such as 2, 3, 5, 7, 11, 13, 17, &c. which cannot be represented by factors, are called *simple*, or *prime numbers;* whereas others, as 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, &c. which may be represented by factors, are called *composite numbers.*

40. *Simple* or *prime numbers* deserve therefore

particular attention, since they do not result from the multiplication of two or more numbers. It is also particularly worthy of observation, that if we write these numbers in succession as they follow each other, thus,

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, &c.*

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* All the prime numbers from 1 to 100000 are to be found in the tables of divisors, which I shall speak of in a succeeding note; But particular tables of the prime numbers from 1 to 101000 have been published at Halle, by M. Kruger, in a German work entitled "Thoughts on Algebra;" M. Kruger had received them from a person called Peter Jaeger, who had calculated them. M. Lambert has continued these tables as far as 102000, and republished them in his supplements to the logarithmic and trigonometrical tables, printed at Berlin in 1770; a work which contains likewise several tables that are of great use in the different branches of mathematics, and explanations which it would be too long to enumerate here.

The Royal Parisian Academy of Sciences is in possession of tables of prime numbers, presented to it by P. Mercastel de l'Oratoire, and by M. du Tour; but they have not been published. They are spoken of in Vol. V. of the Foreign Memoirs, with a reference to a memoir, contained in that volume, by M. Rallier des Ourmes, Honorary Counsellor of the Presidial Court at Rennes, in which the author explains an easy method of finding prime numbers.

In the same volume we find another method by M. Rallier des Ourmes, which is entitled, "A new Method for Division, when the Dividend is a Multiple of the Divisor, and may therefore be divided without a Remainder; and for the Extraction of Roots when the Power is perfect." This method, more curious, indeed, than useful, is almost totally different from the common one: it is very easy, and has this singularity, that, provided we know as many figures on the right of the dividend, or the power, as there are to be in the quotient, or the root, we may pass over the figures which precede them, and thus obtain the quotient.

we can trace no regular order; their increments being sometimes greater, sometimes less; and hitherto no one has been able to discover whether they follow any certain law or not.

41. All *composite* numbers, which may be represented by factors, result from the prime numbers above mentioned; that is to say, all their factors are prime numbers. For, if we find a factor which is not a prime number, it may always be decomposed and represented by two or more prime numbers. When we have represented, for instance, the number 30 by 5×6, it is evident that 6 not being a prime number, but being produced by 2×3, we might have represented 30 by $5 \times 2 \times 3$, or by $2 \times 3 \times 5$; that is to say, by factors which are all prime numbers.

42. If we now consider those composite numbers which may be resolved into prime factors, we shall observe a great difference among them; thus we shall find that some have only two factors, that others have three, and others a still greater number. We have already seen, for example, that

4 is the same as	2×2,	6 is the same as	2×3,
8 - - - -	$2 \times 2 \times 2$,	9 - - - - -	3×3,
10 - - - -	2×5,	12 - - -	$2 \times 3 \times 2$,
14 - - - -	2×7,	15 - - - -	3×5,
16 - -	$2 \times 2 \times 2 \times 2$,	and so on.	

43. Hence it is easy to find a method for analys-

M. Rallier des Ourmes was led to this new method by reflecting on the numbers terminating the numerical expressions of products or powers, a species of numbers which I have remarked also, on other occasions, it would be useful to consider. F. T.

ing any number, or resolving it into its simple factors. Let there be proposed, for instance, the number 360; we shall represent it first by 2×180. Now 180 is equal to 2×90, and

$$\left.\begin{matrix}90\\45\\15\end{matrix}\right\}\text{is the same as}\left\{\begin{matrix}2\times45,\\3\times15,\\3\times5.\end{matrix}\right.\text{ and lastly}$$

So that the number 360 may be represented by these simple factors, $2 \times 2 \times 2 \times 3 \times 3 \times 5$; since all these numbers multiplied together produce 360*.

44. This shows, that prime numbers cannot be divided by other numbers, and, on the other hand, that the simple factors of compound numbers are found most conveniently, and with the greatest certainty, by seeking the simple, or prime numbers, by which those compound numbers are divisible. But for this *division* is necessary; we shall therefore explain the rules of that operation in the following chapter. (*See Appendix, note* 1.)

CHAP. V.

Of the Division of Simple Quantities.

45. When a number is to be separated into two, three, or more equal parts, it is done by means of

* There is a table at the end of a German book of arithmetic, published at Leipsic, by Poetius, in 1728, in which all the numbers from 1 to 10000 are represented in this manner by their simple factors. F. T.

division, which enables us to determine the magnitude of one of those parts. When we wish, for example, to separate the number 12 into three equal parts, we find by division that each of those parts is equal to 4.

The following terms are made use of in this operation. The number which is to be decompounded, or divided, is called the *dividend;* the number of equal parts sought is called the *divisor;* the magnitude of one of those parts, determined by the division, is called the *quotient:* thus, in the above example,

12 is the dividend,
3 is the divisor, and
4 is the quotient.

46. It follows from this, that if we divide a number by 2, or into two equal parts, one of those parts, or the quotient, taken twice, makes exactly the number proposed; and, in the same manner, if we have a number to divide by 3, the quotient taken thrice must give the same number again. In general, the multiplication of the quotient by the divisor must always reproduce the dividend.

47. It is for this reason that division is said to be a rule, which teaches us to find a number or quotient, which, being multiplied by the divisor, will exactly produce the dividend. For example, if 35 is to be divided by 5, we seek for a number which, multiplied by 5, will produce 35. Now this number is 7, since 5 times 7 is 35. The manner of expression employed in this reasoning, is; 5 in 35 goes 7 times; and 5 times 7 make 35.

48. The dividend therefore may be considered as

a product, of which one of the factors is the divisor, and the other the quotient. Thus, supposing we have 63 to divide by 7, we endeavour to find such a product, that, taking 7 for one of its factors, the other factor multiplied by this may exactly give 63. Now 7×9 is such a product, and consequently 9 is the quotient obtained when we divide 63 by 7.

49. In general, if we have to divide a number ab by a, it is evident that the quotient will be b; for a multiplied by b gives the dividend ab. It is clear also, that if we had to divide ab by b, the quotient would be a. And in all examples of division that can be proposed, if we divide the dividend by the quotient, we shall again obtain the divisor; for as 24 divided by 4 gives 6, so 24 divided by 6 will give 4.

50. As the whole operation consists in representing the dividend by two factors, of which one may be equal to the divisor, and the other to the quotient, the following examples will be easily understood. I say first, that the dividend abc, divided by a, gives bc; for a, multiplied by bc, produces abc: in the same manner abc, being divided by b, we shall have ac; and abc, divided by ac, gives b. It is also plain, that $12mn$, divided by $3m$, gives $4n$; for $3m$, multiplied by $4n$, makes $12mn$. But if this same number $12mn$ had been divided by 12, we should have obtained the quotient mn.

51. Since every number a may be expressed by $1a$, or *one* a, it is evident that if we had to divide a, or $1a$, by 1, the quotient would be the same number a. And, on the contrary, if the same number a, or $1a$, is to be divided by a, the quotient will be 1.

52. It often happens that we cannot represent the dividend as the product of two factors, of which one is equal to the divisor; hence, in this case, the division cannot be performed in the manner we have described.

When we have, for example, 24 to divide by 7, it is at first sight obvious, that the number 7 is not a factor of 24; for the product of 7×3 is only 21, and consequently too small; and 7×4 makes 28, which is greater than 24. We discover, however, from this, that the quotient must be greater than 3, and less than 4. In order therefore to determine it exactly, we employ another species of numbers, which are called *fractions*, and which we shall consider in one of the following chapters.

53. Before we proceed to the use of fractions, it is usual to be satisfied with the whole number which approaches nearest to the true quotient, but at the same time paying attention to the *remainder* which is left; thus we say, 7 in 24 goes 3 times, and the remainder is 3, because 3 times 7 produces only 21, which is 3 less than 24. We may also consider the following examples in the same manner:

6)34(5, that is to say, the divisor is 6, the
  30   dividend 34, the quotient 5, and the
  —   remainder 4.
   4

9)41(4, here the divisor is 9, the dividend
  36   41, the quotient 4, and the remain-
  —   der 5.
   5

The following rule is to be observed in examples where there is a remainder.

54. Multiply the divisor by the quotient, and to

the product add the remainder, and the result will be the dividend; this is the method of proving the division, and of discovering whether the calculation is right or not. Thus, in the first of the two last examples, if we multiply 6 by 5, and to the product 30 add the remainder 4, we obtain 34, or the dividend. And in the last example, if we multiply the divisor 9 by the quotient 4, and to the product 36 add the remainder 5, we obtain the dividend 41.

55. Lastly, it is necessary to remark here, with regard to the signs + *plus* and − *minus*, that if we divide $+ab$ by $+a$, the quotient will be $+b$, which is evident. But if we divide $+ab$ by $-a$, the quotient will be $-b$; because $-a \times -b$ gives $+ab$. If the dividend is $-ab$, and is to be divided by the divisor $+a$, the quotient will be $-b$; because it is $-b$ which, multiplied by $+a$, makes $-ab$. Lastly, if we have to divide the dividend $-ab$ by the divisor $-a$, the quotient will be $+b$; for the dividend $-ab$ is the product of $-a$ by $+b$.

56. With regard, therefore, to the signs + and −, division requires the same rules to be observed that we have seen take place in multiplication; viz.

+ by + makes +; + by − makes −;
− by + makes −; − by − makes +:

or, in few words, like signs give *plus*, and unlike signs give *minus*.

57. Thus, when we divide $18pq$ by $-3p$, the quotient is $-6q$. Farther;

$-30xy$ divided by $+6y$ gives $-5x$, and
$-54abc$ divided by $-9b$ gives $+6ac$;

for, in this last example, $-9b$ multiplied by $+6ac$ makes $-6 \times 9abc$, or $-54abc$. But enough has been said on the division of simple quantities; we shall therefore hasten to the explanation of fractions, after having added some farther remarks on the nature of numbers, with respect to their divisors.

CHAP. VI.

Of the Properties of Integers, with respect to their Divisors.

58. As we have seen that some numbers are divisible by certain divisors, while others are not so; it will be proper, in order that we may obtain a more particular knowledge of numbers, that this difference should be carefully observed, both by distinguishing the numbers that are divisible by divisors from those which are not, and by considering the remainder that is left in the division of the latter. For this purpose let us examine the divisors;

2, 3, 4, 5, 6, 7, 8, 9, 10, &c.

59. First, let the divisor be 2; the numbers divisible by it are, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, &c. which, it appears, increase always by two. These numbers, as far as they can be continued, are called *even numbers*. But there are other numbers, viz.

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, &c.

which are uniformly less or greater than the former by unity, and which cannot be divided by 2, without the remainder 1; these are called *odd numbers*.

The even numbers are all comprehended in the general expression $2a$; for they are all obtained by successively substituting for a the integers 1, 2, 3, 4, 5, 6, 7, &c. and hence it follows that the odd numbers are all comprehended in the expression $2a+1$, because $2a+1$ is greater by unity than the even number $2a$.

60. In the second place, let the number 3 be the divisor; the numbers divisible by it are,

3, 6, 9, 12, 15, 18, 21, 24, 27, 30, and so on;

which numbers may be represented by the expression $3a$; for $3a$, divided by 3, gives the quotient a without a remainder. All other numbers which we would divide by 3, will give 1 or 2 for a remainder, and are consequently of two kinds. Those which after the division leave the remainder 1, are,

1, 4, 7, 10, 13, 16, 19, &c.

and are contained in the expression $3a+1$; but the other kind, where the numbers give the remainder 2, are,

2, 5, 8, 11, 14, 17, 20, &c.

which may be generally represented by $3a+2$; so that all numbers may be expressed either by $3a$, or by $3a+1$, or by $3a+2$.

61. Let us now suppose that 4 is the divisor under consideration; then the numbers which it divides are,

4, 8, 12, 16, 20, 24, &c.

which increase uniformly by 4, and are comprehended in the expression $4a$. All other numbers, that is, those which are not divisible by 4, may either leave the remainder 1, or be greater than the former by 1; as,

1, 5, 9, 13, 17, 21, 25, &c.

and consequently may be comprehended in the expression $4a+1$: or they may give the remainder 2; as,

2, 6, 10, 14, 18, 22, 26, &c.

and be expressed by $4a+2$; or, lastly, they may give the remainder 3; as,

3, 7, 11, 15, 19, 23, 27, &c.

and may then be represented by the expression $4a+3$.

All possible integer numbers are therefore contained in one or other of these four expressions;

$$4a,\ 4a+1,\ 4a+2,\ 4a+3.$$

62. It is also nearly the same when the divisor is 5; for all numbers which can be divided by it are comprehended in the expression $5a$, and those which cannot be divided by 5, are reducible to one of the following expressions:

$$5a+1,\ 5a+2,\ 5a+3,\ 5a+4;$$

and in the same manner we may continue, and consider any greater divisor.

63. It is here proper to recollect what has been already said on the resolution of numbers into their simple factors; for every number, among the factors of which is found

2, or 3, or 4, or 5, or 7,

or any other number, will be divisible by those numbers. For example; 60 being equal to $2\times2\times3\times5$, it is evident that 60 is divisible by 2, and by 3, and by 5*.

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* There are some numbers which it is easy to perceive whether they are divisors of a given number or not.

1. A given number is divisible by 2, if the last digit is even; it is divisible by 4, if the two last digits are divisible by 4; it is di-

64. Farther, as the general expression *abcd* is not only divisible by *a*, and *b*, and *c*, and *d*, but also by

ab, *ac*, *ad*, *bc*, *bd*, *cd*, and by
abc, *abd*, *acd*, *bcd*, and lastly by
abcd, that is to say, its own value;

it follows that 60, or $2\times2\times3\times5$, may be divided

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visible by 8, if the three last digits are divisible by 8; and, in general, it is divisible by 2^n, if the *n* last digits are divisible by 2^n.

2. A number is divisible by 3, if the sum of the digits is divisible by 3; it may be divided by 6, if, beside this, the last digit is even; it is divisible by 9, if the sum of the digits may be divided by 9.

3. Every number that has the last digit 0 or 5, is divisible by 5.

4. A number is divisible by 11, when the sum of the first, third, fifth, &c. digits is equal to the sum of the second, fourth, sixth, &c. digits.

It would be easy to explain the reason of these rules, and to extend them to the products of the divisors which we have just now considered. Rules might be devised likewise for some other numbers, but the application of them would in general be longer than an actual trial of the division.

For example, I say that the number 53704689213 is divisible by 7, because I find that the sum of the digits of the number 64004245433 is divisible by 7; and this second number is formed, according to a very simple rule, from the remainders found after dividing by 7 the numbers 10, 100, 1000, &c. 20, 200, 2000, &c. as far as 60, 600, 6000, &c. F.T.

It may not be amiss to explain to the reader the principles upon which these rules are founded.

1. Every number is divisible by 2^n, when the *n* last digits are divisible by 2^n, which includes the other three particular cases.

For every number may be expressed by the form $A\times10^n+B$, where B represents the number expressed by the n last digits.

Thus, for example, $7846144=784614\times10+4$, $=78461\times10^2+44$, $=7846\times10^3+144$, $=784\times10^4+6144$, &c. Now since 10 is divisible by 2, 10^2 is divisible by 4, or 2^2, and generally not only by these simple numbers, but also by those which are composed of two of them; that is to say,

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10^n is divisible by 2^n. And B is also divisible by 2^n by the supposition; consequently the whole number $A \times 10^n + B$ must necessarily be divisible by 2^n, because each of the parts of which it is composed is divisible by that number.

2. Any power of 10 when divided by 3, or 9, leaves a remainder 1, therefore if any power of 10, when multiplied by a given number a (for example $10^n a$), be divided by 3, or 9, it will leave the same remainder as the number a singularly, divided by either of those numbers.

Now every number may be expressed by $10^n a + 10^{n-1} b + 10^{n-2} c + 10^{n-3} d +$, &c., where a, b, c, d, &c. represent the digits of which the number is composed. And since each of those terms, when divided by 3 or 9, leaves the same remainder as its respective digit, therefore the sum of all those terms, that is, the whole number, will leave the same remainder as the sum of its digits $a+b+c+d$, &c. and consequently when the latter is exactly divisible by 3, or 9, the former is so likewise.

Also if the last digit be even, the number is divisible by 2, as well as by 3, or 9, and is therefore likewise divisible by 6 or 18.

3. Every number ending in a 5, or 0, is of one of the form $10A$, or $10A+5$; which is evidently in either case divisible by 5.

4. Every even power of 10, as 10^{2n}, when divided by 11, leaves a remainder $+1$; and every odd power of 10, as 10^{2n+1}, leaves a remainder -1; therefore every number $10^{2n}a$, when divided by 11, leaves a remainder a, and every number $10^{2n+1}b$, divided by the same, leaves a remainder $-b$.

Now we have seen that every number is of the form of $10^n a + 10^{n-1} b + 10^{n-2} c + 10^{n-3} d$, &c. where the powers of 10 are alternately even and odd, and therefore the remainders alternately plus and minus; namely, $+a-b+c-d$, &c. or $-a+b-c+d$, &c.; where a, b, c, d, &c. are the digits that compose the given number, and when $a+c$, &c. $=b+d$, &c. one side being positive and the other negative, they destroy each other, and the whole number is divisible by 11.

The last rule for the number 7 is of no use whatever, but the by 4, 6, 10, 15: and also by those which are composed of three simple factors, that is to say, by 12, 20, 30, and lastly also, by 60 itself.

65. When, therefore, we have represented any number, assumed at pleasure, by its simple factors, it will be very easy to exhibit all the numbers by which it is divisible. For we have only, first, to take the simple factors one by one, and then to multiply them together two by two, three by three, four by four, &c. till we arrive at the number proposed.

66. It must here be particularly observed, that every number is divisible by 1; and also, that every number is divisible by itself; so that every number has at least two factors, or divisors, the number itself and unity: but every number which has no other divisor than these two, belongs to the class of numbers which we have before called *simple*, or *prime numbers*.

Except these simple numbers, all other numbers have, beside unity and themselves, other divisors, as may be seen from the following table, in which are placed under each number all its divisors*.

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principle upon which it is founded is easily demonstrated; it is besides very badly expressed, for it is general for any number, and not peculiar to the number 7, as the other rules are for 3, 9, 11, &c. as it appears to be by the manner in which it is announced.

By referring to the form $10^n a + 10^{n-1} b$, $10^{n-2} c$, &c. under which every number may be expressed, it is evident that if each of those terms, when divided by a given number a, leave respectively the remainders p, q, r, s, &c., and also if the sum of these remainders be divisible by a, it follows that the whole number $10^n a + 10^{n-1} b + 10^{n-2} c$, &c. is divisible by a likewise. ED.

* A similar table for all the divisors of the natural numbers,

TABLE.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	3	2	5	2	7	2	3	2	11	2	13	2	3	2	17	2	19	2
			4		3		4	9	5		3		7	5	4		3		4
					6		8		10		4		14	15	8		6		5
											6				16		9		10
											12						18		20
1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	5	2	6	2	6
P.	P.	P.		P.		P.				P.		P.				P.		P.	

67. Lastly, it ought to be observed that 0, or *nothing*, may be considered as a number which has the property of being divisible by all possible numbers; because by whatever number *a* we divide 0, the quotient is always 0; for it must be remarked, that the multiplication of any number by *nothing* produces nothing, and therefore 0 times *a*, or 0*a*, is 0.

CHAP. VII.

Of Fractions in general.

68. When a number, as 7, for instance, is said not to be divisible by another number, let us suppose

from 1 to 10000, was published at Leyden, in 1767, by M. Henri Anjema. We have likewise another table of divisors, which goes as far as 100000, but it gives only the least divisor of each number. It is to be found in Harris's Lexicon Technicum, the Encyclopédie, and in M. Lambert's Recueil, which we have quoted in the note to p. 15. In this last work, it is continued as far as 102000. F. T.

by 3, this only means, that the quotient cannot be expressed by an integer number; and it must not be thought by any means that it is impossible to form an idea of that quotient. Only imagine a line of 7 feet in length, and nobody can doubt the possibility of dividing this line into 3 equal parts, and of forming a notion of the length of one of those parts.

69. Since therefore we may form a precise idea of the quotient obtained in similar cases, though that quotient may not be an integer number, this leads us to consider a particular species of numbers, called *fractions*, or *broken numbers;* of which the instance adduced furnishes an illustration. For if we have to divide 7 by 3, we easily conceive the quotient which should result, and express it by $\frac{7}{3}$; placing the divisor under the dividend, and separating the two numbers by a stroke, or line.

70. So, in general, when the number a is to be divided by the number b, we represent the quotient by $\frac{a}{b}$, and call this form of expression *a fraction.* We cannot therefore give a better idea of a fraction $\frac{a}{b}$, than by saying that it expresses the quotient resulting from the division of the upper number by the lower. We must remember also, that in all fractions the lower number is called the *denominator*, and that above the line the *numerator*.

71. In the above fraction $\frac{7}{3}$, which we read *seven thirds*, 7 is the numerator, and 3 the denominator. We must also read $\frac{2}{3}$, two thirds; $\frac{3}{4}$, three fourths; $\frac{3}{8}$, three eighths; $\frac{12}{100}$, twelve hundredths; and $\frac{1}{2}$, one half, &c.

72. In order to obtain a more perfect knowledge of the nature of fractions, we shall begin by considering the case in which the numerator is equal to the denominator, as in $\frac{a}{a}$. Now, since this expresses the quotient obtained by dividing a by a, it is evident that this quotient is unity, and that consequently the fraction $\frac{a}{a}$ is equal to 1, or one integer; for the same reason, all the following fractions,

$$\frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6}, \frac{7}{7}, \frac{8}{8}, \text{ \&c.}$$

are equal to one another, each being equal to 1, or one integer.

73. We have seen that a fraction whose numerator is equal to the denominator, is equal to unity. All fractions therefore whose numerators are less than the denominators, have a value less than unity. For if I have a number to divide by another which is greater than itself, the result must necessarily be less than 1: if we cut a line, for example, two feet long, into three parts, one of those parts will undoubtedly be shorter than a foot: it is evident then, that $\frac{2}{3}$ is less than 1, for the same reason; that is, the numerator 2 is less than the denominator 3.

74. If the numerator, on the contrary, be greater than the denominator, the value of the fraction is greater than unity. Thus $\frac{3}{2}$ is greater than 1, for $\frac{3}{2}$ is equal to $\frac{2}{2}$ together with $\frac{1}{2}$. Now $\frac{2}{2}$ is exactly 1; consequently $\frac{3}{2}$ is equal to $1+\frac{1}{2}$, that is, to an integer and a half. In the same manner $\frac{4}{3}$ is equal to $1\frac{1}{3}$, $\frac{5}{3}$ to $1\frac{2}{3}$, and $\frac{7}{3}$ to $2\frac{1}{3}$. And, in general, it is sufficient in such cases to divide the upper number

by the lower, and to add to the quotient a fraction, having the remainder for the numerator and the divisor for the denominator. If the given fraction were, for example, $\frac{43}{12}$, we should have for the quotient 3, and 7 for the remainder; whence we should conclude that $\frac{43}{12}$ is the same as $3\frac{7}{12}$.

75. Thus we see how fractions, whose numerators are greater than the denominators, are resolved into two members; one of which is an integer, and the other a fractional number, having the numerator less than the denominator. Such fractions as contain one or more integers, are called *improper fractions*, to distinguish them from fractions properly so called, which having the numerator less than the denominator, are less than unity, or than an integer.

76. The nature of fractions is frequently considered in another way, which may throw additional light on the subject. If we consider, for example, the fraction $\frac{3}{4}$, it is evident that it is three times greater than $\frac{1}{4}$. Now this fraction $\frac{1}{4}$ means, that if we divide 1 into 4 equal parts, this will be the value of one of those parts; it is obvious then, that by taking 3 of those parts we shall have the value of the fraction $\frac{3}{4}$.

In the same manner we may consider every other fraction; for example, $\frac{7}{12}$; if we divide unity into 12 equal parts, 7 of those parts will be equal to the fraction proposed.

77. From this manner of considering fractions, the expressions *numerator* and *denominator* are derived. For, as in the preceding fraction $\frac{7}{12}$, the number under the line shows that 12 is the number of parts into which unity is to be divided; and as it

may be said to denote, or name, the parts, it has not improperly been called the *denominator*.

Farther, as the upper number, viz. 7, shows that, in order to have the value of the fraction, we must take, or collect, 7 of those parts, and therefore may be said to reckon or number them, it has been thought proper to call the number above the line the *numerator*.

78. As it is easy to understand what $\frac{3}{4}$ is, when we know the signification of $\frac{1}{4}$, we may consider the fractions whose numerator is unity, as the foundation of all others. Such are the fractions,

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \text{ \&c.}$$

and it is observable that these fractions go on continually diminishing: for the more you divide an integer, or the greater the number of parts into which you distribute it, the less does each of those parts become. Thus $\frac{1}{100}$ is less than $\frac{1}{10}$; $\frac{1}{1000}$ is less than $\frac{1}{100}$; and $\frac{1}{10000}$ is less than $\frac{1}{1000}$, &c.

79. As we have seen that the more we increase the denominator of such fractions the less their values become, it may be asked, whether it is not possible to make the denominator so great that the fraction shall be reduced to nothing? I answer, no; for into whatever number of parts unity (the length of a foot, for instance) is divided; let those parts be ever so small, they will still preserve a certain magnitude, and therefore can never be absolutely reduced to nothing.

80. It is true, if we divide the length of a foot into 1000 parts, those parts will not easily fall under the cognizance of our senses; but view them through

a good microscope, and each of them will appear large enough to be still subdivided into more than 100 parts.

At present, however, we have nothing to do with what depends on ourselves, or with what we are capable of performing, and what our eyes can perceive; the question is rather what is possible in itself: and, in this sense of the word, it is certain, that however great we suppose the denominator, the fraction will never entirely vanish, or become equal to 0.

81. We can never therefore arrive completely at 0, or nothing, however great the denominator may be; and consequently as those fractions must always preserve a certain quantity, we may continue the series of fractions in the 78th article without interruption. This circumstance has introduced the expression, that the denominator must be *infinite*, or infinitely great, in order that the fraction may be reduced to 0, or to nothing; hence the word *infinite* in reality signifies here, that we can never arrive at the end of the series of the abovementioned *fractions*.

82. To express this idea, according to the sense of it abovementioned, we make use of the sign ∞, which consequently indicates a number infinitely great; and we may therefore say, that this fraction $\frac{1}{\infty}$ is in reality nothing; because a fraction cannot be reduced to nothing, until the denominator has been increased to *infinity*.

83. It is the more necessary to pay attention to this idea of infinity, as it is derived from the very foundation of our knowledge of this subject, and

more particularly as it will be of the greatest importance in the following part of this treatise.

We may here deduce from it a few consequences that are extremely curious and worthy of attention. The fraction $\frac{1}{\infty}$ represents the quotient resulting from the division of the dividend 1 by the divisor ∞. Now we know, that if we divide the dividend 1 by the quotient $\frac{1}{\infty}$, which is equal to nothing, we obtain again the divisor ∞: hence we acquire a new idea of infinity; and learn that it arises from the division of 1 by 0; so that we are thence entitled to say, that 1 divided by 0 expresses a number infinitely great, or ∞.

84. It may be necessary also in this place to correct the mistake of those who assert that a number infinitely great is not susceptible of increase. This opinion is inconsistent with the just principles which we have laid down; for $\frac{1}{0}$ signifying a number infinitely great, and $\frac{2}{0}$ being incontestably the double of $\frac{1}{0}$, it is evident that a number, though infinitely great, may still become twice, thrice, or any number of times greater*.

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* There are other properties of *nothing* and *infinity* which it may be proper to notice in this place.

1. Nothing, added to or subtracted from any quantity, makes it neither greater nor less.

2. Any quantity multiplied by 0, that is, a quantity taken no times, gives 0 for a product; or $a \times 0 = 0$.

3. $a^0 = 1$, whatever be the numeral value of a. For $a^0 \times a = a^{0+1} = a^1 = a$; but $1 \times a = a$ likewise; therefore $a^0 = 1$.

4. Since $\frac{a}{0} = \infty$, therefore $0 \times \infty = a$; that is, nothing multiplied by infinity produces a finite quantity.

CHAP. VIII.

Of the Properties of Fractions.

85. We have already seen, that each of the fractions,

$$\frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6}, \frac{7}{7}, \frac{8}{8}, \text{\&c.}$$

makes an integer, and that consequently they are all equal to one another. The same equality prevails in the following fractions,

$$\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \frac{10}{5}, \frac{12}{6}, \text{\&c.}$$

each of them making two integers; for the numerator of each, divided by its denominator, give 2. So all the fractions

$$\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \frac{15}{5}, \frac{18}{6}, \text{\&c.}$$

are equal to one another, since 3 is their common value.

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5. Since $a \times 0 = 0$, therefore $\frac{0}{0} = a$; that is, nothing divided by nothing gives for a quotient some finite quantity.

The above subject has been a grand stumbling-block to mathematicians for a considerable time past, and various disputes and controversies have been held in support of this or that opinion. But this is not a proper place to enter into a metaphysical discussion upon the subject; we shall only observe, that having once agreed upon certain symbols for the representation of nothing and infinity, however we may be at a loss to comprehend that which we have represented, yet still, in a mathematical point of view, those symbols are subject to the same laws, in the operations of multiplication, division, &c. as others which represent quantities that are evident to our senses; the business of the mathematician being not so much the consideration of quantities themselves, as the relation subsisting between them, or between those symbols which, by common consent, are made their representatives. Ed.

86. We may likewise represent the value of any fraction in an infinite variety of ways. For if we multiply both the numerator and the denominator of a fraction by the same number, which may be assumed at pleasure, this fraction will still preserve the same value. For this reason all the fractions

$$\tfrac{1}{2}, \tfrac{2}{4}, \tfrac{3}{6}, \tfrac{4}{8}, \tfrac{5}{10}, \tfrac{6}{12}, \tfrac{7}{14}, \tfrac{8}{16}, \tfrac{9}{18}, \tfrac{10}{20}, \text{\&c.}$$

are equal, the value of each being $\tfrac{1}{2}$. Also

$$\tfrac{1}{3}, \tfrac{2}{6}, \tfrac{3}{9}, \tfrac{4}{12}, \tfrac{5}{15}, \tfrac{6}{18}, \tfrac{7}{21}, \tfrac{8}{24}, \tfrac{9}{27}, \tfrac{10}{30}, \text{\&c.}$$

are equal fractions, the value of each of which is $\tfrac{1}{3}$. The fractions

$$\tfrac{2}{3}, \tfrac{4}{6}, \tfrac{8}{12}, \tfrac{10}{15}, \tfrac{12}{18}, \tfrac{14}{21}, \tfrac{16}{24}, \text{\&c.}$$

have likewise all the same value. Hence we may conclude, in general, that the fraction $\frac{a}{b}$ may be represented by any of the following expressions, each of which is equal to $\frac{a}{b}$; viz.

$$\frac{a}{b}, \frac{2a}{2b}, \frac{3a}{3b}, \frac{4a}{4b}, \frac{5a}{5b}, \frac{6a}{6b}, \frac{7a}{7b}, \text{\&c.}$$

87. To be convinced of this, we have only to write for the value of the fraction $\frac{a}{b}$ a certain letter c, representing by this letter c the quotient of the division of a by b; and to recollect that the multiplication of the quotient c by the divisor b must give the dividend. For since c multiplied by b gives a, it is evident that c multiplied by $2b$ will give $2a$, that c multiplied by $3b$ will give $3a$, and that, in general, c multiplied by mb will give ma. Now changing this into an example of division, and dividing the product ma by mb, one of the factors, the quotient must be equal to

the other factor c; but ma divided by mb gives also the fraction $\frac{ma}{mb}$, which is consequently equal to c; which is what was to be proved: for c having been assumed as the value of the fraction $\frac{a}{b}$, it is evident that this fraction is equal to the fraction $\frac{ma}{mb}$, whatever be the value of m.

88. We have thus seen that every fraction may be represented in an infinite number of forms each of which contains the same value; and it is evident that of all these forms, that which is composed of the least numbers, will be most easily understood. For example, we might substitute instead of $\frac{2}{3}$ the following fractions,

$$\frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \frac{12}{18}, \text{ \&c.}$$

but of all these expressions $\frac{2}{3}$ is that of which it is easiest to form an idea. Here therefore a problem arises, how a fraction, such as $\frac{8}{12}$, which is not expressed by the least possible numbers, may be reduced to its simplest form, or to *its least terms*, that is to say, in our present example, to $\frac{2}{3}$.

89. It will be easy to resolve this problem, if we consider that a fraction still preserves its value, when we multiply both its terms, or its numerator and denominator, by the same number. For from this it also follows, that if we divide the numerator and denominator of a fraction by the same number, the fraction will still preserve the same value. This is made more evident by means of the general expression $\frac{ma}{mb}$, for if we divide both the numerator ma and the

denominator *mb* by the number *m*, we obtain the fraction $\frac{a}{b}$, which, as was before proved, is equal to $\frac{ma}{mb}$.

90. In order therefore to reduce a given fraction to its least terms, it is required to find a number by which both the numerator and denominator may be divided. Such a number is called a *common divisor*, and as long as we can find a common divisor to the numerator and the denominator, it is certain that the fraction may be reduced to a lower form; but, on the contrary, when we see that, except unity, no other common divisor can be found, this shows that the fraction is already in its simplest form.

91. To make this more clear, let us consider the fraction $\frac{48}{120}$. We see immediately that both the terms are divisible by 2, and that there results the fraction $\frac{24}{60}$. Also, that it may again be divided by 2, and reduced to $\frac{12}{30}$; and as this likewise has 2 for a common divisor, it is evident that it may be reduced to $\frac{6}{15}$. But now we easily perceive that the numerator and denominator are still divisible by 3; therefore performing this division, we obtain the fraction $\frac{2}{5}$, which is equal to the fraction proposed, and gives the simplest expression to which it can be reduced; for 2 and 5 have no common divisor but 1, which cannot diminish these numbers any farther.

92. This property of fractions preserving an invariable value, whether we divide or multiply the numerator and denominator by the same number, is of the greatest importance, and is the principal foundation of the doctrine of fractions. For example, we can seldom add together two fractions, or subtract them from each other, before we have, by

means of this property, reduced them to other forms, that is to say, to expressions whose denominators are equal. Of this we shall treat in the following chapter.

93. It is necessary however, before we conclude, to remark, that all integers may also be represented by fractions. For example, 6 is the same as $\frac{6}{1}$, because 6 divided by 1 makes 6; we may also, in the same manner, express the number 6 by the fractions $\frac{12}{2}$, $\frac{18}{3}$, $\frac{24}{4}$, $\frac{36}{6}$, and an infinite number of others which have the same value.

CHAP. IX.

Of the Addition and Subtraction of Fractions.

94. When fractions have equal denominators, there is no difficulty in adding and subtracting them; for $\frac{2}{7}+\frac{3}{7}$ is equal to $\frac{5}{7}$, and $\frac{4}{7}-\frac{2}{7}$ is equal to $\frac{2}{7}$. In this case, therefore, either for addition or subtraction, we alter only the numerators, and place the common denominator under the line; thus,

$\frac{7}{100}+\frac{9}{100}-\frac{12}{100}-\frac{15}{100}+\frac{20}{100}$ is equal to $\frac{9}{100}$;

$\frac{24}{50}-\frac{7}{50}-\frac{12}{50}+\frac{31}{50}$ is equal to $\frac{36}{50}$, or $\frac{18}{25}$;

$\frac{16}{20}-\frac{3}{20}-\frac{11}{20}+\frac{14}{20}$ is equal to $\frac{16}{20}$, or $\frac{4}{5}$;

also $\frac{1}{3}+\frac{2}{3}$ is equal to $\frac{3}{3}$, or 1, that is to say, an integer; and

$\frac{2}{4}-\frac{3}{4}+\frac{1}{4}$ is equal to $\frac{0}{4}$, that is to say, nothing, or 0.

95. But when fractions have not equal denominators, we can always change them into other fractions that have the same denominator. For example, when it is proposed to add together the fractions $\frac{1}{2}$ and $\frac{1}{3}$, we must consider that $\frac{1}{2}$ is the same as $\frac{3}{6}$, and that $\frac{1}{3}$ is equivalent to $\frac{2}{6}$; we have therefore, instead of the two fractions proposed, the two following ones, $\frac{3}{6}+\frac{2}{6}$, the sum of which is $\frac{5}{6}$. And if the two fractions were united by the sign *minus*, as $\frac{1}{2}-\frac{1}{3}$, we should have $\frac{3}{6}-\frac{2}{6}$ or $\frac{1}{6}$.

As another example, let the fractions proposed be $\frac{3}{4}+\frac{5}{8}$; then since $\frac{3}{4}$ is the same as $\frac{6}{8}$, this value may be substituted for it, and we may say $\frac{6}{8}+\frac{5}{8}$ makes $\frac{11}{8}$, or $1\frac{3}{8}$.

Suppose farther, that the sum of $\frac{1}{3}$ and $\frac{1}{4}$ were required, I say that it is $\frac{7}{12}$; for $\frac{1}{3}$ makes $\frac{4}{12}$, and $\frac{1}{4}$ makes $\frac{3}{12}$.

96. If we have a greater number of fractions to reduce to a common denominator; for example,

$\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$; in this case the whole depends on finding a number that shall be divisible by all the denominators of those fractions. In this instance 60 is the number which has that property, and which consequently becomes the common denominator. We shall therefore have $\frac{30}{60}$ instead of $\frac{1}{2}$; $\frac{40}{60}$ instead of $\frac{2}{3}$; $\frac{45}{60}$ instead of $\frac{3}{4}$; $\frac{48}{60}$ instead of $\frac{4}{5}$; and $\frac{50}{60}$ instead of $\frac{5}{6}$. If now it be required to add together all these fractions $\frac{30}{60}$, $\frac{40}{60}$, $\frac{45}{60}$, $\frac{48}{60}$, and $\frac{50}{60}$; we have only to add all the numerators, and under the sum place the common denominator 60; that is to say, we shall have $\frac{213}{60}$, or $3\frac{11}{20}$.

97. The whole of this operation consists, as we before stated, in changing fractions whose denominators are unequal into others whose denominators are equal. In order therefore to perform it generally, let $\frac{a}{b}$ and $\frac{c}{d}$ be the fractions proposed. First, multiply the two terms of the first fraction by d, and we shall have the fraction $\frac{ad}{bd}$ equal to $\frac{a}{b}$; next multiply the two terms of the second fraction by b, and we shall have an equivalent value of it expressed by $\frac{bc}{bd}$; thus the two denominators are become equal. Now if the sum of the two proposed fractions be required, we may immediately answer that it is $\frac{ad+bc}{bd}$; and if their difference be asked, we say that it is $\frac{ad \backsim bc}{bd}$. If the fractions $\frac{5}{8}$ and $\frac{7}{9}$, for example,

were proposed, we should obtain in their stead $\frac{45}{72}$ and $\frac{56}{72}$; of which the sum is $\frac{101}{72}$, and the difference $\frac{11}{72}$ *.

98. To this part of the subject belongs also the question, Of two proposed fractions which is the greater or the less? for, to resolve this, we have only to reduce the two fractions to the same denominator. Let us take, for example, the two fractions $\frac{2}{3}$ and $\frac{5}{7}$; when reduced to the same denominator, the first becomes $\frac{14}{21}$, and the second $\frac{15}{21}$, where it is evident that the second, or $\frac{5}{7}$, is the greater, and exceeds the former by $\frac{1}{21}$.

Again, if the two fractions $\frac{3}{5}$ and $\frac{5}{8}$ be proposed, we shall have to substitute for them $\frac{24}{40}$ and $\frac{25}{40}$; whence we may conclude that $\frac{5}{8}$ exceeds $\frac{3}{5}$ only by $\frac{1}{40}$.

99. When it is required to subtract a fraction from an integer, it is sufficient to change one of the units of that integer into a fraction which has the same denominator as that which is to be subtracted; then in the rest of the operation there is no difficulty. If it be required, for example, to subtract $\frac{3}{5}$ from 1, we write

* The rule for reducing fractions to a common denominator, may be concisely expressed thus. Multiply each numerator into every denominator except its own, for a new numerator, and all the denominators together for the common denominator. When this operation has been performed, it will appear that the numerator and denominator of each fraction have been multiplied by the same quantity, and consequently retain the same value.

$\frac{3}{3}$ instead of 1, and say that $\frac{2}{3}$ taken from $\frac{3}{3}$ leaves the remainder $\frac{1}{3}$. So $\frac{5}{12}$, subtracted from 1, leaves $\frac{7}{12}$.

If it were required to subtract $\frac{3}{4}$ from 2, we should write 1 and $\frac{4}{4}$ instead of 2, and should then immediately see that after the subtraction there must remain $1\frac{1}{4}$.

100. It happens also sometimes, that having added two or more fractions together, we obtain more than an integer; that is to say, a numerator greater than the denominator: this is a case which has already occurred, and deserves attention.

We found, for example, article 96, that the sum of the five fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$ was $\frac{213}{60}$, and remarked that the value of this sum was $3\frac{33}{60}$ or $3\frac{11}{20}$. Likewise $\frac{2}{3}+\frac{3}{4}$, or $\frac{8}{12}+\frac{9}{12}$ makes $\frac{17}{12}$, or $1\frac{5}{12}$. We have therefore only to perform the actual division of the numerator by the denominator, to see how many integers there are for the quotient, and to set down the remainder.

Nearly the same must be done to add together numbers compounded of integers and fractions; we first add the fractions, and if the sum produces one or more integers, these are added to the other integers. If it be proposed, for example, to add $3\frac{1}{2}$ and $2\frac{2}{3}$; we first take the sum of $\frac{1}{2}$ and $\frac{2}{3}$, or of $\frac{3}{6}$ and $\frac{4}{6}$, which is $\frac{7}{6}$ or $1\frac{1}{6}$; and thus find the total sum to be $6\frac{1}{6}$.

CHAP. X.

Of the Multiplication and Division of Fractions.

101. The rule for the multiplication of a fraction by an integer, or whole number, is to multiply the numerator only, by the given number, and not to change the denominator: thus,

2 times, or twice $\frac{1}{2}$ makes $\frac{2}{2}$, or 1 integer;

2 times, or twice $\frac{1}{3}$ makes $\frac{2}{3}$; and

3 times, or thrice $\frac{1}{6}$ makes $\frac{3}{6}$, or $\frac{1}{2}$;

4 times $\frac{5}{12}$ makes $\frac{20}{12}$ or $1\frac{8}{12}$, or $1\frac{2}{3}$.

But, instead of this rule, we may use that of dividing the denominator by the given integer, which is preferable, when it can be done, because it shortens the operation. Let it be required, for example, to multiply $\frac{8}{9}$ by 3; if we multiply the numerator by the given integer we obtain $\frac{24}{9}$, which product we must reduce to $\frac{8}{3}$. But if we do not change the numerator, and divide the denominator by the integer, we find immediately $\frac{8}{3}$ or $2\frac{2}{3}$ for the given product; and in the same manner $\frac{13}{24}$ multiplied by 6 gives $\frac{13}{4}$, or $3\frac{1}{4}$.

102. In general, therefore, the product of the multiplication of a fraction $\frac{a}{b}$ by c is $\frac{ac}{b}$; and here it may be remarked, when the integer is exactly equal to

the denominator, that the product must be equal to the numerator.

So that $\begin{cases} \frac{1}{2} \text{ taken twice, gives } 1; \\ \frac{2}{3} \text{ taken thrice, gives } 2; \\ \frac{3}{4} \text{ taken 4 times, gives } 3. \end{cases}$

And, in general, if we multiply the fraction $\frac{a}{b}$ by the number b, the product must be a, as we have already shown; for since $\frac{a}{b}$ expresses the quotient resulting from the division of the dividend a by the divisor b, and because it has been demonstrated that the quotient multiplied by the divisor will give the dividend, it is evident that $\frac{a}{b}$ multiplied by b must produce a.

103. Having thus shown how a fraction is to be multiplied by an integer; let us now consider also how a fraction is to be divided by an integer; this inquiry is necessary before we proceed to the multiplication of fractions by fractions. It is evident, if we have to divide the fraction $\frac{2}{3}$ by 2, that the result must be $\frac{1}{3}$; and that the quotient of $\frac{6}{7}$ divided by 3 is $\frac{2}{7}$. The rule therefore is, to divide the numerator by the integer without changing the denominator. Thus:

$\frac{12}{25}$ divided by 2 gives $\frac{6}{25}$;

$\frac{12}{25}$ divided by 3 gives $\frac{4}{25}$; and

$\frac{12}{25}$ divided by 4 gives $\frac{3}{25}$; &c.

104. This rule may be easily practised, provided the numerator be divisible by the number proposed; but very often it is not: it must therefore be observed that a fraction may be transformed into an infinite number of other expressions, and in that number there must be some by which the numerator might be divided by the given integer. If it were required, for example, to divide $\frac{3}{4}$ by 2, we should change the fraction into $\frac{6}{8}$, and then dividing the numerator by 2, we should immediately have $\frac{3}{8}$ for the quotient sought.

In general, if it be proposed to divide the fraction $\frac{a}{b}$ by c, we change it into $\frac{ac}{bc}$, and then dividing the numerator ac by c, write $\frac{a}{bc}$ for the quotient sought.

105. When therefore a fraction $\frac{a}{b}$ is to be divided by an integer c, we have only to multiply the denominator by that number, and leave the numerator as it is. Thus $\frac{5}{8}$ divided by 3 gives $\frac{5}{24}$, and $\frac{9}{16}$ divided by 5 gives $\frac{9}{80}$.

This operation becomes easier when the numerator itself is divisible by the integer, as we have supposed in article 103. For example, $\frac{9}{16}$ divided by 3 would give, according to our last rule, $\frac{9}{48}$; but by the first rule, which is applicable here, we obtain $\frac{3}{16}$, an expression equivalent to $\frac{9}{48}$, but more simple.

106. We shall now be able to understand how one fraction $\frac{a}{b}$ may be multiplied by another fraction $\frac{c}{d}$.

For this purpose we have only to consider that $\frac{c}{d}$ means that c is divided by d; and on this principle we shall first multiply the fraction $\frac{a}{b}$ by c, which produces the result $\frac{ac}{b}$; after which we shall divide by d, which gives $\frac{ac}{bd}$.

Hence the following rule for multiplying fractions. Multiply the numerators together for a numerator, and the denominators together for a denominator.

Thus $\frac{1}{2}$ by $\frac{2}{3}$ gives the product $\frac{2}{6}$, or $\frac{1}{3}$;
$\frac{2}{3}$ by $\frac{4}{5}$ makes $\frac{8}{15}$;
$\frac{3}{4}$ by $\frac{5}{12}$ produces $\frac{15}{48}$, or $\frac{5}{16}$; &c.

107. It now remains to show how one fraction may be divided by another. Here we remark first, that if the two fractions have the same number for a denominator, the division takes place only with respect to the numerators; for it is evident, that $\frac{3}{12}$ are contained as many times in $\frac{9}{12}$ as 3 is contained in 9, that is to say, three times; and, in the same manner, in order to divide $\frac{8}{12}$ by $\frac{9}{12}$, we have only to divide 8 by 9, which gives $\frac{8}{9}$. We shall also have $\frac{6}{20}$ in $\frac{18}{20}$, 3 times: $\frac{7}{100}$ in $\frac{49}{100}$, 7 times; $\frac{7}{25}$ in $\frac{6}{25}$, $\frac{6}{7}$, &c.

108. But when the fractions have not equal denominators, we must have recourse to the method already mentioned for reducing them to a common denominator. Let there be, for example, the fraction $\frac{a}{b}$ to be divided by the fraction $\frac{c}{d}$: we first reduce

them to the same denominator, and there results $\frac{ad}{bd}$ to be divided by $\frac{cb}{db}$; it is now evident that the quotient must be represented simply by the division of ad by bc; which gives $\frac{ad}{bc}$

Hence the following rule: Multiply the numerator of the dividend by the denominator of the divisor, and the denominator of the dividend by the numerator of the divisor; then the first product will be the numerator of the quotient, and the second will be its denominator.

109. Applying this rule to the division of $\frac{5}{8}$ by $\frac{2}{3}$, we shall have the quotient $\frac{15}{16}$; also the division of $\frac{3}{4}$ by $\frac{1}{2}$ will give $\frac{6}{4}$ or $\frac{3}{2}$, or $1\frac{1}{2}$; and $\frac{25}{48}$ by $\frac{5}{6}$ will give $\frac{150}{240}$, or $\frac{5}{8}$.

110. This rule for division is often expressed in a manner that is more easily remembered, as follows: Invert the terms of the divisor, so that the denominator may be in the place of the numerator, and the latter be written under the line; then multiply the fraction, which is the dividend by this inverted, fraction, and the product will be the quotient sought. Thus $\frac{3}{4}$ divided by $\frac{1}{2}$ is the same as $\frac{3}{4}$ multiplied by $\frac{2}{1}$, which makes $\frac{6}{4}$, or $1\frac{1}{2}$. Also $\frac{5}{8}$ divided by $\frac{2}{3}$ is the same as $\frac{5}{8}$ multiplied by $\frac{3}{2}$, which is $\frac{15}{16}$; or $\frac{25}{48}$ divided by $\frac{5}{6}$ gives the same as $\frac{25}{48}$ multiplied by $\frac{6}{5}$, the product of which is $\frac{150}{240}$, or $\frac{5}{8}$.

We see then, in general, that to divide by the fraction $\frac{1}{2}$ is the same as to multiply by $\frac{2}{1}$, or 2; and that dividing by $\frac{1}{3}$ amounts to multiplying by $\frac{3}{1}$, or by 3, &c.

111. The number 100 divided by $\frac{1}{2}$ will give 200; and 1000 divided by $\frac{1}{3}$ will give 3000. Farther, if it were required to divide 1 by $\frac{1}{1000}$, the quotient would be 1000; and dividing 1 by $\frac{1}{100000}$, the quotient is 100000. This enables us to conceive that, when any number is divided by 0, the result must be a number indefinitely great; for even the division of 1 by the small fraction $\frac{1}{1000000000}$ gives for the quotient the very great number 1000000000.

112. Every number when divided by itself producing unity, it is evident that a fraction divided by itself must also give 1 for the quotient; and the same follows from our rule: for, in order to divide $\frac{3}{4}$ by $\frac{3}{4}$, we must multiply $\frac{3}{4}$ by $\frac{4}{3}$, in which case we obtain $\frac{12}{12}$, or 1; and if it be required to divide $\frac{a}{b}$ by $\frac{a}{b}$, we multiply $\frac{a}{b}$ by $\frac{b}{a}$; where the product $\frac{ab}{ab}$ is also equal to 1.

113. We have still to explain an expression which is frequently used. It may be asked, for example, what is the half of $\frac{3}{4}$; this means, that we must multiply $\frac{3}{4}$ by $\frac{1}{2}$; so likewise, if the value of $\frac{2}{3}$ of $\frac{5}{8}$ were required, we should multiply $\frac{5}{8}$ by $\frac{2}{3}$, which produces $\frac{10}{24}$; and $\frac{3}{4}$ of $\frac{9}{16}$ is the same as $\frac{9}{16}$ multiplied by $\frac{3}{4}$, which produces $\frac{27}{64}$.

114. Lastly, we must here observe, with respect to the signs $+$ and $-$, the same rules that we before laid down for integers. Thus $+\frac{1}{2}$ multiplied by $-\frac{1}{3}$, makes $-\frac{1}{6}$; and $-\frac{2}{3}$ multiplied by $-\frac{4}{5}$, gives $+\frac{8}{15}$.

Farther $-\frac{5}{8}$ divided by $+\frac{2}{3}$, makes $-\frac{15}{16}$; and $-\frac{3}{4}$ divided by $-\frac{3}{4}$, makes $+\frac{12}{12}$, or $+1$.

CHAP. XI.

Of Square Numbers.

115. The product of a number, when multiplied by itself, is called *a square;* and, for this reason, the number, considered in relation to such a product, is called *a square root.* For example, when we multiply 12 by 12, the product 144 is a square, of which the root is 12.

The origin of this term is borrowed from geometry, which teaches us that the content of a square is found by multiplying its side by itself.

116. Square numbers are found therefore by multiplication; that is to say, by multiplying the root by itself; thus 1 is the square of 1, since 1 multiplied by 1 makes 1; likewise, 4 is the square of 2; and 9 the square of 3; 2 also is the root of 4, and 3 is the root of 9.

We shall begin by considering the squares of the natural numbers, and for this purpose shall give the following small table, on the first line of which several numbers, or roots, are ranged, and on the second their squares*.

Numbers.	1	2	3	4	5	6	7	8	9	10	11	12	13
Squares.	1	4	9	16	25	36	49	64	81	100	121	144	169

117. Here it will be readily perceived that the series of square numbers thus arranged has a singular property; namely, that if each of them be subtracted from that which immediately follows, the remainders always increase by 2, and form this series;

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, &c.

which is that of the odd numbers.

118. The squares of fractions are found in the same manner, by multiplying any given fraction by itself. For example, the square of $\frac{1}{2}$ is $\frac{1}{4}$,

$$\text{The square of} \left\{ \begin{matrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{4} \\ \frac{3}{4} \end{matrix} \right\} \text{is} \quad \begin{matrix} \frac{1}{9}; \\ \frac{4}{9}; \\ \frac{1}{16}; \\ \frac{9}{16}, \text{ \&c.} \end{matrix}$$

We have only therefore to divide the square of

* We have very complete tables for the squares of natural numbers, published under the title "Tetragonometria Tabularia, &c. Auct. J. Jobo Ludolfo, Amstelodami, 1690, in 4to." These tables are continued from 1 to 100000, not only for finding those squares, but also the products of any two numbers less than 100000; not to mention several other uses which are explained in the introduction to the work. F. T.

the numerator by the square of the denominator, and the fraction which expresses that division, will be the square of the given fraction; thus, $\frac{25}{64}$ is the square of $\frac{5}{8}$; and reciprocally, $\frac{5}{8}$ is the root of $\frac{25}{64}$.

119. When the square of a mixed number, or a number composed of an integer and a fraction, is required, we have only to reduce it to a single fraction, and then take the square of that fraction. Let it be required, for example, to find the square of $2\frac{1}{2}$; we first express this number by $\frac{5}{2}$, and taking the square of that fraction, we have $\frac{25}{4}$, or $6\frac{1}{4}$, for the value of the square of $2\frac{1}{2}$; also to obtain the square of $3\frac{1}{4}$, we say $3\frac{1}{4}$ is equal to $\frac{13}{4}$; therefore its square is equal to $\frac{169}{16}$, or to $10\frac{9}{16}$. The squares of the numbers between 3 and 4, supposing them to increase by one fourth, are as follows:

Numbers.	3	$3\frac{1}{4}$	$3\frac{1}{2}$	$3\frac{3}{4}$	4
Squares.	9	$10\frac{9}{16}$	$12\frac{1}{4}$	$14\frac{1}{16}$	16

From this small table we may infer, that if a root contain a fraction, its square also contains one. Let the root, for example, be $1\frac{5}{12}$; its square is $\frac{289}{144}$, or $2\frac{1}{144}$; that is to say, a little greater than the integer 2.

120. Let us now proceed to general expressions. First when the root is a, the square must be aa; if the root be $2a$, the square is $4aa$; which shows that by doubling the root, the square becomes 4 times greater; also if the root be $3a$, the square is $9aa$; and if the root be $4a$, the square is $16aa$. Farther, if the root be

ab, the square is *aabb*; and if the root be *abc*, the square is *aabbcc*.

121. Thus, when the root is composed of two, or more factors, we multiply their squares together; and reciprocally, if a square be composed of two, or more factors, of which each is a square, we have only to multiply together the roots of those squares, to obtain the complete root of the square proposed; thus, 2304 is equal to $4 \times 16 \times 36$, the square root of which is $2 \times 4 \times 6$, or 48; and 48 is found to be the true square root of 2304, because 48×48 gives 2304.

122. Let us now consider what must be observed on this subject with regard to the signs + and −. First, it is evident that if the root has the sign +, that is to say, is a positive number, its square must necessarily be a positive number also, because + multiplied by + makes +: hence the square of $+a$ will be $+aa$: but if the root be a negative number, as $-a$, the square is still positive, for it is $+aa$; we may therefore conclude, that $+aa$ is the square both of $+a$ and of $-a$, and that consequently every square has two roots, one positive and the other negative: the square root of 25, for example, is both $+5$ and -5, because -5 multiplied by -5 gives 25, as well as $+5$ by $+5$.

CHAP. XII.

Of Square Roots, and of Irrational Numbers resulting from them.

123. What we have said in the preceding chapter amounts to this; that the square root of a given number is that number whose square is equal to the given number; and that we may put before those roots either the positive or the negative sign.

124. So that when a square number is given, provided we retain in our memory a sufficient number of square numbers, it is easy to find its root; thus, if 196, for example, be the given number, we know that its square root is 14.

Fractions, likewise, are easily managed in the same way; it is evident, for example, that $\frac{5}{7}$ is the square root of $\frac{25}{49}$; to be convinced of which, we have only to take the square root of the numerator and that of the denominator.

If the number proposed be a mixed number, as $12\frac{1}{4}$, we reduce it to a single fraction, which here is $\frac{49}{4}$, and from this we immediately perceive that $\frac{7}{2}$, or $3\frac{1}{2}$ must be the square root of $12\frac{1}{4}$.

125. But when the given number is not a square, as 12, for example, it is not possible to extract its square root, or to find a number which, multiplied by itself, will give the product 12. We know, how-

ever, that the square root of 12 must be greater than 3, because 3×3 produces only 9; and less than 4, because 4×4 produces 16, which is more than 12; we know also, that this root is less than $3\frac{1}{2}$, for we have seen that the square of $3\frac{1}{2}$, or $\frac{7}{2}$, is $12\frac{1}{4}$; and we may approach still nearer to this root, by comparing it with $3\frac{7}{15}$; for the square of $3\frac{7}{15}$, or of $\frac{52}{15}$ is $\frac{2704}{225}$, or $12\frac{4}{225}$; so that this fraction is still greater than the root required, though but very little so, as the difference of the two squares is only $\frac{4}{225}$.

126. We may suppose that as $3\frac{1}{2}$ and $3\frac{7}{15}$ are numbers greater than the root of 12, it might be possible to add to 3 a fraction a little less than $\frac{7}{15}$, and precisely such, that the square of the sum would be equal to 12.

Let us therefore try with $3\frac{3}{7}$, since $\frac{3}{7}$ is a little less than $\frac{7}{15}$. Now $3\frac{3}{7}$ is equal to $\frac{24}{7}$, the square of which is $\frac{576}{49}$, and consequently less by $\frac{12}{49}$ than 12, which may be expressed by $\frac{588}{49}$. It is, therefore, proved that $3\frac{3}{7}$ is less, and that $3\frac{7}{15}$ is greater than the root required. Let us then try a number a little greater than $3\frac{3}{7}$, but yet less than $3\frac{7}{15}$, for example, $3\frac{5}{11}$; this number, which is equal to $\frac{38}{11}$, has for its square $\frac{1444}{121}$; and, by reducing 12 to this denominator, we obtain $\frac{1452}{121}$ which shows that $3\frac{5}{11}$ is still less than the root of 12, viz. by $\frac{8}{121}$; let us therefore substitute for $\frac{5}{11}$ the fraction $\frac{6}{13}$, which is a little greater, and see what will be the result of the comparison of the square of $3\frac{6}{13}$ with the proposed number 12. Here the square of $3\frac{6}{13}$ is $\frac{2025}{169}$; and 12 reduced to the same denominator is $\frac{2028}{169}$; so that $3\frac{6}{13}$ is still too small, though only by $\frac{3}{169}$, whilst $3\frac{7}{15}$ has been found too great.

127. It is evident, therefore, that whatever fraction be joined to 3, the square of that sum must always contain a fraction, and can never be exactly equal to the integer 12; thus, although we know that the square root of 12 is greater than $3\frac{6}{13}$ and less than $3\frac{7}{15}$, yet we are unable to assign an inter-

mediate fraction between these two, which, at the same time, if added to 3, would express exactly the square root of 12; but notwithstanding this, we are not to assert that the square root of 12 is absolutely and in itself indeterminate; it only follows from what has been said, that this root, though it necessarily has a determinate magnitude, cannot be expressed by fractions. (2.)

128. There is therefore a sort of numbers which cannot be assigned by fractions, and which are nevertheless determinate quantities; as, for instance, the square root of 12; and we call this new species of numbers, *irrational numbers;* they occur whenever we endeavour to find the square root of a number which is not a square; thus, 2 not being a perfect square, the square root of 2, or the number which, multiplied by itself, would produce 2, is an irrational quantity. These numbers are also called *surd quantities*, or *incommensurables*.

129. These irrational quantities, though they cannot be expressed by fractions, are nevertheless magnitudes of which we may form an accurate idea; for however concealed the square root of 12, for example, may appear, we are not ignorant that it must be a number which, when multiplied by itself, would exactly produce 12; and this property is sufficient to give us an idea of the number, since it is in our power to approximate towards its value continually.

130. As we are therefore sufficiently acquainted with the nature of the irrational numbers, under our present consideration, a particular sign has been agreed on to express the square roots of all numbers that are not perfect squares; which sign is written

thus $\surd$, and is read *square root*. Thus, $\surd 12$ represents the square root of 12, or the number which, multiplied by itself, produces 12; and $\surd 2$ represents the square root of 2; $\surd 3$ the square root of 3; $\surd\frac{2}{3}$ that of $\frac{2}{3}$; and, in general, $\surd a$ represents the square root of the number a. Whenever, therefore, we would express the square root of a number which is not a square, we need only make use of the mark $\surd$ by placing it before the number.

131. The explanation which we have given of irrational numbers will readily enable us to apply to them the known methods of calculation. For knowing that the square root of 2, multiplied by itself, must produce 2; we know also, that the multiplication of $\surd 2$ by $\surd 2$ must necessarily produce 2; that, in the same manner, the multiplication of $\surd 3$ by $\surd 3$ must give 3; that $\surd 5$ by $\surd 5$ makes 5; that $\surd\frac{2}{3}$ by $\surd\frac{2}{3}$ makes $\frac{2}{3}$; and, in general, that $\surd a$ multiplied by $\surd a$ produces a.

132. But when it is required to multiply $\surd a$ by $\surd b$, the product is $\surd ab$; for we have already shown, that if a square has two or more factors, its root must be composed of the roots of those factors; we therefore find the square root of the product ab, which is $\surd ab$, by multiplying the square root of a, or $\surd a$, by the square root of b, or $\surd b$; &c. It is evident from this, that if b were equal to a, we should have $\surd aa$ for the product of $\surd a$ by $\surd b$. But $\surd aa$ is evidently a, since aa is the square of a.

133. In division, if it were required, for example, to divide $\sqrt{a}$, by $\sqrt{b}$, we obtain $\sqrt{\frac{a}{b}}$; and in this instance the irrationality may vanish in the quotient; thus, having to divide $\sqrt{18}$ by $\sqrt{8}$, the quotient is $\sqrt{\frac{18}{8}}$, which is reduced to $\sqrt{\frac{9}{4}}$, and consequently to $\frac{3}{2}$, because $\frac{9}{4}$ is the square of $\frac{3}{2}$.

134. When the number before which we have placed the radical sign $\sqrt{}$, is itself a square, its root is expressed in the usual way; thus $\sqrt{4}$ is the same as 2; $\sqrt{9}$ is the same as 3; $\sqrt{36}$ the same as 6; and $\sqrt{12\frac{1}{4}}$ the same as $\frac{7}{2}$, or $3\frac{1}{2}$. In these instances the irrationality is only apparent, and vanishes of course.

135. It is easy also to multiply irrational numbers by ordinary numbers; thus for example, 2 multiplied by $\sqrt{5}$ makes $2\sqrt{5}$; and 3 times $\sqrt{2}$ makes $3\sqrt{2}$; in the second example, however, as 3 is equal to $\sqrt{9}$, we may also express 3 times $\sqrt{2}$ by $\sqrt{9}$ multiplied by $\sqrt{2}$, or by $\sqrt{18}$; also $2\sqrt{a}$ is the same as $\sqrt{4a}$, and $3\sqrt{a}$ the same as $\sqrt{9a}$; and, in general, $b\sqrt{a}$ has the same value as the square root of bba, or $\sqrt{bba}$; whence we infer reciprocally, that when the number which is preceded by the radical sign contains a square, we may take the root of that square and put it before the sign, as we should do in writing $b\sqrt{a}$ instead of $\sqrt{bba}$. After this, the following reductions will be easily understood:

$$\left.\begin{array}{l}\sqrt{8}, \text{ or } \sqrt{2.4} \\ \sqrt{12}, \text{ or } \sqrt{3.4} \\ \sqrt{18}, \text{ or } \sqrt{2.9} \\ \sqrt{24}, \text{ or } \sqrt{6.4} \\ \sqrt{32}, \text{ or } \sqrt{2.16} \\ \sqrt{75}, \text{ or } \sqrt{3.25}\end{array}\right\} \text{is equal to} \left\{\begin{array}{l}2\sqrt{2}; \\ 2\sqrt{3}; \\ 3\sqrt{2}; \\ 2\sqrt{6}; \\ 4\sqrt{2}; \\ 5\sqrt{3};\end{array}\right.$$

and so on.

136. Division is founded on the same principles, as $\sqrt{a}$ divided by $\sqrt{b}$ gives $\frac{\sqrt{a}}{\sqrt{b}}$, or $\sqrt{\frac{a}{b}}$. In the same manner,

$$\left.\begin{array}{l}\frac{\sqrt{8}}{\sqrt{2}} \\ \frac{\sqrt{18}}{\sqrt{2}} \\ \frac{\sqrt{12}}{\sqrt{3}}\end{array}\right\} \text{is equal to} \left\{\begin{array}{l}\sqrt{\frac{8}{2}}, \text{ or } \sqrt{4}, \text{ or } 2; \\ \sqrt{\frac{18}{2}}, \text{ or } \sqrt{9}, \text{ or } 3; \\ \sqrt{\frac{12}{3}}, \text{ or } \sqrt{4}, \text{ or } 2.\end{array}\right.$$

Farther

$$\left.\begin{array}{l}\frac{2}{\sqrt{2}} \\ \frac{3}{\sqrt{3}} \\ \frac{12}{\sqrt{6}}\end{array}\right\} \text{is equal to} \left\{\begin{array}{l}\frac{\sqrt{4}}{\sqrt{2}}, \text{ or } \sqrt{\frac{4}{2}}, \text{ or } \sqrt{2}; \\ \frac{\sqrt{9}}{\sqrt{3}}, \text{ or } \sqrt{\frac{9}{3}}, \text{ or } \sqrt{3}. \\ \frac{\sqrt{144}}{\sqrt{6}}, \text{ or } \sqrt{\frac{144}{6}}, \text{ or } \sqrt{24},\end{array}\right.$$

or $\sqrt{6 \times 4}$, or lastly $2\sqrt{6}$.

137. There is nothing in particular to be observed in addition and subtraction, because we only connect the numbers by the signs $+$ and $-$: for example, $\sqrt{2}$ added to $\sqrt{3}$ is written $\sqrt{2}+\sqrt{3}$; and $\sqrt{3}$ subtracted from $\sqrt{5}$ is written $\sqrt{5}-\sqrt{3}$.

138. We may observe lastly, that in order to distinguish the irrational numbers, we call all other

numbers, both integral and fractional, *rational numbers;* so that, whenever we speak of rational numbers, we understand integers or fractions.

CHAP. XIII.

Of Impossible, or Imaginary Quantities, which arise from the same source.

139. We have already seen that the squares of numbers, negative as well as positive, are always positive, or affected by the sign $+$; having shown that $-a$ multiplied by $-a$ gives $+aa$, the same as the product of $+a$ by $+a$; wherefore, in the preceding chapter, we supposed that all the numbers, of which it was required to extract the square roots, were positive.

140. When it is required, therefore, to extract the root of a negative number, a great difficulty arises; since there is no assignable number, the square of which would be a negative quantity; suppose, for example, that we wished to extract the root of -4; we here require such a number as, when multiplied by itself, would produce -4; now this number is neither $+2$ nor -2, because the square both of $+2$ and of -2, is $+4$, and not -4.

141. We must therefore conclude, that the square root of a negative number cannot be either a positive

number or a negative number, since the squares of negative numbers also take the sign *plus*; consequently the root in question must belong to an entirely distinct species of numbers; since it cannot be ranked either among positive or among negative numbers.

142. Now we before remarked, that positive numbers are all greater than nothing, or 0, and that negative numbers are all less than nothing, or 0; so that whatever exceeds 0 is expressed by positive numbers, and whatever is less than 0 is expressed by negative numbers: the square roots of negative numbers, therefore, are neither greater nor less than nothing; yet we cannot say, that they are 0; for 0 multiplied by 0 produces 0, and consequently does not give a negative number.

143. And, since all numbers which it is possible to conceive are either greater or less than 0, or are 0 itself, it is evident that we cannot rank the square root of a negative number amongst possible numbers; and we must therefore say that it is an impossible quantity. In this manner we are led to the idea of numbers which from their nature are impossible, which numbers are usually called *imaginary quantities*, because they exist merely in the imagination.

144. All such expressions as $\surd -1$, $\surd -2$, $\surd -3$, $\surd -4$, &c. are consequently impossible, or imaginary numbers, since they represent roots of negative quantities: and of such numbers we may truly assert that, they are neither nothing, nor greater than nothing, nor less than nothing; which necessarily constitutes them imaginary, or impossible.

145. But notwithstanding this, these numbers pre-

sent themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; since we know that by $\sqrt{-4}$ is meant a number which, multiplied by itself, produces $-$; for this reason also, nothing prevents us from making use of these imaginary numbers, and employing them in calculation.

146. The first idea that occurs on the present subject is, that the square of $\sqrt{-3}$, for example, or the product of $\sqrt{-3}$ by $\sqrt{-3}$, must be -3; that the product of $\sqrt{-1}$ by $\sqrt{-1}$ is -1; and, in general, that by multiplying $\sqrt{-a}$ by $\sqrt{-a}$, or by taking the square of $\sqrt{-a}$ we obtain $-a$.

147. Now, as $-a$ is equal to $+a$ multiplied by -1, and as the square root of a product is found by multiplying together the roots of its factors, it follows that the root of a times -1, or $\sqrt{-a}$, is equal to $\sqrt{a}$ multiplied by $\sqrt{-1}$; but $\sqrt{a}$ is a possible or real number, consequently the whole impossibility of an imaginary quantity may be always reduced to $\sqrt{-1}$; for this reason, $\sqrt{-4}$ is equal to $\sqrt{4}$ multiplied by $\sqrt{-1}$, or equal to $2\sqrt{-1}$, because $\sqrt{4}$ is equal to 2; likewise -9 is reduced to $\sqrt{9}\times\sqrt{-1}$, or $3\sqrt{-1}$; and $\sqrt{-16}$ is equal to $4\sqrt{-1}$.

148. Moreover, as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$; and $\sqrt{4}$, or 2, for the value of the product of $\sqrt{-1}$ by $\sqrt{-4}$; thus we see, therefore, that two imaginary numbers, multiplied together, produce a real, or possible one.

But, on the contrary, a possible number, multiplied by an impossible number, gives always an imaginary product: thus, $\sqrt{-3}$ by $\sqrt{+5}$ gives $\sqrt{-15}$.

149. It is the same with regard to division; for $\sqrt{a}$ divided by $\sqrt{b}$ making $\sqrt{\frac{a}{b}}$, it is evident that $\sqrt{-4}$ divided by $\sqrt{-1}$ will make $\sqrt{+4}$, or 2; that $\sqrt{+3}$ divided by $\sqrt{-3}$ will give $\sqrt{-1}$; and that 1 divided by $\sqrt{-1}$ gives $\sqrt{\frac{+1}{-1}}$, or $\sqrt{-1}$; because 1 is equal to $\sqrt{+1}$.

150. We have before observed, that the square root of any number has always two values, one positive and the other negative; that $\sqrt{4}$, for example, is both $+2$ and -2, and that, in general, we may take $-\sqrt{a}$ as well as $+\sqrt{a}$ for the square root of a. This remark applies also to imaginary numbers; the square root of $-a$ is both $+\sqrt{-a}$ and $-\sqrt{-a}$; but we must not confound the signs $+$ and $-$, which are before the radical sign $\sqrt{}$, with the sign which comes after it.

151. It still remains for us to remove any doubt which may be entertained concerning the utility of the numbers of which we have been speaking; for those numbers being impossible, it would not be surprising if they were thought entirely useless, and the object only of an unfounded speculation; this, however, would be a mistake; for the calculation of imaginary quantities is of the greatest importance, as questions frequently arise, of which we cannot immediately say whether they include any thing real and possible, or not; but when the solution of such a question leads to imaginary numbers, we are certain that what is required is impossible.

In order to illustrate what we have said by an example, suppose it were proposed to divide the number 12 into two such parts, that the product of those parts may be 40. If we resolve this question by the ordinary rules, we find for the parts sought $6+\sqrt{-4}$ and $6-\sqrt{-4}$; but these numbers being imaginary, we conclude, that it is impossible to resolve the question.

The difference will be easily perceived, if we suppose the question had been to divide 12 into two parts which multiplied together would produce 35; for it is evident that those parts must be 7 and 5.

CHAP. XIV.

Of Cubic Numbers.

152. When a number has been multiplied twice by itself, or, which is the same thing, when the square of a number has been multiplied once more by that number, we obtain a product which is called a *cube*, or a *cubic number*. Thus, the cube of a is aaa, since it is the product obtained by multiplying a by itself, or by a, and that square aa again by a.

The cubes of the natural numbers, therefore, succeed each other in the following order*:

* We are indebted to a mathematician of the name of J. Paul Buchner, for tables published at Nuremberg in 1701, in which are to be found the cubes, as well as the squares, of all numbers from 1 to 12000. F. T.

Numbers	1	2	3	4	5	6	7	8	9	10
Cubes	1	8	27	64	125	216	343	512	729	1000

153. If we consider the differences of those cubes, as we did of the squares, by subtracting each cube from that which comes after it, we obtain the following series of numbers:

7, 19, 37, 61, 91, 127, 169, 217, 271.

Where we do not at first observe any regularity in them; but if we take the respective differences of these numbers, we find the following series:

12, 18, 24, 30, 36, 42, 48, 54, 60;

in which the terms, it is evident, increase always by 6.

154. After the definition we have given of a cube, it will not be difficult to find the cubes of fractional numbers; thus, $\frac{1}{8}$ is the cube of $\frac{1}{2}$; $\frac{1}{27}$ is the cube of $\frac{1}{3}$; and $\frac{8}{27}$ is the cube of $\frac{2}{3}$: and in the same manner, we have only to take the cube of the numerator and that of the denominator separately, and we shall have $\frac{27}{64}$ for the cube of $\frac{3}{4}$.

155. If it be required to find the cube of a mixed number, we must first reduce it to a single fraction, and then proceed in the manner that has been described. To find, for example, the cube of $1\frac{1}{2}$, we must take that of $\frac{3}{2}$, which is $\frac{27}{8}$, or $3\frac{3}{8}$; also the cube of $1\frac{1}{4}$, or of the single fraction $\frac{5}{4}$, is $\frac{125}{64}$, or

$1\frac{61}{64}$; and the cube of $3\frac{1}{4}$, or of $\frac{13}{4}$, is $\frac{2197}{64}$, or $34\frac{21}{64}$.

156. Since *aaa* is the cube of *a*, that of *ab* will be *aaabbb*; whence we see, that if a number has two or more factors, we may find its cube by multiplying together the cubes of those factors; for example, as 12 is equal to 3×4, we multiply the cube of 3, which is 27, by the cube of 4, which is 64, and we obtain 1728, the cube of 12; and farther, the cube of $2a$ is $8aaa$, and consequently 8 times greater than the cube of *a*; and likewise, the cube of $3a$ is $27aaa$, that is to say, 27 times greater than the cube of *a*.

157. Let us attend here also to the signs + and −. It is evident that the cube of a positive number $+a$ must also be positive, that is $+aaa$; but if it be required to cube a negative number $-a$, it is found by first taking the square, which is $+aa$, and then multiplying, according to the rule, this square by $-a$, which gives for the cube required $-aaa$. In this respect, therefore, it is not the same with cubic numbers as with squares, since the latter are always positive; whereas the cube of -1 is -1, that of -2 is -8, that of -3 is -27, and so on.

CHAP. XV.

Of Cube Roots, and of Irrational Numbers resulting from them.

158. As we can, in the manner already explained, find the cube of a given number, so, when

a number is proposed, we may also reciprocally find a number, which, multiplied twice by itself, will produce that number. The number here sought is called, with relation to the other, *the cube root*; so that the cube root of a given number is the number whose cube is equal to that given number.

159. It is easy therefore to determine the cube root, when the number proposed is a real cube, such as in the examples in the last chapter; for we easily perceive that the cube root of 1 is 1; that of 8 is 2; that of 27 is 3; that of 64 is 4, and so on. And, in the same manner, the cube root of -27 is -3; and that of -125 is -5.

Farther, if the proposed number be a fraction, as $\frac{8}{27}$, the cube root of it must be $\frac{2}{3}$; and that of $\frac{64}{343}$ is $\frac{4}{7}$. Lastly, the cube root of a mixed number $2\frac{10}{27}$ must be $\frac{4}{3}$, or $1\frac{1}{3}$; because $2\frac{10}{27}$ is equal to $\frac{64}{27}$.

160. But if the proposed number be not a cube, its cube root cannot be expressed either in integers, or in fractional numbers; for example, 43 is not a cubic number; therefore it is impossible to assign any number, either integer or fractional, whose cube shall be exactly 43; we may however affirm, that the cube root of that number is greater than 3, since the cube of 3 is only 27; and less than 4, because the cube of 4 is 64; we know therefore, that the cube root required is necessarily contained between the numbers 3 and 4.

161. Since the cube root of 43 is greater than 3, if we add a fraction to 3, it is certain that we may

approximate still nearer and nearer to the true value of this root; but we can never assign the number which expresses the value exactly; because the cube of a mixed number can never be perfectly equal to an integer, such as 43; if we were to suppose, for example, $3\frac{1}{2}$, or $\frac{7}{2}$ to be the cube root required, the error would be $\frac{1}{8}$; for the cube of $\frac{7}{2}$ is only $\frac{343}{8}$, or $42\frac{7}{8}$.

162. This therefore shows, that the cube root of 43 cannot be expressed in any way, either by integers or by fractions; still however we have a distinct idea of the magnitude of this root; which induces us to use, in order to represent it, the sign $\sqrt[3]{}$, which we place before the proposed number, and which is read *cube root*, to distinguish it from the square root, which is often called simply the root; thus $\sqrt[3]{43}$ means the cube root of 43, that is to say, the number whose cube is 43, or which, multiplied by itself, and then by itself again, produces 43.

163. Now it is evident that such expressions cannot belong to rational quantities, but that they rather form a particular species of irrational quantities: they have nothing in common with square roots, and it is not possible to express such a cube root by a square root; as, for example, by $\sqrt{12}$; for the square of $\sqrt{12}$ being 12, its cube will be $12\sqrt{12}$, consequently still irrational, and cannot therefore be equal to 43.

164. If the proposed number be a real cube, our expressions become rational; thus, $\sqrt[3]{1}$ is equal to 1;

$\sqrt[3]{8}$ is equal to 2; $\sqrt[3]{27}$ is equal to 3; and, generally, $\sqrt[3]{aaa}$ is equal to a.

165. If it were proposed to multiply one cube root, $\sqrt[3]{a}$, by another, $\sqrt[3]{b}$, the product must be $\sqrt[3]{ab}$; for we know that the cube root of a product ab is found by multiplying together the cube roots of the factors. Hence, also, if we divide $\sqrt[3]{a}$ by $\sqrt[3]{b}$, the quotient will be $\sqrt[3]{\frac{a}{b}}$.

166. We farther perceive, that $2\sqrt[3]{a}$ is equal to $\sqrt[3]{8a}$, because 2 is equivalent to $\sqrt[3]{8}$; that $3\sqrt[3]{a}$ is equal to $\sqrt[3]{27a}$, and $b\sqrt[3]{a}$ is equal to $\sqrt[3]{abbb}$; and, reciprocally, if the number under the radical sign has a factor which is a cube, we may make it disappear by placing its cube root before the sign; for example, instead of $\sqrt[3]{64a}$ we may write $4\sqrt[3]{a}$; and $5\sqrt[3]{a}$ instead of $\sqrt[3]{125a}$: hence $\sqrt[3]{16}$ is equal to $2\sqrt[3]{2}$, because 16 is equal to 8×2.

167. When a number proposed is negative, its cube root is not subject to the same difficulties that occurred in treating of square roots; for, since the cubes of negative numbers are negative, it follows that the cube roots of negative numbers are also negative; thus $\sqrt[3]{-8}$ is equal to -2, and $\sqrt[3]{-27}$ to -3; it follows also, that $\sqrt[3]{-12}$ is the same as $-\sqrt[3]{12}$, and that $\sqrt[3]{-a}$ may be expressed by $-\sqrt[3]{a}$; whence we see that the sign $-$, when it is found after the sign of the cube root, might also have been placed before it. We are not therefore led here to impossible or imaginary numbers, which happened in considering the square roots of negative numbers.

CHAP. XVI.

Of Powers in general.

168. The product which we obtain by multiplying a number once or several times by itself, is called *a power*. Thus, a square which arises from the multiplication of a number by itself, and a cube which we obtain by multiplying a number twice by itself, are powers. We say also in the former case, that the number is raised to the second degree, or to the second power; and in the latter, that the number is raised to the third degree, or to the third power.

169. We distinguish those powers from one another by the number of times that the given number has been multiplied by itself. For example, a square is called the second power, because a certain given number has been multiplied by itself; and if a number has been multiplied twice by itself we call the product the third power, which therefore means the same as the cube; also if we multiply a number three times by itself we obtain its fourth power, or what is commonly called the *biquadrate:* and thus it will be easy to understand what is meant by the fifth, sixth, seventh, &c. power of a number. I shall only add, that powers, after the fourth degree, cease to have any other but these numeral distinctions.

170. To illustrate this still better, we may observe, in the first place, that the powers of 1 remain always the same; because, whatever number of times we multiply 1 by itself, the product is found to be always

1. We shall therefore begin by representing the powers of 2 and of 3, which succeed each other as in the following order:

Powers.	Of the number 2.	Of the number 3.
1st	2	3
2d	4	9
3d	8	27
4th	16	81
5th	32	243
6th	64	729
7th	128	2187
8th	256	6561
9th	512	19683
10th	1024	59049
11th	2048	177147
12th	4096	531441
13th	8192	1594323
14th	16384	4782969
15th	32768	14348907
16th	65536	43046721
17th	131072	129140163
18th	262144	387420489

But the powers of the number 10 are the most remarkable; it being on these powers that the system of our arithmetic is founded; a few of them ranged in order, and beginning with the first power, are as follow:

1st	2d	3d	4th	5th	6th	
10,	100,	1000,	10000,	100000,	1000000,	&c.

171. In order to illustrate this subject, and to consider it in a more general manner, we may observe, that the powers of any number, a, succeed each other in the following order:

1st	2d	3d	4th	5th	6th	
a,	*aa*,	*aaa*,	*aaaa*,	*aaaaa*,	*aaaaaa*,	&c.

But we soon feel the inconvenience attending this manner of writing the powers, which consists in the necessity of repeating the same letter very often, to express high powers; and the reader also would have no less trouble, if he were obliged to count all the letters, to know what power is intended to be represented. The hundredth power, for example, could not be conveniently written in this manner; and it would be equally difficult to read it.

172. To avoid this inconvenience, a much more commodious method of expressing such powers has been devised, which from its extensive use deserves to be carefully explained. Thus, for example, to express the hundredth power, we simply write the number 100 above the quantity whose hundredth power we would express, and a little towards the right-hand; thus a^{100} represents *a* raised to the 100th power, or the hundredth power of *a*. It must here also be observed, that the name *exponent* is given to the number written above that whose power, or degree, it represents, and which in the present instance is 100.

173. In the same manner, a^2 signifies *a* raised to the 2d power, or the second power of *a*, which we represent sometimes also by *aa*, because both these expressions are written and understood with equal facility; but to express the cube, or the third power *aaa*, we write a^3, according to the rule, that we may occupy less room; so a^4 signifies the fourth, a^5 the fifth, and a^6 the sixth power of *a*.

174. In a word, the different powers of *a* will be represented by a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, &c.

Hence we see that in this manner we might very properly have written a^1 instead of a for the first term, to show the order of the series more clearly: in fact a^1 is no more than a, as this unit shows that the letter a is to be written only once. Such a series of powers is called also a geometrical progression, because each term is one-time greater than the preceding.

175. As in this series of powers each term is found by multiplying the preceding term by a, which increases the exponent by 1; so when any term is given, we may also find the preceding one if we divide by a, because this diminishes the exponent by 1. This shows that the term which precedes the first term a^1 must necessarily be $\frac{a}{a}$, or 1; and, if we proceed according to the exponents, we immediately conclude, that the term which precedes the first must be a^0; and hence we deduce this remarkable property, that a^0 is always equal to 1, however great or small the value of the number a may be, and even when a is nothing; that is to say, a^0 is equal to 1.

176. We may also continue our series of powers in a retrograde order, and that in two different ways; first, by dividing always by a; and secondly, by diminishing the exponent by unity; and it is evident that, whether we follow the one or the other, the terms are still perfectly equal. This decreasing series is represented in both forms in the following table, which must be read backwards, or from right to left.

	$\frac{1}{aaaaaa}$	$\frac{1}{aaaaa}$	$\frac{1}{aaaa}$	$\frac{1}{aaa}$	$\frac{1}{aa}$	$\frac{1}{a}$	1	a
1st.	$\frac{1}{a^6}$	$\frac{1}{a^5}$	$\frac{1}{a^4}$	$\frac{1}{a^3}$	$\frac{1}{a}$	$\frac{1}{a^1}$		
2d.	a^{-6}	a^{-5}	a^{-4}	a^{-3}	a^{-2}	a^{-1}	a^0	a^1

177. We are now come to the knowledge of powers whose exponents are negative, and are enabled to assign the precise value of those powers. Thus, from what has been said, it appears that

$$\left.\begin{array}{l} a^0 \\ a^{-1} \\ a^{-2} \\ a^{-3} \\ a^{-4} \end{array}\right\} \text{is equal to} \left\{\begin{array}{l} 1; \\ \frac{1}{a}; \\ \frac{1}{aa}, \text{ or } \frac{1}{a^2}; \\ \frac{1}{a^3}; \\ \frac{1}{a^4}, \&c. \end{array}\right.$$

178. It will also be easy, from the foregoing notation, to find the powers of a product, ab; for they must evidently be ab, or a^1b^1, a^2b^2, a^3b^3, a^4b^4, a^5b^5, &c. and the powers of fractions will be found in the same manner; for example, those of $\frac{a}{b}$ are

$$\frac{a^1}{b^1}, \frac{a^2}{b^2}, \frac{a^3}{b^3}, \frac{a^4}{b^4}, \frac{a^5}{b^5}, \frac{a^6}{b^6}, \frac{a^7}{b^7}, \&c.$$

179. Lastly, we have to consider the powers of negative numbers. In which case suppose the given number to be $-a$; then its powers will form the following series:

$$-a, +a^2, -a^3, +a^4, -a^5, +a^6, \&c.$$

Where we may observe, that those powers only become negative, whose exponents are odd numbers, and that, on the contrary, all the powers, which have an even number for the exponent, are positive. So that the third, fifth, seventh, ninth, &c. powers have all the sign —; and the second, fourth, sixth, eighth, &c. powers are affected by the sign +.

CHAP. XVII.

Of the Calculation of Powers.

180. We have nothing particular to observe with regard to the addition and subtraction of powers; for we only represent those operations by means of the signs + and —, when the powers are different; for example, a^3+a^2 is the sum of the second and third powers of a; and a^5-a^4 is what remains when we subtract the fourth power of a from the fifth; and neither of these results can be abridged; but when we have powers of the same kind or degree, it is evidently unnecessary to connect them by signs; as a^3+a^3 becomes $2a^3$, &c.

181. But in the multiplication of powers, several circumstances require attention.

First, when it is required to multiply any power of a by a, we obtain the succeeding power; that is to say, the power whose exponent is greater by an unit; thus a^2, multiplied by a, produces a^3; and a^3, multiplied by a, produces a^4: and, in the same manner, when it is required to multiply by a the powers of that number which have negative exponents, we must add 1 to the exponent; thus a^{-1} multiplied by a produces a^0 or 1; which is made more evident by considering that a^{-1} is equal to $\frac{1}{a}$, and that the product of $\frac{1}{a}$ by a being $\frac{a}{a}$, it is consequently equal to 1; likewise a^{-2} multiplied by a, produces a^{-1}, or $\frac{1}{a}$; and a^{-10} multiplied by a, gives a^{-9}, and so on.

182. Next, if it be required to multiply a power of a by a^2, or the second power, I say that the exponent becomes greater by 2; thus, the product of a^2 by a^2 is a^4; that of a^2 by a^3 is a^5; that of a^4 by a^2 is a^6; and, more generally, a^n multiplied by a^2 makes a^{n+2}. With regard to negative exponents, we shall have a^1, or a, for the product of a^{-1} by a^2; for a^{-1} being equal to $\frac{1}{a}$, it is the same as if we had divided aa by a; consequently the product required is $\frac{aa}{a}$, or a; also a^{-2}, multiplied by a^2, produces a^0, or 1; and a^{-3}, multiplied by a^2, produces a^{-1}.

183. It is no less evident, that to multiply any power of a by a^3, we must increase its exponent by three units; and that consequently the product of a^n by a^3 is a^{n+3}. And whenever it is required to multiply together two powers of a, the product will be also a power of a, and such that its exponent will be the sum of those of the two given powers. For

example, a^4 multiplied by a^5 will make a^9, and a^{12} multiplied by a^7 will produce a^{19}, &c.

184. From these considerations we may easily determine the highest powers. To find, for instance, the twenty-fourth power of 2, I multiply the twelfth power by the twelfth power, because 2^{24} is equal to $2^{12} \times 2^{12}$. Now we have already seen that 2^{12} is 4096; I say therefore that the number 16777216, or the product of 4096 by 4096, expresses the power required, 2^{24}.

185. Let us now proceed to division; where we shall remark, in the first place, that to divide a power of a by a, we must subtract 1 from the exponent, or diminish it by unity; thus, a^5 divided by a gives a^4; and a^0, or 1, divided by a, is equal to a^{-1} or $\frac{1}{a}$; also a^{-3} divided by a, gives a^{-4}.

186. If we have to divide a given power of a by a^2, we must diminish the exponent by 2; and if by a^3, we must subtract 3 units from the exponent of the power proposed; and, in general, whatever power of a it is required to divide by any other power of a, the rule is always to subtract the exponent of the second from the exponent of the first of those powers: thus a^{15} divided by a^7 will give a^8; a^6 divided by a^7 will give a^{-1}; and a^{-} divided by a^4 will give a^{-7}.

187. From what has been said above, it is easy to understand the method of finding the powers of powers, this being done by multiplication. When we seek, for example, the square, or the second power of a^3, we find a^6; and in the same manner we find a^{12} for the third power, or the cube, of a^4; and to obtain the square of a power, we have only to double

its exponent; for its cube, we must triple the exponent; and so on; thus the square of a^n is a^{2n}; the cube of a^n is a^{3n}; the seventh power of a^n is a^{7n}, &c.

188. The square of a^2, or the square of the square of a, being a^4, we see why the fourth power is called the *bi-quadrate:* also, the square of a^3 being a^6; the sixth power has therefore received the name of *the square-cubed.*

Lastly, the cube of a^3 being a^9, we call the ninth power the *cubo-cube:* after this, no other denominations of this kind have been introduced for powers, and indeed the two last are very little used.

CHAP. XVIII.

Of Roots, with relation to Powers in general.

189. Since the square root of a given number is a number whose square is equal to that given number; and since the cube root of a given number is a number whose cube is equal to that given number; it follows that any number whatever being given, we may always suppose such roots of it, that their fourth, or their fifth, or any other power of that root, may be equal to the given number. To distinguish these different kinds of roots better, we shall call the square root, *the second root;* and the cube root, *the third root;* because according to this denomination we may call *the fourth root*, that whose biquadrate is

equal to a given number; and *the fifth root*, that whose fifth power is equal to a given number, &c.

190. As the square, or second root, is marked by the sign $\sqrt{\ }$, and the cubic or third root by the sign $\sqrt[3]{\ }$, so the fourth root is represented by the sign $\sqrt[4]{\ }$; the fifth root by the sign $\sqrt[5]{\ }$; and so on. It is evident therefore that according to this method of expression, the sign of the square root ought to be $\sqrt[2]{\ }$; but as of all roots this occurs most frequently, it has been agreed, for the sake of brevity, to omit the number 2 in the sign of this root. So that when a radical sign has no number prefixed to it, this always shows that the square root is meant.

191. To explain this matter still better, we shall here exhibit the different roots of the number a, with their respective values:

$$\left.\begin{array}{l}\sqrt{a}\\ \sqrt[3]{a}\\ \sqrt[4]{a}\\ \sqrt[5]{a}\\ \sqrt[6]{a}\end{array}\right\}\text{is the}\left\{\begin{array}{l}\text{2d}\\ \text{3d}\\ \text{4th}\\ \text{5th}\\ \text{6th}\end{array}\right\}\text{root of}\left\{\begin{array}{l}a,\\ a,\\ a,\\ a,\\ a,\ \text{and so on.}\end{array}\right.$$

So that conversely,

$$\left.\begin{array}{l}\text{The 2d}\\ \text{The 3d}\\ \text{The 4th}\\ \text{The 5th}\\ \text{The 6th}\end{array}\right\}\text{power of}\left\{\begin{array}{l}\sqrt{a}\\ \sqrt[3]{a}\\ \sqrt[4]{a}\\ \sqrt[5]{a}\\ \sqrt[6]{a}\end{array}\right\}\text{is equal to}\left\{\begin{array}{l}a,\\ a,\\ a,\\ a,\\ a,\end{array}\right.$$

and so on.

192. Whether the number a therefore be great or small, we know what value to affix to all these roots of different degrees.

It must be remarked also, that if we substitute unity for a, all those roots remain constantly 1; because all the powers of 1 have unity for their value.

If the number a be greater than 1, all its roots will also exceed unity. Lastly, if that number be less than 1, all its roots will also be less than unity.

193. When the number a is positive, we know from what was before said of the square and cube roots, that all the other roots may also be determined, and will be real and possible numbers.

But if the number a be negative, its second, fourth, sixth, and all its even roots, become impossible, or imaginary numbers; because all the powers of an even order, whether of positive or of negative numbers, are affected by the sign $+$: whereas the third, fifth, seventh, and all its odd roots, become negative, but rational; because the odd powers of negative numbers are also negative.

194. We have here also an inexhaustible source of new kinds of surds, or irrational quantities; for whenever the number a is not really such a power, as some one of the foregoing indices represents, or seems to require, it is impossible to express that root either in whole numbers or in fractions; and consequently it must be classed among the numbers which are called irrational.

CHAP. XIX.

Of the Method of representing Irrational Numbers by Fractional Exponents.

195. We have shown in the preceding chapter, that the square of any power is found by doubling the exponent of that power; or that, in general, the square, or the second power, of a^n, is a^{2n}; and the converse also follows, viz. that the square root of the power a^{2n} is a^n, which is found by taking half the exponent of that power, or dividing it by 2.

196. Thus the square root of a^2 is a^1, or a; that of a^4 is a^2; that of a^6 is a^3; and so on: and as this is general, the square root of a^3 must necessarily be $a^{\frac{3}{2}}$, and that of a^5 is $a^{\frac{5}{2}}$; consequently, we shall have in the same manner $a^{\frac{1}{2}}$ for the square root of a^1; whence we see that $a^{\frac{1}{2}}$ is equal to $\sqrt{a}$; which new method of representing the square root demands particular attention.

197. We have also shown, that to find the cube of a power as a^n, we must multiply its exponent by 3, and consequently that cube is a^{3n}.

Hence conversely, when it is required to find the third, or cube root, of the power a^{3n}, we have only to divide that exponent by 3, and may therefore with certainty conclude, that the root required is a^n: consequently a^1, or a, is the cube root of a^3; a^2 is that of a^6; a^3 that of a^9; and so on.

198. There is nothing therefore to prevent us from

applying the same reasoning to those cases in which the exponent is not divisible by 3, or from concluding that the cube root of a^2 is $a^{\frac{2}{3}}$, and that the cube root of a^4 is $a^{\frac{4}{3}}$, or $a^{1\frac{1}{3}}$; consequently, the third, or cube root of a, or a^1, must be $a^{\frac{1}{3}}$: whence also it appears that $a^{\frac{1}{3}}$ is the same as $\sqrt[3]{a}$.

199. It is likewise the same with roots of a higher degree: thus the fourth root of a will be $a^{\frac{1}{4}}$, which expression has the same value as $\sqrt[4]{a}$; the fifth root of a will be $a^{\frac{1}{5}}$, which is consequently equivalent to $\sqrt[5]{a}$; and the same observation may be extended to all roots of a higher degree.

200. We might therefore entirely reject the radical signs at present made use of, and employ in their stead the fractional exponents which we have explained; but as we have been long accustomed to those signs, and meet with them in most books of algebra, it might be wrong to banish them entirely from calculation; there is, however, sufficient reason also to employ, as is now frequently done, the other method of notation, because it manifestly corresponds with the nature of the thing: in fact, we see immediately that $a^{\frac{1}{2}}$ is the square root of a, because we know that the square of $a^{\frac{1}{2}}$, that is to say, $a^{\frac{1}{2}}$ multiplied by $a^{\frac{1}{2}}$, is equal to a^1, or a.

201. What has been now said is sufficient to show how we are to understand all other fractional exponents that may occur: thus if we have, for example, $a^{\frac{4}{3}}$, this means, that we must first take the fourth power

of a, and then extract its cube, or third root; so that $a^{\frac{4}{3}}$ is the same as the common expression $\sqrt[3]{a^4}$. Hence, to find the value of $a^{\frac{3}{4}}$, we must first take the cube, or the third power of a, which is a^3, and then extract the fourth root of that power; so that $a^{\frac{3}{4}}$ is the same as $\sqrt[4]{a^3}$, and $a^{\frac{4}{5}}$ is equal to $\sqrt[5]{a^4}$, &c.

202. When the fraction which represents the exponent exceeds unity, we may express the value of the given quantity in another way: for instance, suppose it to be $a^{\frac{5}{2}}$; this quantity is equivalent to $a^{2\frac{1}{2}}$, which is the product of a^2 by $a^{\frac{1}{2}}$: now $a^{\frac{1}{2}}$ being equal to $\sqrt{a}$, it is evident that $a^{\frac{5}{2}}$ is equal to $a^2\sqrt{a}$: also $a^{\frac{10}{3}}$, or $a^{3\frac{1}{3}}$, is equal to $a^3\sqrt[3]{a}$; and $a^{\frac{15}{4}}$, that is, $a^{3\frac{3}{4}}$, expresses $a^3\sqrt[4]{a^3}$. These examples are sufficient to illustrate the great utility of fractional exponents.

203. Their use extends also to fractional numbers: for if there be given $\frac{1}{\sqrt{a}}$, we know that this quantity is equal to $\frac{1}{a^{\frac{1}{2}}}$; and we have seen already that a fraction of the form $\frac{1}{a^n}$ may be expressed by a^{-n}; so that instead of $\frac{1}{\sqrt{a}}$ we may use the expression $a^{-\frac{1}{2}}$; and in the same manner, $\frac{1}{\sqrt[3]{a}}$ is equal to $a^{-\frac{1}{3}}$: again, if the quantity $\frac{a^2}{\sqrt[4]{a^3}}$ be proposed; let it be transformed into this, $\frac{a^2}{a^{\frac{3}{4}}}$, which is the product of a^2 by $a^{-\frac{3}{4}}$; then

this product is equivalent to $a^{\frac{5}{4}}$, or to $a^{1\frac{1}{4}}$, or lastly, to $a\sqrt[4]{a}$. And practice will render similar reductions equally easy.

204. We shall observe, in the last place, that each root may be represented in a variety of ways: for $\sqrt{a}$ being the same as $a^{\frac{1}{2}}$, and $\frac{1}{2}$ being transformable into the fractions, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, &c. it is evident that $\sqrt{a}$ is equal to $\sqrt[4]{a^2}$, or to $\sqrt[6]{a^3}$, or to $\sqrt[8]{a^4}$, and so on: and in the same manner, $\sqrt[3]{a}$, which is equal to $a^{\frac{1}{3}}$, will be equal to $\sqrt[6]{a^2}$, or to $\sqrt[9]{a^3}$, or to $\sqrt[12]{a^4}$. Hence also we see that the number a, or a^1, might be represented by the following radical expressions:

$$\sqrt[2]{a^2},\ \sqrt[3]{a^3},\ \sqrt[4]{a^4},\ \sqrt[5]{a^5},\ \&c.$$

205. This property is of great use in multiplication and division; for if we have, for example, to multiply $\sqrt[2]{a}$ by $\sqrt[3]{a}$, we write $\sqrt[6]{a^3}$ for $\sqrt[2]{a}$, and $\sqrt[6]{a^2}$ instead of $\sqrt[3]{a}$; so that in this manner we obtain the same radical sign for both, and the multiplication being now performed, gives the product $\sqrt[6]{a^5}$: and the same result is also deduced from $a^{\frac{1}{2}+\frac{1}{3}}$, which is the product of $a^{\frac{1}{2}}$ multiplied by $a^{\frac{1}{3}}$; for $\frac{1}{2}+\frac{1}{3}$ is $\frac{5}{6}$, and consequently the product required is $a^{\frac{5}{6}}$, or $\sqrt[6]{a^5}$.

On the contrary, if it were required to divide $\sqrt[2]{a}$, or $a^{\frac{1}{2}}$, by $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, we should have for the quotient $a^{\frac{1}{2}-\frac{1}{3}}$, or $a^{\frac{3}{6}-\frac{2}{6}}$, that is to say, $a^{\frac{1}{6}}$, or $\sqrt[6]{a}$.

CHAP. XX.

Of the different Methods of Calculation, and of their mutual Connexion.

206. Hitherto we have only explained the different methods of calculation: namely, addition, subtraction, multiplication, and division; the involution of powers, and the extraction of roots. It will not be improper therefore, in this place, to trace back the origin of these different methods, and to explain the connexion which subsists among them; in order that we may satisfy ourselves whether it be possible or not for other operations of the same kind to exist; which inquiry will throw new light on the subjects that we have considered.

In prosecuting this design, we shall make use of a new character which may be employed instead of the expression that has been so often repeated, *is equal to;* this sign is $=$, which is read *is equal to:* thus, when I write $a=b$, this means that a is equal to b: as, for example $3\times5=15$.

207. The first mode of calculation which presents itself to the mind, is undoubtedly addition, by which we add two numbers together and find their sum: let therefore a and b be the two given numbers, and let their sum be expressed by the letter c, then we shall have $a+b=c$; so that when we know the two numbers a and b, addition teaches us to find the number c.

208. Preserving this comparison $a+b=c$, let us reverse the question by asking, how we are to find the number b, when we know the numbers a and c.

It is here required therefore to know what number must be added to a, in order that the sum may be the number c: suppose, for example, $a=3$ and $c=8$; so that we must have $3+b=8$; then b will evidently be found by subtracting 3 from 8; and, in general, to find b, we must subtract a from c, whence arises $b=c-a$; for by adding a to both sides again, we have $b+a=c-a+a$, that is to say $=c$, as we supposed.

209. Subtraction therefore takes place, when we invert the question which gives rise to addition. But the number which it is required to subtract may happen to be greater than that from which it is to be subtracted; as for example, if it were required to subtract 9 from 5: this instance therefore furnishes us with the idea of a new kind of numbers, which we call negative numbers, because $5-9=-4$.

210. When several numbers are to be added together which are all equal, their sum is found by multiplication, and is called a product; thus ab means the product arising from the multiplication of a by b, or from the addition of the number a, b number of times; and if we represent this product by the letter c, we shall have $ab=c$; thus multiplication teaches us how to determine the number c, when the numbers a and b are known.

211. Let us now propose the following question: the numbers a and c being known, to find the number b. Suppose for example, $a=3$ and $c=15$, so that $3b=15$, and let us inquire by what number 3 must be multiplied, in order that the product may be

15, for the question proposed is reduced to this condition which is division: hence the number required is found by dividing 15 by 3; and therefore, in general, the number b is found by dividing c by a; from which results the equation $b=\frac{c}{a}$.

212. Now, as it frequently happens that the number c cannot be really divided by the number a, while the letter b must however have a determinate value, another new kind of numbers presents itself, which are fractions: for example, suppose $a=4$, and $c=3$, so that $4b=3$; then it is evident that b cannot be an integer, but a fraction, and that we shall have $b=\frac{3}{4}$.

213. Hence we have seen that multiplication arises from addition, that is to say, from the addition of several equal quantities; and if we now proceed farther, we shall perceive that from the multiplication of several equal quantities together powers are derived; which powers are represented in a general manner by the expression a^b, which signifies that the number a must be multiplied as many times by itself *minus* 1 as is indicated by the number b. And we know from what has been already said, that in the present instance, a is called the root, b the exponent, and a^b the power.

214. Farther, if we represent this power also by the letter c, we have $a^b=c$, an equation in which three letters a, b, c, are found; and we have shown in treating of powers, how to find the power itself, that is, the letter c, when a root a and its exponent b are given. Suppose, for example, $a=5$, and $b=3$, so that $c=5^3$: then it is evident that we must take the third power of 5, which is 125, so that in this case $c=125$.

215. We have now seen how to determine the power c, by means of the root a and the exponent b; but if we wish to reverse the question, we shall find that this may be done in two ways, and that there are two different cases to be considered: for if two of these three numbers a, b, c, were given, and it were required to find the third, we should immediately perceive that this question would admit of three different suppositions, and consequently of three solutions. Now we have considered the case in which a and b were the given numbers, we may therefore suppose farther that c and a, or c and b, are known, and that it is required to determine the third letter; but before we proceed any farther, let us point out a very essential distinction between involution and the two operations which lead to it. When, in addition, we reversed the question, it could be done only in one way; it was a matter of indifference whether we took c and a, or c and b, for the given numbers, because we might indifferently write $a+b$, or $b+a$; and it was also the same with multiplication; we could at pleasure take the letters a and b for each other, the equation $ab=c$ being exactly the same as $ba=c$: but in the calculation of powers, the same thing does not take place, and we can by no means write b^a instead of a^b; as a single example will be sufficient to illustrate: for let $a=5$, and $b=3$; then we shall have $a^b=5^3=125$; but $b^a=3^5=243$: which are two very different results.

216. It is evident then, that we may propose two questions more: one, to find the root a by means of the given power c, and the exponent b; the other,

to find the exponent b, supposing the power c and the root a to be known.

217. It may be said, indeed, that the former of these questions has been resolved in the chapter on the extraction of roots; since if $b = 2$, for example, and $a^2 = c$, we know by this means, that a is a number whose square is equal to c, and consequently that $a = \sqrt{c}$; and in the same manner, if $b = 3$ and $a^3 = c$, we know that the cube of a must be equal to the given number c, and consequently that $a = \sqrt[3]{c}$. It is therefore easy to conclude generally from this how to determine the letter a by means of the letters c and b; for we must necessarily have $a = \sqrt[b]{c}$.

218. We have already remarked also the consequence which follows, when the given number is not a real power; a case which very frequently occurs; namely, that then the required root a can neither be expressed by integers, nor by fractions; yet since this root must necessarily have a determinate value, the same consideration led us to a new kind of numbers, which, as we observed, are called *surd* or *irrational* numbers; and which we have seen are divisible into an infinite number of different sorts, on account of the great variety of roots: lastly, by the same inquiry we were led to the knowledge of another particular kind of numbers, which have been called *imaginary numbers*.

219. It remains now to consider the second question, which was to determine the exponent by means of the power c and the root a, both being known; and on this question, which has not yet occurred, is founded the important theory of logarithms, the use of which is so

extensive through the whole compass of mathematics, that scarcely any long calculation can be carried on without their assistance; and we shall find, in the folling chapter, for which we reserve this theory, that it will lead us to another kind of numbers entirely new, as they cannot be ranked among the irrational numbers before mentioned.

CHAP. XXI.

Of Logarithms in general.

220. Resuming the equation $a^b = c$, we shall begin by remarking that, in the doctrine of logarithms, we assume for the root a, a certain number taken at pleasure, and suppose this root to preserve invariably its assumed value; and this being laid down, we take the exponent b such, that the power a^b becomes equal to a given number c; in which case this exponent b is said to be the *logarithm* of the number c: and to express this we shall use the letter L. or the initial letters *log.*: thus, by $b = \text{L}. c$, or $b = log. c$, we mean that b is equal to the logarithm of the number c, or that the logarithm of c is b.

221. We see then, that the value of the root a being once established, the logarithm of any number c is nothing more than the exponent of that power of a, which is equal to c: so that c being $= a^b$, b is the logarithm of the power a^b. If for the present we sup-

pose $b=1$, we have 1 for the logarithm of a^1, and consequently $log.\,a=1$; but if we suppose $b=2$, we have 2 for the logarithm of a^2; that is to say, $log.\,a^2=2$, and we may in the same manner obtain $log.\,a^3=3$; $log.\,a^4=4$; $log.\,a^5=5$, and so on.

222. If we make $b=0$, it is evident that 0 will be the logarithm of a^0; but $a^0=1$; consequently $log.1=0$, whatever be the value of the root a.

Suppose $b=-1$, then -1 will be the logarithm of a^{-1}; but $a^{-1}=\frac{1}{a}$; so that we have $log.\frac{1}{a}=-1$, and in the same manner, we shall have $log.\frac{1}{a^2}=-2$; $log.\frac{1}{a^3}=-3$; $log.\frac{1}{a^4}=-4$, &c.

223. It is evident, then, how we may represent the logarithms of all the powers of a, and even those of fractions which have unity for the numerator, and for the denominator a power of a; we see also, that in all those cases the logarithms are integers; but it must be observed, that if b were a fraction, it would be the logarithm of an irrational number: if we suppose, for example, $b=\frac{1}{2}$, it follows that $\frac{1}{2}$ is the logarithm of $a^{\frac{1}{2}}$, or of $\sqrt{a}$; consequently we have also $log.\sqrt{a}=\frac{1}{2}$; and we shall find, in the same manner, that $log.\sqrt[3]{a}=\frac{1}{3}$, $log.\sqrt[4]{a}=\frac{1}{4}$, &c.

224. But if it be required to find the logarithm of another number c, it will readily be perceived that it can neither be an integer nor a fraction; yet there must be such an exponent b, that the power

a^b may become equal to the number proposed; we have therefore $b = log. c$; and generally, $a^{L.c} = c$.

225. Let us now consider another number d, whose logarithm has been represented in a similar manner by $log.d$; so that $a^{L.d} = d$. Here if we multiply this expression by the preceding one $a^{L.c} = c$, we shall have $a^{L.c+L.d} = cd$; hence the exponent is always the logarithm of the power; consequently $log.c + log.d = log.cd$. But if, instead of multiplying, we divide the former expression by the latter, we shall obtain $a^{L.c-L.d} = \frac{c}{d}$; and consequently $log.c - log.d = log.\frac{c}{d}$.

226. This leads us to the two principal properties of logarithms, which are contained in the equations $log.c + log.d = log.cd$, and $log.c - log.d = log.\frac{c}{d}$. Now the former of these equations teaches us, that the logarithm of a product, as cd, is found by adding together the logarithms of the factor; and the latter shows us, that the logarithm of a fraction may be determined by subtracting the logarithm of the denominator from that of the numerator.

227. It also follows from this, that when it is required to multiply or divide two numbers by one another, we have only to add or subtract their logarithms; and this is what constitutes the singular utility of logarithms in calculation; for it is evidently much easier to add or subtract, than to multiply or divide, particularly when the question involves large numbers.

228. But logarithms are attended with still greater advantages in the involution of powers and the ex-

traction of roots; for if $d=c$, we have by the first property $log.c+log.c=log.cc$; consequently $log.cc=2\,log.c$; and in the same manner we obtain $log.c^3=3log.c$; $log.c^4=4log.c$; and, generally, $log.c^n=n\,log.c$: and if we now substitute fractional numbers for n, we shall have, for example, $log.c^{\frac{1}{2}}$, that is to say, $log.\sqrt{c}$, $=\frac{1}{2}log.c$; and lastly, if we suppose n to represent negative numbers, we shall have $log.c^{-1}$, or $log.\frac{1}{c}$, $=-log.c$; $log.c^{-2}$, or $log.\frac{1}{cc}$, $=-2\,log.c$, and so on; which follows not only from the equation $log.c^n=n\,log.c$, but also from $log.1=0$, as we have already seen.

229. If therefore we had tables, in which logarithms should be calculated for all numbers, we might certainly derive from them very great assistance in performing the most prolix calculations; such, for instance, as require frequent multiplications, divisions, involutions, and extractions of roots; for, in such tables, we should have not only the logarithms of all numbers, but also the numbers answering to all logarithms. If it were required, for example, to find the square root of the number c, we must first find the logarithm of c, that is, $log.c$, and next taking the half of that logarithm, or $\frac{1}{2}log.c$, we should have the logarithm of the square root required: we have therefore only to look in the tables for the number answering to that logarithm, in order to obtain the root required.

230. We have seen above, that the numbers 1, 2, 3, 4, 5, 6, &c. that is to say, all positive numbers,

are logarithms of the root a, and of its positive powers; consequently logarithms of numbers greater than unity; and, on the contrary, that the negative numbers, as -1, -2, &c. are logarithms of the fractions $\frac{1}{a}$, $\frac{1}{aa}$, &c. which are less than unity, but yet greater than nothing.

Hence it follows, that, if the logarithm be positive, the number is always greater than unity; but if the logarithm be negative, the number is always less than unity, and yet greater than 0; consequently we cannot express the logarithms of negative numbers, and must therefore conclude, that the logarithms of negative numbers are impossible, and that they belong to the class of imaginary quantities.

231. In order to illustrate this more fully, it will be proper to fix on a determinate number for the root a. Let us make choice of that, on which the common logarithmic tables are formed, that is, the number 10, which has been preferred, because it is the foundation of our arithmetic. But it is evident that any other number, provided it were greater than unity, would answer the same purpose: and the reason why we cannot suppose $a=1$, is manifest; as all the powers a^b would then be constantly equal to unity, and could never become equal to another given number c.

CHAP. XXII.

Of the Logarithmic Tables that are now in use.

232. In those tables, as we have already mentioned, we begin with the supposition, that the root a is $=10$; so that the logarithm of any number c is the exponent to which we must raise the number 10 in order that the power resulting from it may be equal to the number c; or, if we denote the logarithm of c by $L.c$, we shall always have $10^{L.c}=c$.

233. We have already observed, that the logarithm of the number 1 is always 0; and we have also $10^0=1$; consequently, $log.1=0$; $log.10=1$; $log.100=2$; $log.1000=3$; $log.10000=4$; $log.100000=5$; $log.1000000=6$: farther, $log.\frac{1}{10}=-1$; $log.\frac{1}{100}=-2$; $log.\frac{1}{1000}=-3$; $log.\frac{1}{10000}=-4$; $log.\frac{1}{100000}=-5$; $log.\frac{1}{1000000}=-6$.

234. The logarithms of the principal numbers, therefore, are easily determined; but it is much more difficult to find the logarithms of all the other numbers, yet they must be inserted in the tables: this however is not the place to lay down all the rules that are necessary for such an inquiry; we shall therefore at present content ourselves with a general view only of the subject.

235. First, since $log.1=0$ and $log.10=1$, it is evident that the logarithms of all numbers between 1 and 10 must be included between 0 and unity, and consequently be greater than 0, and less than 1.

It will therefore be sufficient to consider the single number 2; the logarithm of which is certainly greater than 0, but less than unity; and if we represent this logarithm by the letter x, so that $log. 2=x$, the value of that letter must be such as to give exactly $10^x=2$.

We easily perceive also, that x must be considerably less than $\frac{1}{2}$, or which amounts to the same thing, that $10^{\frac{1}{2}}$ is greater than 2; for if we square both sides, the square of $10^{\frac{1}{2}}=10$ and the square of $2=4$; now this latter is much less than the former: and in the same manner we see that x is even less than $\frac{1}{3}$; that is to say, $10^{\frac{1}{3}}$ is greater than 2: for the cube of $10^{\frac{1}{3}}$ is 10, and that of 2 is only 8. But, on the contrary, by making $x=\frac{1}{4}$ we give it too small a value, because the fourth power of $10^{\frac{1}{4}}$ being 10, and that of 2 being 16, it is evident that $10^{\frac{1}{4}}$ is less than 2: thus we see that x, or the $log.2$, is less than $\frac{1}{3}$, but greater than $\frac{1}{4}$: and in the same manner we may determine, with respect to every fraction contained between $\frac{1}{4}$ and $\frac{1}{3}$, whether it be too great or too small.

For example, $\frac{2}{7}$ is a fraction less than $\frac{1}{3}$ and greater than $\frac{1}{4}$; now $10^{\frac{2}{7}}$ is less than 2: the seventh power of $10^{\frac{2}{7}}$ is 10^2, or 100, and the seventh power of 2 is 128, which is consequently greater than the former. We see therefore that $\frac{2}{7}$ is less than $log. 2$, and that *log.* 2, which was found less than $\frac{1}{3}$, is however greater than $\frac{2}{7}$.

Let us try another fraction, which, in consequence of what we have already found, must be contained between $\frac{2}{7}$ and $\frac{1}{3}$; $\frac{3}{10}$ is a fraction between these limits, and it is therefore required to find whether $10^{\frac{3}{10}}=2$; if this be the case, the tenth powers of those numbers are also equal; now the tenth power of $10^{\frac{3}{10}}$ is $10^3=1000$, and the tenth power of 2 is 1024; we conclude therefore, that $10^{\frac{3}{10}}$ is less than 2, and consequently that $\frac{3}{10}$ is too small a fraction, and therefore the *log.* 2, though less than $\frac{1}{3}$, is yet greater than $\frac{3}{10}$.

236. This discussion serves to prove, that *log.* 2 has a determinate value, since we know that it is certainly greater than $\frac{3}{10}$, but less than $\frac{1}{3}$; we shall not however proceed any farther in this investigation at present. Being therefore still ignorant of its true value, we shall represent it by x, so that $log.\ 2=x$; and endeavour to show how, if it were known, we could deduce from it the logarithms of an infinity of other numbers. For this purpose we shall make use of the equation already mentioned, namely, $log.\ cd=log.\ c+log.\ d$, which comprehends the property, that the logarithm of a product is found by adding together the logarithms of the factors.

237. First, as $log.\ 2=x$, and $log.\ 10=1$, we shall have $log.\ 20=x+1$, $log.\ 200=x+2$
$log.\ 2000=x+3$, $log.\ 20000=x+4$
$log.\ 200000=x+5$, $log.\ 2000000=x+6$, &c.

238. Farther, as $log.\ c^2=2\ log.\ c$, and $log.\ c^3=3\ log.\ c$, and $log.\ c^4=4\ log\ c$, &c. we have

∴ $log.\,4=2x$; $log.\,8=3x$; $log.\,16=4x$; $log.\,32=5x$; $log.\,64=6x$, &c. Hence we find also, that

$log.\,40=2x+1$, $log.\,400=2x+2$
$log.\,4000=2x+3$, $log.\,40000=2x+4$, &c.
$log.\,80=3x+1$, $log.\,800=3x+2$
$log.\,8000=3x=3$, $log.\,80000=3x+4$, &c.
$log.\,160=4x+1$, $log.\,1600=4x+2$
$log.\,16000=4x+3$, $log.\,160000=4x+4$, &c.

239. Let us resume also the other fundamental equation, $log.\,\frac{c}{d}=log.\,c-log.\,d$, and let us suppose $c=10$, and $d=2$; since $log.\,10=1$, and $log.\,2=x$, we shall have $log.\,\frac{10}{2}$ or $log.\,5=1-x$, and shall deduce from hence the following equations:

$log.\,50=2-x$, $log.\,500=3-x$
$log.\,5000=4-x$, $log.\,50000=5-x$, &c.
$log.\,25=2-2x$, $log.\,125=3-3x$
$log.\,625=4-4x$, $log.\,3125=5-5x$, &c.
$log.\,250=3-2x$, $log.\,2500=4-2x$
$log.\,25000=5-2x$, $log.\,250000=6-2x$, &c.
$log.\,1250=4-3x$, $log.\,12500=5-3x$
$log.\,125000=6-3x$, $log.\,1250000=7-3x$, &c.
$log.\,6250=5-4x$, $log.\,62500=6-4x$
$log.\,625000=7-4x$, $log.\,6250000=8-4x$, &c.

and so on.

240. If we knew the logarithm of 3, this would be the means also of determining a number of other logarithms; as appears from the following examples. Let the $log.\,3$ be represented by the letter y: then,

$log.\,30=y+1$, $log.\,300=y+2$
$log.\,3000=y+3$, $log.\,30000=y+4$, &c.

$log. 9 = 2y$, $log. 27 = 3y$, $log. 81 = 4y$, &c. we shall have also,

$log. 6 = x + y$, $log. 12 = 2x + y$, $log. 18 = x + 2y$,
$log. 15 = log. 3 + log. 5 = y + 1 - x$.

241. We have already seen that all numbers arise from the multiplication of prime numbers. If therefore we only knew the logarithms of all the prime numbers, we could find the logarithms of all the other numbers by simple additions. The number 210, for example, being formed by the factors 2, 3, 5, 7, its logarithm will be $log. 2 + log. 3 + log. 5 + log. 7$. In the same manner, since $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$, we have $log. 360 = 3\, log. 2 + 2\, log. 3 + log. 5$. It is evident, therefore, that by means of the logarithms of the prime numbers we may determine those of all others; and that we must first apply to the determination of the former, if we would construct tables of logarithms.

CHAP. XXIII.

Of the Method of expressing Logarithms.

242. We have seen that the logarithm of 2 is greater than $\frac{3}{10}$, and less than $\frac{1}{3}$, and that consequently the exponent of 10 must fall between those two fractions, in order that the power may become 2. Now although we know this, yet whatever fraction we assume on this condition, the power resulting from it

will be always an irrational number, greater or less than 2; and consequently the logarithm of 2 cannot be accurately expressed by such a fraction; therefore we must content ourselves with determining the value of that logarithm by such an approximation as may render the error of little or no importance; for which purpose we employ what are called *decimal fractions*, the nature and properties of which ought to be explained as clearly as possible.

243. It is well known that, in the ordinary way of writing numbers by means of the ten figures, or characters,

0, 1, 2, 3, 4, 5, 6, 7, 8, 9,

the first figure on the right alone has its natural signification; that the figures in the second place have ten times the value which they would have had in the first; that the figures in the third place have a hundred times the value; and those in the fourth a thousand times, and so on: so that as they advance towards the left, they acquire a value ten times greater than they had in the preceding rank; thus, in the number 1765, the figure 5 is in the first place on the right and is just equal to 5; in the second place is 6; but this figure, instead of 6, represents 10×6, or 60: the figure 7 is in the third place, and represents 100×7, or 700; and lastly, the 1, which is in the fourth row, becomes 1000; so that we read the given number thus;

One thousand, seven hundred, and sixty five.

244. As the value of figures becomes always ten times greater, as we go from the right towards the left, and as it consequently becomes continually ten times less as we go from the left towards the right; we may

in conformity to this law advance still farther towards the right, and obtain figures whose value will continue to become ten times less than in the preceding place; but it must be observed, that the place where the figures have their natural value is marked by a point. So that if we meet, for example, with the number 36.54892, it is to be understood in this manner; the figure 6, in the first place, has its natural value; and the figure 3, which is in the second place to the left means 30. But the figure 5 which comes after the point, expresses only $\frac{5}{10}$; and the 4 is equal only to $\frac{4}{100}$; the figure 8 is equal to $\frac{8}{1000}$; the figure 9 is equal to $\frac{9}{10000}$; and the figure 2 corresponds to $\frac{2}{100000}$. We see then, that the more those figures advance towards the right, the more their values diminish, and at last, those values become so small, that they may be considered as nothing *.

245. This is the kind of numbers which we call *decimal fractions*, and in this manner logarithms are represented in the tables. The logarithm of 2, for

* The operations of arithmetic are employed on decimal fractions in the same manner as on whole numbers; some precautions only are necessary, after the operation, to place the point properly, which separates the whole numbers from the decimals. On this subject, we may consult almost any of the treatises on arithmetic. In the multiplication of these fractions, when the multiplicand and multiplier contain a great number of decimals, the operation would become too long, and would give the result much more exact than is for the most part necessary; but it may be simplified by a method which is not to be found in many authors, and which is pointed out by M. Marie in his edition of the mathematical lessons of M. de la Caille, where he likewise explains a similar method for the division of decimals. F. T.

The method alluded to in this note is clearly explained in Bonnycastle's Arithmetic.

example, is expressed by 0·3010300; in which we see, 1st. That since there is 0 before the point, this logarithm does not contain an integer; 2dly, that its value is $\frac{3}{10}+\frac{0}{100}+\frac{1}{1000}+\frac{0}{10000}+\frac{3}{100000}+\frac{0}{1000000}+\frac{0}{10000000}$. We might have left out the two last ciphers, but they serve to show that the logarithm in question contains none of those parts which have 1000000 and 10000000 for the denominator. It is however to be understood, that by continuing the series we still might have found smaller parts; but with regard to these, they are neglected on account of their extreme minuteness.

246. The logarithm of 3 is expressed in the table by 0·4771213; we see, therefore, that it contains no integer, and that it is composed of the following fractions: $\frac{4}{10}+\frac{7}{100}+\frac{7}{1000}+\frac{1}{10000}+\frac{2}{100000}+\frac{1}{1000000}+\frac{3}{10000000}$. Not that the logarithm is thus expressed with the utmost exactness; we are however certain that the error is less than $\frac{1}{10000000}$, which is certainly so small, that it may very well be neglected in most calculations.

247. According to this method of expressing logarithms, that of 1 must be represented by 0·0000000, since it is really $=0$: the logarithm of 10 is 1·0000000, where it evidently is exactly $=1$: the logarithm of 100 is 2·0000000, or 2. And hence we may conclude, that the logarithms of all numbers, which are included between 10 and 100, and

consequently composed of two figures, are comprehended between 1 and 2, and therefore must be expressed by 1 *plus* a decimal fraction, as *log*. 50 $=1\cdot6989700$; its value therefore is unity, *plus* $\frac{6}{10}+\frac{9}{100}+\frac{8}{1000}+\frac{9}{10000}+\frac{7}{100000}$: and it will be also easily perceived, that the logarithms of numbers, between 100 and 1000, are expressed by the integer 2 with a decimal fraction: those of numbers between 1000 and 10000, by 3 *plus* a decimal fraction: those of numbers between 10000 and 100000, by 4 integers *plus* a decimal fraction, and so on: thus the *log*. 800, for example, is 2·9030900; that of 2290 is 3·3598355, &c.

248. On the other hand, the logarithms of numbers which are less than 10, or expressed by a single figure, do not contain an integer, and for this reason we find 0 before the point: so that we have two parts to consider in a logarithm. First, that which precedes the point, or the integral part; and the other, the decimal fractions that are to be added to the former. The integral part of a logarithm, which is usually called the *characteristic*, is easily determined from what we have said in the preceding article. Thus it is 0, for all the numbers which have but *one figure;* it is 1, for those which have *two;* it is 2, for those which have *three;* and, in general, it is always one less than the number of figures. If, therefore, the logarithm of 1766 be required, we already know that the first part, or that of the integers, is necessarily 3.

249. So reciprocally, we know at the first sight of the integer part of a logarithm, how many figures compose the number answering to that logarithm; since

the number of those figures always exceed the integer part of the logarithm by unity. Suppose, for example, the number answering to the logarithm 6·4771213 were required, we know immediately that that number must have seven figures, and be greater than 1000000. And in fact this number is 3000000; for *log*. $3000000 = log. 3 + log. 10000000$. Now *log*. $3 = 0·4771213$, and *log*. $1000000 = 6$, and the sum of those two logarithms is 6·4771213.

250. The principal consideration therefore with respect to each logarithm is, the decimal fraction which follows the point, and even that, when once known, serves for several numbers. In order to prove this, let us consider the logarithm of the number 365; its first part is undoubtedly 2; with respect to the other, or the decimal fraction, let us at present represent it by the letter x; we shall have *log*. $365 = 2 + x$; then multiplying continually by 10, we shall have *log*. $3650 = 3 + x$; *log*. $36500 = 4 + x$; *log*. $365000 = 5 + x$, and so on.

But we can also go back, and continually divide by 10; which will give us *log*. $36·5 = 1 + x$; *log*. $3·65 = 0 + x$; *log*. $0·365 = -1 + x$; *log*. $0·0365 = -2 + x$; *log*. $0·00365 = -3 + x$, and so on.

251. All those numbers then which arise from the figures 365, whether preceded, or followed, by ciphers, have always the same decimal fraction for the second part of the logarithm: and the whole difference lies in the integer before the point, which, as we have seen, may become negative; namely, when the number proposed is less than 1. But as beginners find a difficulty in managing negative numbers, it is usual, in those cases, to increase the in-

tegers of the logarithm by 10, that is, to write 10 instead of 0 before the point; so that instead of −1 we have 9; instead of −2 we have 8; instead of −3 we have 7, &c.; but then we must remember, that the characteristic has been taken ten units too great, and by no means suppose that the number consists of 10, 9, or 8 figures. It is likewise easy to conceive, that, if in the case we speak of, this characteristic be less than 10, we must write the figures of the number after a point, to show that they are decimals: for example, if the characteristic be 9, we must begin at the first place after a point; if it be 8, we must also place a cipher in the first row, and not begin to write the figures till the second: thus 9·5622929 would be the logarithm of 0·365, and 8·5622929 the log. of 0·0365. But this manner of writing logarithms is principally employed in tables of sines.

252. In the common tables, the decimals of logarithms are usually carried to seven places or figures, the last of which consequently represents the $\frac{1}{10000000}$ part, and we are sure that they are never erroneous by the whole of this part, and that therefore the error cannot be of any importance. There are, however, calculations in which we require still greater exactness; and then we employ the large tables of Vlacq, where the logarithms are calculated to ten decimal places*.

* The most valuable set of tables we are acquainted with are those published by Dr. Hutton, late Professor of Mathematics at the Royal Military Academy, Woolwich, under the title of,

253. As the first part, or characteristic of a logarithm, is subject to no difficulty, it is seldom expressed in the tables; the second part only is written, or the seven figures of the decimal fraction. There is a set of English tables in which we find the logarithms of all numbers from 1 to 100000, and even those of greater numbers; for small additional tables show what is to be added to the logarithms, in proportion to the figures which the proposed numbers have more than those in the tables. We easily find, for example, the logarithm of 379456, by means of that of 37945 and the small tables of which we speak*.

"Mathematical Tables; containing common, hyperbolic, and logistic logarithms. Also sines, tangents, &c. to which is prefixed a large and original history of the discoveries and writings relating to those subjects."

* The English tables spoken of in the text are those which were published by Sherwin at the beginning of this century, and have been several times reprinted; they are likewise to be found in the tables of Gardener, which are commonly made use of by astronomers, and which have been reprinted at Avignon. With respect to these tables it is proper to remark, that as they do not carry logarithms farther than seven places, independent of the characteristic, we cannot use them with perfect exactness except on numbers that do not exceed six digits; but when we employ the great tables of Vlacq, which carry the logarithms as far as ten decimal places, we may, by taking the proportional parts, work, without error, upon numbers that have as many as nine digits. The reason of what we have said, and the method of employing these tables in operations upon still greater numbers, is well explained in Saunderson's "Elements of Algebra," Book IX. Part II.

It is farther to be observed, that these tables only give the logarithms answering to given numbers, so that when we wish to get the numbers answering to given logarithms, it is seldom that we

254. From what has been said, it will easily be perceived, how we are to obtain from the tables the number corresponding to any logarithm which may occur: Thus in multiplying the numbers 343 and 2401; since we must add together the logarithms of those numbers, the calculation will be as follows:

$$\left.\begin{array}{l} log.\ 343=2\cdot5352941 \\ log.\ 2401=3\cdot3803922 \end{array}\right\}\text{added}$$

$$\begin{array}{rl} 5\cdot9156863 & \text{their sum.} \\ log.\ 823540=5\cdot9156847 & \text{nearest tabular log.} \\ \hline 16 & \text{difference,} \end{array}$$

which in the Table of Differences answers to 3; this therefore being used instead of the ciphers, gives 823543 for the product sought: for the sum is the logarithm of the product required; its characteristic 5 shows that the product is composed of 6 figures; which are found as above.

255. But it is in the extraction of roots that logarithms are of the greatest service, we shall therefore give an example of the manner in which they are used

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find in the tables the precise logarithms that are given, and we are for the most part under the necessity of seeking for these numbers in an indirect way, by the method of interpolation. In order to supply this defect, another set of tables was published at London in 1742, under the title of "The Anti-logarithmic Canon, &c. by James Dodson;" he has arranged the decimals of logarithms from 0,0001 to 1,0000, and opposite to them in order the corresponding numbers carried as far as eleven places; and has likewise given the proportional parts necessary for determining the numbers which answer to the intermediate logarithms that are not to be found in the table. F. T.

in calculations of this kind. Suppose, for example, it were required, to extract the square root of 10. Here we have only to divide the logarithm of 10, which is 1·0000000 by 2; and the quotient 0·5000000 is the logarithm of the root required; and the number in the tables which answers to that logarithm, is 3·16228, the square of which is very nearly equal to 10, being only one hundred thousandth part too great*.

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* In the same manner we may extract any other root, by dividing the log. of the number by the denominator of the index of the root to be extracted; that is, to extract the cube root, divide the log. by 3, the fourth root by 4, and so on for any other extraction. Thus, for example, if the 5th root of 2 were required,

The log. of 2 is 0·3010300: therefore

5)0·3010300

0·0602060 is the log. of the root, which by the tables is found to correspond to 1·1497; and hence we have $\sqrt[5]{2}=1·1497$. But here it may be proper to observe, that when the index, or characteristic of the log. is negative, and not divisible by the denominator of the index of the root to be extracted; then as many units must be borrowed as will make it exactly divisible, carrying those units to the next figure, as in common division. ED.

SECTION II.

Of the different Methods of calculating Compound Quantities.

CHAP. I.

Of the Addition of Compound Quantities.

256. When two or more expressions, consisting of several terms, are to be added together, the operation is frequently represented merely by signs, placing each expression between two parentheses, and connecting it with the rest by means of the sign $+$. Thus, for example, if it be required, to add the expressions $a+b+c$ and $d+e+f$, we represent the sum by

$$(a+b+c)+(d+e+f).$$

257. It is evident that this is not to perform addition, but only to represent it; we see, however, at the same time, that in order to perform it actually, we have only to leave out the parentheses; for as the number $d+e+f$ is to be added to $a+b+c$, we know that this is done by joining to it first $+d$, then $+e$, and then $+f$; which therefore gives the sum $a+b+c+d+e+f$; and the same method is to be observed, if any of the terms are affected by the sign $-$; as they must be connected in the same way, by means of their proper sign.

258. To make this more evident, we shall consider an example in pure numbers, proposing to add the

expression $15-6$ to $12-8$. Here, if we begin by adding 15, we shall have $12-8+15$; but this is adding too much, since we had only to add $15-6$, and it is evident that 6 is the number which we have added too much; let us therefore take this 6 away by writing it with the negative sign, and we shall have the true sum,

$$12-8+15-6;$$

which shows that the sums are found by writing all the terms, each with its proper sign.

259. If it were required therefore to add the expression $d-e-f$ to $a-b+c$, we should express the sum thus,

$$a-b+c+d-e-f;$$

remarking however that it is of no consequence in what order we write these terms; for their places may be changed at pleasure, provided their signs be preserved; so that this sum might have been written thus,

$$c-e+a-f+d-b.$$

260. It is evident, therefore, that addition is attended with no difficulty, whatever be the form of the terms to be added: thus, if it were necessary to add together the expressions $2a^3+6\sqrt{b}-4\,log.\,c$ and $5\sqrt[5]{a}-7c$, we should write them

$$2a^3+6\sqrt{b}-4\,log.\,c+5\sqrt[5]{a}-7c,$$

either in this or in any other order of the terms; for if the signs are not changed, the sum will always be the same.

261. But it frequently happens that the sums represented in this manner may be considerably abridged, as is the case when two or more terms destroy each other; that is, when we find in the same sum

the terms $+a-a$, or $3a-4a+a$: also when two or more terms may be reduced to one, &c. Thus in the following examples:

$$3a+2a=5a, \qquad 7b-3b=+4b$$
$$-6c+10c=+4c; \qquad 4d-2d=2d$$
$$5a-8a=-3a, \qquad -7b+b=-6b$$
$$-3c-4c=-7c, \qquad -3d-5d=-8d$$
$$2a-5a+a=-2a, -3b-5b+2b=-6b.$$

Whenever two or more terms, therefore, are entirely the same with regard to letters, their sum may be abridged; but those cases must not be confounded with such as these, $2a^2+3a$, or $2b^3-b^4$, which admit of no abridgment.

262. Let us consider now some other examples of reduction, as the following, which will lead us immediately to an important truth. Suppose it were required to add together the expressions $a+b$ and $a-b$; our rule gives $a+b+a-b$; now $a+a=2a$, and $b-b=0$; the sum therefore is $2a$: consequently if we add together the sum of two numbers $(a+b)$ and their difference $(a-b)$, we obtain the double of the greater of those two numbers.

This will be perhaps better understood from the following examples:

$$\begin{array}{l} 3a-2b-c \\ 5b-6c+a \\ \hline 4a+3b-7c \end{array} \qquad \begin{array}{l} a^3-2a^2b+2ab^2 \\ \quad -a^2b+2ab^2-b^3 \\ \hline a^3-3a^2b+4ab^2-b^3 \end{array}$$

$$\begin{array}{l} 4a^2-3b+2c \\ 3a^2+2b-12c \\ \hline 7a^2-b-10c \end{array} \qquad \begin{array}{l} a^4+2ab+b^3 \\ a^4-2a^2b+3b^3 \\ \hline 2a^2b+2ab+4b^3 \end{array}$$

CHAP. II.

Of the Subtraction of Compound Quantities.

263. If we wish merely to represent subtraction, we inclose each expression within two parentheses, joining, by the sign $-$, the expression which is to be subtracted, to that from which we have to subtract it.

When we subtract, for example, the expression $d-e+f$ from the expression $a-b+c$, we write the remainder thus:

$$(a-b+c)-(d-e+f);$$

and this method of representing it sufficiently shows which of the two expressions is to be subtracted from the other.

264. But if we wish to perform the actual subtraction, we must observe, first, that when we subtract a positive quantity $+b$ from another quantity a, we obtain $a-b$: and secondly, when we subtract a negative quantity $-b$ from a, we obtain $a+b$; as has been before shown.

265. Suppose now it were required to subtract the expression $b-d$ from $a-c$, we first take away b, which gives $a-c-b$: but this is taking too much away by the quantity d, since we had to subtract only $b-d$; we must therefore restore the value of d, and shall then have

$$a-c-b+d;$$

whence it is evident that the terms of the expression to be subtracted must change their signs, and then be connected to the terms of the other expression.

266. Subtraction is therefore easily performed by this rule, since we have only to write the expression from which we are to subtract, connecting the other to it without any change beside that of the signs. Thus, in the first example, where it was required to subtract the expression $d-e+f$ from $a-b+c$, we obtain $a-b+c-d+e-f$.

An example in numbers will render this still more clear; for if we subtract $6-2+4$ from $9-3+2$, we evidently obtain

$$9-3+2-6+2-4=0;$$

for $9-3+2=8$; also, $6-2+4=8$; now $8-8=0$.

267. Subtraction being therefore subject to no difficulty, we have only to remark, that if there are found in the remainder two or more terms which are entirely similar with regard to the letters, that remainder may be reduced to an abridged form, by the same rules which we have given in addition.

268. Suppose we have to subtract $a-b$ from $a+b$; that is, to take the difference of two numbers from their sum: we shall then have $a+b-a+b$; but $a-a=0$, and $b+b=2b$; the remainder sought is therefore $2b$, that is to say, the double of the less of the two quantities.

269. The following examples will supply the place of farther illustrations:

a^2+ab+b^2	$3a-4b+5c$	$a^3+3a^2b+3ab^2+b^3$	$\sqrt{a}+2\sqrt{b}$
$-a^2+ab+b^2$	$2b+4c-6a$	$a^3-3a^2b+3ab^2-b^3$	$\sqrt{a}-3\sqrt{b}$
$2a^2$.	$9a-6b+c$.	$6a^2b+2b^3$.	$+5\sqrt{b}$.

CHAP. III.

Of the Multiplication of Compound Quantities.

270. When it is only required to represent multiplication, we put each of the expressions, that are to be multiplied together, within two parentheses, and join them to each other, sometimes without any sign, and sometimes placing the sign × between them. Thus for example, to represent the product of the two expressions $a-b+c$ and $d-e+f$, we write

$$(a-b+c)\times(d-e+f)$$

or barely by $(a-b+c)\ (d-e+f)$

which method of expressing products is much used, because it immediately exhibits the factors of which they are composed.

271. But in order to show how multiplication is actually performed, we may remark, in the first place, that to multiply, for example, a quantity, such as $a-b+c$, by 2, each term of it is separately multiplied by that number; so that the product is

$$2a-2b+2c.$$

And the same thing takes place with regard to all other numbers; for if d were the number by which it was required to multiply the same expression, we should obtain

$$ad-bd+cd.$$

272. In the last article we have supposed d to be

a positive number; but if the multiplier were a negative number, as $-e$, the rule formerly given must be applied; namely, that unlike signs multiplied together produce $-$, and like signs $+$. Thus we should have

$$-ae+be-ce.$$

273. Now in order to show how a quantity, A, is to be multiplied by a compound quantity, $d-e$; let us first consider an example in numbers, supposing that A is to be multiplied by $7-3$. Here it is evident, that we are required to take the quadruple of A: for if we first take A seven times, it will then be necessary to subtract 3A from that product.

In general, therefore, if it be required to multiply A by $d-e$, we multiply the quantity A first by d and then by e, and subtract this last product from the first: whence results $d\text{A}-e\text{A}$.

If we now suppose $\text{A}=a-b$, and that this is the quantity to be multiplied by $d-e$; we shall have

$$\begin{array}{l} d\text{A}=ad-bd \\ e\text{A}=ae-be \\ \hline \end{array}$$

whence $d\text{A}-e\text{A}=ad-bd-ae+be$ is the product required.

274. Since therefore we know accurately the product $(a-b)\times(d-e)$, we shall now exhibit the same example of multiplication under the following form:

$$\begin{array}{c} a-b \\ d-e \\ \hline ad-bd-ae+be. \end{array}$$

Which shows, that we must multiply each term of the upper expression by each term of the lower, and that, with regard to the signs, we must strictly observe the rule before given; a rule which this circumstance would completely confirm, if it admitted of the least doubt.

275. It will be easy, therefore, according to this method, to calculate the following example, which is, to multiply $a+b$ by $a-b$;

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline \end{array}$$

Product a^2-b^2.

276. Now we may substitute for a and b any numbers whatever; so that the above example will furnish the following theorem; viz. The sum of two numbers, multiplied by their difference, is equal to the difference of the squares of those numbers: which theorem may be expressed thus:

$$(a+b)\times(a-b)=a^2-b^2.$$

And from this another theorem may be derived; namely, The difference of two square numbers is always a product, and divisible both by the sum and by the difference of the roots of those two squares; consequently, the difference of two squares can never be a prime number*.

* This theorem is not general, for when the difference of the two numbers is 1, and their sum is a prime, it is evident that the difference of the two squares is also a prime: thus $6^2-5^2=11$, $7^2-6^2=13$, $9^2-8^2=17$, &c. In fact, every prime number is the difference of two integral squares. Ed.

277. Let us now calculate some other examples:

$$\begin{array}{l}
2a-3\\
\quad a+2\\
\hline
2a^2-3a\\
\qquad +4a-6\\
\hline
2a^2+\ a-6\\
\hline
\end{array}
\qquad\qquad
\begin{array}{l}
4a^2-6a+9\\
\quad 2a+3\\
\hline
8a^3-12a^2+18a\\
\qquad +12a^2-18a+27\\
\hline
8a^3+27\\
\hline
\end{array}$$

$$\begin{array}{l}
3a^2-2ab\\
2a\ -4b\\
\hline
6a^3-4a^2b\\
-12a^2b+8ab^2\\
\hline
6a^3-16a^2b+8ab^2\\
\hline
\end{array}
\qquad\qquad
\begin{array}{l}
a^2+ab^3\\
a^4-a^3b^3\\
\hline
a^6+a^5b^3\\
\quad -a^5b^3-a^4b^6\\
\hline
a^6-a^4b^6\\
\hline
\end{array}$$

$$\begin{array}{l}
a^2+2ab+2b^2\\
a^2-2ab+2b^2\\
\hline
a^4+2a^3b+2a^2b^2\\
\quad -2a^3b-4a^2b^2-4ab^2\\
\qquad\qquad +2a^2b^2+4ab^3+4b^4\\
\hline
\qquad a^4+b^4\\
\hline
\end{array}$$

$$\begin{array}{l}
2a^2-3ab-4b^2\\
3a^2-2ab+b^2\\
\hline
6a^4-9a^3b-12a^2b^2\\
\quad -4a^3b+6a^2b^2+8ab^3\\
\qquad\qquad +2a^2b^2-3ab^3-4b^4\\
\hline
6a^4-13a^3b-4a^2b^2+5ab^3-4b^4\\
\hline
\end{array}$$

$$\begin{array}{l} a^2+b^2+c^2-ab-ac-bc \\ a+b+c \\ \hline a^3+ab^2+ac^2-a^2b-a^2c-abc \\ \quad a^2b+b^3+bc^2-ab^2-abc-b^2c \\ \quad\quad a^2c+b^2c+c^3-abc-ac^2-bc^2 \\ \hline a^3-3abc+b^3+c^3 \end{array}$$

278. When we have more than two quantities to multiply together, it will easily be understood that, after having multiplied two of them together, we must then multiply that product by one of those which remain, and so on: but it is indifferent what order is observed in those multiplications.

Let it be proposed, for example, to find the value, or product, of the four following factors, *viz.*

I.	II.	III.	IV.
$(a+b)$	(a^2+ab+b^2)	$(a-b)$	(a^2-ab+b^2).

1st. The product of the factors I. and II.

$$\begin{array}{l} a^2+ab+b^2 \\ a+b \\ \hline a^3+a^2b+ab^2 \\ \quad +a^2b+ab^2+b^3 \\ \hline a^3+2a^2b+2ab^2+b^3 \end{array}$$

2d. The product of the factors III. and IV.

$$\begin{array}{l} a^2-ab+b^2 \\ a-b \\ \hline a^3-a^2b+ab^2 \\ \quad -a^2b+ab^2-b^3 \\ \hline a^3-2a^2b+2ab^2-b^3 \end{array}$$

It remains now to multiply the first product I. II. by this second product III. IV.

$$\begin{array}{l} a^3+2a^2b+2ab^2+b^3 \\ a^3-2a^2b+2ab^2-b^3 \\ \hline a^6+2a^5b+2a^4b^2+a^3b^3 \\ \quad -2a^5b-4a^4b^2-4a^3b^3-2a^2b^4 \\ \qquad\qquad 2a^4b^2+4a^3b^3+4a^2b^4+2ab^5 \\ \qquad\qquad\qquad -a^3b^3-2a^2b^4-2ab^5-b^6 \\ \hline \qquad\qquad a^6-b^6 \\ \hline \end{array}$$

which is the product required.

279. Now let us resume the same example, but change the order of it, first multiplying the factors I. and III. and then II. and IV. together.

$$\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline a^2-b^2 \\ \hline \end{array} \qquad \begin{array}{l} a^2+ab+b^2 \\ a^2-ab+b^2 \\ \hline a^4+a^3b+a^2b^2 \\ \quad -a^3b-a^2b^2-ab^3 \\ \qquad\qquad a^2b^2+ab^3+b^4 \\ \hline a^4+a^2b^2+b^4 \\ \hline \end{array}$$

Then multiplying the two products I. III. and II. IV.

$$\begin{array}{l} a^4+a^2b^2+b^4 \\ a^2-b^2 \\ \hline a^6+a^4b^2+a^2b^4 \\ \quad -a^4b^2-a^2b^4-b^6 \\ \hline a^6-b^6 \\ \hline \end{array}$$

which is the product required.

280. We may even perform this calculation in a manner still more concise, by first multiplying the I[st]. factor by the IV[th]. and then the II[d]. by the III[d].

$$\begin{array}{ll} a^2-ab+b^2 & a^2+ab+b^2 \\ a+b & a-b \\ \hline a^3-a^2b+ab^2 & a^3+a^2b+ab^2 \\ \quad a^2b-ab^2+b^3 & \quad -a^2b-ab^2-b^3 \\ \hline a^3+b^3 & a^3-b^3 \end{array}$$

It remains to multiply the product I. IV. and II. III.

$$\begin{array}{l} a^3+b^3 \\ a^3-b^3 \\ \hline a^6+a^3b^3 \\ \quad -a^3b^3-b^6 \\ \hline a^6-b^6 \end{array}$$

the same result as before.

281. It will be proper to illustrate this example by a numerical application. For this purpose, let us make $a=3$ and $b=2$, we shall have $a+b=5$, and $a-b=1$; farther, $a^2=9$, $ab=6$, $b^2=4$: therefore $a^2+ab+b^2=19$, and $a^2-ab+b^2=7$: so that the product required is that of $5\times19\times1\times7$, which is 665.

Now $a^6=729$, and $b^6=64$, consequently the product required is $a^6-b^6=665$, as we have already seen.

CHAP. IV.

Of the Division of Compound Quantities.

282. When we wish simply to represent division, we make use of the usual mark of fractions, which is, to write the denominator under the numerator, separating them by a line; or to inclose each quantity between parentheses, placing two points between the divisor and dividend, and a line between them. Thus if it were required, for example, to divide $a+b$ by $c+d$ we should represent the quotient thus $\frac{a+b}{c+d}$, according to the former method; and thus,

$$(a+b)\div(c+d)$$

according to the latter, where each expression is read $a+b$ *divided by* $c+d$.

283. When it is required to divide a compound quantity by a simple one, we divide each term separately, as in the following examples:

$$(6a-8b+4c)\div 2=3a-4b+2c$$
$$(a^2-2ab)\div a=a-2b$$
$$(a^3-2a^2b+3ab^2)\div a=a^2-2ab+3b^2$$
$$(4a^2-6a^2c+8abc)\div 2a=2a-3ac+4bc$$
$$(9a^2bc-12ab^2c+15abc^2)\div 3abc=3a-4b+5c.$$

284. If it should happen that a term of the dividend is not divisible by the divisor, the quotient is represented by a fraction: thus,

$$(2+b)\div 2=1+\frac{b}{2}$$

$$(a^2+ab+b^2)\div a^2=1+\frac{b}{a}+\frac{b^2}{a^2}$$

$$(2a+b+c)\div 2 = a+\frac{b}{2}+\frac{c}{2}.$$

And here we may write $\frac{1}{2}b$, instead of $\frac{b}{2}$, because $\frac{1}{2}$ times b is equal to $\frac{b}{2}$; and in the same manner $\frac{b}{3}$ is the same as $\frac{1}{3}b$, and $\frac{2b}{3}$ the same as $\frac{2}{3}b$, &c.

285. But when the divisor is itself a compound quantity, division becomes more difficult, which frequently occurs where we least expect it; and when it cannot be performed, we must content ourselves with representing the quotient by a fraction, in the manner that we have already described. But at present we will only consider some cases in which actual division succeeds.

286. Suppose, for example, it were required to divide $ac-bc$ by $a-b$, the quotient must here be such as, when multiplied by the divisor $a-b$, will produce the dividend $ac-bc$. Now it is evident, that this quotient must include c, since without it we could not obtain ac; in order therefore to try whether c is the whole quotient, we have only to multiply it by the divisor, and see if that multiplication produces the whole dividend, or only a part of it. In the present case, if we multiply $a-b$ by c, we have $ac-bc$, which is exactly the dividend; so that c is the whole quotient. It is no less evident, that

$$(a^2+ab)\div(a+b)=a;$$
$$(3a^2-2ab)\div(3a-2b)=a;$$
$$(6a^2-9ab)\div(2a-3b)=3a, \text{ \&c.}$$

287. We cannot fail, in this way, to find a part of the quotient; if, therefore, what we have found, when

multiplied by the divisor, does not yet exhaust the dividend, we have only to divide the remainder again by the divisor, in order to obtain a second part of the quotient; and to continue the same method, until we have found the whole.

Let us, as an example, divide $a^2+3ab+2b^2$ by $a+b$; it is evident, in the first place, that the quotient will include the term a, since otherwise we should not obtain a^2. Now, from the multiplication of the divisor $a+b$ by a, arises a^2+ab; which quantity being subtracted from the dividend, leaves a remainder $2ab+2b^2$; and this remainder must also be divided by $a+b$, where it is evident that the quotient of this division must contain the term $2b$: again, $2b$, multiplied by $a+b$, produces $2ab+2b^2$; consequently $a+2b$ is the quotient required; which, multiplied by the divisor $a+b$, ought to produce the dividend $a^2+3ab+2b^2$. See the work at length:

$$
\begin{array}{r}
a+b)\,a^2+3ab+2b^2\,(a+2b \\
a^2+\ ab\qquad\quad \\
\hline
2ab+2b^2 \\
2ab+2b^2 \\
\hline
0.\quad
\end{array}
$$

288. This operation will be considerably facilitated by choosing one of the terms of the divisor which contains the highest power to be written first, and then, in arranging the terms of the dividend, begin with the highest powers of that first term of the divisor, continuing it according to the powers of that letter: which term in the preceding example was a; but the following examples will render the operation more perspicuous.

$$
\begin{array}{l}
a-b)a^3-3a^2b+3ab^2-b^3(a^2-2ab+b^2 \\
\quad a^3-a^2b \\
\hline
\quad -2a^2b+3ab^2 \\
\quad -2a^2b+2ab^2 \\
\hline
\qquad ab^2-b^3 \\
\qquad ab^2-b^3 \\
\hline
\qquad 0.
\end{array}
$$

$$
\begin{array}{l}
a+b)a^2-b^2(a-b \\
\quad a^2+ab \\
\hline
\quad -ab-b^2 \\
\quad -ab-b^2 \\
\hline
\quad 0.
\end{array}
$$

$$
\begin{array}{l}
3a-2b)18a^2-8b^2(6a+4b \\
\quad 18a^2-12ab \\
\hline
\qquad 12ab-8b^2 \\
\qquad 12ab-8b^2 \\
\hline
\qquad 0.
\end{array}
$$

$$
\begin{array}{l}
a+b)a^3+b^3(a^2-ab+b^2 \\
\quad a^3+a^2b \\
\hline
\quad -a^2b+b^3 \\
\quad -a^2b-ab^2 \\
\hline
\qquad ab^2+b^3 \\
\qquad ab^2+b^3 \\
\hline
\qquad 0.
\end{array}
$$

$$\begin{array}{l}
2a-b)8a^3-b^3(4a^2+2ab+b^2 \\
\quad 8a^3-4a^2b \\
\hline
\quad\quad 4a^2b-b^3 \\
\quad\quad 4a^2b-2ab^2 \\
\hline
\quad\quad\quad 2ab^2-b^3 \\
\quad\quad\quad 2ab^2-b^3 \\
\hline
\quad\quad\quad\quad 0.
\end{array}$$

$$\begin{array}{l}
a^2-2ab+b^2)a^4-4a^3b+6a^2b^2-4ab^3+b^4(a^2-2ab+b^2 \\
\quad a^4-2a^3b+a^2b^2 \\
\hline
\quad\quad -2a^3b+5a^2b^2-4ab^3 \\
\quad\quad -2a^3b+4a^2b^2-2ab^3 \\
\hline
\quad\quad\quad a^2b^2-2ab^3+b^4 \\
\quad\quad\quad a^2b^2-2ab^3+b^4 \\
\hline
\quad\quad\quad\quad 0.
\end{array}$$

$$\begin{array}{l}
a^2-2ab+4b^2)a^4+4a^2b^2+16b^4(a^2+2ab+4b^2 \\
\quad a^4-2a^3b+4a^2b^2 \\
\hline
\quad\quad 2a^3b+16b^4 \\
\quad\quad 2a^3b-4a^2b^2+8ab^3 \\
\hline
\quad\quad\quad 4a^2b^2-8ab^3+16b^4 \\
\quad\quad\quad 4a^2b^2-8ab^3+16b^4 \\
\hline
\quad\quad\quad\quad 0.
\end{array}$$

$$a^2-2ab+2b^2)a^4+4b^4(a^2+2ab+2b^2$$
$$a^4-2a^3b+2a^2b^2$$
$$2a^3b-2a^2b^2+4b^4$$
$$2a^3b-4a^2b^2+4ab^3$$
$$2a^2b^2-4ab^3+4b^4$$
$$2a^2b^2-4ab^3+4b^4$$
$$0.$$

$$1-2x+x^2)1-5x+10x^2-10x^3+5x^4-x^5(1-3x+3x^2-x^3$$
$$1-2x+x^2$$
$$-3x+9x^2-10x^3$$
$$-3x+6x^2-\ 3x^3$$
$$3x^2-7x^3+5x^4$$
$$3x^2-6x^3+3x^4$$
$$-x^3+2x^4-x^5$$
$$-x^3+2x^4-x^5$$
$$0.$$

CHAP. V.

Of the Resolution of Fractions into Infinite Series.*

289. When the dividend is not divisible by the divisor, the quotient is expressed, as we have already

* The Theory of Series is one of the most important in all the mathematics. The series considered in this chapter were discovered by Mercator, about the middle of the last century; and

observed, by a fraction: thus, if we have to divide 1 by $1-a$, we obtain the fraction $\frac{1}{1-a}$: this, however, does not prevent us from attempting the division according to the rules that have been given, nor from continuing it as far as we please, and we shall thus not fail to find the true quotient, though under different forms.

290. To prove this, let us actually divide the dividend 1 by the divisor $1-a$, thus:

$$\begin{array}{l} 1-a)1 \quad (1+\frac{a}{1-a} \\ \quad\;\; 1-a \\ \hline \text{remainder } a \end{array}$$

$$\begin{array}{l} \text{or, } 1-a)1 \quad (1+a+\frac{a^2}{1-a} \\ \quad\;\; 1-a \\ \hline \quad\quad\quad a \\ \quad\quad\quad a-a^2 \\ \hline \text{remainder } a^2 \end{array}$$

soon after, Newton discovered those derived from the extraction of roots, which are treated of in Chapter XII. of this section. This theory has gradually received improvements from several other distinguished mathematicians. The works of James Bernoulli, and the second part of the "Differential Calculus" of Euler, are the books in which the fullest information is to be obtained on these subjects. There is likewise in the Memoirs of Berlin for 1768, a new method by M. de la Grange for resolving, by means of infinite series, all literal equations of any dimension whatever. F. T.

To find a greater number of forms, we have only to continue dividing the remainder a^2 by $1-a$;

$$1-a)a^2\ (a^2+\frac{a^3}{1-a}$$
$$\frac{a^2-a^3}{a^3}$$

then, $$1-a)a^3\ (a^3+\frac{a^4}{1-a}$$
$$\frac{a^3-a^4}{a^4}$$

and again, $$1-a)a^4\ (a^4+\frac{a^5}{1-a}$$
$$\frac{a^4-a^5}{a^5}, \&c.$$

291. This shows that the fraction $\frac{1}{1-a}$ may be exhibited under all the following forms:

$$\frac{1}{1-a}=1+\frac{a}{1-a}; =1+a+\frac{a^2}{1-a};$$
$$=1+a+a^2+\frac{a^3}{1-a}; =1+a+a^2+a^3+\frac{a^4}{1-a};$$
$$=1+a+a^2+a^3+a^4+\frac{a^5}{1-a}, \&c.$$

Now, by considering the first of these expressions, which is $1+\frac{a}{1-a}$, and remembering that 1 is the same as $\frac{1-a}{1-a}$, we have

$$1+\frac{a}{1-a}=\frac{1-a}{1-a}+\frac{a}{1-a}=\frac{1-a+a}{1-a}=\frac{1}{1-a}.$$

If we follow the same process with regard to the second expression, $1+a+\frac{a^2}{1-a}$, that is to say, if we reduce the integral part $1+a$ to the same denominator, $1-a$, we shall have $\frac{1-a^2}{1-a}$, to which if we add $+\frac{a^2}{1-a}$, we shall have $\frac{1-a^2+a^2}{1-a}$, that is to say, $\frac{1}{1-a}$.

In the third expression, $1+a+a^2+\frac{a^3}{1-a}$, the integers reduced to the denominator $1-a$ make $\frac{1-a^3}{1-a}$; and if we add to that the fraction $\frac{a^3}{1-a}$, we have $\frac{1}{1-a}$, as before; therefore all these expressions are equal in value to $\frac{1}{1-a}$, the proposed fraction.

292. This being the case, we may continue the series as far as we please, without being under the necessity of performing any more calculations; and thus we shall have

$$\frac{1}{1-a}=1+a+a^2+a^3+a^4+a^5+a^6+a^7+\frac{a^8}{1-a};$$

or we might continue this farther, and still go on without end; for which reason it may be said that the proposed fraction has been resolved into an infinite series, which is, $1+a+a^2+a^3+a^4+a^5+a^6+a^7+a^8+a^9+a^{10}+a^{11}+a^{12}$, &c. to infinity: and there are sufficient grounds to maintain, that the value of this infinite series is the same as that of the fraction $\frac{1}{1-a}$.

293. What we have said may at first appear strange; but the consideration of some particular cases will make it easily understood. Let us, for instance, suppose, in the first place, $a=1$; our series will become $1+1+1+1+1+1+1$, &c.; and the fraction $\frac{1}{1-a}$, to which it must be equal, becomes $\frac{1}{0}$. Now we have before remarked, that $\frac{1}{0}$ is a number infinitely great; which is therefore here confirmed in a satisfactory manner.

Again, if we suppose $a=2$, our series becomes $1+2+4+8+16+32+64$, &c. to infinity, and its value must be the same as $\frac{1}{1-2}$, that is to say $\frac{1}{-1}$ $=-1$; which at first sight will appear absurd. But it must be remarked, that if we wish to stop at any term of the above series, we cannot do so without joining the fraction which remains; suppose, for example, we were to stop at 64, after having written $1+2+4+8+16+32+64$, we must join the fraction $\frac{128}{1-2}$, or $\frac{128}{-1}$, or -128; we shall therefore have $127-128$, that is in fact -1.

294. These are the considerations which are necessary, when we assume for a numbers greater than unity; but if we suppose a less than 1, the whole becomes more intelligible: for example, Let $a=\frac{1}{2}$; and

we shall have $\frac{1}{1-a}=\frac{1}{1-\frac{1}{2}}=\frac{1}{\frac{1}{2}}=2$, which will be equal to the following series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}$, &c. to infinity. Now, if we take only two terms of this series, we shall have $1+\frac{1}{2}$, and it wants $\frac{1}{2}$ of being equal to $\frac{1}{1-a}=2$; if we take three terms, it wants $\frac{1}{4}$; for the sum is $1\frac{3}{4}$; if we take four terms, we have $1\frac{7}{8}$, and the deficiency is only $\frac{1}{8}$: therefore, the more terms we take, the less the difference becomes; and, consequently, if we continue the series to infinity, there will be no difference at all between its sum and the value of the fraction $\frac{1}{1-a}$, or 2.

295. Let $a=\frac{1}{3}$; and our fraction $\frac{1}{1-a}$ will be $=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}=1\frac{1}{2}$, which, reduced to an infinite series, becomes $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}$, &c. which is consequently equal to $\frac{1}{1-a}$.

Here, if we take two terms, we have $1\frac{1}{3}$, and there wants $\frac{1}{6}$; if we take three terms, we have $1\frac{4}{9}$, and there will still be wanting $\frac{1}{18}$; if we take four terms, we shall have $1\frac{13}{27}$, and the difference will be $\frac{1}{54}$; since, therefore, the error always becomes three times less, it must evidently vanish at last.

296. Suppose $a=\frac{2}{3}$; we shall have $\frac{1}{1-a}=\frac{1}{1-\frac{2}{3}}$ $=3$, $=1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}+\frac{32}{243}$, &c. to infinity, and here, by taking first $1\frac{2}{3}$, the error is $1\frac{1}{3}$; taking three terms, which make $2\frac{1}{9}$, the error is $\frac{8}{9}$; taking four terms, we have $2\frac{11}{27}$, and the error is $\frac{16}{27}$.

297. If $a=\frac{1}{4}$, the fraction is $\frac{1}{1-\frac{1}{4}}=\frac{1}{\frac{3}{4}}=1\frac{1}{3}$; and the series becomes $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$, &c. The two first terms are equal to $1\frac{1}{4}$, which gives $\frac{1}{12}$ for the error; and taking one term more, we have $1\frac{5}{15}$, that is to say, only an error of $\frac{1}{48}$.

298. In the same manner, we may resolve the fraction $\frac{1}{1+a}$, into an infinite series by actually dividing the numerator 1 by the denominator $1+a$, which, after a certain number of terms have been obtained, will give the law by which the following terms are formed, so that the series may be carried to any length without the trouble of continual division, as is shown in the following example.

$$
\begin{array}{l}
1+a)\ 1\ (1-a+a^2-a^3+a^4 \\
\quad 1+a \\
\quad \overline{\quad -a} \\
\quad -a-a^2 \\
\quad \overline{\qquad a^2} \\
\qquad a^2+a^3 \\
\qquad \overline{\quad -a^3} \\
\qquad -a^3-a^4 \\
\qquad \overline{\qquad a^4} \\
\qquad\quad a^4+a^5 \\
\qquad\quad \overline{\quad -a^5}, \&c.
\end{array}
$$

Whence it follows, that the fraction $\frac{1}{1+a}$ is equal to the series,

$$1-a+a^2-a^3+a^4-a^5+a^6-a^7, \&c.$$

299. If we make $a=1$, we have this remarkable comparison:

$\frac{1}{1+a}=\frac{1}{2}=1-1+1-1+1-1+1-1$, &c. to infinity; which appears rather contradictory; for if we stop at -1, the series gives 0; and if we finish at $+1$, it gives 1; but this is precisely what solves the difficulty; for since we must go on to infinity, without stopping either at -1 or at $+1$, it is evident, that the sum can neither be 0 nor 1 but that this result must lie between these two, and therefore be $\frac{1}{2}$*.

* The author seems here to have assumed too much, for it does not follow because the series is neither equal to 0, nor 1,

300. Let us now make $a=\frac{1}{2}$, and our fraction will be $\frac{1}{1+\frac{1}{2}}=\frac{2}{3}$, which must therefore express the value of the series $1-\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\frac{1}{64}$, &c. to infinity; here if we take only the two leading terms of this series, we have $\frac{1}{2}$, which is too small by $\frac{1}{6}$; if we take three terms, we have $\frac{3}{4}$, which is too much by $\frac{1}{12}$; if we take four terms, we have $\frac{5}{8}$, which is too small by $\frac{1}{24}$, &c.

301. Suppose again $a=\frac{1}{3}$, our fraction will be $=\frac{1}{1+\frac{1}{3}}=\frac{3}{4}$, which must be equal to this series $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}-\frac{1}{243}+\frac{1}{729}$, &c. continued to infinity. Now, by considering only two terms, we have $\frac{2}{3}$, which is too small by $\frac{1}{12}$; three terms make $\frac{7}{9}$, which

---

that it must necessarily be equal to $\frac{1}{2}$. This difficulty, however, is easily obviated, by observing that no infinite series is in reality equal to the fraction from which it is derived without the remainder be considered, which, in the present case, is alternately $+\frac{1}{2}$ and $-\frac{1}{2}$; that is, $+\frac{1}{2}$ when the series is 0, and $-\frac{1}{2}$ when the series is 1, which still gives the same value for the whole expression. ED.

is too much by $\frac{1}{36}$; four terms give $\frac{20}{27}$, which is too small by $\frac{1}{108}$, and so on.

302. The fraction $\frac{1}{1+a}$ may also be resolved into an infinite series another way; namely, by dividing 1 by $a+1$, as follows:

$$
\begin{array}{l}
a+1)\ 1\ \ \left(\frac{1}{a}-\frac{1}{a^2}+\frac{1}{a^3}-\frac{1}{a^4}+\frac{1}{a^5}\right. \\
\quad 1+\frac{1}{a} \\
\quad \overline{\quad -\frac{1}{a}} \\
\quad -\frac{1}{a}-\frac{1}{a^2} \\
\quad \overline{\qquad \frac{1}{a^2}} \\
\qquad \frac{1}{a^2}+\frac{1}{a^3} \\
\qquad \overline{\quad -\frac{1}{a^3}}
\end{array}
$$

It is however unnecessary to carry the actual division any farther, as we are enabled already to continue the series to any length, from the law which may be observed in those terms we have obtained; namely, the signs are alternately *plus* and *minus*, and each term is equal to the preceding one multiplied by $\frac{1}{a}$.

Consequently, our fraction $\frac{1}{a+1}$, is equal to the infinite series $\frac{1}{a}-\frac{1}{a^2}+\frac{1}{a^3}-\frac{1}{a^4}+\frac{1}{a^5}-\frac{1}{a^6}$, &c. Let us make $a=1$, and we shall have the series $1-1+1-1+1-1$, &c. $=\frac{1}{2}$, as before: and if we suppose $a=2$, we shall have the series $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-\frac{1}{64}$, &c. $=\frac{1}{3}$.

303. In the same manner, by resolving the general fraction $\frac{c}{a+b}$ into an infinite series, we shall have,

$$a+b)\ c\ \left(\frac{c}{a}-\frac{bc}{a^2}+\frac{b^2c}{a^3}-\frac{b^3c}{a^4}\right.$$

$$c+\frac{bc}{a}$$

$$-\frac{bc}{a}$$

$$-\frac{bc}{a}-\frac{b^2c}{a^2}$$

$$\frac{b^2c}{a^2}$$

$$\frac{b^2c}{a^2}+\frac{b^3c}{a^3}$$

$$-\frac{b^3c}{a^3}$$

And here again the law of continuation is manifest; the signs being alternately + and −, and each succeeding term is formed by multiplying the foregoing one by $\frac{b}{a}$.

Whence it appears, that we may compare $\frac{c}{a+b}$ with the series $\frac{c}{a}-\frac{bc}{a^2}+\frac{b^2c}{a^3}-\frac{b^3c}{a^4}$, &c. to infinity.

Let $a=2$, $b=4$, $c=3$, and we shall have

$$\frac{c}{a+b}=\frac{3}{2+4}=\frac{3}{6}=\frac{1}{2}=\frac{3}{2}-3+6-12, \text{ \&c.}$$

If $a=10$, $b=1$, and $c=11$, we shall have

$$\frac{c}{a+b}=\frac{11}{10+1}=1=\frac{11}{10}-\frac{11}{100}+\frac{11}{1000}-\frac{11}{10000}, \text{ \&c.}$$

Here if we consider only one term of the series, we have $\frac{11}{10}$, which is too much by $\frac{1}{10}$; if we take two terms, we have $\frac{99}{100}$, which is too small by $\frac{1}{100}$; if we take three terms, we have $\frac{1001}{1000}$, which is too much by $\frac{1}{1000}$, &c.

304. When there are more than two terms in the divisor, we may also continue the division to infinity in the same manner. Thus if the fraction $\frac{1}{1-a+a^2}$ were proposed, the infinite series, to which it is equal, will be found by dividing the numerator by the denominator till the particular law of the series be observed, as in the following operation.

$$
\begin{array}{l}
1-a+a^2)\ 1\ (1+a-a^3-a^4+a^6+a^7, \text{ \&c.} \\
\quad 1-a+a^2 \\
\quad\hline \\
\qquad a-a^2 \\
\qquad a-a^2+a^3 \\
\qquad\hline \\
\qquad\quad -a^3 \\
\qquad\quad -a^3+a^4-a^5 \\
\qquad\quad\hline \\
\qquad\qquad -a^4+a^5 \\
\qquad\qquad -a^4+a^5-a^6 \\
\qquad\qquad\hline \\
\qquad\qquad\qquad a^6 \\
\qquad\qquad\qquad a^6-a^7+a^8 \\
\qquad\qquad\qquad\hline \\
\qquad\qquad\qquad\quad a^7-a^8 \\
\qquad\qquad\qquad\quad a^7-a^8+a^9 \\
\qquad\qquad\qquad\quad\hline \\
\qquad\qquad\qquad\qquad -a^9
\end{array}
$$

We have therefore the equation

$\frac{1}{1-a+a^2}=1+a-a^3-a^4+a^6+a^7$, &c.; where, if we make $a=1$, we have $1=1+1-1-1+1+1-1-1$, &c. which series contains twice the series found above $1-1+1-1+1$, &c. Now, as we have found this to be $\frac{1}{2}$, it is not extraordinary that we should find $\frac{2}{2}$, or 1, for the value of that which we have just determined.

By making $a=\frac{1}{2}$, we shall have the equation

$\frac{1}{\frac{3}{4}}=\frac{4}{3}=1+\frac{1}{2}-\frac{1}{8}-\frac{1}{16}+\frac{1}{64}+\frac{1}{128}-\frac{1}{512}$, &c.

If $a=\frac{1}{3}$, we shall have the equation $\frac{1}{\frac{7}{9}}=\frac{9}{7}=1+\frac{1}{3}-\frac{1}{27}-\frac{1}{81}+\frac{1}{729}$, &c. and if we take the four leading terms of this series, we have $\frac{104}{81}$, which is only $\frac{1}{567}$ less than $\frac{9}{7}$.

Suppose again $a=\frac{2}{3}$, we shall have $\frac{1}{\frac{7}{9}}=\frac{9}{7}=1+\frac{2}{3}-\frac{8}{27}-\frac{16}{81}+\frac{64}{729}$, &c. this series is therefore equal to the preceding one; and by subtracting one from the other, we obtain $\frac{1}{3}-\frac{7}{27}-\frac{15}{81}+\frac{63}{729}$, &c. which is necessarily $=0$.

305. The method, which we have here explained, serves to resolve, generally, all fractions into infinite series; which is often found to be of the greatest utility; it is also remarkable, that an infinite series, though it never ceases, may have a determinate value. It should likewise be observed, that, from this branch of mathematics, inventions of the utmost importance have been derived, on which account the subject deserves to be studied with the greatest attention.

CHAP. VI.

Of the Squares of Compound Quantities.

306. When it is required to find the square of a compound quantity, we have only to multiply it by itself, and the product will be the square required.

For example, the square of $a+b$ is found in the following manner:

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad\;\; ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

307. So that when the root consists of two terms added together, as $a+b$, the square comprehends, 1st. the squares of each term, namely a^2 and b^2; 2dly, twice the product of the two terms, namely $2ab$; so that the sum $a^2+2ab+b^2$ is the square of $a+b$: let, for example, $a=10$ and $b=3$, that is to say, let it be required to find the square of 13, we shall have $100+60+9$, or 169.

308. We may easily find, by means of this formula, the squares of numbers, however great, if we divide them into two parts: thus, for example, the square of 57, if we consider that this number is the same as $50+7$, will be found $=2500+700+49=3249$.

309. Hence it is evident, that the square of $a+1$ will be a^2+2a+1: and since the square of a is a^2,

we find the square of $a+1$ by adding to that square $2a+1$; and it must be observed, that this $2a+1$ is the sum of the two roots a and $a+1$.

Thus, as the square of 10 is 100, that of 11 will be 100+21: the square of 57 being 3249, that of 58 is 3249+115=3364; the square of 59=3364+117=3481; the square of 60=3481+119=3600, &c.

310. The square of a compound quantity, as $a+b$, is represented in this manner $(a+b)^2$; we have therefore $(a+b)^2=a^2+2ab+b^2$, whence we deduce the following equations:

$(a+1)^2=a^2+2a+1$; $(a+2)^2=a^2+4a+4$;
$(a+3)^2=a^2+6a+9$; $(a+4)^2=a^2+8a+16$;
&c.

311. If the root be $a-b$, the square of it is $a^2-2ab+b^2$, which contains also the squares of the two terms, but in such a manner that we must take from their sum twice the product of those two terms: let, for example, $a=10$ and $b=-1$, then the square of 9 will be found equal to 100−20+1=81.

312. Since we have the equation $(a-b)^2=a^2-2ab+b^2$, we shall have $(a-1)^2=a^2-2a+1$; the square of $a-1$ is found, therefore, by subtracting from a^2 the sum of the two roots a and $a-1$, namely, $2a-1$; thus, for example, if $a=50$, we have $a^2=2500$, and $2a-1=99$; therefore $49^2=2500-99=2401$.

313. What we have said here may be also confirmed and illustrated by fractions; for if we take as the root $\frac{3}{5}+\frac{2}{5}=1$, the square will be,

$$\frac{9}{25}+\frac{4}{25}+\frac{12}{25}=\frac{25}{25}=1.$$

Farther, the square of $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ will be $\frac{1}{4}-\frac{1}{3}+\frac{1}{9}$ $=\frac{1}{36}$.

314. When the root consists of a greater number of terms, the method of determining the square is the same. Let us find, for example, the square of $a+b+c$:

$$\begin{array}{l} a+b+c \\ a+b+c \\ \hline a^2+ab+ac \\ \quad\quad ab+b^2+bc \\ \quad\quad\quad\quad ac+bc+c^2 \\ \hline a^2+2ab+2ac+b^2+2bc+c^2 \end{array}$$

So that it includes, first, the square of each term of the root, and beside that, the double products of those terms multiplied two by two.

315. To illustrate this by an example, let us divide the number 256 into three parts, $200+50+6$; its square will then be composed of the following parts:

$$\begin{array}{rr} 200^2= & 40000 \\ 50^2= & 2500 \\ 6^2= & 36 \\ 2.50.200= & 20000 \\ 2.\ 6.200= & 2400 \\ 2.\ 6.\ 50= & 600 \\ \hline & 65536=256\times256, \text{ or } 256^2. \end{array}$$

316. When some terms of the root are negative, the square is still found by the same rule; only we must be careful what signs we prefix to the double pro-

ducts: thus $(a-b-c)^2 = a^2+b^2+c^2-2ab-2ac+2bc$; and if we represent the number 256 by $300-40-4$, we shall have,

Positive Parts.	Negative Parts.
$300^2 = 90000$	$2.40.300 = 24000$
$40^2 = 1600$	$2.4.300 = 2400$
$2.40.4 = 320$	
$4^2 = 16$	-26400
$+91936$	
-26400	

65536, the square of 256 as before.

CHAP. VII.

Of the Extraction of Roots applied to Compound Quantities.

317. In order to give a certain rule for this operation, we must consider attentively the square of the root $a+b$, which is $a^2+2ab+b^2$, in order that we may reciprocally find the root of a given square.

318. We must consider therefore, first, that as the square $a^2+2ab+b^2$ is composed of several terms, it is certain that the root also will comprise more than one term; and that if we write the square in such a manner that the powers of one of the letters, as a, may go on continually diminishing, the first term will be the square of the first term of the root; and since,

in the present case, the first term of the square is a^2, it is certain that the first term of the root is a.

319. Having therefore found the first term of the root, that is to say a, we must consider the rest of the square, namely $2ab+b^2$, to see if we can derive from it the second part of the root, which is b: now this remainder $2ab+b^2$ may be represented by the product, $(2a+b)b$; wherefore the remainder having two factors $2a+b$ and b, it is evident that we shall find the latter, b, which is the second part of the root, by dividing the remainder $2ab+b^2$ by $2a+b$.

320. So that the quotient, arising from the division of the above remainder by $2a+b$, is the second term of the root required; and in this division we observe, that $2a$ is the double of the first term a, which is already determined; so that although the second term is yet unknown, and it is necessary, for the present, to leave its place empty, we may nevertheless attempt the division, since in it we attend only to the first term $2a$; but as soon as the quotient is found, which in the present case is b, we must put it in the vacant place, and thus render the division complete.

321. The calculation, therefore, by which we find the root of the square $a^2+2ab+b^2$, may be represented thus:

$$\begin{array}{rl} & a^2+2ab+b^2(a+b \\ & a^2 \\ \hline 2a+b) & 2ab+b^2 \\ & 2ab+b^2 \\ \hline & 0. \end{array}$$

322. We may, also, in the same manner, find the

square root of other compound quantities, provided they are squares, as will appear from the following examples:

$$
\begin{array}{r|l}
 & a^2+6ab+9b^2\ (a+3b \\
 & a^2 \\
\hline
2a+3b) & 6ab+9b^2 \\
 & 6ab+9b^2 \\
\hline
 & \quad 0.
\end{array}
$$

$$
\begin{array}{r|l}
 & 4a^2-4ab+b^2\ (2a-b \\
 & 4a^2 \\
\hline
4a-b) & -4ab+b^2 \\
 & -4ab+b^2 \\
\hline
 & \quad 0.
\end{array}
$$

$$
\begin{array}{r|l}
 & 9p^2+24pq+16q^2\ (3p+4q \\
 & 9p^2 \\
\hline
6p+4q) & 24pq+16q^2 \\
 & 24pq+16q^2 \\
\hline
 & \quad 0.
\end{array}
$$

$$
\begin{array}{r|l}
 & 25x^2-60x+36\ (5x-6 \\
 & 25x^2 \\
\hline
10x-6) & -60x+36 \\
 & -60x+36 \\
\hline
 & \quad 0.
\end{array}
$$

323. When there is a remainder after the division, it is a proof that the root is composed of more than two terms; and we must in that case consider the

two terms already found as forming the first part, and endeavour to derive the other from the remainder, in the same manner as we found the second term of the root from the first. The following examples will render this operation more clear:

$$
\begin{array}{rl}
 & a^2+2ab-2ac-2bc+b^2+c^2 \;(a+b-c \\
 & a^2 \\
\hline
2a+b) & 2ab-2ac-2bc+b^2+c^2 \\
 & 2ab \qquad\qquad\quad +b^2 \\
\hline
2a+2b-c) & -2ac-2bc+c^2 \\
 & -2ac-2bc+c^2 \\
\hline
 & 0.
\end{array}
$$

$$
\begin{array}{rl}
 & a^4+2a^3+3a^2+2a+1 \;(a^2+a+ \\
 & a^4 \\
\hline
2a^2+a) & 2a^3+3a^2 \\
 & 2a^3+\;a^2 \\
\hline
2a^2+2a+1) & 2a^2+2a+1 \\
 & 2a^2+2a+1 \\
\hline
 & 0.
\end{array}
$$

$$
\begin{array}{rl}
 & a^4-4a^3b+8ab^3+4b^4 \;(a^2-2ab-2b^2 \\
 & a^4 \\
\hline
2a^2-2ab) & -4a^3b+8ab^3+4b^4 \\
 & -4a^3b+4a^2b^2 \\
\hline
2a^2-4ab-2b^2) & -4a^2b^2+8ab^3+4b^4 \\
 & -4a^2b^2+8ab^3+4b^4 \\
\hline
 & 0.
\end{array}
$$

$$
\begin{array}{rl}
 & a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6 \quad (a^3-3a^2b+3ab^2-b^3 \\
 & a^6 \\
\hline
2a^3-3a^2b) & -6a^5b+15a^4b \\
 & -6a^5b+\ 9a^4b^2 \\
\hline
2a^3-6a^2b+3ab^2) & 6a^4b^2-20a^3b^3+15a^2b^4 \\
 & 6a^4b^2-18a^3b^3+\ 9a^2b^4 \\
\hline
2a^3-6a^2b+6ab^2-b^3) & -\ 2a^3b^3+6a^2b^4-6ab^5+b^6 \\
 & -\ 2a^3b^3+6a^2b^4-6ab^5+b^6 \\
\hline
 & 0.
\end{array}
$$

324. We easily deduce from the rule which we have explained, the method which is taught in books of arithmetic for the extraction of the square root, as will appear by attending to the following examples in numbers:

$$
\begin{array}{rl}
 & \dot{5}2\dot{9}\ (23 \\
 & 4 \\
\hline
43) & 129 \\
 & 129 \\
\hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & 2\dot{3}0\dot{4}\ (48 \\
 & 16 \\
\hline
88) & 704 \\
 & 704 \\
\hline
 & 0.
\end{array}
$$

$$
\begin{array}{rl}
 & 4\dot{0}9\dot{6}\ (64 \\
 & 36 \\
\hline
124) & 496 \\
 & 496 \\
\hline
 & 0.
\end{array}
\qquad
\begin{array}{rl}
 & 9\dot{6}0\dot{4}\ (98 \\
 & 81 \\
\hline
188) & 1504 \\
 & 1504 \\
\hline
 & 0.
\end{array}
$$

```
       · · ·
       15625 (125
       1
      -------
  22)  56
       44
      -------
 245)  1225
       1225
      -------
       0.
```

```
        · · ·
        998001 (999
        81
       -------
  189)  1880
        1701
       -------
 1989)  17901
        17901
       -------
        0.
```

325. But when there is a remainder after all the figures have been used, it is a proof that the number proposed is not a square, and consequently that its root cannot be assigned; in such cases, the radical sign, which we before employed, is made use of, which is written before the quantity, and the quantity itself is placed between parentheses, or under a line: thus the square root of a^2+b^2 is represented by $\surd(a^2+b^2)$, or by $\surd\overline{a^2+b^2}$; and $\surd(1-x^2)$, or $\surd\overline{1-x^2}$, expresses the square root of $1-x^2$; or instead of this radical sign, we may use the fractional exponent $\frac{1}{2}$, and represent the square root of a^2+b^2, for instance, by $(a^2+b^2)^{\frac{1}{2}}$, or by $\overline{a^2+b^2}|^{\frac{1}{2}}$.

CHAP. VIII.

Of the Calculation of Irrational Quantities.

326. When it is required to add together two or more irrational quantities, this is to be done, according to the method before laid down, by writing all the terms in succession, each with its proper sign: and with regard to abbreviations, we must remark that, instead of $\sqrt{a}+\sqrt{a}$, for example, we may write $2\sqrt{a}$; and that $\sqrt{a}-\sqrt{a}=0$, because these two terms destroy one another; thus the quantities $3+\sqrt{2}$ and $1+\sqrt{2}$, added together, make $4+2\sqrt{2}$, or $4+\sqrt{8}$; the sum of $5+\sqrt{3}$ and $4-\sqrt{3}$, is 9; and that of $2\sqrt{3}+3\sqrt{2}$ and $\sqrt{3}-\sqrt{2}$ is $3\sqrt{3}+2\sqrt{2}$.

327. Subtraction also is very easy, since we have only to add the proposed numbers, after having changed their signs; as will be readily seen in the following example, by subtracting the lower line from the upper one:

$$\begin{array}{l} 4-\sqrt{2}+2\sqrt{3}-3\sqrt{5}+4\sqrt{6} \\ 1+2\sqrt{2}-2\sqrt{3}-5\sqrt{5}+6\sqrt{6} \\ \hline 3-3\sqrt{2}+4\sqrt{3}+2\sqrt{5}-2\sqrt{6}. \end{array}$$

328. In multiplication, we must recollect that $\sqrt{a}$ multiplied by $\sqrt{a}$ produces a; and that if the numbers which follow the sign $\sqrt{}$ are different, as a and b, we have $\sqrt{ab}$ for the product of $\sqrt{a}$ multiplied by $\sqrt{b}$; and thus it will be easy to calculate the following examples:

$$\begin{array}{l} 1+\sqrt{2} \\ 1+\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad +\sqrt{2}+2 \\ \hline 1+2\sqrt{2}+2=3+2\sqrt{2}. \end{array} \qquad \begin{array}{l} 4+2\sqrt{2} \\ 2-\sqrt{2} \\ \hline 8+4\sqrt{2} \\ \quad -4\sqrt{2}-4 \\ \hline 8-4=4. \end{array}$$

329. What we have said applies also to imaginary quantities; we shall only observe farther, that $\sqrt{-a}$ multiplied by $\sqrt{-a}$ produces $-a$. Also if it were required to find the cube of $-1+\sqrt{-3}$, we should first take the square of that number, and then multiply that square by the same number; as in the following operation:

$$\begin{array}{l} -1+\sqrt{-3} \\ -1+\sqrt{-3} \\ \hline \quad 1-\sqrt{-3} \\ \qquad -\sqrt{-3}-3 \\ \hline 1-2\sqrt{-3}-3=-2-2\sqrt{-3} \\ \qquad\qquad\qquad -1+\sqrt{-3} \\ \hline \qquad\qquad\qquad 2+2\sqrt{-3} \\ \qquad\qquad\qquad\quad -2\sqrt{-3}+6 \\ \hline \qquad\qquad\qquad 2+6=8. \end{array}$$

330. In the division of surds, we have only to express the proposed quantities in the form of a fraction; which may be then changed into another expression having a rational denominator; for if the denominator be $a+\sqrt{b}$, for example, and we multiply both this and the numerator by $a-\sqrt{b}$, the new denominator will be a^2-b, in which there is no radical sign. Let it be proposed, for example, to divide

$3+2\sqrt{2}$ by $1+\sqrt{2}$: we shall first have $\frac{3+2\sqrt{2}}{1+\sqrt{2}}$; then multiplying the two terms of the fraction by $1-\sqrt{2}$, we shall have for the numerator:

$$\begin{array}{l} 3+2\sqrt{2} \\ 1-\sqrt{2} \\ \hline 3+2\sqrt{2} \\ \quad -3\sqrt{2}-4 \\ \hline 3-\sqrt{2}-4=-\sqrt{2}-1; \end{array}$$

and for the denominator:

$$\begin{array}{l} 1+\sqrt{2} \\ 1-\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad -\sqrt{2}-2 \\ \hline 1-2=-1. \end{array}$$

Our new fraction therefore is $\frac{-\sqrt{2}-1}{-1}$; and if we again multiply the two terms by -1, we shall have for the numerator $\sqrt{2}+1$, and for the denominator $+1$. Now it is easy to show that $\sqrt{2}+1$ is equal to the proposed fraction $\frac{3+2\sqrt{2}}{1+\sqrt{2}}$; for $\sqrt{2}+1$ being multiplied by the divisor $1+\sqrt{2}$, gives

$$\begin{array}{l} 1+\sqrt{2} \\ 1+\sqrt{2} \\ \hline 1+\sqrt{2} \\ \quad +\sqrt{2}+2 \\ \hline \end{array}$$

we have $1+2\sqrt{2}+2=3+2\sqrt{2}$.

Again: $8-5\sqrt{2}$ divided by $3-2\sqrt{2}$ is in the first instance $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}$; and multiplying the two terms of this fraction by $3+2\sqrt{2}$, we have for the numerator,

$$\begin{array}{l} 8-5\sqrt{2} \\ 3+2\sqrt{2} \\ \hline 24-15\sqrt{2} \\ \quad +16\sqrt{2}-20 \\ \hline 24+\sqrt{2}-20=4+\sqrt{2}; \end{array}$$

and for the denominator,

$$\begin{array}{l} 3-2\sqrt{2} \\ 3+2\sqrt{2} \\ \hline 9-6\sqrt{2} \\ \quad +6\sqrt{2}-8 \\ \hline 9-8=+1. \end{array}$$

Consequently the quotient will be $4+\sqrt{2}$: the truth of this may be proved, as before, by multiplication; thus,

$$\begin{array}{l} 4+\sqrt{2} \\ 3-2\sqrt{2} \\ \hline 12+3\sqrt{2} \\ \quad -8\sqrt{2}-4 \\ \hline 12-5\sqrt{2}-4=8-5\sqrt{2}. \end{array}$$

331. In the same manner we may transform irrational fractions into others that have rational denominators: if we have, for example, the fraction $\frac{1}{5-2\sqrt{6}}$, and multiply its numerator and denomina-

tor by $5+2\sqrt{6}$; we transform it into this, $\frac{5+2\sqrt{6}}{1}$ $=5+2\sqrt{6}$; in like manner the fraction $\frac{2}{-1+\sqrt{-3}}$ assumes this form, $\frac{2+2\sqrt{-3}}{-4}=\frac{1+\sqrt{-3}}{-2}$; also $\frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}-\sqrt{5}}=\frac{11+2\sqrt{30}}{1}=11+2\sqrt{30}$.

332. When the denominator contains several terms, we may in the same manner make the radical signs in it vanish one by one: thus if the fraction $\frac{1}{\sqrt{10}-\sqrt{2}-\sqrt{3}}$ be proposed; we first multiply these two terms by $\sqrt{10}+\sqrt{2}+\sqrt{3}$, and obtain the fraction $\frac{\sqrt{10}+\sqrt{2}+\sqrt{3}}{5-2\sqrt{6}}$; then multiplying its numerator and denominator by $5+2\sqrt{6}$, we have $5\sqrt{10}+11\sqrt{2}+9\sqrt{3}+2\sqrt{60}$.

CHAP. IX.

Of Cubes, and of the Extraction of Cube Roots.

333. To find the cube of $a+b$, we have only to multiply its square $a^2+2ab+b^2$ again by the quantity itself, thus,

$$\begin{array}{l} a^2+2ab+b^2 \\ \underline{a+b} \\ a^3+2a^2b+ab^2 \\ \quad\;\; \underline{a^2b+2ab^2+b^3} \\ a^3+3a^2b+3ab^2+b^3 \end{array}$$

which gives the cube required.

We see therefore that it contains the cubes of the two parts of the root, *plus* $3a^2b+3ab^2$, which quantity is equal to $(3ab)\times(a+b)$; that is, the triple product of the two parts a and b, multiplied by their sum.

334. So that whenever a root is composed of two terms, it is easy to find its cube by this rule: for example, the number $5=3+2$; its cube is therefore $27+8+18\times5=125$.

And if $7+3=10$ be the root; then the cube will be $343+27+63\times10=1000$.

To find the cube of 36, let us suppose the root $36=30+6$, and we have for the cube required, $27000+216+540\times36=46656$.

335. But if, on the other hand, the cube be given, namely, $a^3+3a^2b+3ab^2+b^3$, and it be required to find its root, we must premise the following remarks:

First, when the cube is arranged according to the powers of one letter, we easily know by the leading term a^3, the first term a of the root, since the cube of it is a^3; if, therefore, we subtract that cube from the cube proposed, we obtain the remainder, $3a^2b + 3ab^2 + b^3$, which must furnish the second term of the root.

336. But as we already know, from Art. 333, that the second term is $+b$, we have principally to discover how it may be derived from the above remainder. Now that remainder may be expressed by two factors, thus $(3a^2+3ab+b^2)\times(b)$; if, therefore, we divide by $3a^2+3ab+b^2$, we obtain the second part of the root $+b$, which is required.

337. But as this second term is supposed to be unknown, the divisor also is unknown; nevertheless we have the first term of that divisor, which is sufficient; for it is $3a^2$, that is, thrice the square of the first term already found; and by means of this, it is not difficult to find also the other part, b, and then to complete the divisor before we perform the division; for this purpose, it will be necessary to join to $3a^2$ thrice the product of the two terms, or $3ab$, and b^2, or the square of the second term of the root.

338. Let us apply what we have said to two examples of other given cubes.

$$\begin{array}{rl} & a^3+12a^2+48a+64\ (a+4 \\ & a^3 \\ \hline 3a^2+12a+16) & 12a^2+48a+64 \\ & 12a^2+48a+64 \\ \hline & 0. \end{array}$$

$$
\begin{array}{rl}
 & a^6-6a^5+15a^4-20a^3+15a^2-6a+1\,(a^2-2a+1 \\
 & a^6 \\
 \hline
3a^4-6a^3+4a^2) & -6a^5+15a^4-20a^3 \\
 & -6a^5+12a^4-8a^3 \\
 \hline
3a^4-12a^3+12a^2+3a^2-6a+1) & 3a^4-12a^3+15a^2-6a+1 \\
 & 3a^4-12a^3+15a^2-6a+1 \\
 \hline
 & 0.
\end{array}
$$

339. The analysis which we have given is the foundation of the common rule for the extraction of the cube root in numbers; see the following example of the operation in the number 2197:

$$
\begin{array}{r|l}
 & \dot{2}197\dot{}(10+3=13 \\
 & 1000 \\
 \hline
300 & 1197 \\
90 & \\
9 & \\
\hline
399 & 1197 \\
\hline
 & 0.
\end{array}
$$

Let us also extract the cube root of 34965783:

$$
\begin{array}{r|l}
 & 34965783\,(300+20+7 \\
 & 27000000 \\
 \hline
270000 & 7965783 \\
18000 & \\
400 & \\
\hline
288400 & 5768000 \\
\hline
307200 & 2197783 \\
6720 & \\
49 & \\
\hline
313969 & 2197783 \\
\hline
 & 0.
\end{array}
$$

CHAP. X.

Of the higher Powers of Compound Quantities.

340. After squares and cubes, we must consider higher powers, or powers of a greater number of degrees; which are generally represented by exponents in the manner which we before explained: we have only to remember, when the root is compound, to enclose it in a parenthesis: thus $(a+b)^5$ means that $a+b$ is raised to the fifth power, and $(a-b)^6$ represents the sixth power of $a-b$, and so on. We shall in this chapter explain the nature of these powers.

341. Let $a+b$ be the root, or the first power, and the higher powers will be found by multiplication in the following manner:

$$
\begin{array}{rl}
(a+b)^1 = & a+b \\
 & a+b \\
\hline
 & a^2+ab \\
 & \quad +ab+b^2 \\
\hline
(a+b)^2 = & a^2+2ab+b^2 \\
 & a+b \\
\hline
 & a^3+2a^2b+ab^2 \\
 & \quad + a^2b+2ab^2+b^3 \\
\hline
(a+b)^3 = & a^3+3a^2b+3ab^2+b^3 \\
 & a+b \\
\hline
 & a^4+3a^3b+3a^2b^2+ab^3 \\
 & \quad + a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
 & a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{array}
$$

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$a+b$$

$$a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4$$
$$+a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5$$

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$
$$a+b$$

$$a^6+5a^5b+10a^4b^2+10a^3b^3+5a^2b^4+a^5b$$
$$+a^5b+5a^4b^2+10a^3b^3+10a^2b^4+5ab^5+b^6$$

$$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6, \text{ \&c.}$$

342. The powers of the root $a-b$ are found in the same manner; and we shall immediately perceive that they do not differ from the preceding, excepting that the 2d, 4th, 6th, &c. terms are affected by the sign *minus*.

$$(a-b)^1=a-b$$
$$a-b$$

$$a^2-ab$$
$$-ab+b^2$$

$$(a-b)^2=a^2-2ab+b^2$$
$$a-b$$

$$a^3-2a^2b+ab^2$$
$$-a^2b+2ab^2-b^3$$

$$(a-b)^3=a^3+3a^2b+3ab^2-b^3$$
$$a-b$$

$$a^4-3a^3b+3a^2b^2-ab^3$$
$$-a^3b+3a^2b^2-3ab^3+b^4$$

$$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$$
$$a-b$$

$$a^5-4a^4b+6a^3b^2-4a^2b^3+ab^4$$
$$-a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5$$

$$a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$$

$$
\begin{array}{l}
(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5 \\
\qquad\qquad a - b \\
\hline
\qquad\qquad a^6 - 5a^5b + 10a^4b^2 - 10a^3b^3 + 5a^2b^4 - ab^5 \\
\qquad\qquad\quad - a^5b + 5a^4b^2 - 10a^3b^3 + 10a^2b^4 - 5ab^5 + b^6 \\
\hline
(a-b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6, \text{ \&c.}
\end{array}
$$

Here we see that all the odd powers of b have the sign $-$, while the even powers retain the sign $+$; the reason of which is evident; for since $-b$ is a term of the root, the powers of that letter will ascend in the following series, $-b$, $+b^2$, $-b^3$, $+b^4$, $-b^5$, $+b^6$, &c. which clearly shows that the even powers must be affected by the sign $+$, and the odd ones by the contrary sign $-$.

343. An important question occurs in this place; namely, how we may find, without being obliged always to perform the same calculation, all the powers either of $a+b$, or $a-b$?

We must remark, in the first place, that if we can assign all the powers of $a+b$, those of $a-b$ are also found, since we have only to change the signs of the even terms, that is to say, of the second, the fourth, the sixth, &c. The business then is to establish a rule, by which any power of $a+b$, however high, may be determined without the necessity of calculating all the preceding ones.

344. Now if from the powers which we have already determined we ta[illegible]away the numbers that precede each term, w[illegible] called the *coefficients*, we observe in all t[illegible]a singular order; first, we see the first t[illegible] root raised to the power which is requir[illegible]ing terms the powers of a diminis[illegible]and the powers of b increas[illegible]; so that the sum

of the exponents of a and of b is always the same, and always equal to the exponent of the power required; and, lastly, we find the term b by itself raised to the same power: if therefore the tenth power of $a+b$ were required, we are certain that the terms, without their coefficients, would succeed each other in the following order; a^{10}, a^9b, a^8b^2, a^7b^3, a^6b^4, a^5b^5, a^4b^6, a^3b^7, a^2b^8, ab^9, b^{10}.

345. It remains therefore to show how we are to determine the coefficients which belong to those terms, or the numbers by which they are to be multiplied. Now, with respect to the first term, its coefficient is always unity; and, as to the second, its coefficient is constantly the exponent of the power; but with regard to the other terms, it is not so easy to observe any order in their coefficients; yet, if we continue those coefficients, we shall not fail to discover the law by which they are formed; as will appear from the following table.

Powers.	Coefficients.
1st	1, 1
2d	1, 2, 1
3d	1, 3, 3, 1
4th	1, 4, 6, 4, 1
5th	1, 5, 10, 10, 5, 1
6th	1, 6, 15, 20, 15, 6, 1
7th	1, 7, 21, 35, 35, 21, 7, 1
8th	1, 8, 28, 56, 70, 56, 28, 8, 1
9th	1, 9, 36, 84, 126, 126, 84, 36, 9, 1
10th	1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1

then that the tenth power of $a+b$ will be,
$b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5$
$+ 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$.

346. Now with regard to the coefficients, it must be observed, that for each power their sum must be equal to the number 2 raised to the same power; for let $a=1$ and $b=1$, then each term, without the coefficients, will be 1; consequently, the value of the power will be simply the sum of the coefficients; this sum, in the preceding example, is 1024, and accordingly $(1+1)^{10}=2^{10}=1024$; and it is the same with all other powers; thus we have for the

1st $1+1=2=2^1$,
2d $1+2+1=4=2^2$,
3d $1+3+3+1=8=2^3$,
4th $1+4+6+4+1=16=2^4$,
5th $1+5+10+10+5+1=32=2^5$,
6th $1+6+15+20+15+6+1=64=2^6$,
7th $1+7+21+35+35+21+7+1=128=2^7$.

347. Another necessary remark, with regard to the coefficients, is, that they increase from the beginning to the middle, and then decrease in the same order; and in the even powers, the greatest coefficient is exactly in the middle; but in the odd powers, two coefficients, equal and greater than the others, are found in the middle, belonging to the mean terms.

The order of the coefficients likewise deserves particular attention; for it is in this order that we discover the means of determining them for any power whatever, without calculating all the preceding powers. We shall here explain this method, reserving the demonstration however for the next chapter.

348. In order to find the coefficients of any power proposed, the seventh for example, let us write the following fractions one after the other:

$$\frac{7}{1}, \frac{6}{2}, \frac{5}{3}, \frac{4}{4}, \frac{3}{5}, \frac{2}{6}, \frac{1}{7}.$$

In this arrangement we perceive that the numerators begin by the exponent of the power required, and that they diminish successively by unity: while the denominators follow in the natural order of the numbers, 1, 2, 3, 4, &c. Now the first coefficient being always 1, the first fraction gives the second coefficient; the product of the two first fractions, multiplied together, represents the third coefficient; the product of the three first fractions represents the fourth coefficient, and so on. Thus the

1st coefficient is	1	$= 1$
2d - - - - -	$\frac{7}{1}$	$= 7$
3d - - - - -	$\frac{7.6}{1.2}$	$= 21$
4th - - - - -	$\frac{7.6.5}{1.2.3}$	$= 35$
5th - - - - -	$\frac{7.6.5.4}{1.2.3.4}$	$= 35$
6th - - - - -	$\frac{7.6.5.4.3}{1.2.3.4.5}$	$= 21$
7th - - - - -	$\frac{7.6.5.4.3.2}{1.2.3.4.5.6}$	$= 7$
8th - - - - -	$\frac{7.6.5.4.3.2.1}{1.2.3.4.5.6.7}$	$= 1$

349. And hence we readily find that the coefficients of the second power, or square, are 1, 2, and 1.

The third power furnishes the fractions $\frac{3}{1}, \frac{2}{2}, \frac{1}{3}$; wherefore the

1st coefficient $= 1$. 2d $= \frac{3}{1} = 3$.

3d $= 3 \cdot \frac{2}{2} = 3$. 4th $= \frac{3}{1} \cdot \frac{2}{2} \cdot \frac{1}{3} = 1$.

We have, likewise, for the fourth power, the fractions $\frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}$; hence the

1st coefficient $= 1$

2d $\frac{4}{1} = 4$. 3d $\frac{4}{1} \cdot \frac{3}{2} = 6$.

4th $\frac{4}{1} \cdot \frac{3}{2} \cdot \frac{2}{3} = 4$. 5th $\frac{4}{1} \cdot \frac{3}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} = 1$.

350. This rule evidently renders it unnecessary for us to find the coefficients of the preceding powers, as it enables us to discover immediately the coefficients which belong to any one proposed. Thus, for the tenth power, we write the fractions $\frac{10}{1}, \frac{9}{2}, \frac{8}{3}, \frac{7}{4}, \frac{6}{5}, \frac{5}{6}, \frac{4}{7}, \frac{3}{8}, \frac{2}{9}, \frac{1}{10}$, by means of which we find the

1st coefficient $= 1$,

2d $= \frac{10}{1} = 10$. 7th $= 252 \cdot \frac{5}{6} = 210$.

3d $= 10 \cdot \frac{9}{2} = 45$. 8th $= 210 \cdot \frac{4}{7} = 120$.

4th $= 45 \cdot \frac{8}{3} = 120$. 9th $= 120 \cdot \frac{3}{8} = 45$.

5th $= 120 \cdot \frac{7}{4} = 210$. 10th $= 45 \cdot \frac{2}{9} = 10$.

6th $= 210 \cdot \frac{6}{5} = 252$. 1th $= 10 \cdot \frac{1}{10} = 1$.

351. We may also write these fractions as they are, without computing their value; and in this manner it is easy to express any power of $a+b$. Thus, $(a+b)^{100}=a^{100}+\frac{100}{1}.a^{99}b+\frac{100.99}{1.2}+a^{98}b^2$ $+\frac{100.99.98}{1.2.3}a^{97}b^3+\frac{100.99.98.97}{1.2.3.4}a^{96}b^4+$, &c. Or, which is a more general mode of expression,

$$(a+b)^n=a^n+\frac{n}{1}a^{n-1}b+\frac{n.\overline{n-1}}{1.2}a^{n-2}b^2$$

$$+\frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}a^{n-3}b^3+\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1.2.3.4}$$

$$a^{n-4}b^4 \text{ \&c. } - - \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}----1}{1.2.3.4----n}.$$

This elegant theorem for the involution of a compound quantity of two terms, evidently includes all powers whatever; and we shall afterwards show how the same may be applied to the extraction of roots.

CHAP. XI.

Of the Transposition of the Letters, on which the demonstration of the preceding Rule is founded.

352. If we trace back the origin of the coefficients which we have been considering, we shall find, that each term is presented, as many times as it is possible to transpose the letters, of which that term is composed; or, to express the same thing differently, the coefficient of each term is equal to

the number of transpositions that the letters composing that term admit of. In the second power, for example, the term *ab* is taken twice, that is to say, its coefficient is 2; and in fact we may change the order of the letters which compose that term twice, since we may write *ab* and *ba*; the term *aa*, on the contrary, is found only once, and here the order of the letters can undergo no change, or transposition; in the third power of $a+b$, the term *aab* may be written in three different ways, thus *aab*, *aba*, *baa*; so likewise the coefficient is 3; in the fourth power, the term a^3b or *aaab* admits of four different arrangements, *aaab*, *aaba*, *abaa*, *baaa*; and its coefficient is also 4; the term *aabb* admits of six transpositions, *aabb*, *abba*, *baba*, *abab*, *bbaa*, *baab*, and its coefficient is 6; and so on for all other cases.

353. In fact, if we consider that the fourth power, for example, of any root consisting of more than two terms, as $(a+b+c+d)^4$, is found by the multiplication of the four factors, $(a+b+c+d)$ $(a+b+c+d)$ $(a+b+c+d)$ $(a+b+c+d)$, we readily see, that each letter of the first factor must be multiplied by each letter of the second, then by each letter of the third, and, lastly, by each letter of the fourth. So that every term is not only composed of four letters, but it also presents itself, or enters into the sum, as many times as those letters can be differently arranged with respect to each other, and hence arises its coefficient.

354. It is therefore of great importance to know, in how many different ways a given number of letters may be arranged; but, in this inquiry, we

must particularly consider, whether the letters in question are the same, or different; for when they are the same, there can be no transposition of them, and for this reason the simple powers, as a^2, a^3, a^4, &c. have all unity for their coefficients.

355. Let us first suppose all the letters different; and, beginning with the simplest case of two letters, or *ab*, we immediately discover that two transpositions may take place, namely, *ab* and *ba*.

If we have three letters, *abc*, to consider, we observe that each of the three may take the first place, while the two others will admit of two transpositions; thus if *a* be the first letter, we have two arrangements *abc*, *acb*; if *b* be in the first place, we have the arrangements *bac*, *bca*; lastly, if *c* occupy the first place, we have also two arrangements, namely *cab*, *cba*; consequently the whole number of arrangements is $3\times2=6$.

If there be four letters, *abcd*, each may occupy the first place; and in every case the three others may form six different arrangements, as we have just seen, therefore the whole number of transpositions is $4\times6=24=4\times3\times2\times1$.

If we have five letters, *abcde*, each of the five may be the first, and the four others will admit of twenty-four transpositions; so that the whole number of transpositions will be $5\times24=120=5\times4\times3\times2\times1$.

356. Consequently, however great the number of letters may be, it is evident, provided they are all different, that we may easily determine the number of transpositions, and that we may for this purpose make use of the following table:

Number of Letters.	Number of Transpositions.
1	- - 1=1.
2	- - - - 2 . 1=2.
3	- - 3 . 2 . 1=6.
4	- - - 4 . 3 . 2 . 1=24.
5	- - - 5 . 4 . 3 . 2 . 1=120.
6	- - 6 . 5 . 4 . 3 . 2 . 1=720.
7	- - 7 . 6 . 5 . 4 . 3 . 2 . 1=5040.
8	- 8 . 7 . 6 . 5 . 4 . 3 . 2 . 1=40320.
9	- 9 . 8 . 7 . 6 . 5 . 4 . 3 . 2 . 1=362880.
10	10 . 9 . 8 . 7 . 6 . 5 . 4 . 3 . 2 . 1=3628800.

357. But, as we have intimated, the numbers in this table can be made use of only when all the letters are different; for if two or more of them are alike, the number of transpositions becomes much less; and if all the letters are the same, we have only one arrangement; we shall therefore now show how the numbers in the table are to be diminished, according to the number of letters that are alike.

358. When two letters are given, and those letters are the same, the two arrangements are reduced to one, and consequently the number, which we have found above, is reduced to the half; that is to say, it must be divided by 2; if we have three letters alike, the six transpositions are reduced to one; whence it follows that the numbers in the table must be divided by 6=3 . 2 . 1; and for the same reason, if four letters are alike, we must divide the numbers found by 24, or 4 . 3 . 2 . 1, &c.

It is easy therefore to find how many transpositions the letters *aaabbc*, for example, may undergo. They are in number 6, and consequently, if they were all different, they would admit of 6 . 5 . 4 . 3 . 2 . 1 transpositions; but since a is found thrice in those letters, we must divide that number of transpositions by 3 . 2 . 1; and since b occurs twice, we must again divide it by 2 . 1; the number of transpositions required will therefore be $\frac{6 . 5 . 4 . 3 . 2 . 1}{3 . 2 . 1 . 2 . 1} = 5 . 4 . 3 = 60.$

359. We may now readily determine the coefficients of all the terms of any power; as for example of the seventh power $(a+b)^7$.

The first term is a^7, which occurs only once; and as all the other terms have each seven letters, it follows that the number of transpositions for each term would be 7 . 6 . 5 . 4 . 3 . 2 . 1, if all the letters were different; but since in the second term, a^6b, we find six letters alike, we must divide the above product by 6 . 5 . 4 . 3 . 2 . 1, whence it follows that the coefficient is $\frac{7 . 6 . 5 . 4 . 3 . 2 . 1}{6 . 5 . 4 . 3 . 2 . 1} = \frac{7}{1}.$

In the third term, a^5b^2, we find the same letter a five times, and the same letter b twice; we must therefore divide that number first by 5 . 4 . 3 . 2 . 1, and then by 2 . 1; whence results the coefficient $\frac{7 . 6 . 5 . 4 . 3 . 2 . 1}{5 . 4 . 3 . 2 . 1 . 2 . 1} = \frac{7 . 6}{1 . 2}.$

The fourth term a^4b^3 contains the letter a four times, and the letter b thrice; consequently, the whole number of the transpositions of the seven letters, must be divided, in the first place, by 4 . 3 .

2 . 1, and, secondly, by 3 . 2 . 1, and the coefficient becomes $=\frac{7 . 6 . 5 . 4 . 3 . 2 . 1}{4 . 3 . 2 . 1 . 3 . 2 . 1}=\frac{7 . 6 . 5}{1 . 2 . 3}$.

In the same manner, we find $\frac{7 . 6 . 5 . 4}{1 . 2 . 3 . 4}$ for the coefficient of the fifth term, and so of the rest; by which the rule before given is demonstrated *.

360. These considerations carry us farther, and show us also, how to find all the powers of roots

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* From the *Theory of Combinations,* also, are frequently deduced the rules that have just been considered for determining the coefficients of terms of the power of a binomial; and this is perhaps attended with some advantage, as the whole is then reduced to a single formula..

In order to perceive the difference between *permutations* and *combinations,* it may be observed, that in the former we enquire in how many different ways the letters, which compose a certain formula may change places; whereas, in combinations it is only necessary to know how many times these letters may be taken or multiplied together, one by one, two by two, three by three, &c.

Let us take the formula *abc;* here we know that the letters which compose it admit of six permutations, namely *abc, acb, bac, bca, cab, cba:* but as for combinations, it is evident that by taking these three letters one by one, we have three combinations, namely *a, b,* and *c;* if two by two, we have the three combinations, *ab, ac,* and *bc;* lastly, if we take them three by three, we have only the single combination *abc.*

Now, in the same manner as we prove that *n* different things admit of $1\times2\times3\times4\text{--}n$ different permutations, and that if *r* of these *n* things are equal, the number of permutations is $\frac{1\times2\times3\times4\text{--}n}{1\times2\times3\times\text{ --}r}$; so likewise we prove that *n* things may be taken *r* by *r*, $\frac{n\times(n-1)\times(n-2)\cdot(n-r+1)}{1\times2\times3\text{--}r}$ number of times; or that we composed of more than two terms *. We shall apply them to the third power of $a+b+c$; the terms of which must be formed by all the possible combinations of three letters, each term having for its coefficient the number of its transpositions, as above.

Here without performing the multiplication, the third power of $(a+b+c)$ will be, $a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2+3bc^2+c^3$.

Now suppose $a=1$, $b=1$, $c=1$, the cube of $1+1+1$, or of 3, will be $1+3+3+3+6+3+1+3+3+1=27$; which result is accurate, and confirms the rule. But if we had supposed $a=1$, $b=1$,

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may take r of these n things in so many different ways. Hence, if we call n the exponent of the power to which we wish to raise the binomial $a+b$, and r the exponent of the letter b in any term, the coefficient of that term is always expressed by the formula $\frac{n\times(n-1)\times(n-2)\cdots(n-r+1)}{1\times 2\times 3\cdots r}$. Thus, in the example, article 359, where $n=7$, we have a^5b^2 for the third term, the exponent $r=2$, and consequently the coefficient $=\frac{7\times 6}{1\times 2}$; for the fourth term we have $r=3$ and the coefficient $=\frac{7\times 6\times 5}{1\times 2\times 3}$, and so on; which are evidently the same results as the permutations.

For complete and exensive treatises on the theory of combinations, we are indebted to *Frenicle*, *De Montmort*, *James Bernoulli*, &c. The two last have investigated this theory, with a view to its great utility in the calculation of probabilities. F. T.

* Roots, or quantities, composed of more than two terms, are called *polynomials*, in order to distinguish them from *binomials*, or quantities composed of two terms. F. T.

and $c=-1$, we should have found for the cube of $1+1-1$, that is of 1,

$1+3-3+3-6+3+1-3+3-1=1$, which is a still further confirmation of the rule.

CHAP. XII.

Of the Expression of Irrational Powers by Infinite Series.

361. As we have shown the method of finding any power of the root $a+b$, however great the exponent may be, we are able to express, generally, the power of $a+b$, whose exponent is undetermined; for it is evident that if we represent that exponent by n, we shall have by the rule already given (art. 348 and the following):

$$(a+b)^n=a^n+\frac{n}{1}a^{n-1}b+\frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}a^{n-4}b^4+\text{ \&c.}$$

362. If the same power of the root $a-b$ were required, we need only change the signs of the second, fourth, sixth, &c. terms, and should have

$$(a-b)^n=a^n-\frac{n}{1}a^{n-1}b+\frac{n}{1}\cdot\frac{n-1}{2}a^{n-2}b^2-\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3+\frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}a^{n-4}b^4-\text{ \&c.}$$

363. These formulas are remarkably useful, since they serve also to express all kinds of radicals; for we have shown that all irrational quantities may assume the form of powers whose exponents are fractional, and that $\sqrt[2]{a}=a^{\frac{1}{2}}$, $\sqrt[3]{a}=a^{\frac{1}{3}}$, and $\sqrt[4]{a}=a^{\frac{1}{4}}$, &c.: we have, therefore,

$$\sqrt[2]{(a+b)}=(a+b)^{\frac{1}{2}};\ \sqrt[3]{(a+b)}=(a+b)^{\frac{1}{3}};$$
$$\text{and } \sqrt[4]{(a+b)}=(a+b)^{\frac{1}{4}},\ \&c.$$

Consequently, if we wish to find the square root of $a+b$, we have only to substitute for the exponent n the fraction $\frac{1}{2}$, in the general formula, art. 361, and we shall have first, for the coefficients,

$\frac{n}{1}=\frac{1}{2}$; $\frac{n-1}{2}=-\frac{1}{4}$; $\frac{n-2}{3}=-\frac{3}{6}$; $\frac{n-3}{4}=-\frac{5}{8}$; $\frac{n-4}{5}=-\frac{7}{10}$; $\frac{n-5}{6}=-\frac{9}{12}$. Then, $a^n=a^{\frac{1}{2}}=\sqrt{a}$ and $a^{n-1}=\frac{1}{\sqrt{a}}$; $a^{n-2}=\frac{1}{a\sqrt{a}}$; $a^{n-3}=\frac{1}{a^2\sqrt{a}}$, &c. or we might express those powers of a in the following manner: $a^n=\sqrt{a}$; $a^{n-1}=\frac{\sqrt{a}}{a}$; $a^{n-2}=\frac{a^n}{a^2}=\frac{\sqrt{a}}{a^2}$; $a^{n-3}=\frac{a^n}{a^3}=\frac{\sqrt{a}}{a^3}$; $a^{n-4}=\frac{a^n}{a^4}=\frac{\sqrt{a}}{a^4}$, &c.

364. This being laid down, the square root of $a+b$ may be expressed in the following manner:

$$\sqrt{(a+b)}=\sqrt{a}+\frac{1}{2}b\frac{\sqrt{a}}{a}-\frac{1}{2}\cdot\frac{1}{4}b^2\frac{\sqrt{a}}{aa}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}b^3\frac{\sqrt{a}}{a^3}$$
$$-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\cdot\frac{5}{8}b^4\frac{\sqrt{a}}{a^4},\ \&c.$$

365. If a therefore be a square number, we may

assign the value of $\sqrt{a}$, and, consequently, the square root of $a+b$ may be expressed by an infinite series, without any radical sign.

Let, for example, $a=c^2$, we shall have $\sqrt{a}=c$; then

$$\sqrt{(c^2+b)}=c+\frac{1}{2}\cdot\frac{b}{c}-\frac{1}{8}\cdot\frac{b^2}{c^3}+\frac{1}{16}\cdot\frac{b^3}{c^5}-\frac{5}{128}\cdot\frac{b^4}{c^7},$$
&c.

We see, therefore, that there is no number, whose square root we may not extract in this manner; since every number may be resolved into two parts, one of which is a square represented by c^2. If, for example, the square root of 6 be required, we make $6=4+2$, consequently $c^2=4$, $c=2$, $b=2$, whence results

$$\sqrt{6}=2+\frac{1}{2}-\frac{1}{16}+\frac{1}{64}-\frac{5}{1024}, \text{ \&c.}$$

If we take only the two leading terms of this series, we shall have $2\frac{1}{2}=\frac{5}{2}$, the square of which, $\frac{25}{4}$, is $\frac{1}{4}$ greater than 6; but if we consider three terms, we have $2\frac{7}{16}=\frac{39}{16}$, the square of which, $\frac{1521}{256}$, is still $\frac{15}{256}$ too small.

366. Since, in this example, $\frac{5}{2}$ approaches very nearly to the true value of $\sqrt{6}$, we shall take for 6 the equivalent quantity $\frac{25}{4}-\frac{1}{4}$; thus $c^2=\frac{25}{4}$; $c=$

$\frac{5}{2}$; $b=\frac{1}{4}$; and calculating only the two leading terms, we find $\sqrt{6}=\frac{5}{2}+\frac{1}{2}\cdot\frac{-\frac{1}{4}}{\frac{5}{2}}=\frac{5}{2}-\frac{1}{2}\cdot\frac{\frac{1}{4}}{\frac{5}{2}}=\frac{5}{2}-\frac{1}{20}=\frac{49}{20}$; the square of which fraction being $\frac{2401}{400}$, it exceeds the square of $\sqrt{6}$ only by $\frac{1}{400}$.

Now, making $6=\frac{2401}{400}-\frac{1}{400}$, so that $c=\frac{49}{20}$ and $b=-\frac{1}{400}$; and still taking only the two leading terms, we have $\sqrt{6}=\frac{49}{20}+\frac{1}{2}\cdot\frac{-\frac{1}{400}}{\frac{49}{20}}=\frac{49}{20}-\frac{1}{2}\cdot\frac{\frac{1}{400}}{\frac{49}{20}}=\frac{49}{20}-\frac{1}{1960}=\frac{4801}{1960}$, the square of which is $\frac{23049601}{3841600}$; and 6, when reduced to the same denominator, is $=\frac{23049600}{3841600}$; the error therefore is only $\frac{1}{3841600}$.

367. In the same manner, we may express the cube root of $a+b$ by an infinite series; for since $\sqrt[3]{(a+b)}=(a+b)^{\frac{1}{3}}$, we shall have in the general formula, $n=\frac{1}{3}$, and for the coefficients, $\frac{n}{1}=\frac{1}{3}$; $\frac{n-1}{2}=-\frac{1}{3}$; $\frac{n-2}{3}=-\frac{5}{9}$; $\frac{n-3}{4}=-\frac{2}{3}$; $\frac{n-4}{5}=-\frac{11}{15}$, &c. and with regard to the powers of a, we shall have $a^n=\sqrt[3]{a}$; $a^{n-1}=\frac{\sqrt[3]{a}}{a}$; $a^{n-2}=$

$\frac{\sqrt[3]{a}}{a^2}$; $a^{n-3} = \frac{\sqrt[3]{a}}{a^3}$, &c. then $\sqrt[3]{(a+b)} = \sqrt[3]{a} + \frac{1}{3} \cdot b\frac{\sqrt[3]{a}}{a} - \frac{1}{9} \cdot b^2\frac{\sqrt[3]{a}}{a^2} + \frac{5}{81} \cdot b^3\frac{\sqrt[3]{a}}{a^3} - \frac{10}{243} \cdot b^4\frac{\sqrt[3]{a}}{a^4}$, &c.

368. If a therefore be a cube, or $a = c^3$, we have $\sqrt[3]{a} = c$, and the radical signs will vanish; for we shall have

$$\sqrt[3]{(c^3+b)} = c + \frac{1}{3} \cdot \frac{b}{c^2} - \frac{1}{9} \cdot \frac{b^2}{c^5} + \frac{5}{81} \cdot \frac{b^3}{c^8} - \frac{10}{243} \cdot \frac{b^4}{c^{11}}$$

$+$, &c.

369. We have therefore arrived at a formula, which will enable us to find, *by approximation*, the cube root of any number; since every number may be resolved into two parts, as c^3+b, the first of which is a cube.

If we wish, for example, to determine the cube root of 2, we represent 2 by $1+1$, so that $c=1$ and $b=1$, consequently $\sqrt[3]{2} = 1 + \frac{1}{3} - \frac{1}{9} + \frac{5}{81}$, &c. The two leading terms of this series make $1\frac{1}{3} = \frac{4}{3}$, the cube of which $\frac{64}{27}$ is too great by $\frac{10}{27}$: let us therefore make $2 = \frac{64}{27} - \frac{10}{27}$, we have $c = \frac{4}{3}$ and $b = -\frac{10}{27}$, and consequently $\sqrt[3]{2} = \frac{4}{3} + \frac{1}{3} \cdot \frac{-\frac{10}{27}}{\frac{16}{9}}$: these two terms give $\frac{4}{3} - \frac{5}{72} = \frac{91}{72}$, the cube of which is $\frac{753571}{373248}$: but, $2 = \frac{746496}{373248}$, so that the error is $\frac{7075}{373248}$; and in this way we might still approximate, the faster in proportion as we take a greater number of terms *.

CHAP. XIII.

Of the Resolution of Negative Powers.

370. We have already shown, that $\frac{1}{a}$ may be expressed by a^{-1}; we may therefore also express $\frac{1}{a+b}$ by $(a+b)^{-1}$; so that the fraction $\frac{1}{a+b}$ may be considered as a power of $a+b$, namely, that power whose exponent is -1; from which it follows, that the series already found as the value of $(a+b)^n$ extends also to this case.

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* In the Philosophical Transactions for 1694, Dr. Halley has given a very elegant and general method for extracting roots of any degree whatever by approximation; where he demonstrates this general formula,

$$\sqrt[m]{a^m \pm b} = \frac{m-2}{m-1}a + \sqrt{\frac{a^2}{(m-1)^2} \pm \frac{2b}{(m^2-m)a^{m-1}}}.$$

Those who have not an opportunity of consulting the Philosophical Transactions, will find the formation and the use of this formula explained in the new edition of Leçons Elementaires de Mathematiques by M. L'Abbé de la Caille, published by M. L'Abbé Marie. F. T. See also Dr. Hutton's Dictionary.

371. Since, therefore, $\frac{1}{a+b}$ is the same as $(a+b)^{-1}$, let us suppose, in the general formula, $n=-1$; and we shall first have for the coefficients $\frac{n}{1}=-1$; $\frac{n-1}{2}=-1$; $\frac{n-2}{3}=-1$; $\frac{n-3}{4}=-1$, &c. and for the powers of a we have $a^n=a^{-1}=\frac{1}{a}$; $a^{n-1}=a^{-2}=\frac{1}{a^2}$; $a^{n-2}=\frac{1}{a^3}$; $a^{n-3}=\frac{1}{a^4}$ &c.: so that $(a+b)^{-1}=\frac{1}{a+b}=\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}-\frac{b^5}{a^6}$, &c. which is the same series that we found before by division.

372. Farther, $\frac{1}{(a+b)^2}$ being the same with $(a+b)^{-2}$, let us reduce this quantity also to an infinite series: for this purpose we must suppose $n=-2$, and we shall thus have for the coefficients $\frac{n}{1}=-\frac{2}{1}$; $\frac{n-1}{2}=-\frac{3}{2}$; $\frac{n-2}{3}=-\frac{4}{3}$; $\frac{n-3}{4}=-\frac{5}{4}$, &c.; and for the powers of a we obtain $a^n=\frac{1}{a^2}$; $a^{n-1}=\frac{1}{a^3}$; $a^{n-2}=\frac{1}{a^4}$; $a^{n-3}=\frac{1}{a^5}$, &c.: we have therefore $(a+b)^{-2}=\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2.b}{1.a^3}+\frac{2.3.b^2}{1.2.a^4}-\frac{2.3.4b^3}{1.2.3a^5}+\frac{2.3.4.5.b^4}{1.2.3.4.a^6}$, &c. Now, $\frac{2}{1}=2$; $\frac{2.3}{1.2}=3$; $\frac{2.3.4}{1.2.3}=4$; $\frac{2.3.4.5}{1.2.3.4}=5$, &c. and consequently,

$$\frac{1}{(a+b)^2}=\frac{1}{a^2}-2\frac{b}{a^3}+3\frac{b^2}{a^4}-4\frac{b^3}{a^5}5+\frac{b^4}{a^6}-6\frac{b^5}{a^7}+7\frac{b^6}{a^8},$$

&c.

373. Let us proceed and suppose $n = -3$, and we shall have a series expressing the value of $\frac{1}{(a+b)^3}$, or of $(a+b)^{-3}$. Here the coefficients will be $\frac{n}{1} = -\frac{3}{1}$; $\frac{n-1}{2} = -\frac{4}{2}$; $\frac{n-2}{3} = -\frac{5}{3}$, &c. and the powers of a become, $a^n = \frac{1}{a^3}$; $a^{n-1} = \frac{1}{a^4}$; $a^{n-2} = \frac{1}{a^5}$, &c. which gives

$$\frac{1}{(a+b)^3} = \frac{1}{a^3} - \frac{3.b}{1.a^4} + \frac{3.4.b^2}{1.2.a^5} - \frac{3.4.5b^3}{1.2.3a^6} + \frac{3.4.5.6b}{1.2.3.4a^7} =$$

$$\frac{1}{a^3} - 3\frac{b}{a^4} + 6\frac{b^2}{a^5} - 10\frac{b^3}{a^6} + 15\frac{b^4}{a^7} - 21\frac{b^5}{a^8} + 28\frac{b^6}{a^9}, \text{ \&c.}$$

If now we make $n = -4$; we shall have for the coefficients $\frac{n}{1} = -\frac{4}{1}$; $\frac{n-1}{2} = -\frac{5}{2}$; $\frac{n-2}{3} = -\frac{6}{3}$; $\frac{n-3}{4} = -\frac{7}{4}$, &c. and for the powers, $a^n = \frac{1}{a^4}$; $a^{n-1} = \frac{1}{a^5}$; $a^{n-2} = \frac{1}{a^6}$; $a^{n-3} = \frac{1}{a^7}$; $a^{n-4} = \frac{1}{a^8}$, whence we obtain,

$$\frac{1}{(a+b)^4} = \frac{1}{a^4} - \frac{4b}{1a^5} + \frac{4.5.b^2}{1.2.a^6} - \frac{4.5.6b^3}{1.2.3a^7}, \text{ \&c.}$$

$$= \frac{1}{a^4} - 4\frac{b}{a^5} + 10\frac{b^2}{a^6} - 20\frac{b^3}{a^7} + 35\frac{b^4}{a^8} - 56\frac{b^5}{a^9} +, \text{ \&c.}$$

374. The different cases that have been considered enable us to conclude with certainty, that we shall have, generally, for any negative power of $a+b$;

$$\frac{1}{(a+b)^m} = \frac{1}{a^m} - \frac{mb}{a^{m+1}} + \frac{m.\overline{m+1}.b^2}{2.a^{m+2}} - \frac{m.\overline{m+1}.\overline{m+2}b^3}{2.3.a^{m+3}},$$

&c. And, by means of this formula, we may transform all such fractions into infinite series, substituting fractions also, or fractional exponents, for m, in order to express irrational quantities.

375. The following considerations will illustrate this subject still farther: for we have seen that,

$$\frac{1}{a+b}=\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a}-\frac{b^5}{a^6}+, \&c.$$

If, therefore, we multiply this series by $a+b$, the product ought to give 1; and this is found to be true, as will be seen by performing the multiplication:

$$\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}-\frac{b^5}{a^6}+, \&c.$$
$$a+b$$

$$1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+\frac{b^4}{a^4}-\frac{b^5}{a^5}+, \&c.$$
$$+\frac{b}{a}-\frac{b^2}{a^2}+\frac{b^3}{a^3}-\frac{b^4}{a^4}+\frac{b^5}{a^5}-, \&c.$$

where all the terms but the first cancel each other.

376. We have also found, that

$$\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}-\frac{6b^5}{a^7}, \&c.$$

if, therefore, we multiply this series by $(a+b)^2$, the product ought also to be equal to 1; now $(a+b)^2=a^2+2ab+b^2$, and

$$\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}-\frac{6b^5}{a^7}+a, \&c.$$
$$a^2+2ab+b^2$$

$$1-\frac{2b}{a}+\frac{3b^2}{a^2}-\frac{4b^3}{a^3}+\frac{5b^4}{a^4}-\frac{6b^5}{a^5}+, \&c.$$
$$+\frac{2b}{a}-\frac{4b^2}{a^2}+\frac{6b^3}{a^3}-\frac{8b^4}{a^4}+\frac{10b^5}{a^5}-, \&c.$$
$$+\frac{b^2}{a^2}-\frac{2b^3}{a^3}+\frac{3b^4}{a^4}-\frac{4b^5}{a^5}+, \&c.$$

which gives 1 for the product as the nature of the thing required.

377. If we multiply the series which we found for the value of $\frac{1}{(a+b)^2}$, by $a+b$ only, the product ought to answer to the fraction $\frac{1}{a+b}$, or be equal to the series already found, namely, $\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}$, &c. and this the actual multiplication will confirm.

$$\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}, \text{ \&c.}$$

$$a+b$$

$$\frac{1}{a}-\frac{2b}{a^2}+\frac{3b^2}{a^3}-\frac{4b^3}{a^4}+\frac{5b^4}{a^5}, \text{ \&c.}$$

$$+\frac{b}{a^2}-\frac{2b^2}{a^3}+\frac{3b^3}{a^4}-\frac{4b^4}{a^5}, \text{ \&c.}$$

$$\frac{1}{a}-\frac{b}{a^2}+\frac{b^2}{a^3}-\frac{b^3}{a^4}+\frac{b^4}{a^5}-, \text{ \&c. as required.}$$

SECTION III.

Of Ratios and Proportions.

CHAP. I.

Of Arithmetical Ratio, or of the Difference between two Numbers.

378. Two quantities are either equal to one another, or they are not. In the latter case, where one is greater than the other, we may consider their inequality under two different points of view: we may ask, *how much* one of the quantities is greater than the other? Or we may ask, *how many times* the one is greater than the other? The results which constitute the answers to these two questions are both called *relations*, or *ratios*; but we call the former an *arithmetical ratio*, and the latter a *geometrical ratio*, without however these denominations having any connexion with the subject itself, the adoption of the expressions having been entirely arbitrary.

379. It is evident, that the quantities of which we speak must be of one and the same kind; otherwise we could not determine any thing with regard to their equality or inequality; for it would be absurd to ask if two pounds and three ells are equal quantities. So that in what follows, quantities of the same kind only are to be considered; and as they may always be expressed by numbers, it is of num-

bers only that we shall treat, as was mentioned at the beginning.

380. When of two given numbers, therefore, it is required how much one is greater than the other, the answer to this question determines the arithmetical ratio of the two numbers; and since this answer consists in giving the difference of the two numbers, it follows, that an arithmetical ratio is nothing but the *difference* between two numbers; and as this appears to be a better expression, we shall reserve the words *ratio* and *relation* to express geometrical ratios.

381. As the difference between two numbers is found by subtracting the less from the greater, nothing can be easier than resolving the question how much one is greater than the other; so that when the numbers are equal, the difference being nothing, if it be required how much one of the numbers is greater than the other, we answer, by nothing; for example, 6 being equal to 2×3, the difference between 6 and 2×3 is 0.

382. But when the two numbers are not equal, as 5 and 3, and it is required how much 5 is greater than 3, the answer is, 2; which is obtained by subtracting 3 from 5; likewise 15 is greater than 5 by 10; and 20 exceeds 8 by 12.

383. We have therefore three things to consider on this subject; 1st. the greater of the two numbers; 2d. the less; and 3d. the difference; which three quantities are so connected together, that any two of the three being given, we may always determine the third.

Let the greater number be a, the less b, and the difference d; then d will be found by subtracting b

from a, so that $d=a-b$; whence we see how to find d, when a and b are given.

384. But if the difference and the less of the two numbers, that is, if d and b were given, we might determine the greater number by adding together the difference and the less number, which gives $a=b+d$; for if we take from $b+d$ the less number b, there remains d, which is the known difference: suppose, for example, the less number is 12, and the difference 8, then the greater number will be 20.

385. Lastly, if beside the difference d, the greater number a be given, the other number b is found by subtracting the difference from the greater number, which gives $b=a-d$; for if the number $a-d$ be taken from the greater number a, there remains d, which is the given difference.

386. The connexion, therefore, among the numbers a, b, d, is of such a nature as to give the three following results: 1st. $d=a-b$; 2d. $a=b+d$; 3d. $b=a-d$; and if one of these three comparisons be just, the others must necessarily be so also: therefore, generally, if $z=x+y$, it necessarily follows, that $y=z-x$, and $x=z-y$.

387. With regard to these arithmetical ratios we must remark, that if we add to the two numbers a and b any number c, assumed at pleasure, or subtract it from them, the difference remains the same; that is, if d is the difference between a and b, that number d will also be the difference between $a+c$ and $b+c$, and between $a-c$ and $b-c$; thus, for example, the difference between the numbers 20 and 12 being 8, that difference will remain the same, whatever number we add to or subtract from the numbers 20 and 12.

388. The proof of which is evident: for if $a-b=d$ we have also $(a+c)-(b+c)=d$; and likewise $(a-c)-(b-c)=d$.

389. And if we double the two numbers a and b, the difference will also become double; thus, when $a-b=d$, we shall have $2a-2b=2d$; and, generally, $na-nb=nd$, whatever value we give to n.

CHAP. II.

Of Arithmetical Proportion.

390. When two arithmetical ratios, or relations, are equal, this equality is called an *arithmetical proportion*.

Thus, when $a-b=d$ and $p-q=d$, so that the difference is the same between the numbers p and q as between the numbers a and b, we say that these four numbers form an arithmetical proportion; which we write thus, $a-b=p-q$, expressing clearly by this, that the difference between a and b is equal to the difference between p and q.

391. An arithmetical proportion consists therefore of four terms, which must be such, that if we subtract the second from the first, the remainder is the same as when we subtract the fourth from the third; thus, the four numbers 12, 7, 9, 4, form an arithmetical proportion, because $12-7=9-4$.

392. When we have an arithmetical proportion, as $a-b=p-q$, we may make the second and third terms change places, writing $a-p=b-q$: and this equality will be no less true; for, since $a-b=p-q$, add b to both sides, and we have $a=b+p-q$: then subtract p from both sides, and we have $a-p=b-q$.

In the same manner, as $12-7=9-4$, so also $12-9=7-4$.

393. We may in every arithmetical proportion put the second term also in the place of the first, if we make the same transposition of the third and fourth; that is, if $a-b=p-q$, we have also $b-a=q-p$; for $b-a$ is the negative of $a-b$, and $q-p$ is also the negative of $p-q$; and thus, since $12-7=2-4$, we have also, $7-12=4-9$.

394. But the most interesting property of every arithmetical proportion is this, that the sum of the second and third term is always equal to the sum of the first and fourth. This property, which we must particularly consider, is expressed also by saying that the sum of the *means* is equal to the sum of the *extremes*; thus, since $12-7=9-4$, we have $7+9=12+4$; the sum being in both cases 16.

395. In order to demonstrate this principal property, let $a-b=p-q$; then if we add to both $b+q$, we have $a+q=b+p$; that is, the sum of the first and fourth terms is equal to the sum of the second and third: and inversely, if four numbers, a, b, p, q, are such that the sum of the second and third is equal to the sum of the first and fourth, that is, if $b+p=a+q$, we conclude, without a possibility of mistake, that those numbers are in arithmetical proportion, and that $a-b=p-q$; for, since $a+q=b+p$, if we

subtract from both sides $b+q$, we obtain $a-b=p-q$.

Thus the numbers 18, 13, 15, 10, being such that the sum of the means ($13+15=28$) is equal to the sum of the extremes ($18+10=28$), it is certain that they also form an arithmetical proportion; and consequently, that $18-13=15-10$.

396. It is easy, by means of this property, to resolve the following question. The three first terms of an arithmetical proportion being given, to find the fourth? Let a, b, p, be the three first terms, and let us express the fourth by q, which it is required to determine, then $a+q=b+p$; by subtracting a from both sides, we obtain $q=b+p-a$.

Thus the fourth term is found by adding together the second and third, and subtracting the first from that sum. Suppose, for example, that 19, 28, 13, are the three first given terms, the sum of the second and third is 41; and taking from it the first, which is 19, there remains 22 for the fourth term sought, and the arithmetical proportion will be represented by $19-28=13-22$, or by $28-19=22-13$, or, lastly, by $28-22=19-13$.

397. When in an arithmetical proportion the second term is equal to the third, we have only three numbers; the property of which is this, that the first, *minus* the second, is equal to the second, *minus* the third; or that the difference between the first and second number is equal to the difference between the second and third; the three numbers 19, 15, 11, are of this kind, since $19-15=15-11$.

398. Three such numbers are said to form a con-

tinued arithmetical proportion, which is sometimes written thus, 19 : 15 : 11. Such proportions are also called *arithmetical progressions*, particularly if a greater number of terms follow each other according to the same law.

An arithmetical progression may be either *increasing*, or *decreasing*; the former distinction is applied when the terms go on increasing, that is to say, when the second exceeds the first, and the third exceeds the second by the same quantity; as in the numbers 4, 7, 10; and the decreasing progression is that in which the terms go on always diminishing by the same quantity, such as the numbers 9, 5, 1.

399. Let us suppose the numbers a, b, c, to be in arithmetical progression; then $a-b=b-c$, whence it follows, from the equality between the sum of the extremes and that of the means, that $2b=a+c$; and if we subtract a from both, we have $2b-a=c$.

400. So that when the two first terms a, b, of an arithmetical progression are given, the third is found by taking the first from twice the second. Let 1 and 3 be the two first terms of an arithmetical progression, the third will be $2\times3-1=5$; and these three numbers 1, 3, 5, give the proportion

$$1-3=3-5.$$

401. By following the same method, we may pursue the arithmetical progression as far as we please; we have only to find the fourth term by means of the second and third, in the same manner as we determined the third by means of the first and second, and so on. Let a be the first term, and b the second, the third will be $2b-a$, the fourth $4b-2a-b=$

$3b-2a$, the fifth $6b-4a-2b+a=4b-3a$, the sixth $8b-6a-3b+2a=5b-4a$, the seventh $10b-8a-4b+3a=6b-5a$, &c.

CHAP. III.

Of Arithmetical Progressions.

402. We have already remarked, that a series of numbers composed of any number of terms, which always increase, or decrease, by the same quantity, is called an *arithmetical progression.*

Thus, the natural numbers written in their order, as 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. form an arithmetical progression, because they constantly increase by unity; and the series 25, 22, 19, 16, 13, 10, 7, 4, 1, &c. is also such a progression, since the numbers constantly decrease by 3.

403. The number, or quantity, by which the terms of an arithmetical progression become greater or less, is called the *difference;* so that when the first term and the difference are given, we may continue the arithmetical progression to any length.

For example, let the first term be 2, and the difference 3, and we shall have the following increasing progression: 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, &c. in which each term is found by adding the difference to the preceding one.

404. It is usual to write the natural numbers, 1,

2, 3, 4, 5, &c. above the terms of such an arithmetical progression, in order that we may immediately perceive the rank which any term holds in the progression, which numbers, when written above the terms, are called *indices;* thus the above example will be written as follows:

Indices.	1	2	3	4	5	6	7	8	9	10	
Arith. Prog.	2,	5,	8,	11,	14,	17,	20,	23,	26,	29,	&c.

where we see that 29 is the tenth term.

405. Let a be the first term, and d the difference, the arithmetical progression will go on in the following order:

1	2	3	4	5	6	7	
a,	$a \pm d$,	$a \pm 2d$,	$a \pm 3d$,	$a \pm 4d$,	$a \pm 5d$,	$a \pm 6d$,	&c.

according as the series is increasing or decreasing, whence it appears that any term of the progression might be easily found, without the necessity of finding all the preceding ones, by means only of the first term a and the difference d; thus, for example, the tenth term will be $a \pm 9d$, the hundredth term $a \pm 99d$, and, generally, the nth term will be $a \pm (n-1)d$.

406. When we stop at any point of the progression, it is of importance to attend to the first and the last term, since the index of the last will represent the number of terms; if, therefore, the first term be a, the difference d, and the number of terms n, we shall have for the last term $a \pm (n-1)d$, according as the series is increasing or decreasing, which is consequently found by multiplying the difference by the number of terms *minus* one, and adding or subtracting that product from the first term. Suppose, for example, in an ascending arithmetical progression of a hundred terms, the first term is 4,

and the difference 3; then the last term will be $99 \times 3 + 4 = 301$.

407. When we know the first term a and the last z, with the number of terms n, we can find the difference d; for, since the last term $z = a \pm (n-1)d$, if we subtract a from both sides, we obtain $z \backsim a = (n-1)d$. So that by taking the difference between the first and last term, we have the product of the difference multiplied by the number of terms *minus* 1; we have therefore only to divide $z \backsim a$ by $n-1$ in order to obtain the required value of the difference d, which will be $\frac{z \backsim a}{n-1}$; and this result furnishes the following rule: Divide the difference of the first and last term by the number of terms *minus* 1, and the quotient will be the common difference: by means of which we may write the whole progression.

408. Suppose, for example, that we have an increasing arithmetical progression of nine terms, whose first is 2 and last 26, and that it is required to find the difference; here we must subtract the first term 2 from the last 26, and divide the remainder, which is 24, by $9-1$, that is by 8; and the quotient 3 will be equal to the difference required, and the whole progression will be:

1	2	3	4	5	6	7	8	9
2,	5,	8,	11,	14,	17,	20,	23,	26.

To give another example, let us suppose that the first term is 1, the last 2, the number of terms 10, and that the arithmetical progression, answering to these suppositions, is required; we shall immediately have for the difference $\frac{2-1}{10-1}=\frac{1}{9}$, and thence conclude that the progression is:

1	2	3	4	5	6	7	8	9	10
1,	$1\frac{1}{9}$,	$1\frac{2}{9}$,	$1\frac{3}{9}$,	$1\frac{4}{9}$,	$1\frac{5}{9}$,	$1\frac{6}{9}$,	$1\frac{7}{9}$,	$1\frac{8}{9}$,	2.

Let now the first term be $2\frac{1}{3}$, the last term $12\frac{1}{2}$, and the number of terms 7; the difference will be $\frac{12\frac{1}{2}-2\frac{1}{3}}{7-1}=\frac{10\frac{1}{6}}{6}=\frac{61}{36}=1\frac{25}{36}$, and consequently the progression:

1	2	3	4	5	6	7
$2\frac{1}{3}$,	$4\frac{1}{36}$,	$5\frac{13}{18}$,	$7\frac{5}{12}$,	$9\frac{1}{9}$,	$10\frac{29}{36}$,	$12\frac{1}{2}$.

409. If now the first term a, the last term z, and the difference d, are given, we may from them find the number of terms n; for since $z \backsim a=(n-1)d$, by dividing both sides by d, we have $\frac{z \backsim a}{d}=n-1$; also n being greater by 1 than $n-1$, we have

$n=\frac{z \backsim a}{d}+1$; consequently the number of terms is found by dividing the difference between the first and the last term, or $z \backsim a$, by the difference of the progression, and adding unity to the quotient.

For example, let the first term be 4, the last 100, and the difference 12, the number of terms will be $\frac{100-4}{12}+1=9$; and these nine terms will be,

1	2	3	4	5	6	7	8	9
4,	16,	28,	40,	52,	64,	76,	88,	100.

If the first term be 2, the last 6, and the difference $\frac{1}{3}$, the number of terms will be $\frac{4}{1\frac{1}{3}}+1=4$; and these four terms will be,

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2, & 3\frac{1}{3}, & 4\frac{2}{3}, & 6. \end{array}$$

Again, let the first term be $3\frac{1}{3}$, the last $7\frac{2}{3}$, and the difference $1\frac{4}{9}$, the number of terms will be $\frac{7\frac{2}{3}-3\frac{1}{3}}{1\frac{4}{9}}+1=4$; which are,

$$3\tfrac{1}{3}, \quad 4\tfrac{7}{9}, \quad 6\tfrac{2}{9}, \quad 7\tfrac{2}{3}.$$

410. It must be observed, however, that as the number of terms is necessarily an integer, if we had not obtained such a number for n, in the examples of the preceding article, the questions would have been absurd.

Whenever we do not obtain an integer number for the value of $\frac{z \backsim a}{d}$, it will be impossible to resolve the question; and, consequently, in order that questions of this kind may be possible, $z \backsim a$ must be divisible by d.

411. From what has been said, it may be concluded, that we have always four quantities, or things, to consider in an arithmetical progression:

1st The first term a.

2d The last term z.

3d The difference d.

4th The number of terms n.

And the relations of these quantities to each other are such, that if we know three of them, we are able to determine the fourth; for,

1. If a, d, and n, are known, we have $z=a\pm(n-1)d$.

2. If z, d, and n, are known, we have

$$a = z \backsim (n-1)d.$$

3. If a, z, and n, are known, we have $d = \frac{z \backsim a}{n-1}$.

4. If a, z, and d, are known, we have $n = \frac{z \backsim a}{d} + 1$.

CHAP. IV.

Of the Summation of Arithmetical Progressions.

412. It is often necessary also to find the sum of an arithmetical progression; which might be done by adding all the terms together; but as the addition would be very tedious when the progression consisted of a great number of terms, a rule has been devised by which the sum may be more readily obtained.

413. We shall first consider a particular given progression, such that the first term is 2, the difference 3, the last term 29, and the number of terms 10;

1	2	3	4	5	6	7	8	9	10
2,	5,	8,	11,	14,	17,	20,	23,	26,	29.

In this progression we see that the sum of the first and last term is 31; the sum of the second and the last but one 31; the sum of the third and the last but two 31, and so on; and thence we conclude that the sum of any two terms equally distant, the one from the first, and the other from the last term, is always equal to the sum of the first and the last term.

414. The reason of this may be easily traced; for if we suppose the first to be a, the last z, and the difference d, the sum of the first and the last term is $a+z$; and the second term being $a+d$, and the last but one $z-d$, the sum of these two terms is also $a+z$. Farther, the third term being $a+2d$, and the last but two $z-2d$, it is evident that these two terms also, when added together, make $a+z$; and the demonstration may be easily extended to any other two terms equally distant from the first and last.

415. To determine, therefore, the sum of the progression proposed, let us write the same progression term by term, inverted, and add the corresponding terms together, as follows:

$$\begin{array}{l} 2+5+8+11+14+17+20+23+26+29 \\ 29+26+23+20+17+14+11+8+5+2 \\ \hline 31+31+31+31+31+31+31+31+31+31 \end{array}$$

This series of equal terms is evidently equal to twice the sum of the given progression: now the number of those equal terms is 10, as in the progression, and their sum consequently is equal to $10\times31=310$. Hence as this sum is twice the sum of the arithmetical progression, the sum required must be 155.

416. If we proceed in the same manner with respect to any arithmetical progression, the first term of which is a, the last z, and the number of terms n; writing under the given progression the same progression inverted, and adding term to term, we shall have a series of n terms, each of which will be ex-

pressed by $a+z$; therefore the sum of this series will be $n(a+z)$, which is twice the sum of the proposed arithmetical progression; the latter, therefore, will be represented by $\frac{n(a+z)}{2}$.

417. This result furnishes an easy method of finding the sum of any arithmetical progression; and may be reduced to the following rule:

Multiply the sum of the first and the last term by the number of terms, and half the product will be the sum of the whole progression. Or, which amounts to the same, multiply the sum of the first and the last term by half the number of terms. Or, multiply half the sum of the first and the last term by the whole number of terms.

418. It will be necessary to illustrate this rule by some examples.

First, let it be required to find the sum of the progression of the natural numbers, 1, 2, 3, &c. to 100. This will be, by the first rule, $\frac{100\times101}{2}=50\times101=5050$.

If it were required to tell how many strokes a clock strikes in twelve hours; we must add together the numbers 1, 2, 3, as far as 12; now this sum is found immediately to be $\frac{12\times13}{2}=6\times13=78$. If we wished to know the sum of the same progression continued to 1000, we should find it to be 500500; and the sum of this progression, continued to 10000, would be 50005000.

419. Suppose a person buys a horse, on condition that for the first nail he shall pay 5 pence, for the

second 8 pence, for the third 11 pence, and so on, always increasing 3 pence more for each nail, the whole number of which is 32; required the purchase of the horse?

In this question it is required to find the sum of an arithmetical progression, the first term of which is 5, the difference 3, and the number of terms 32; we must therefore begin by determining the last term; which is found by the rule in articles 406 and 411 to be $5+\overline{31\times 3}=98$; after which the sum required is easily found to be $\frac{103\times 32}{2}=103\times 16$; whence we conclude that the horse costs 1648 pence, or 6*l*. 17*s*. 4*d*.

420. Generally, let the first term be a, the difference d, and the number of terms n; and let it be required to find, by means of these data, the sum of the whole progression. As the last term must be $a\pm(n-1)d$, the sum of the first and the last will be $2a\pm(n-1)d$; and multiplying this sum by the number of terms n, we have $2na\pm n(n-1)d$; the sum required therefore will be $na\pm\frac{n(n-1)d}{2}$.

Now this formula, if applied to the preceding example, or to $a=5$, $d=3$, and $n=32$, gives $5\times 32+\frac{32\cdot 31\cdot 3}{2}=160+1488=1648$; the same sum that we obtained before.

421. If it be required to add together all the natural numbers from 1 to n, we have, for finding this sum, the first term 1, the last term n, and the number of terms n; therefore the sum required is $\frac{n^2+n}{2}$

$=\frac{n(n+1)}{2}$. If we make $n=1766$, the sum of all the numbers, from 1 to 1766, will be $883\times1767=1560261$.

422. Let the progression of uneven numbers be proposed, 1, 3, 5, 7, &c. continued to n terms, and let the sum of it be required. Here the first term is 1, the difference 2, the number of terms n; the last term will therefore be $1+(n-1)2=2n-1$, and consequently the sum required n^2.

The whole therefore consists in multiplying the number of terms by itself; so that whatever number of terms of this progression we add together, the sum will be always a square, namely, the square of the number of terms; which we shall exemplify as follows:

Indices,	1	2	3	4	5	6	7	8	9	10	&c.
Progress.	1,	3,	5,	7,	9,	11,	13,	15,	17,	19,	&c.
Sum,	1,	4,	9,	16,	25,	36,	49,	64,	81,	100,	&c.

423. Let the first term be 1, the difference 3, and the number of terms n; we shall have the progression 1, 4, 7, 10, &c. the last term of which will be $1+(n-1)3=3n-2$; wherefore the sum of the first and the last term is $3n-1$, and consequently the sum of this progression is equal to $\frac{n(3n-1)}{2}=\frac{3n^2-n}{2}$; and if we suppose $n=20$, the sum will be $10\times59=590$.

424. Again, let the first term be 1, the difference d, and the number of terms n; then the last term will be $1+(n-1)d$; to which adding the first, we have $2+(n-1)d$, and multiplying by the number of

terms, we have $2n+n(n-1)d$; whence we deduce the sum of the progression $n+\frac{n(n-1)d}{2}$.

And by making d successively equal to 1, 2, 3, 4, &c., we obtain the following particular values, as shown in the table below.

If $d=1$, the sum is $n+\frac{n(n-1)}{2}=\frac{n^2+n}{2}$

$d=2$, - - - $n+\frac{2n(n-1)}{2}=n^2$

$d=3$, - - - $n+\frac{3n(n-1)}{2}=\frac{3n^2-n}{2}$

$d=4$, - - - $n+\frac{4n(n-1)}{2}=2n^2-n$

$d=5$, - - - $n+\frac{5n(n-1)}{2}=\frac{5n^2-3n}{2}$

$d=6$, - - - $n+\frac{6n(n-1)}{2}=3n^2-2n$

$d=7$, - - - $n+\frac{7n(n-1)}{2}=\frac{7n^2-5n}{2}$

$d=8$, - - - $n+\frac{8n(n-1)}{2}=4n^2-3n$

$d=9$, - - - $n+\frac{9n(n-1)}{2}=\frac{9n^2-7n}{2}$

$d=10$, - - - $n+\frac{10n(n-1)}{2}=5n^2-4n$

CHAP. V.

Of Figurate, or Polygonal Numbers.*

425. The summation of arithmetical progressions, which begin by 1, and the difference of which is 1,

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* The French translator has justly observed, in his note at the conclusion of this chapter, that algebraists make a distinction between figurate and polygonal numbers, but as he has not entered far upon this subject, the following illustration may not be unacceptable.

It will be immediately perceived in the following table, that each series is derived immediately from the foregoing one, being the sum of all its terms from the beginning to that place, and hence also the law of continuation, and the general term of each series, will be readily discovered.

Natural	1, 2, 3, 4, 5 - - -	n general term
Triangular	1, 3, 6, 10, 15 - - -	$\frac{n.\overline{n+1}}{2}$
Pyramidal	1, 4, 10, 20, 35 - - -	$\frac{n.\overline{n+1}.\overline{n+2}}{2.3}$
Triangular-pyramidal	1, 5, 15, 35, 70 - - -	$\frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}}{2.3.4}$

And in general the figurate number of any order m will be expressed by the formula

$$\frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}\;.\;-\;-\;\overline{n+m-1}}{1.2\;.\;3\;.\;4\;-\;-\;m}.$$

Now one of the principal properties of these numbers, and which Fermat considered as very interesting, (*see his notes on Diophantus, page* 16), is this: that if from the nth term of any series the $\overline{n-1}$ term of the same series be subtracted, the remainder will be the nth term of the preceding series. Thus, in the third series above given, the nth term is $\frac{n.\overline{n+1}.\overline{n+2}}{2.3}$; and

2, 3, or any other integer, leads us to the theory of *polygonal numbers*, which are formed by adding together the terms of any such progression.

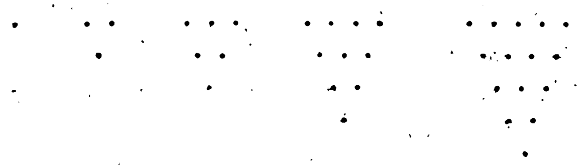
426. Suppose the difference to be 1; then, since the first term is 1 also, we shall have the arithmetical progression, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c. and if in this progression we take the sum of one, of two, of three, &c. terms, the following series of numbers will arise:

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, \&c.$$

for $1=1$, $1+2=3$, $1+2+3=6$, $1+2+3+4=10$, &c.

Which numbers are called *triangular*, or *trigonal* numbers, because we may always arrange as many points in the form of a triangle as they contain units, thus:

1 3 6 10 15



427. In all these triangles we see how many points each side contains. In the first triangle there is only one point; in the second there are two; in the third there are three; in the fourth there are

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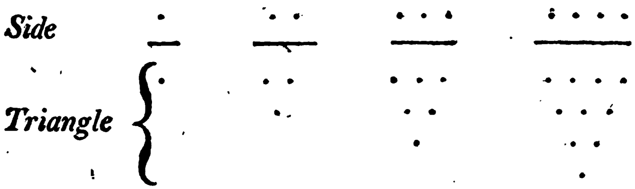
consequently the $\overline{n-1}$ term, by substituting $\overline{n-1}$ instead of n, is $\frac{\overline{n-1}\,.\,n\,.\,\overline{n+1}}{2.3}$; and if the latter be subtracted from the former, the remainder is $\frac{n.\overline{n-1}}{2}$, which is the nth term of the preceding order of numbers. And exactly the same law will be observed between two consecutive terms of any one of these sums.

four, &c.: so that the triangular numbers, or the number of points, which is simply called the *triangle*, are arranged according to the number of points that the side contains, which number is called the *side*; that is, the third triangular number, or the third triangle, is that whose side has three points; the fourth, that whose side has four; and so on; which may be represented thus:



428. A question therefore presents itself here, which is, how to determine the triangle when the side is given? and, after what has been said, this may be easily resolved.

For if the side be n, the triangle will be $1+2+3+4+\cdots n$. Now the sum of this progression is $\frac{n^2+n}{2}$; consequently the value of the triangle is $\frac{n^2+n}{2}$ *.

Thus if $\left\{\begin{array}{l} n=1, \\ n=2, \\ n=3, \\ n=4, \end{array}\right\}$ the triangle is $\left\{\begin{array}{l} 1, \\ 3, \\ 6, \\ 10, \end{array}\right.$

and so on: and when $n=100$, the triangle will be 5050.

* M. de Joncourt published at the Hague, in 1762, a table of trigonal numbers answering to all the natural numbers from 1 to 20000; which tables are found useful in facilitating a great number of arithmetical operations, as the author shows in a very long introduction. F. T.

429. This formula $\frac{n^2+n}{2}$ is called the general formula of triangular numbers; because by it we find the triangular number, or the triangle, which answers to any side indicated by n.

This may be transformed into $\frac{n(n+1)}{2}$; which serves also to facilitate the calculation; since one of the two numbers n, or $n+1$, is always an even number, and consequently divisible by 2.

So, if $n=12$, the triangle is $\frac{12\times13}{2}=6\times13=78$; and if $n=15$, the triangle is $\frac{15\times16}{2}=15\times8=120$, &c.

430. Let us now suppose the difference to be 2, and we shall have the following arithmetical progression:

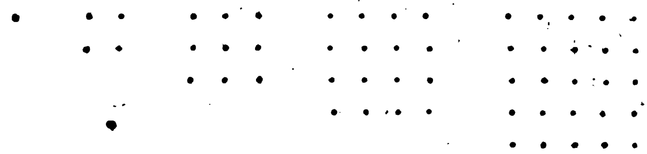
1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, &c.

the sums of which, taking successively one, two, three, four terms, &c. form the following series:

1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, &c.

the terms of which are called *quadrangular* numbers, or *squares*; since they in fact represent the squares of the natural numbers, as we found them before; and this denomination is the more suitable from this circumstance, that we can always form a square with the number of points which those terms indicate, thus:

1, 4, 9, 16, 25,



431. We see here, that the side of any square contains precisely the number of points which the square root indicates; thus, for example, the side of the square 16 consists of 4 points; that of the square 25 consists of 5 points; and, in general, if the side be n, that is, if the number of the terms of the progression, 1, 3, 5, 7, &c. which we have taken, be expressed by n, the square, or the quadrangular number, will be equal to the sum of those terms, that is to n^2, as we have already seen, Article 422; but it is unnecessary to extend our consideration of square numbers any farther, having already treated of them at length.

432. If now we call the difference 3, and take the sums in the same manner as before, we obtain numbers which are called *pentagons*, or *pentagonal* numbers, though they cannot be so well represented by points *.

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* It is not, however, that we are unable to represent, by points, polygons of any number of sides; but the rule which I am going to explain for this purpose, seems to have escaped all the writers on algebra whom I have consulted.

I begin with drawing a small polygon that has the number of sides required; this number remains constant for one and the same series of polygonal numbers, and it is equal to 2 *plus* the difference of the arithmetical progression from which the series is produced. I then choose one of its angles, in order to draw from the angular point all the diagonals of this polygon, which, with the two sides containing the angle that has been taken, are to be indefinitely produced; after that, I take these two sides, and the diagonals of the first polygon on the indefinite lines, each as often as I choose; and draw, from the corresponding points marked by the compass, lines parallel to the sides of the first polygon; and divide them into as many equal parts, or by as

Indices,	1	2	3	4	5	6	7	8	9	&c.
Arith. prog.	1,	4,	7,	10,	13,	16,	19,	22,	25,	&c.
Pentagon,	1,	5,	12,	22,	35,	51,	70,	92,	117,	&c.

the indices showing the side of each pentagon.

433. It follows from this, that if we make the side n, the pentagonal number will be $\frac{3n^2-n}{2}=\frac{n(3n-1)}{2}$.

Let, for example, $n=7$, the pentagon will be 70; and if the pentagon, whose side is 100, be required, we make $n=100$, and obtain 14950 for the number sought.

434. If we suppose the difference to be 4, we arrive at *hexagonal* numbers, as we see by the following progressions:

Indices,	1	2	3	4	5	6	7	8	9	&c.
Arithm. prog.	1,	5,	9,	13,	17,	21,	25,	29,	33,	&c.
Hexagon,	1,	6,	15,	28,	45,	66,	91,	120,	153,	&c.

where the indices still show the side of each hexagon.

435. So that when the side is n, the hexagonal number is $2n^2-n=n(2n-1)$; and we have farther to remark, that all the hexagonal numbers are also triangular; since, if we take of these last the first,

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many points as there are actually in the diagonals and the two sides produced. This rule is general, from the triangle up to the polygon of an infinite number of sides: and the division of these figures into triangles might furnish matter for many curious considerations, and for elegant transformations of the general formulæ, by which the polygonal numbers are expressed in this chapter; but it is unnecessary to dwell on them at present. F. T.

The ingenuity of the reader will readily lead him to the formation of the figures above described.

the third, the fifth, &c. we have precisely the series of hexagons.

436. In the same manner we may find the numbers which are heptagonal, octagonal, &c.; and thus it is easy to construct the following table of formulæ for all numbers that are comprehended under the general name of *polygonal* numbers.

Supposing the side to be represented by n, we have for the

triangle $\frac{n^2+n}{2}=\frac{n(n+1)}{2}$.

square - $\frac{2n^2+0n}{2}=n^2$.

V gon - $\frac{3n^2-n}{2}=\frac{n(3n-1)}{2}$.

VI gon - $\frac{4n^2-2n}{2}=2n^2-n=n(2n-1)$.

VII gon $\frac{5n^2-3n}{2}=\frac{n(5n-3)}{2}$.

VIII gon $\frac{6n^2-4n}{2}=3n^2-2n=n(3n-2)$.

IX gon - $\frac{7n^2-5n}{2}=\frac{n(7n-5)}{2}$.

X gon - $\frac{8n^2-6n}{2}=4n^2-3n=n(4n-3)$.

XI gon - $\frac{9n^2-7n}{2}=\frac{n(9n-7)}{2}$.

XII gon $\frac{10n^2-8n}{2}=5n^2-4n=n(5n-4)$.

XX gon - $\frac{18n^2-16n}{2}=9n^2-8n=n(9n-8)$.

xxv gon $\frac{23n^2-21n}{2}=\frac{n(23n-21)}{2}$.

m gon - $\frac{(m-2)n^2-(m-4)n}{2}$*.

437. So that the side being n, the m gonal number will be generally represented by $\frac{(m-2)n^2-(m-4)n}{2}$; whence we may deduce all the possible polygonal numbers which have the side n: thus, for example, if the bigonal numbers were required, we should have $m=2$, and consequently the number sought $=n$; that is to say, the bigonal numbers are the natural numbers, 1, 2, 3, &c.

If we make $m=3$, we have $\frac{n^2+n}{2}$ for the triangular number required.

If we make $m=4$, we have the square number n^2, &c.

438. To illustrate this rule by examples, suppose that the xxv gonal number, whose side is 36, were required; we look first in the table for the xxv gonal number, whose side is n, and it is found to be

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* The general expression for the $m$gonal number is easily derived from the summation of an arithmetical progression, whose first term is 1, common difference d, and number of terms n; as in the following series; viz. $1+(1+d)+(1+2d)+$, &c. $(1+\overline{n-1}.d)$, the sum of which is expressed by $\frac{(2+\overline{n-1}.d)n}{2}$; but in all cases $d=m-2$, therefore substituting this value for d, the expression becomes $\frac{2n+(n^2-n).(m-2)}{2}=\frac{(m-2)n^2-(m-4)n}{2}$ as in the formula. ED.

$\frac{23n^2 - 21n}{2}$. Then, making $n = 36$, we find 14526 for the number sought.

439. Suppose, for example, that a person bought a house, and being asked how much he paid for it, he answers, that the 365th gonal number of 12 is the number of crowns which it cost him.

In order therefore to find this number, we make $m = 365$, and $n = 12$; and substituting these values in the general formula, we find for the price of the house 23970 crowns*.

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* This chapter is intitled "Of Figurate or Polygonal Numbers." It is not however without foundation that some algebraists make a distinction between *figurate* numbers and *polygonal* numbers. For the numbers commonly called *figurate* are all derived from a single arithmetical progression, and each series of numbers is formed from it by adding together the terms of the series which goes before. On the other hand, every series of *polygonal* numbers is produced from a different arithmetical progression. Hence, in strictness, we cannot speak of a single series of figurate numbers, as being at the same time a series of polygonal numbers. This will be made more evident by the following tables.

TABLE OF FIGURATE NUMBERS.

Constant numbers - - -	1.	1.	1.	1.	1.	1.	&c.
Natural - - - - - - -	1.	2.	3.	4.	5.	6.	&c.
Triangular - - - - -	1.	3.	6.	10.	15.	21.	&c.
Pyramidal - - - - -	1.	4.	10.	20.	35.	56.	&c.
Triangular-pyramidal - -	1.	5.	15.	35.	70.	126.	&c.

TABLE OF POLYGONAL NUMBERS.

Diff. of the progr.	Numbers						
1	triangular	1.	3.	6.	10.	15.	&c.
2	square - -	1.	4.	9.	16.	25.	&c.
3	pentagon -	1.	5.	12.	22.	35.	&c.
4	hexagon -	1.	6.	15.	28.	45.	&c.

Powers likewise form particular series of numbers. The two

CHAP. VI.

Of Geometrical Ratio.

440. The *geometrical ratio* of two numbers is found by resolving this question, *How many times* is one of those numbers greater than the other? which is done by dividing one by the other; and the quotient will express the ratio required.

441. We have here therefore three things to consider; 1st, the first of the two given numbers, which is called the *antecedent;* 2dly, the other number, which is called the *consequent;* 3dly, the ratio of the two numbers, or the quotient arising from the division of the antecedent by the consequent; thus, for example, if the relation of the numbers 18 and 12 be required, 18 is the antecedent, 12 is the consequent;

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first are to be found among the figurate numbers, and the third among the polygonal; which will appear by successively substituting for a the numbers 1, 2, 3, &c.

TABLE OF POWERS.

a^0 - - - - -	1.	1.	1.	1.	1.	&c.
a^1 - - - - -	1.	2.	3.	4.	5.	&c.
a^2 - - - - -	1.	4.	9.	16.	25.	&c.
a^3 - - - - -	1.	8.	27.	64.	125.	&c.
a^4 - - - - -	1.	16.	81.	256.	625.	&c.

The algebraists of the sixteenth and seventeenth centuries paid great attention to these different kinds of numbers and their mutual connexion, and they discovered in them a great variety of curious properties; but as their utility is not great, they are now seldom introduced into the systems of mathematics. F. T.

and the ratio will be $\frac{18}{12} = 1\frac{1}{2}$; whence we see that the antecedent contains the consequent one and a half times.

442. It is usual to represent geometrical relation by two points, placed one above the other, between the antecedent and the consequent; thus $a:b$ means the geometrical relation of these two numbers, or the ratio of a to b.

We have already remarked that this sign is employed to represent division*, and for this reason we make use of it here; because, in order to know the ratio, we must divide a by b; the relation expressed by this sign being read simply, *a is to b*.

443. Relation therefore is expressed by a fraction whose numerator is the antecedent, and whose denominator is the consequent; but perspicuity requires that this fraction should be always reduced to its lowest terms; which is done, as we have already shown, by dividing both the numerator and denominator by their greatest common divisor: thus the fraction $\frac{18}{12}$ becomes $\frac{3}{2}$, by dividing both terms by 6.

444. So that relations only differ according as their ratios are different; and there are as many different kinds of geometrical relations as we can conceive different ratios.

The first kind is undoubtedly that in which the ratio becomes unity; this case happens when the two numbers are equal, as in $3:3::4:4::a:a$; the ratio is here 1, and for this reason we call it the relation of equality.

* It will be observed that in the present edition we have made use of the symbol $\div$ for division, as is now usually done in books on this subject.

Next follow those relations in which the ratio is another whole number; thus 4:2 the ratio is 2, and is called *double* ratio; 12:4 the ratio is 3, and is called *triple* ratio; 24:6 the ratio is 4, and is called *quadruple* ratio, &c.

We may next consider those relations whose ratios are expressed by fractions; such as 12:9, where the ratio is $\frac{4}{3}$ or $1\frac{1}{3}$; and 18:27, where the ratio is $\frac{2}{3}$, &c. We may also distinguish those relations in which the consequent contains exactly twice, thrice, &c. the antecedent: such are the relations 6:12, 5:15, &c. the ratio of which some call *subduple*, *subtriple*, &c. ratios.

Farther, we call that ratio *rational* which is an expressible number; the antecedent and consequent being integers, such as 11:7, 8:15, &c. and we call that an *irrational* or *surd* ratio, which can neither be exactly expressed by integers, nor by fractions, such as $\sqrt{5}:8$, or $4:\sqrt{3}$.

445. Let a be the antecedent, b the consequent, and d the ratio, we know already that a and b being given, we find $d=\frac{a}{b}$: if the consequent b were given with the ratio, we should find the antecedent $a=bd$, because bd divided by b gives d: and lastly, when the antecedent a is given, and the ratio d, we find the consequent $b=\frac{a}{d}$; for, dividing the antecedent a by the consequent $\frac{a}{d}$, we obtain the quotient d, that is to say, the ratio.

446. Every relation $a:b$ remains the same, if we multiply or divide the antecedent and consequent by

the same number, because the ratio is the same: thus, for example, let d be the ratio of $a:b$, we have $d=\frac{a}{b}$; now the ratio of the relation $na:nb$ is also $\frac{na}{nb}=d$, and that of the relation $\frac{a}{n}:\frac{b}{n}$ is likewise $\frac{na}{nb}=d$.

447. When a ratio has been reduced to its lowest terms, it is easy to perceive and enunciate the relation: for example, when the ratio $\frac{a}{b}$ has been reduced to the fraction $\frac{p}{q}$, we say $a:b=p:q$, or $a:b::p:q$, which is read, a is to b as p is to q: thus, the ratio of 6:3 being $\frac{2}{1}$, or 2, we say 6:3::2:1; we have likewise 18:12::3:2, and 24:18::4:3, and 30:45 ::2:3, &c.; but if the ratio cannot be abridged, the relation is already expressed in its simplest form; for we do not simplify the relation by saying 9:7::9:7.

448. On the other hand, we may sometimes change the relation of two very great numbers into one that shall be more simple and evident, by reducing both to their lowest terms; thus, for example, we can say 28844:14422::2:1; or, 10566:7044 ::3:2; or, 57600:25200::16:7.

449. In order, therefore, to express any relation in the clearest manner, it is necessary to reduce it to the smallest possible numbers; which is easily done, by dividing the two terms by their greatest common divisor; thus, to reduce the relation 57600:25200 to that of 16:7, we have only to perform the single

operation of dividing the numbers 57600 and 25200 by 3600, which is their greatest common divisor.

450. It is important, therefore, to know how to find the greatest common divisor of two given numbers; but this requires a rule, which we shall explain in the following chapter.

CHAP. VII.

Of the Greatest Common Divisor of two given Numbers.

451. There are some numbers which have no other common divisor than unity, and when the numerator and denominator of a fraction are of this nature, it cannot be reduced to a more convenient form*. The two numbers 48 and 35, for example, have no common divisor, though each has its own divisors; for which reason we cannot express the relation 48:35 more simply, because the division of two numbers by 1 does not diminish them.

452. But when the two numbers have a common divisor, it is found, and even the greatest which they have, by the following rule:

Divide the greater of the two numbers by the less;

* In this case, the two numbers are said to be prime to each other.

next, divide the preceding divisor by the remainder; what remains in this second division will afterwards become a divisor for a third division, in which the remainder of the preceding divisor will be the dividend; which operation must be continued till we arrive at a division that leaves no remainder; and this last divisor will be the greatest common divisor of the two given numbers.

Thus for the two numbers 576 and 252.

```
252) 576 (2
     504
     ---
      72) 252 (3
          216
          ---
           36) 72 (2
               72
               --
                0.
```

So that in this instance the greatest common divisor is 36.

453. It will be proper to illustrate this rule by some other examples; and for this purpose let the greatest common divisor of the numbers 504 and 312 be required.

```
312) 504 (1
     312
     ---
     192) 312 (1
          192
          ---
          120) 192 (1
               120
               ---
                72
```

```
72) 120 (1
    72
    --
    48) 72 (1
        48
        --
        24) 48 (2
            48
            --
             0.
```

So that 24 is the greatest common divisor, and consequently the relation 504 : 312 is reduced to the form 21 : 13.

454. Let the relation 625 : 529 be given, and the greatest common divisor of these two numbers be required.

```
529) 625 (1
     529
     ---
     96) 529 (5
         480
         ---
         49) 96 (1
             49
             --
             47) 49 (1
                 47
                 --
                 2) 47 (23
                    46
                    --
                    1) 2 (2
                       2
                       -
                       0.
```

Wherefore 1 is, in this case, the greatest common divisor, and consequently we cannot express the relation $625:529$ by less numbers, nor reduce it to simpler terms.

455. It may be necessary, in this place, to give a demonstration of the foregoing rule; and in order to this, let a be the greater, and b the less of the given numbers; and let d be one of their common divisors; then it is evident that a and b being divisible by d, we may also divide the quantities $a-b$, $a-2b$, $a-3b$, and, in general, $a-nb$ by d.

456. The converse is no less true: that is, if the numbers b and $a-nb$ are divisible by d, the number a will also be divisible by d; for nb being divisible by d, we could not divide $a-nb$ by d, if a were not also divisible by d.

457. We observe farther, that if d be the *greatest* common divisor of two numbers, b and $a-nb$, it will also be the greatest common divisor of the two numbers a and b; for if a greater common divisor than d could be found for these numbers a and b, that number would also be a common divisor of b and $a-nb$; and consequently d would not be the greatest common divisor of these two numbers; but we have supposed d the greatest divisor common to b and $a-nb$; therefore d must also be the greatest common divisor of a and b.

458. These things being laid down, let us divide, according to the rule, the greater number a by the less b; and let us suppose the quotient to be n; then the remainder will be $a-nb$, which must necessarily be less than b; and this remainder $a-nb$ having the same greatest common divisor with b, as

the given numbers a and b, we have only to repeat the division, dividing the preceding divisor b by the remainder $a-nb$; and the new remainder which we obtain will still have, with the preceding divisor, the same greatest common divisor, and so on.

459. We proceed in the same manner till we arrive at a division without a remainder; that is, in which the remainder is nothing; let therefore p be the last divisor, contained exactly a certain number of times in its dividend; and this dividend will evidently be divisible by p, and will have the form mp; so that the numbers p and mp are both divisible by p; and it is also evident that they have no greater common divisor, because no number can actually be divided by a number greater than itself; consequently this last divisor is also the greatest common divisor of the given numbers a and b.

460. We may now give another example of the same rule, requiring the greatest common divisor of the numbers 1728 and 2304: the operation is as follows:

```
1728) 2304 (1
      1728
      ----
       576) 1728 (3
            1728
            ----
               0.
```

Hence it follows that 576 is the greatest common divisor, and that the relation 1728:2304 is reduced to 3:4; that is to say, 1728 is to 2304 the same as 3 is to 4.

CHAP. VIII.

Of Geometrical Proportions.

461. Two geometrical relations are equal when their ratios are equal; and this equality of two relations is called a *geometrical proportion*; thus, for example, we write $a:b=c:d$, or $a:b::c:d$, to indicate that the relation $a:b$ is equal to the relation $c:d$; but this is more simply expressed by saying a is to b as c to d; as in the following proportion, $8:4::12:6$; when the ratio of the relation $8:4$ is $\frac{2}{1}$, which is also the ratio of the relation $12:6$.

462. So that $a:b::c:d$ being a geometrical proportion, the ratio must be the same on both sides, consequently $\frac{a}{b}=\frac{c}{d}$; and reciprocally, if the fractions $\frac{a}{b}=\frac{c}{d}$, we have $a:b::c:d$.

463. A geometrical proportion consists therefore of four terms, such that the first divided by the second gives the same quotient as the third divided by the fourth; and hence we deduce an important property, common to all geometrical proportion, which is, that the product of the first and the last term is always equal to the product of the second and third; or, more simply, that the product of the extremes is equal to the product of the means.

464. In order to demonstrate this property, let us take the geometrical proportion $a:b::c:d$, so that

$\frac{a}{b}=\frac{c}{d}$. Here if we multiply both these fractions by b we obtain $a=\frac{bc}{d}$, and multiplying both sides farther by d, we have $ad=bc$; but ad is the product of the extreme terms, and bc is that of the means, which two products are found to be equal.

465. Reciprocally if the four numbers a, b, c, d, are such that the product of the two extremes a and d is equal to the product of the two means b and c, we are certain that they form a geometrical proportion; for, since $ad=bc$, we have only to divide both sides by bd, which gives us $\frac{ad}{bd}=\frac{bc}{bd}$, or $\frac{a}{b}=\frac{c}{d}$, and consequently $a:b::c:d$.

466. The four terms of a geometrical proportion, as $a:b::c:d$, may be transposed in different ways, without destroying the proportion; for the rule being always, that the product of the extremes is equal to the product of the means, or $ad=bc$, we may say,

1st. $b:a::d:c$; 2dly. $a:c::b:d$;
3dly. $d:b::c:a$; 4thly. $d:c::b:a$.

467. Beside these four geometrical proportions, we may deduce some others from the same proportion, $a:b::c:d$; for we may say, $a+b:a::c+d:c$, or the first term *plus* the second is to the first as the third *plus* the fourth is to the third; that is, $a+b:a::c+d:c$.

We may farther say, the first *minus* the second is to the first as the third *minus* the fourth is to the third, or $a-b:a::c-d:c$. For if we take the product of the extremes and the means, we have

$ac-bc=ac-ad$, which evidently leads to the equality $ad=bc$.

And in the same manner we may demonstrate that $a+b:b::c+d:d$; and that $a-b:b::c-d:d$.

468. All the proportions which we have deduced from $a:b::c:d$ may be represented generally as follows:

$$ma+nb:pa+qb::mc+nd:pc+qd.$$

For the product of the extreme terms is $mpac+npbc+mqad+nqbd$; which, since $ad=bc$, becomes $mpac+npbc+mqbc+nqbd$; also the product of the mean terms is $mpac+mqbc+npad+nqbd$; or, since $ad=bc$, it is $mpac+mqbc+npbc+nqbd$; so that the two products are equal.

469. It is evident, therefore, that a geometrical proportion being given, for example, 6:3 :: 10:5, an infinite number of others may be deduced from it; we shall however give only a few:

3:6::5:10;	6:10::3:5;	9:6::15:10;
3:3::5: 5;	9:15::3:5;	9:3::15: 5.

470. Since in every geometrical proportion the product of the extremes is equal to theproduct of the means, we may, when the three first terms are known, find the fourth from them; thus let the three first terms be 24:15::40 to the fourth term: here, as the product of the means is 600, the fourth term multiplied by the first, that is by 24, must also make 600; consequently by dividing 600 by 24 the quotient 25 will be the fourth term required, and the whole proportion will be 24:15::40:25. In general, therefore, if the three first terms are $a:b::c$; we put d for the unknown fourth letter; and since $ad=bc$, we

divide both sides by a, and have $d=\frac{bc}{a}$; so that the fourth term is $\frac{bc}{a}$, which is found by multiplying the second term by the third, and dividing that product by the first.

471. This is the foundation of the celebrated *Rule of Three* in arithmetic; for in that rule we suppose three numbers given, and seek a fourth, which is in geometrical proportion with those three; so that the first may be to the second as the third is to the fourth.

472. But here it will be necessary to pay attention to some particular circumstances.

First, if in two proportions the first and the third terms are the same, as in $a:b::c:d$, and $a:f::c:g$, then the two second and the two fourth terms will also be in geometrical proportion, so that $b:d::f:g$; for the first proportion being transformed into this, $a:c::b:d$, and the second into this, $a:c::f:g$, it follows that the relations $b:d$ and $f:g$ are equal, since each of them is equal to the relation $a:c$; thus, for example, if $5:100::2:40$, and $5:15::2:6$, we must have $100:40::15:6$.

473. But if the two proportions are such that the mean terms are the same in both, I say that the first terms will be in an inverse proportion to the fourth terms; that is, if $a:b::c:d$, and $f:b::c:g$, it follows that $a:f::g:d$. Let the proportions be, for example, $24:8::9:3$, and $6:8::9:12$, we have $24:6::12:3$; the reason is evident; for the first proportion gives $ad=bc$; and the second gives $fg=bc$; therefore $ad=fg$, and $a:f::g:d$, or $a:g::f:d$.

474. Two proportions being given, we may always produce a new one by separately multiplying the first term of the one by the first term of the other, the second by the second, and so on with respect to the other terms; thus the proportions $a:b::c:d$ and $e:f::g:h$ will furnish this, $ae:bf::cg:dh$; for the first giving $ad=bc$, and the second giving $eh=fg$, we have also $adeh=bcfg$; but now $adeh$ is the product of the extremes, and $bcfg$ is the product of the means in the new proportion; so that the two products being equal, the proportion is true.

475. Let now the two proportions be $6:4::15:10$ and $9:12::15:20$, their combination will give the proportion $6 . 9:4 . 12::15 . 15:10 . 20$,

$$\text{or } 54:48::225:200,$$
$$\text{or } 9:8::9:8.$$

476. We shall observe, lastly, that if two products are equal, $ad=bc$, we may reciprocally convert this equality into a geometrical proportion; for we shall always have one of the factors of the first product, in the same proportion to one of the factors of the second product, as the other factor of the second product is to the other factor of the first product: that is, in the present case, $a:c::b:d$, or $a:b::c:d$. Let $3\times8=4\times6$, and we may form from it this proportion, $8:4::6:3$, or this, $3:4::6:8$; likewise, if $3\times5=1\times15$, we shall have $3:15::1:5$, or $5:1::15:3$, or $3:1::15:5$.

CHAP. IX.

Observations on the Rules of Proportion and their Utility.

477. This theory is so useful in the common occurrences of life, that scarcely any person can do without it; there being always proportion between prices and commodities; and when different kinds of money are the subject of exchange, the whole consists in determining their mutual relations; and the examples furnished by these reflections will be very proper for illustrating the principles of proportion, and showing their utility by the application of them.

478. If we wished to know, for example, the relation between two kinds of money; suppose an old *louis d'or* and a *ducat*: we must first know the value of those pieces when compared with others of the same kind; thus, an old louis being, at Berlin, worth 5 rixdollars and 8 drachms, and a ducat being worth 3 rixdollars, we may reduce these two values to one denomination; either to rixdollars, which gives the proportion $1L:1D::5\frac{1}{3}R:3R.$ or $::16:9$; or to drachms, in which case we have $1L:1D::128:72::16:9$; which proportions evidently give the true relation of the old louis to the ducat; for the equality of the products of the extremes and the means gives, in both cases, 9 louis = 16 ducats; and, by means of this comparison, we may change any sum of old louis into ducats, and vice-versa. Thus, suppose it were

required to find how many ducats there are in 1000 old louis, we have this proportion:

Lou. Lou. Duc. Duc.

As 9 : 1000 :: 16 : $1777\frac{7}{9}$, the number sought.

If, on the contrary, it were required to find how many old louis d'or there are in 1000 ducats, we have the following proportion:

Duc. Duc. Lou.

As 16 : 1000 :: 9 : $562\frac{1}{2}$ louis. Ans.

479. At Petersburgh the value of the ducat varies, and depends on the course of exchange; which course determines the value of the ruble in stivers, or Dutch halfpence, 105 of which make a ducat. So that when the exchange is at 45 stivers per ruble, we have this proportion:

As 45 : 105 :: 3 : 7;

and hence this equality, 7 rubles = 3 ducats.

Hence again we shall find the value of a ducat in rubles; for

Du. Du. Ru.

As 3 : 1 :: 7 : $2\frac{1}{3}$ rubles;

that is, 1 ducat is equal to $2\frac{1}{3}$ rubles.

But if the exchange were at 50 stivers, the proportion would be,

As 50 : 105 :: 10 : 21;

which would give 21 rubles = 10 ducats; whence 1 ducat = $2\frac{1}{10}$ rubles. Lastly, when the exchange is at 44 stivers, we have

As 44 : 105 :: 1 : $2\frac{17}{44}$ rubles;

which is equal to 2 rubles $38\frac{7}{11}$ copecks.

480. It follows also from this, that we may compare different kinds of money, which we have fre-

quently occasion to do in bills of exchange. Suppose, for example, that a person of this place has 1000 rubles to be paid to him at Berlin, and that he wishes to know the value of this sum in ducats at Berlin.

The exchange is here at $47\frac{1}{2}$; that is to say, one ruble makes $47\frac{1}{2}$ stivers; and in Holland, 20 stivers make a florin; $2\frac{1}{2}$ Dutch florins make a Dutch dollar: also, the exchange of Holland with Berlin is at 142; that is to say, for 100 Dutch dollars, 142 dollars are paid at Berlin; and lastly, the ducat is worth 3 dollars at Berlin.

481. To resolve the question proposed, we may proceed step by step: let us therefore begin with the stivers: since 1 ruble $= 47\frac{1}{2}$ stivers, or 2 rubles $= 95$ stivers, we shall have

Ru. Ru. Stiv.
As 2 : 1000 :: 95 : 47500 stivers;

then again,

Stiv. Stiv. Flor.
As 20 : 47500 :: 1 : 2375 florins.

Also, since $2\frac{1}{2}$ florins $= 1$ Dutch dollar, or 5 florins $= 2$ Dutch dollars; we shall therefore have

Flor. Flor. D.D.
As 5 : 2375 :: 2 : 950 Dutch dollars.

Then taking the dollars of Berlin, according to the exchange, at 142, we shall have

D.D. D.D. Dollars.
As 100 : 950 :: 142 : 1349 dollars of Berlin.

And lastly,

Dol. Dol. Du.
As 3 : 1349 :: 1 : $449\frac{2}{3}$ ducats,

which is the number sought.

482. Now in order to render these calculations

still more complete, let us suppose that the Berlin banker refuses, under some pretext or other, to pay this sum, and to accept the bill of exchange without five per cent. discount; that is, paying only 100 instead of 105. In that case we must make use of the following proportion:

As $105 : 449\frac{2}{3} :: 100 : 428\frac{16}{63}$ ducats;

which is the answer under those conditions.

483. We have shown that six operations are necessary in making use of the Rule of Three; but we can greatly abridge those calculations by a rule which is called the *Rule of Reduction*, or *Double Rule of Three*. To explain which, we shall first consider the two antecedents of each of the six preceding operations:

1st.	2 rubles	:	95 stivers.
2d.	20 stivers	:	1 Dutch flor.
3d.	5 Dutch flor.	:	2 Dutch doll.
4th.	100 Dutch doll.	:	142 dollars.
5th.	3 dollars	:	1 ducat.
6th.	105 ducats	:	100 ducats.

If we now look over the preceding calculations, we shall observe, that we have always multiplied the given sum by the third terms, or second antecedents, and divided the products by the first; it is evident, therefore, that we shall arrive at the same results by multiplying at once the sum proposed by the product of all the third terms, and dividing by the product of all the first terms: or, which amounts to the same thing, that we have only to make the following proportion: As the product of all the first terms, is to the given number of rubles, so is the product of all the second terms, to the number of ducats payable at Berlin.

484. This calculation is abridged still more, when amongst the first terms some are found that have common divisors with the second or third terms; for, in this case, we destroy those terms, and substitute the quotient arising from the division by that common divisor. The preceding example will, in this manner, assume the following form.

As $(2.20.5.100.3.105):1000::(95.2.142.100)$ $:\frac{1000.95.2.142.100}{2.20.5.100.3.105}$; and after canceling the common divisors in the numerator and denominator, this will become

$$\frac{19.71.20}{3.21}=\frac{26980}{63}=428\tfrac{16}{63}$$ ducats, as before.

485. The method which must be observed in using the Rule of Reduction is this: we begin with the kind of money in question, and compare it with another which is to begin the next relation, in which we compare this second kind with a third, and so on. Each relation, therefore, begins with the same kind as the preceding relation ended with; and the operation is continued till we arrive at the kind of money which the answer requires; at the end of which we must reckon the fractional remainders.

486. Let us give some other examples, in order to facilitate the practice of this calculation.

If ducats gain at Hamburgh 1 per cent. on two dollars banco; that is to say, if 50 ducats are worth, not 100, but 101 dollars banco; and if the exchange between Hamburgh and Konigsberg is 119 drachms of Poland; that is, if 1 dollar banco is equal to 119 Polish drachms; how many Polish florins are equivalent to 1000 ducats?

It being understood that 30 Polish drachms make 1 Polish florin,

Here	1 : 1000	::	2 dollars banco
	100	—	101 dollars banco
	1	—	119 Polish drs.
	30	—	1 Polish flor.

therefore,

$$(100.30):1000::(2.101.119):\frac{1000.2.101.119}{100.30}$$

$$=\frac{2.101.119}{3}=8012\tfrac{2}{3}\text{ Polish florins. Ans.}$$

487. We will propose another example, which may still farther illustrate this method.

Ducats of Amsterdam are brought to Leipsick, having in the former city the value of 5 flor. 4 stivers current; that is to say, 1 ducat is worth 104 stivers, and 5 ducats are worth 26 Dutch florins: if, therefore, the *agio of the bank* at Amsterdam is 5 per cent.; that is, if 105 currency are equal to 100 banco; and if the exchange from Leipsick to Amsterdam, in bank money, is 133¼ per cent.; that is, if for 100 dollars we pay at Leipsick 133¼ dollars; and lastly, 2 Dutch dollars making 5 Dutch florins; it is required to determine how many dollars we must pay at Leipsick, according to these exchanges, for 1000 ducats?

By the rule

5 : 1000	::	26 flor. Dutch curr.
105	—	100 flor. Dutch banco
400	—	533 doll. of Leipsick
5	—	2 doll. banco;

therefore,

As $(5.105.400.5):1000::(26.100.533.2)$

$: \frac{1000.26.100.533.2}{5.105.400.5} = \frac{4.26.533}{21} = 2639\frac{1}{21}$ dollars, the number sought.

And exactly in the same manner we may proceed with other examples of this kind.

CHAP. X.

Of Compound Relations.

488. *Compound Relations* are obtained by multiplying the terms of two or more relations, the antecedents by the antecedents, and the consequents by the consequents; we say then, that the relation between those two products is *compounded* of the relations given.

Thus the relations $a:b$, $c:d$, $e:f$, give the compound relation $ace:bdf$.

489. A relation continuing always the same, when we divide both its terms by the same number, in order to abridge it, we may greatly facilitate the above composition by comparing the antecedents and the consequents, for the purpose of making such reductions as we performed in the last chapter.

For example, we find the compound relation of the following given relations thus:

Relations given.

12 : 25, 28 : 33, and 55 : 56.

Which becomes

$$(12.28.55):(25.33.56)=2:5$$

by cancelling the common divisors.

So that $2:5$ is the compound relation required.

490. The same operation is to be performed, when it is required to calculate generally by letters; and the most remarkable case is that in which each antecedent is equal to the consequent of the preceding relation. If the given relations are

$$\begin{array}{c} a : b \\ b : c \\ c : d \\ d : e \\ e : a \end{array}$$

the compound relation is $1:1$.

491. The utility of these principles will be perceived, when it is observed, that the relation between two square fields is compounded of the relations of the lengths and the breadths.

Let the two fields, for example, be A and B; A having 500 feet in length by 60 feet in breadth; the length of B being 360 feet, and its breadth 100 feet; the relation of the lengths will be $500:360$, and that of the breadths $60:100$. So that we have

$$(500.60):(360.100)=5:6$$

Wherefore the field A is to the field B, as 5 to 6.

492. Again, let the field A be 720 feet long, 88 feet broad; and let the field B be 660 feet long, and 90 feet broad; the relations will be compounded in the following manner:

Relation of the lengths	720 : 660
Relation of the breadths	88 : 90

and by canceling, the

Relation of the fields A and B 16 : 15.

493. Farther, if it be required to compare two rooms with respect to the space or contents, we observe, that that relation is compounded of three relations; namely, that of the lengths, breadths, and heights. Let there be, for example, a room A, whose length is 36 feet, breadth 16 feet, and height 14 feet, and a room B, whose length is 42 feet, breadth 24 feet, and height 10 feet; we shall have these three relations:

For the length	36	:	42
For the breadth	16	:	24
For the height	14	:	10

And canceling the common measures, these become 4:5. So that the contents of the room A is to the contents of the room B, as 4 to 5.

494. When the relations which we compound in this manner are equal, there result multiplicate relations. Namely, two equal relations give a *duplicate ratio*, or *ratio of the squares*; three equal relations produce the *triplicate ratio*, or *ratio of the cubes*; and so on. For example, the relations $a:b$ and $a:b$ give the compound relation $a^2:b^2$; wherefore we say, that the squares are in the duplicate ratio of their roots. And the ratio $a:b$ multiplied twice, giving the ratio $a^3:b^3$, we say that the cubes are in the triplicate ratio of their roots.

495. Geometry teaches, that two circular spaces are in the duplicate relation of their diameters; this means, that they are to each other as the squares of their diameters.

Let A be such a space, having its diameter 45 feet, and B another circular space, whose diameter is 30 feet; the first space will be to the second as 45×45 to 30×30; or, compounding these two equal relations, as $9 : 4$.

Therefore the two areas are to each other as 9 to 4.

496. It is also demonstrated, that the solid contents of spheres are in the ratio of the cubes of their diameters: so that the diameter of a globe, A, being 1 foot, and the diameter of a globe, B, being 2 feet, the solid content of A will be to that of B, as $1^3 : 2^3$; or as 1 to 8. If, therefore, the spheres are formed of the same substance, the latter will weigh 8 times as much as the former.

497. It is evident that we may in this manner find the weight of cannon balls, their diameters and the weight of one being given. For example, let there be the ball A, whose diameter is 2 inches, and weight 5 pounds; and if the weight of another ball be required, whose diameter is 8 inches, we have this proportion,

$$2^3 : 8^3 :: 5 : 320 \text{ pounds},$$

which gives the weight of the ball B; and for another ball C, whose diameter is 15 inches, we should have,

$$2^3 : 15^3 :: 5 : 2109\tfrac{3}{8}\text{lb. Ans.}$$

498. When the ratio of two fractions, as $\frac{a}{b} : \frac{c}{d}$, is required, we may always express it in integer numbers; for we have only to multiply the two fractions

by bd, in order to obtain the ratio $ad : bc$, which is equal to the other; and from hence results the proportion $\frac{a}{b} : \frac{c}{d} :: ad : bc$. If, therefore, ad and bc have common divisors, the ratio may be reduced to fewer terms. Thus $\frac{15}{24} : \frac{25}{36} :: (15.36) : (24.25) :: 9 : 10$.

499. If we wished to know the ratio of the fractions $\frac{1}{a}$ and $\frac{1}{b}$, it is evident that we should have $\frac{1}{a} : \frac{1}{b} :: b : a$; which is expressed by saying, that two fractions, which have unity for their numerator, are in the *reciprocal* or *inverse* ratio of their denominators: and the same thing is said of two fractions which have any common numerator; for $\frac{c}{a} : \frac{c}{b} :: b : a$. But if two fractions have their denominators equal, as $\frac{a}{c} : \frac{b}{c}$, they are in the *direct ratio* of the numerators; namely, as $a : b$. Thus $\frac{3}{8} : \frac{3}{16} :: \frac{6}{16} : \frac{3}{16} :: 6 : 3 :: 2 : 1$, and $\frac{10}{7} : \frac{15}{7} :: 10 : 15 :: 2 : 3$.

500. It has been observed, in the free descent of bodies, that a body falls about 16 English feet in a second, that in two seconds of time it falls from the height of 64 feet, and in three seconds it falls 144 feet. Hence it is concluded, that the heights are to each other as the squares of the times; and reciprocally, that the times are in the subduplicate

ratio of the heights, or as the square roots of the heights*.

If, therefore, it be required to determine how long a stone will be in falling from the height of 2304 feet; we have 16 : 2304 :: 1 : 144, the square of the time; and consequently the time required is 12 seconds.

501. If it be required to determine how far, or through what height, a stone will pass by descending for the space of an hour, or 3600 seconds; we must say,

$$\text{As } 1^2 : 3600^2 :: 16 : 207360000 \text{ feet},$$

the height required.

Which being reduced is found equal to 39272 miles; and consequently nearly five times greater than the diameter of the earth.

502. It is the same with regard to the price of precious stones, which are not sold in the proportion of their weight; every body knows that their prices follow a much greater ratio. The rule for diamonds is, that the price is in the duplicate ratio of the weight; that is to say, the ratio of the prices is equal to the square of the ratio of the weights. The weight of diamonds is expressed in carats, and a carat is equivalent to 4 grains; if, therefore, a diamond of one carat is worth 10 livres, a diamond of 100 carats will be worth as many times 10 livres as the

* The space descended by a heavy body, in the latitude of London, in the first second of time, has been found by experiment to be $16\frac{1}{12}$ English feet; but in calculations where great accuracy is not required, the fraction may be omitted. Ed.

square of 100 contains 1; so that we shall have, according to the Rule of Three,

As $1:10000::10:100000$ liv. Ans.

There is a diamond in Portugal which weighs 1680 carats; its price will be found, therefore, by making

$1^2:1680^2::10:28224000$ livres.

503. The posts, or mode of travelling, in France, furnish sufficient examples of compound ratios; because the price is regulated by the compound ratio of the number of horses, and the number of leagues, or posts. Thus, for example, if one horse cost 20 sous per post, it is required to find how much must be paid for 28 horses for $4\frac{1}{2}$ posts.

We write first the ratio of the horses - - - - 1 : 28
Under this ratio we put that of the stages - - 2 : 9

And, compounding the two ratios, we have - 2 : 252
Or, $1:126::1$ liv. to 126 fr. or 42 crowns.

Again, If I pay a ducat for eight horses for 3 miles, how much must I pay for thirty horses for four miles? The calculation is as follows:

$$\begin{array}{c} 8:30 \\ 3:\ 4 \end{array}$$

$1:5::1$ duc. : 5 ducats; the sum required.

504. The same composition occurs when workmen are to be paid, since those payments generally follow the ratio compounded of the number of workmen and that of the days which they have been employed.

If, for example, 25 sous per day be given to one mason, and it is required what must be paid to 24 masons who have worked for 50 days: we state the calculation thus:

1 : 24
1 : 50

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1 : 1200 :: 25 : 1500 francs.

In these examples, five things being given, the rule which serves to resolve them is called, in books of arithmetic, The Rule of Five, or Double Rule of Three.

CHAP. XI.

Of Geometrical Progressions.

505. A series of numbers, which are always becoming a certain number of times greater or less, is called a *geometrical progression*, because each term is constantly to the following one in the same geometrical ratio: and the number which expresses how many times each term is greater than the preceding, is called the *exponent*, or *ratio*. Thus, when the first term is 1 and the exponent 2, the geometrical progression becomes,

Terms	1	2	3	4	5	6	7	8	9	&c.
Prog.	1,	2,	4,	8,	16,	32,	64,	128,	256,	&c.

The numbers 1, 2, 3, &c. always marking the place which each term holds in the progression.

506. If we suppose, in general, the first term to be a, and the ratio b, we have the following geometrical progression:

$$\begin{array}{lllllllll} & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8 \ldots n. \\ \textit{Prog.} & a, & ab, & ab^2, & ab^3, & ab^4, & ab^5, & ab^6, & ab^7 \ldots ab^{n-1}. \end{array}$$

So that, when this progression consists of n terms, the last term is ab^{n-1}: we must, however, remark here, that if the ratio b be greater than unity, the terms increase continually; if $b=1$, the terms are all equal; lastly, if b be less than 1, or a fraction, the terms continually decrease. Thus, when $a=1$, and $b=\frac{1}{2}$, we have this geometrical progression:

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \text{ \&c.}$$

507. Here therefore we have to consider:

1. The first term, which we have called a.
2. The exponent, which we call b.
3. The number of terms, which we have expressed by n.
4. And the last term, which, we have already seen, is expressed by ab^{n-1}.

So that, when the three first of these are given, the last term is found by multiplying the $n-1$ power of b, or b^{n-1}, by the first term a.

If, therefore, the 50th term of the geometrical progression 1, 2, 4, 8, &c. were required, we should have $a=1$, $b=2$, and $n=50$; consequently the 50th term would be 2^{49}; and as $2^9=512$, we shall have $2^{10}=1024$; wherefore the square of 2^{10}, or 2^{20}, $=1048576$, and the square of this number, which is 1099511627776, $=2^{40}$. Multiplying therefore this value of 2^{40} by 2^9, or by 512, we have 2^{49} equal to 562949953421312 for the 50th term.

508. One of the principal questions which occur on this subject, is to find the sum of all the terms of a geometrical progression; we shall therefore explain the method of doing this. In order to which let there be given, first, the following progression, consisting of ten terms:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512,$$

the sum of which we shall represent by s, so that $s = 1+2+4+8+16+32+64+128+256+512$; now doubling both sides, we shall have

$2s = 2+4+8+16+32+64+128+256+512+1024$; and subtracting from this progression that represented by s, there remains $s = 1024 - 1 = 1023$; wherefore the sum required is 1023.

509. Suppose now, in the same progression, that the number of terms is undetermined, that is, let them be generally represented by n, so that the sum in question, or

$$s, = 1+2+2^2+2^3+2^4 \ldots . 2^{n-1};$$

and if we multiply by 2, we have

$$2s = 2+2^2+2^3+2^4 \ldots . . 2^n,$$

then subtracting from this equation the preceding one, we have $s = 2^n - 1$. It is evident, therefore, that the sum required is found, by multiplying the last term, 2^{n-1}, by the exponent 2, in order to have 2^n, and subtracting unity from that product.

510. This is made still more evident by the following examples, in which we substitute successively, for n, the numbers 1, 2, 3, 4, &c.

$1 = 1$; $1+2 = 3$; $1+2+4 = 7$; $1+2+4+8 = 15$; $1+2+4+8+16 = 31$; $1+2+4+8+16+32 = 63$, &c.

511. On this subject the following question is generally proposed. A man offers to sell his horse upon the following condition, that is, he demands 1 penny for the first nail, 2 for the second, 4 for the third, 8 for the fourth, and so on, doubling the price of each succeeding nail. It is required to find the price of the horse, the nails being 32 in number?

This question is evidently reduced to finding the sum of all the terms of the geometrical progression 1, 2, 4, 8, 16, &c. continued to the 32d term. Now, that last term is 2^{31}; and, as we have already found $2^{20}=1048576$, and $2^{10}=1024$, we shall have $2^{20}\times 2^{10}=2^{30}=1073741824$; and multiplying again by 2, the last term $2^{31}=2147483648$; doubling therefore this number, and subtracting unity from the product, the sum required becomes 4294967295 pence; which being reduced, we have 17895697*l.* 1*s.* 3*d.* for the price of the horse.

512. Let the ratio now be 3, and suppose it be required to find the sum of the geometrical progression 1, 3, 9, 27, 81, 243, 729, consisting of 7 terms.

Calling the sum s as before, we have

$$s=1+3+9+27+81+243+729.$$

And multiplying by 3,

$$3s=3+9+27+81+243+729+2187.$$

Then subtracting the former series from the latter, we have $2s=2187-1=2186$; so that the double of the sum is 2186, and consequently the sum required is 1093.

513. In the same progression, let the number of terms be n, and the sum s; so that

$$s = 1 + 3 + 3^2 + 3^3 + 3^4 + \ldots \ldots 3^{n-1}.$$

If now we multiply by 3, we have

$$3s = 3 + 3^2 + 3^3 + 3^4 + \ldots \ldots 3^n.$$

Then subtracting from this expression the value of s, as before, we shall have $2s = 3^n - 1$; therefore $s = \frac{3^n - 1}{2}$. So that the sum required is found by multiplying the last term by 3, subtracting 1 from the product, and dividing the remainder by 2; as will appear, also, from the following particular cases:

1 - - - -	$\frac{1 \times 3 - 1}{2}$	$=$	1
$1+3$ - - -	$\frac{3 \times 3 - 1}{2}$	$=$	4
$1+3+9$ - - -	$\frac{3 \times 9 - 1}{2}$	$=$	13
$1+3+9+27$ - -	$\frac{3 \times 27 - 1}{2}$	$=$	40
$1+3+9+27+81$ -	$\frac{3 \times 81 - 1}{2}$	$=$	$121.$

514. Let us now suppose, generally, the first term to be a, the ratio b, the number of terms n, and their sum s, so that

$$s = a + ab + ab^2 + ab^3 + ab^4 + \ldots \ldots ab^{n-1}.$$

If we multiply by b, we have

$$bs = ab + ab^2 + ab^3 + ab^4 + ab^5 + \ldots \ldots ab^n,$$

and taking the difference between this and the above equation, there remains $(b \backsim 1)\, s = ab^n \backsim a$; whence we easily deduce the sum required $s=\frac{a.(b^n \backsim 1)}{b \backsim 1}$.

Consequently, the sum of any geometrical progression is found, by multiplying the last term by the ratio, and dividing the difference between this product and the first term, by the difference between 1 and the ratio.

515. Let there be a geometrical progression of seven terms, of which the first is 3; and let the ratio be 2: we shall then have $a=3$, $b=2$, and $n=7$; therefore the last term is 3×2^6, or $3\times 64, =$ 192; and the whole progression will be

$$3,\ 6,\ 12,\ 24,\ 48,\ 96,\ 192.$$

Farther, if we multiply the last term 192 by the ratio 2, we have 384; subtracting the first term, there remains 381; and dividing this by $b-1$, or by 1, we have 381 for the sum of the whole progression.

516. Again, let there be a geometrical progression of six terms, of which the first is 4; and let the ratio be $\frac{3}{2}$: then the progression is

$$4,\ 6,\ 9,\ \frac{27}{2},\ \frac{81}{4},\ \frac{243}{8}.$$

If, now we multiply the last term by the ratio, we shall have $\frac{729}{16}$; and subtracting the first term, the remainder is $\frac{665}{16}$, which, divided by $b-1=\frac{1}{2}$, gives $\frac{665}{8}=83\frac{1}{8}$ for the sum of the series.

517. When the exponent is less than 1, and, consequently, when the terms of the progression continually diminish, the sum of such a decreasing progression, carried on to infinity, may be accurately expressed.

For example, let the first term be 1, the ratio $\frac{1}{2}$, and the sum s, so that:

$$s=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+, \text{ \&c.}$$

ad infinitum.

If we multiply by 2, we have

$$2s=2+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+, \text{ \&c.}$$

ad infinitum: and, subtracting the preceding progression, there remains $s=2$ for the sum of the proposed infinite progression.

518. If the first term be 1, the ratio $\frac{1}{3}$, and the sum s; so that

$$s=1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+, \text{ \&c. ad infinitum:}$$

Then multiplying the whole by 3, we have

$$3s=3+1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+, \text{ \&c. ad infinitum;}$$

and subtracting the value of s, there remains $2s=3$; wherefore the sum $s=1\frac{1}{2}$.

519. Let there be a progression whose sum is s, first term 2, and ratio $\frac{3}{4}$; so that

$$s=2+\frac{3}{2}+\frac{9}{8}+\frac{27}{32}+\frac{81}{128}+, \text{ \&c. ad infinitum.}$$

Now multiplying by $\frac{4}{3}$, we have

$$\frac{4}{3}s=\frac{8}{3}+2+\frac{3}{2}+\frac{9}{8}+\frac{27}{32}+\frac{81}{128}+, \text{ \&c. ad infinitum;}$$

and subtracting from this progression s, there remains $\frac{1}{3}s=\frac{8}{3}$; wherefore the sum required is 8.

520. If we suppose, in general, the first term to be a, and the ratio of the progression to be $\frac{b}{c}$, so that this fraction may be less than 1, and consequently c greater than b; the sum of the progression, carried on ad infinitum, will be found thus:

Make $s=a+\frac{ab}{c}+\frac{ab^2}{c^2}+\frac{ab^3}{c^3}+\frac{ab^4}{c^4}+$, &c.

Then multiplying by $\frac{b}{c}$, we shall have

$$s\frac{b}{c}=\frac{ab}{c}+\frac{ab^2}{c^2}+\frac{ab^3}{c^3}+\frac{ab^4}{c^4}+, \text{ \&c. ad infinitum;}$$

and, subtracting this equation from the preceding, there remains $(1-\frac{b}{c})s=a$.

Consequently $s=\frac{a}{1-\frac{b}{c}}=\frac{ac}{c-b}$.

The sum of the infinite geometrical progression proposed is, therefore, found by dividing the first term a by 1 minus the ratio, or by multiplying the first term a by the denominator of the ratio, and dividing the product by the same denominator diminished by the numerator of the ratio *.

521. In the same manner we find the sums of progressions, the terms of which are alternately affected by the signs + and −. Suppose, for example,

$$s=a-\frac{ab}{c}+\frac{ab^2}{c^2}-\frac{ab^3}{c^3}+\frac{ab^4}{c^4}-, \text{ \&c.}$$

and multiplying by $\frac{b}{c}$, we have,

$$\frac{b}{c}s=\frac{ab}{c}-\frac{ab^2}{c^2}+\frac{ab^3}{c^3}-\frac{ab^4}{c^4}, \text{ \&c.}$$

Now, adding this equation to the preceding, we obtain $(1+\frac{b}{c})s=a$: whence we deduce the sum required, $s=\frac{a}{1+\frac{b}{c}}$, or $s=\frac{ac}{c+b}$.

522. It is evident, therefore, that if the first term $a=\frac{3}{5}$, and the ratio be $\frac{2}{5}$, that is to say, $b=2$ and $c=5$; we shall find the sum of the progression $\frac{3}{5}+\frac{6}{25}+\frac{12}{125}+\frac{24}{625}+$, &c. $=1$; since, by subtracting the ratio from 1, there remains $\frac{3}{5}$, and by dividing the first term by that remainder, the quotient is 1.

It is also evident, if the terms be alternately

* This particular case is included in the general rule, Art. 514.

positive and negative, and the progression assume this form:

$$\frac{3}{5}-\frac{6}{25}+\frac{12}{125}-\frac{24}{625}+, \text{ \&c.}$$

that the sum will be

$$\frac{a}{1+\frac{b}{c}}=\frac{\frac{3}{5}}{\frac{7}{5}}=\frac{3}{7}.$$

523. Again: let there be proposed the infinite progression,

$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}+\frac{3}{100000}+, \text{ \&c.}$$

The first term is here $\frac{3}{10}$, and the ratio is $\frac{1}{10}$; therefore subtracting this last from 1, there remains $\frac{9}{10}$, and, if we divide the first term by this fraction, we have $\frac{1}{3}$ for the sum of the given progression. So that taking only one term of the progression, namely, $\frac{3}{10}$, the error would be $\frac{1}{10}$.

And taking two terms, $\frac{3}{10}+\frac{3}{100}=\frac{33}{100}$, there would still be wanting $\frac{1}{100}$ to make the sum, which we have seen is $\frac{1}{3}$.

524. Let there now be given the infinite progression,

$$9+\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}+, \text{ \&c.}$$

The first term is 9, the ratio is $\frac{1}{10}$. So that 1 minus the ratio is $\frac{9}{10}$; and $\frac{9}{\frac{9}{10}}=10$, the sum required: which series is expressed by a decimal fraction, thus, 9·9999999, &c.

CHAP. XII.

Of Infinite Decimal Fractions.

525. We have already seen, in logarithmic calculations, that decimal fractions are employed instead of vulgar fractions: the same are also advantageously employed in other calculations; it will therefore be very necessary to show how a vulgar fraction may be transformed into a decimal fraction; and, conversely, how we may express the value of a decimal by a vulgar fraction.

526. Let it be required, in general, to change the fraction $\frac{a}{b}$, into a decimal; as this fraction expresses the quotient of the division of the numerator a by the denominator b, let us write, instead of a, the quantity a·0000000, whose value does not at all differ from that of a, since it contains neither tenth parts, hundredth parts, nor any other parts whatever. If we now divide this quantity by the

number b, according to the common rules of division, observing to put the point in the proper place, which separates the decimal and the integers, we shall obtain the decimal sought. Thus in the following examples:

Let there be given first the fraction $\frac{1}{2}$, the division in decimals will assume this form:

$$\frac{2)1{\cdot}0000000}{0{\cdot}5000000} = \frac{1}{2}.$$

Hence it appears, that $\frac{1}{2}$ is equal to 0·5000000 or to 0·5; which is sufficiently evident, since this decimal fraction represents $\frac{5}{10}$, which is equivalent to $\frac{1}{2}$.

527. Let now $\frac{1}{3}$ be the given fraction, and we have

$$\frac{3)1{\cdot}0000000}{0{\cdot}3333333} = \frac{1}{3}.$$

This shows, that the decimal fraction whose value is $\frac{1}{3}$, cannot, strictly, ever be discontinued, and that it goes on ad infinitum, repeating always the number 3; which agrees with what has been already shown, namely, that the fractions

$$\frac{3}{10}+\frac{3}{100}+\frac{3}{1000}+\frac{3}{10000}, \text{ \&c. } \textit{ad infinitum}, = \frac{1}{3}.$$

The decimal fraction which expresses the value of $\frac{2}{3}$, is also continued ad infinitum; for we have,

$$\frac{3)2{\cdot}0000000}{0{\cdot}6666666} = \frac{2}{3}.$$

Which is also evident from what we have just said, because $\frac{2}{3}$ is the double of $\frac{1}{3}$.

528. If $\frac{1}{4}$ be the fraction proposed, we have

$$\frac{4)1\cdot0000000}{0\cdot2500000} = \frac{1}{4}.$$

So that $\frac{1}{4}$ is equal to $0\cdot2500000$, or to 0.25: which is evidently true, since $\frac{2}{10} + \frac{5}{100} = \frac{25}{100} = \frac{1}{4}$.

In like manner, we should have for the fraction $\frac{3}{4}$,

$$\frac{4)\ 3\cdot0000000}{0\cdot7500000} = \frac{3}{4}.$$

So that $\frac{3}{4} = 0\cdot75$: and in fact

$$\frac{7}{10} + \frac{5}{100} = \frac{75}{100} = \frac{3}{4}.$$

The fraction $\frac{5}{4}$ is changed into a decimal fraction, by making

$$\frac{4)\ 5\cdot0000000}{1\cdot2500000} = \frac{5}{4}.$$

Now $1 + \frac{25}{100} = \frac{5}{4}$.

529. In the same manner, $\frac{1}{5}$ will be found equal to $0\cdot2$; $\frac{2}{5} = 0\cdot4$; $\frac{3}{5} = 0\cdot6$; $\frac{4}{5} = 0\cdot8$; $\frac{5}{5} = 1$; $\frac{6}{5} = 1\cdot2$, &c.

When the denominator is 6, we find $\frac{1}{6} = 0\cdot1666666$, &c. which is equal to $0\cdot666666 - 0\cdot5$; now $0\cdot666666 = \frac{2}{3}$ and $0\cdot5 = \frac{1}{2}$, wherefore $0\cdot1666666 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$.

We find, also, $\frac{2}{6}=0{\cdot}333333$, &c. $=\frac{1}{3}$; but $\frac{3}{6}$ becomes $0{\cdot}5000000=\frac{1}{2}$; also, $\frac{5}{6}=0{\cdot}833333=0{\cdot}333333+0{\cdot}5$, that is to say, $\frac{1}{3}+\frac{1}{2}=\frac{5}{6}$.

530. When the denominator is 7, the decimal fractions become more complicated: thus for example, we find $\frac{1}{7}=0{\cdot}142857$; however it must be observed, that these six figures are continually repeated. To be convinced, therefore, that this decimal fraction precisely expresses the value of $\frac{1}{7}$, we may transform it into a geometrical progression, whose first term is $\frac{142857}{1000000}$, the ratio being $\frac{1}{1000000}$; and consequently, the sum $=\frac{\frac{142857}{1000000}}{1-\frac{1}{1000000}}=\frac{142857}{1000000}=\frac{1}{7}$.

531. We may prove, in a manner still more easy, that the decimal fraction which we have found is exactly equal to $\frac{1}{7}$; for by substituting for its value the letter s, we have

$$
\begin{aligned}
s&=0{\cdot}142857142857142857, \text{ \&c.}\\
10s&=1{\cdot}\ 42857142857142857, \text{ \&c.}\\
100s&=14{\cdot}\ 2857142857142857, \text{ \&c.}\\
1000s&=142{\cdot}\ 857142857142857, \text{ \&c.}\\
10000s&=1428{\cdot}\ 57142857142857, \text{ \&c.}\\
100000s&=14285{\cdot}\ 7142857142857, \text{ \&c.}\\
1000000s&=142857{\cdot}\ 142857142857, \text{ \&c.}\\
\text{Subtract } s&=\quad 0{\cdot}\ 142857142857, \text{ \&c.}\\
\hline
999999s&=142857.
\end{aligned}
$$

And, dividing by 999999, we have $s=\frac{142857}{999999}$ $=\frac{1}{7}$. Wherefore the decimal fraction, which was represented by s, is $=\frac{1}{7}$.

532. In the same manner $\frac{2}{7}$ may be transformed into a decimal fraction, which will be 0·28571428, &c. and this enables us to find more easily the value of the decimal fraction which we have represented by s; because 0·28571428, &c. must be the double of it, and, consequently, $=2s$. Now we have seen that

$$100s=14{\cdot}28571428571,\ \&c.$$

So that subtracting $2s=\ \ 0{\cdot}28571428571,\ \&c.$

there remains $98s=14$

wherefore $s=\frac{14}{98}=\frac{1}{7}$.

We also find $\frac{3}{7}=0{\cdot}42857142857$, &c. which, according to our suppposition, must be equal to $3s$; and we have found that

$$10s=1{\cdot}42857142857,\ \&c.$$

So that subtracting $3s=0{\cdot}42857142857,\ \&c.$

we have $7s=1$, wherefore $s=\frac{1}{7}$.

533. When a proposed fraction, therefore, has the denominator 7, the decimal fraction is infinite, and 6 figures are continually repeated; the reason of which is easy to perceive, namely, that when we continue the division a remainder must return, sooner or

later, which we have had already. Now, in this division, 6 different numbers only can form the remainder, namely 1, 2, 3, 4, 5, 6; so that, at least after the sixth division, the same figures must return; but when the denominator is such as to lead to a division without remainder, these cases do not happen.

534. Suppose now that 8 is the denominator of the fraction proposed: we shall find the following decimal fractions:

$\frac{1}{8}=0{\cdot}125$; $\frac{2}{8}=0{\cdot}25$; $\frac{3}{8}=0{\cdot}375$; $\frac{4}{8}=0{\cdot}5$; $\frac{5}{8}=0{\cdot}625$; $\frac{6}{8}=0{\cdot}75$; $\frac{7}{8}=0{\cdot}875$, &c.

If the denominator be 9, we have

$\frac{1}{9}=0{\cdot}111$, &c. $\frac{2}{9}=0{\cdot}222$, &c. $\frac{3}{9}=0{\cdot}333$, &c.

And if the denominator be 10, we have $\frac{1}{10}=0{\cdot}1$, $\frac{2}{10}=0{\cdot}2$, $\frac{3}{10}=0{\cdot}3$. This is evident from the nature of decimals, as also that $\frac{1}{100}=0{\cdot}01$; $\frac{37}{100}=0{\cdot}37$; $\frac{256}{1000}=0{\cdot}256$; $\frac{24}{10000}=0{\cdot}0024$, &c.

536. If 11 be the denominator of the given fraction, we shall have $\frac{1}{11}=0{\cdot}0909090$, &c. Now, suppose it were required to find the value of this decimal fraction: let us call it s, and we shall have

$$s = 0{\cdot}090909,$$
$$10s = 00{\cdot}909090,$$
$$100s = 9{\cdot}09090.$$

If, therefore, we subtract from the last the value of s, we shall have $99s = 9$, and consequently $s = \frac{9}{99} = \frac{1}{11}$: thus, also,

$$\frac{2}{11} = 0{\cdot}181818, \text{ \&c.}$$
$$\frac{3}{11} = 0{\cdot}272727, \text{ \&c.}$$
$$\frac{6}{11} = 0{\cdot}545454, \text{ \&c.}$$

537. There are a great number of decimal fractions, therefore, in which one, two, or more figures constantly recur, and which continue thus to infinity. Such fractions are curious, and we shall show how their values may be easily found *.

---

* These recurring decimals furnish many interesting researches; I had entered upon them, before I saw the present *Algebra*, and should perhaps have prosecuted my inquiry, had I not likewise found a Memoir in the *Philosophical Transactions* for 1769, intitled *The Theory of circulating Fractions*. I shall content myself with stating here the reasoning with which I began.

Let $\frac{n}{d}$ be any real fraction irreducible to lower terms. And suppose it were required to find how many decimal places we must reduce it to, before the same terms will return again. In order to determine this, I begin by supposing that $10n$ is greater than d; if that were not the case, and only $100n$ or $1000n > d$, it would be necessary to begin with trying to reduce $\frac{10n}{d}$ or $\frac{100n}{d}$, &c. to less terms, or to a fraction $\frac{n^{\text{I}}}{d^{\text{I}}}$.

This being established, I say that the same period can return

Let us first suppose, that a single figure is constantly repeated, and let us represent it by a, so that $s = 0{\cdot}aaaaaaa$. We have

$$10s = a{\cdot}aaaaaaa$$

and subtracting $s = 0{\cdot}aaaaaaa$

we have $9s = a$; wherefore $s = \frac{a}{9}$.

When two figures are repeated, as ab, we have $s = 0{\cdot}abababa$. Therefore $100s = ab{\cdot}ababab$; and

---

only when the same remainder n returns in the continual division. Suppose that when this happens we have added s cyphers, and that q is the integral part of the quotient; then abstracting from the point, we shall have $\frac{n \times 10^s}{d} = q + \frac{n}{d}$; wherefore $q = \frac{n}{d} \times (10^s - 1)$. Now as q must be an integer number, it is required to determine the least integer number for s, such that $\frac{n}{d} \times (10^s - 1)$, or only $\frac{10^s - 1}{d}$, may be an integer number.

This problem requires several cases to be distinguished: the first is that in which d is a divisor of 10, or of 100, or of 1000, &c. and it is evident that in this case there can be no circulating fraction. For the second case we shall take that in which d is an odd number, and not a factor of any power of 10; in this case the value of s may rise to $d-1$, but frequently it is less. A third case is that in which d is even, and consequently, without being a factor of any power of 10, has nevertheless a common divisor with one of those powers: this common divisor can only be a number of the form 2^c; so that if $\frac{d}{2^c} = e$, I say, the period will be the same as for the fraction $\frac{n}{e}$, but will not commence before the figure represented by c. This case comes to the same therefore with the second case, on which it is evident the theory depends. F. T. See Appendix, note 3.

if we subtract s from it, there remains $99s = ab$; consequently, $s = \frac{ab}{99}$.

When three figures, as abc, are found repeated, we have $s = 0{\cdot}abcabcabc$; consequently, $1000s = abc{\cdot}abcabc$; and subtract s from it, there remains $999s = abc$; wherefore $s = \frac{abc}{999}$, and so on.

Whenever, therefore, a decimal fraction of this kind occurs, it is easy to find its value. Let there be given, for example, 0·296296: its value will be $\frac{296}{999} = \frac{8}{27}$, by dividing both its terms by 37.

This fraction ought to give again the decimal fraction proposed; and we may easily be convinced that this is the real result, by dividing 8 by 9, and then that quotient by 3, because $27 = 3 \times 9$: thus we have

9) 8·0000000

3) 0·8888888

0·2962962, &c.

which is the decimal fraction that was proposed.

539. Suppose it was required to reduce the fraction $\frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10}$, to a decimal. The operation is as follows:

2) 1·00000000000000

3) 0·50000000000000

4) 0·16666666666666

5) 0·041666666666666

6) 0·008333333333333

7) 0·001388888888888

8) 0·000198412698411

9) 0·000024801587301

10) 0·0000027557319

0·00000027557319.

CHAP. XIII.

Of the Calculation of Interest *.

540. We are accustomed to express the interest of any principal by *per cents.*, signifying how much

* The theory of the calculation of interest owes its first improvements to Leibneitz, who delivered the principal elements of it in the *Acta Eruditorum* of Leipsic for 1683. It was afterwards the subject of several detached dissertations written in a very interesting manner. It has been most indebted to those mathematicians who have cultivated political arithmetic; in which are combined, in a manner truly useful, the calculation of probabilities, the calculation of interest, and the data furnished by the bills of mortality. We are still in want of a good elementary treatise of political arithmetic, though this extensive branch of science has been much attended to in England, France, and Holland. F. T.

interest is annually paid for the sum of 100 pounds. And it is very usual to put out the principal sum at 5 per cent., that is, on such terms, that we receive 5 pounds interest for every 100 pounds principal: nothing therefore is more easy than to calculate the interest for any sum; for we have only to say, according to the rule of three:

As 100 is to the principal proposed, so is the rate per cent. to the interest required. Let the principal, for example, be 860*l.*, its annual interest is found by this proportion;

$$\text{As } 100 : 860 :: 5 : 43.$$

Therefore 43*l.* is the interest required.

541. We shall not dwell any longer on examples of simple interest, but pass on immediately to the calculation of *compound interest;* where the chief subject of enquiry is, to what sum does a given principal amount, after a certain number of years, the interest being annually added to the principal? In order to resolve this question, we begin with the consideration, that 100*l.* placed out at 5 per cent. become, at the end of a year, a principal of 105*l.*: therefore let the principal be a; its amount, at the end of the year, will be found, by saying; as 100 is to a, so is 105 to the answer, which gives

$$\frac{105a}{100} = \frac{21a}{20} = \frac{21}{20} \times a, = a + \frac{1}{20} \times a.$$

542. So that, when we add to the original principal its twentieth part, we obtain the amount of the principal at the end of the first year: and adding to this its twentieth part, we know the amount

of the given principal at the end of two years, and so on. It is easy, therefore, to compute the successive and annual increases of the principal, and to continue this calculation to any length.

543. Suppose, for example, that a principal, which is at present 1000*l.*, is put out at five per cent. and that the interest is added every year to the principal; to find its amount at any time. As this calculation must lead to fractions, we shall employ decimals, but without carrying them farther than the thousandth parts of a pound, since smaller parts do not at present enter into consideration.

The given principal of 1000*l.* will be worth

after 1 year - - -	1050*l.*
	52·5,
after 2 years - - -	1102·5
	55·125,
after 3 years - - -	1157·625
	57·881,
after 4 years - - -	1215·506
	60·775,
after 5 years - - -	1276·281, &c.

which sums are formed by always adding $\frac{1}{20}$ of the preceding principal.

544. We may continue the same method, for any number of years; but when this number is very great, the calculation becomes long and tedious; but it may always be abridged, in the following manner:

Let the present principal be *a*, and since a principal of 20*l.* amounts to 21*l.* at the end of a year, the principal a will amount to $\frac{21}{20}.a$ at the end of a year: and the same principal will amount, the following year, to $\frac{21^2}{20^2}.a=\left(\frac{21}{20}\right)^2.a$: also, this principal of two years will amount to $\left(\frac{21}{20}\right)^3.a$, the year after: which will therefore be the principal of three years; and still increasing in the same manner, the given principal will amount to $\left(\frac{21}{20}\right)^4.a$ at the end of four years; to $\left(\frac{21}{20}\right)^5.a$, at the end of five years; and after a century, it will amount to $\left(\frac{21}{20}\right)^{100}.a$; so that, in general, $\left(\frac{21}{20}\right)^n.a$ will be the amount of this principal, after n years; and this formula will serve to determine the amount of the principal, after any number of years.

545. The fraction $\frac{21}{20}$, which is used in this calculation, depends on the interest having been reckoned at 5 per cent., and on $\frac{21}{20}$ being equal to $\frac{105}{100}$. But if the interest were estimated at 6 per cent. the principal a would amount to $\frac{106}{100}.a$, at the end of a year; to $\left(\frac{106}{100}\right)^2.a$, at the end of two years; and to $\frac{106^n}{100}.a$, at the end of n years.

If the interest is only at 4 per cent. the principal a will amount only to $\left(\frac{104}{100}\right)^n . a$, after n years.

546. Now when the principal a, as well as the number of years, is given, it is easy to resolve these formulæ by logarithms. For if the question be according to our first supposition, we shall take the logarithm of $\left(\frac{21}{20}\right)^n . a$, which is $= log. \left(\frac{21}{20}\right)^n + log. a$; because the given formula is the product of $\left(\frac{21}{20}\right)^n$ and a. Also, as $\left(\frac{21}{20}\right)^n$ is a power, we shall have $log. \left(\frac{21}{20}\right)^n = n\ log. \frac{21}{20}$: so that the logarithm of the principal required is $n\ log. \frac{21}{20} + log. a$; and farther, the logarithm of the fraction $\frac{21}{20} = log. 21 - log. 20$.

547. Let now the principal be 1000*l.* and let it be required to find how much this principal will amount to at the end of 100 years, reckoning the interest at 5 per cent.

Here we have $n = 100$; and consequently, the logarithm of the principal required will be $100\ log. \frac{21}{20} + log. 1000$, which quantity is calculated thus:

$$log. 21 = 1{\cdot}3222193$$
$$\text{subtracting } log. 20 = 1{\cdot}3010300$$

$$log. \frac{21}{20} = 0{\cdot}0211893$$

$$\begin{array}{lr} \text{multiplying by} & 100 \\ \hline 100\ log.\ \frac{21}{20} & = 2{\cdot}1189300 \\ \text{adding } log.\ 1000 & = 3{\cdot}0000000 \\ \hline \text{gives} & 5{\cdot}1189300 \end{array}$$

which is the logarithm of the principal required.

We perceive, from the characteristic of this logarithm, that the principal required will be a number consisting of six figures, and it is found to be 131501*l.*

548. Again, suppose a principal of 3452*l.* was put out at 6 per cent., what will it amount to at the end of 64 years?

We have here $a = 3452$, and $n = 64$. Wherefore the logarithm of the principal sought is

$64\ log.\ \frac{53}{50} + log.\ 3452$, which is calculated thus:

$$\begin{array}{lr} log.\ 53 & = 1.7242759 \\ \text{subtracting } log.\ 50 & = 1.6989700 \\ \hline log.\ \frac{53}{50} & = 0{\cdot}0253059 \\ \text{multiplying by} & 64 \\ \hline 64\ log.\ \frac{53}{50} & = 1{\cdot}6195776 \\ log.\ 3452 & = 3{\cdot}5380708 \\ \hline \text{which gives} & 5{\cdot}1576484. \end{array}$$

And taking the number of this logarithm, we find the principal required equal to 143763*l.*

549. When the number of years is very great, as it is required to multiply this number by the loga-

rithm of a fraction, a considerable error might arise from the logarithms in the tables not being calculated beyond 7 figures of decimals; for which reason, it will be necessary to employ logarithms carried to a greater number of figures, as in the following example.

A principal of 1*l.* being placed at 5 per cent., compound interest, for 500 years, it is required to find to what sum this principal will amount, at the end of that period.

We have here $a=1$ and $n=500$; and consequently the logarithm of the principal sought is equal to 500 $log. \frac{21}{20}+log.\ 1$, which produces this calculation:

$$log.\ 21=1\cdot 322219294733919$$
subtracting $$log.\ 20=1\cdot 301029995663981$$

$$log.\ \frac{21}{20}=0\cdot 021189299069938$$
multiply by 500

$$500\ log.\ \frac{21}{20}=10\cdot 594649534969000$$

Which is, therefore, the logarithm of the principal required, and will be found to correspond to 39323200000*l.*

550. If we not only add the interest annually to the principal, but also increase it every year by a new sum b, the original principal, which we call a, would increase each year in the following manner:

after 1 year, $\frac{21}{20}a+b$,

after 2 years, $\left(\frac{21}{20}\right)^2 a+\frac{21}{20}b+b$, after 3 years, $\left(\frac{21}{20}\right)^3 a+\left(\frac{21}{20}\right)^2 b+\frac{21}{20}b+b$,

after 4 years, $\left(\frac{21}{20}\right)^4 a+\left(\frac{21}{20}\right)^3 b+\left(\frac{21}{20}\right)^2 b+\frac{21}{20}b+b$,

after n years, $\left(\frac{21}{20}\right)^n a+\left(\frac{21}{20}\right)^{n-1} b+\left(\frac{21}{20}\right)^{n-2} b+\frac{21}{20}b$, &c.

This principal consists, as is evident, of two parts of which the first is $\left(\frac{21}{20}\right)^n a$; and the other, taken inversely, forms the series $b+\frac{21}{20}b+\left(\frac{21}{20}\right)^2 b+\left(\frac{21}{20}\right)^3 b+ \ldots\ldots \left(\frac{21}{20}\right)^{n-1} b$; which series is evidently a geometrical progression, the ratio of which is equal to $\frac{21}{20}$, and we shall therefore find its sum, by first multiplying the last term $\left(\frac{21}{20}\right)^{n-1} b$ by the exponent $\frac{21}{20}$; which gives $\left(\frac{21}{20}\right)^n b$: then, subtracting the first term b, there remains $\left(\frac{21}{20}\right)^n b-b$; and, lastly, dividing by the exponent *minus* 1, that is to say by $\frac{1}{20}$, we shall find the sum required to be $20\left(\frac{21}{20}\right)^n b-20b$; therefore the principal sought is, $\left(\frac{21}{20}\right)^n a+20\left(\frac{21}{20}\right)^n b-20b$ $=\left(\frac{21}{20}\right)^n+(a+20b)-20b$.

551. The resolution of this formula requires us to

calculate, separately, its first term $\left(\frac{21}{20}\right)^n \times (a+20b)$, which is n *log.* $\frac{21}{20}$ + *log.* $(a+20b)$; for the number which answers to this logarithm in the tables, will be the first term; and if from this we subtract $20b$, we shall have the principal sought.

552. A person has a principal of 1000*l.* placed out at five per cent., compound interest, to which he adds annually 100*l.* beside the interest: what will be the amount of this principal at the end of twenty-five years?

We have here $a=1000$; $b=100$; $n=25$; the operation is therefore as follows:

$$log.\ \frac{21}{20}=0{\cdot}021189299$$

Multiplying by 25 we have

$$25\ log.\ \frac{21}{20}=0{\cdot}5297324750$$

$$log.\ (a+20b)=3{\cdot}4771213135$$

And the sum $=4{\cdot}0068537885$.

So that the first part, or the number which answers to this logarithm, is 10159·1, and if we subtract $20b=2000$, we find that the principal in question, after twenty-five years, will amount to 8159·1*l.*

553. Since then this principal of 1000*l.* is always increasing, and after twenty-five years amounts to $8159\frac{1}{10}$*l.* we may require, in how many years it will amount to 1000000*l.*

Let n be the number of years required: and, since

$a=1000$, $b=100$, the principal will be, at the end of n years;

$\left(\frac{21}{20}\right)^n \cdot (3000)-2000$, which sum must make 1000000; from it therefore results this equation;

$$3000\left(\frac{21}{20}\right)^n-2000=1000000$$

And adding 2000 to both sides, we have

$$3000\left(\frac{21}{20}\right)^n=1002000$$

Then dividing both sides by 3000, we have $\left(\frac{21}{20}\right)^n=334$.

And taking the logarithms, $n\,log.\,\frac{21}{20}=log.\,334$;

Then dividing by $log.\,\frac{21}{20}$, we obtain $n=\frac{log.\,334}{log.\,\frac{21}{20}}$.

Now $log.\,334=2{\cdot}5237465$, and $log.\,\frac{21}{20}=0{\cdot}0211893$;

therefore $n=\frac{2{\cdot}5237465}{0{\cdot}0211893}$; and if, lastly, we multiply the two terms of this fraction by 10000000, we shall have $n=\frac{25237465}{211893}$, $=19$ years, 1 month, 7 days; and this is the time, in which the principal of 1000*l.* will be increased to 1000000*l.*

554. But if we supposed that a person, instead of annually increasing his principal by a certain fixed sum, diminished it, by spending a certain sum every year, we should have the following gradations, as the values of that principal *a*, year after year, supposing it put out at 5 per cent., compound interest, and re-

presenting the sum which is annually taken from it by b:

after 1 year, it would be $\frac{21}{20}a-b$,

after 2 years, $\left(\frac{21}{20}\right)^2 a-\frac{21}{20}b-b$,

after 3 years, $\left(\frac{21}{20}\right)^3 a-\left(\frac{21}{20}\right)^2 b-\frac{21}{20}b-b$,

after n years, $\left(\frac{21}{20}\right)^n a-\left(\frac{21}{20}\right)^{n-1} b-\left(\frac{21}{20}\right)^{n-2} b \ldots\ldots$

$-\left(\frac{21}{20}\right)b-b$.

555. This principal consists of two parts, one of which is $\left(\frac{21}{20}\right)^n . a$, and the other, which must be subtracted from it, taking the terms inversely, forms the following geometrical progression:

$$b+\left(\frac{21}{20}\right)b+\left(\frac{21}{20}\right)^2 b+\left(\frac{21}{20}\right)^3 b+\ \ldots\ldots\ \left(\frac{21}{20}\right)^{n-1} b.$$

Now we have already found that the sum of this progression is $20\left(\frac{21}{20}\right)^n b-20b$; if, therefore, we subtract this quantity from $\left(\frac{21}{20}\right)^n a$, we shall have for the principal required, after n years,

$$\left(\frac{21}{20}\right)^n (a-20b)+20b.$$

556. We might have also deduced this formula immediately from that of Art. 550. For, in the same manner as we annually added the sum b, in the former supposition; so, in the present, we subtract the same sum b every year. We have there-

fore only to put in the former formula, $-b$ every where instead of $+b$. But it must here be particularly remarked, that if $20b$ is greater than a, the first part becomes negative, and consequently, the principal will continually diminish, and this will be easily perceived; for if we annually take away from the principal more than is added to it by the interest, it is evident that this principal must continually become less, and at last must be absolutely reduced to nothing; as will appear from the following example:

557. A person puts out a principal of 100000*l.* at 5 per cent. interest; but he spends annually 6000*l.*; which is more than the interest of his principal, the latter being only 5000*l.*; consequently, the principal will continually diminish; and it is required to determine, in what time it will be all spent? Let us suppose the number of years to be n, and since $a=100000$ and $b=6000$, we know that after n years the amount of the principal will be

$$-20000\left(\frac{21}{20}\right)^n+120000, \text{ or } 120000-20000\left(\frac{21}{20}\right)^n.$$

So that the principal will become nothing, when $20000\left(\frac{21}{20}\right)^n$ amounts to 120000; or when $20000\left(\frac{21}{20}\right)^n=120000$. Now dividing both sides by 20000, we have $\left(\frac{21}{20}\right)^n=6$; and taking the logarithm, we have $n\,log.\,\frac{21}{20}=log.\,6$; then dividing by

$log.\ \frac{21}{20}$, we have $n = \frac{log.\ 6}{log.\ \frac{21}{20}}$, or $n = \frac{0\cdot7781513}{0\cdot0211893}$; and, consequently, $n =$ 36 years, 8 months, 22 days; at the end of which time, no part of the principal will remain.

558. It will here be proper also to show how, from the same principles, we may calculate interest for times shorter than whole years; for this purpose we make use of the formula $\left(\frac{21}{20}\right)^n . a$ already found, which expresses the amount of a principal, at 5 per cent., compound interest, at the end of n years; for if the time be less than a year, the exponent n becomes a fraction, and the calculation is performed by logarithms as before. If, for example, the amount of a principal at the end of one day were required, we should make $n = \frac{1}{365}$; if after two days, $n = \frac{2}{365}$, and so on.

Suppose, for example, the amount of 100000*l.* for 8 days were required, the interest being at 5 per cent.

Here $a = 100000$, and $n = \frac{8}{365}$, consequently the principal sought is $\left(\frac{21}{20}\right)^{\frac{8}{365}} \times 100000$; the logarithm of which quantity is $log.\ \left(\frac{21}{20}\right)^{\frac{8}{365}} + log.\ 100000 = \frac{8}{365} log. \frac{20}{21} + log.\ 100000$. Now $log. \frac{21}{20} = 0\cdot0211893$, which, multiplied by $\frac{8}{365}$, gives 0·0004644, to which

adding $log.$ 100000 =	5·0000000
the sum is	5·0004644.

and the natural number of this logarithm is found to be 100107. So that, in the first eight days, the interest of the principal is 107*l.*

560. To this subject belong also the questions for calculating the present value of a sum of money, which is payable only after a term of years. For as 20*l.*, in ready money, amounts to 21*l.* in a year; so reciprocally, a sum of 21*l.*, which cannot be received till the end of one year, is really worth only 20*l.* If, therefore, we express, by a, a sum whose payment is due at the end of a year, the present value of this sum is $\frac{20}{21}a$; and therefore to find the present worth of a principal a, payable a year hence, we must multiply it by $\frac{20}{21}$; to find its value two years before the time of payment, we multiply it by $\left(\frac{20}{21}\right)^2 a$; and in general, its value, n years before the time of payment, will be expressed by $\left(\frac{20}{21}\right)^n a$.

561. Suppose, for example, a man has to receive for five successive years, an annual rent of 100*l.* and that he wishes to give it up for ready money, the interest being at 5 per cent.; it is required to find how much he is to receive. Here the calculations may be made in the following manner:

For 100*l.* due

after 1 year, he receives	95·239
after 2 years - - - - - - - -	90·704
after 3 years - - - - - - - -	86·385
after 4 years - - - - - - - -	82·272
after 5 years - - - - - - - -	78·355

Sum of the 5 terms = 432·955

So that the possessor of the rent can claim, in ready money, only 432·955*l*.

562. If such a rent were to last a greater number of years, the calculation, in the manner we have performed it, would become very tedious; but in that case it may be facilitated as follows:

Let the annual rent be a, commencing at present and lasting n years, it will be actually worth

$$a+\left(\frac{20}{21}\right)a+\left(\frac{20}{21}\right)^2 a+\left(\frac{20}{21}\right)^3 a+\left(\frac{20}{21}\right)^4 a \ldots\ldots +\left(\frac{24}{21}\right) a.$$

Which is a geometrical progression, and the whole is reduced to finding its sum. We therefore multiply the last term by the exponent, the product of which is $\left(\frac{20}{21}\right)^{n+1} a$; then, subtracting the first term, there remains $\left(\frac{20}{21}\right)^{n+1} a-a$; and lastly, dividing by the exponent *minus* 1, that is, by $-\frac{1}{21}$, or, which amounts to the same, multiplying by -21, we shall have the sum required,

$$-21.\left(\frac{20}{21}\right)^{n+1}.a+21a, \text{ or, } 21a-21.\left(\frac{20}{21}\right)^{n+1}.a;$$

and the value of the second term, which it is required to subtract, is easily calculated by logarithms.

SECTION IV.

Of Algebraic Equations, and of the Resolution of those Equations.

CHAP. I.

Of the Solution of Problems in general.

563. The principal object of Algebra, as well as of all the other branches of the Mathematics, is to determine the value of quantities which were before unknown; and this is obtained by considering attentively the conditions given, which are always expressed in known numbers: for which reason Algebra has been defined, *The science which teaches how to determine unknown quantities by means of those that are known.*

564. The above definition agrees with all that has been hitherto laid down: we have always seen that the knowledge of certain quantities lead to that of other quantities, which before might have been considered as unknown.

Of this, Addition will readily furnish an example; for, in order to find the sum of two or more given numbers, we had to seek for an unknown number which should be equal to those known numbers taken together. And in Subtraction, we sought for a number which should be equal to the difference of two known numbers. And a multitude of other examples are presented by multiplication, division, the involution of powers, and the extraction of roots; the question being always reduced to finding, by means of known quantities, other quantities which are unknown.

565. In the last section, also, different questions were resolved, in which it was required to determine a number that could not be deduced from the knowledge of other given numbers, except under certain conditions. Yet all those questions were reduced to finding, by the aid of some given numbers, a new number which should have a certain connexion with them; and this connexion was determined by certain conditions, or properties, which were to agree with the quantity sought.

566. In Algebra, when we have a question to resolve, we represent the number sought by one of the last letters of the alphabet, and then consider in what manner the given conditions can form an equality between two quantities; which equality is represented by a kind of formula, called an *equation*, that enables us finally to determine the value of the number sought, and consequently to resolve the question. Sometimes several numbers are sought; but they are found in the same manner by equations.

567. Let us endeavour to explain this farther by an example; by supposing that the following question, or *problem*, was proposed:

Twenty persons, men and women, dine at a tavern; the share of the reckoning for one man is 8 shillings, for one woman is 7 shillings, and the whole reckoning 7*l.* 5*s.*: required the number of men and women separately?

In order to resolve this question, let us suppose that the number of men is x; and, considering this number as known, we shall proceed in the same manner as if we wished to try whether it corresponded with the conditions of the question. Now the number of men being x, and the men and women making together twenty persons, it is easy to determine the number of the women, having only to subtract that of the men from 20, that is to say, the number of women is $20-x$.

But each man spends 8 shillings; therefore x men must spend $8x$ shillings.

And, since each woman spends 7 shillings, $20-x$ women must spend $140-7x$ shillings.

So that adding together $8x$ and $140-7x$, we see that the whole 20 persons must spend $140+x$ shillings. And we know already how much they have spent; namely, 7*l.* 5*s.* or 145*s.*; there must be an equality, therefore, between $140+x$ and 145; that is to say, we have the equation $140+x=145$, and thence we easily deduce $x=5$, and consequently $20-x=20-5=15$; so that the company consisted of 5 men and 15 women.

568. Again, suppose twenty persons, men and women, go to a tavern; the men spend 24 shillings, and the women as much; but it is found that the men have spent 1 shilling each more than the women. Required the number of men and women separately?

Let the number of men be represented by x.

Then the women will be $20-x$.

Now the x men having spent 24 shillings, the share of each man is $\frac{24}{x}$: also the $20-x$ women having also spent 24 shillings, the share of each woman is $\frac{24}{20-x}$.

But we know that the share of each woman is one shilling less than that of each man; if, therefore, we subtract 1 from the share of a man, we must obtain that of a woman; and consequently $\frac{24}{x}-1=\frac{24}{20-x}$. This, therefore, is the equation from which we are to deduce the value of x; which value is not found with the same ease as in the preceding question; but we shall afterwards see that $x=8$, which value corresponds to the equation; for $\frac{24}{8}-1=\frac{24}{12}$ includes the equality $2=2$.

569. It is evident therefore how essential it is, in all problems, to consider the circumstances of the question attentively, in order to deduce from it an equation that shall express by letters the numbers sought, or unknown. After that, the whole art consists in resolving those equations, or deriving from them the values of the unknown numbers; and this shall be the subject of the present section.

570. We must remark, in the first place, the diversity which subsists among the questions themselves. In some, we seek only for one unknown quantity; in others, we have to find two, or more; and, it is to be observed, with regard to this last case, that in order to determine them all, we must deduce from the circumstances, or the conditions of the pro-

blem, as many equations as there are unknown quantities.

571. It must have already been perceived, that an equation consists of two parts separated by the sign of equality, =, to show that those two quantities are equal to one another; and we are often obliged to perform a great number of transformations on those two parts, in order to deduce from them the value of the unknown quantity; but these transformations must be all founded on the following principles: That two equal quantities remain equal, whether we add to them, or subtract from them, equal quantities; whether we multiply them, or divide them, by the same number; whether we raise them both to the same power, or extract their roots of the same degree; or lastly, if we take the logarithms of those quantities, as we have already done in the preceding section.

572. The equations which are most easily resolved, are those in which the unknown quantity does not exceed the first power, after the terms of the equation have been properly arranged; and these are called *simple equations*, or *equations of the first degree*. But if, after having reduced an equation, we find in it the square, or the second power, of the unknown quantity, it is called an *equation of the second degree*, which is more difficult to resolve. *Equations of the third degree* are those which contain the cube of the unknown quantity, and so on; all of which we shall treat of in the present section.

CHAP. II.

Of the Resolution of Simple Equations, or Equations of the First Degree.

573. When the number sought, or the unknown quantity, is represented by the letter x, and the equation we have obtained is such, that one side contains only that x, and the other simply a known number, as, for example, $x=25$, the value of x is already known: we must always endeavour, therefore, to arrive at such a form, however complicated the equation may be when first obtained; and, in the course of this section, the rules shall be explained which serve to facilitate these reductions.

574. Let us begin with the simplest cases, and suppose, first, that we have arrived at the equation $x+9=16$; here we see immediately that $x=7$: and, in general, if we have found $x+a=b$, where a and b express any known numbers, we have only to subtract a from both sides, to obtain the equation $x=b-a$, which indicates the value of x.

575. If we had the equation $x-a=b$, we must add a to both sides, and obtain the value of $x=b+a$; and we must proceed in the same manner, if the equation has this form, $x-a=a^2+1$: for we shall have immediately $x=a^2+a+1$.

In the equation $x-8a=20-6a$, we find

$$x=20-6a+8a, \text{ or } x=20+2a:$$

And in this, $x+6a=20+3a$, we have

$$x=20+3a-6a, \text{ or } x=20-3a.$$

576. If the original equation has this form, $x-a+b=c$, we may begin by adding a to both sides, which will give $x+b=c+a$; and then subtracting b from both sides, we shall find $x=c+a-b$: but we might also add $+a-b$ at once to both sides; and thus obtain immediately $x=c+a-b$.

So likewise in the following examples:

If $x-2a+3b=0$, we have $x=2a-3b$.

If $x-3a+2b=25+a+2b$, we have $x=25+4a$.

If $x-9+6a=25+2a$, we have $x=34-4a$.

577. When the given equation has the form $ax=b$, we only divide the two sides by a, to obtain $x=\frac{b}{a}$. But if the equation has the form $ax+b-c=d$, we must first make the terms that accompany ax vanish, by adding to both sides $-b+c$; and then dividing the new equation $ax=d-b+c$ by a, by which is obtained $x=\frac{d-b+c}{a}$.

And the same value of x would have been found by subtracting $+b-c$ from the given equation; that is, we should have had, in the same form, $ax=d-b+c$, and $x=\frac{d-b+c}{a}$. Hence,

If $2x+5=17$, we have $2x=12$, and $x=6$.

If $3x-8=7$, we have $3x=15$, and $x=5$.

If $4x-5-3a=15+9a$, we have $4x=20+12a$, and consequently $x=5+3a$.

578. When the first equation has the form $\frac{x}{a}=b$, we multiply both sides by a, in order to have $x=ab$.

But if it is $\frac{x}{a}+b-c=d$, we must first make

$\frac{x}{a}=d-b+c$; after which we find

$x=(d-b+c)\,a=ad-ab+ac$.

Let $\frac{1}{2}x-3=4$, then $\frac{1}{2}x=7$, and $x=14$.

Let $\frac{1}{3}x-1+2a=3+a$, the $\frac{1}{3}x=4-a$, and $x=12-3a$.

Let $\frac{x}{a-1}-1=a$, then $\frac{x}{a-1}=a+1$, and $x=a^2-1$.

579. When we have arrived at such an equation as $\frac{ax}{b}=c$, we first multiply by b, in order to have $ax=bc$, and then dividing by a, we find $x=\frac{bc}{a}$.

If $\frac{ax}{b}-c=d$, we begin by giving the equation this form $\frac{ax}{b}=d+c$, after which, we obtain the value of $ax=bd+bc$, and then that of $x=\frac{bd+bc}{a}$.

Let $\frac{2}{3}x-4=1$, then $\frac{2}{3}x=5$, and $2x=15$; whence $x=\frac{15}{2}$, $=7\frac{1}{2}$.

If $\frac{3}{4}x+\frac{1}{2}=5$, we have $\frac{3}{4}x=5-\frac{1}{2}=\frac{9}{2}$; whence $3x=18$, and $x=6$.

580. Let us now consider the case, which may frequently occur, that is, in which two or more terms contain the letter x, either on one side of the equation or on both.

If those terms are all on the same side, as in the equation $x+\frac{1}{2}x+5=11$, we have $x+\frac{1}{2}x=6$, or $3x=12$, and lastly, $x=4$.

Let $x+\frac{1}{2}x+\frac{1}{3}x=44$, be an equation, in which the value of x is required: if we first multiply by 3, we have $4x+\frac{3}{2}x=132$; then multiplying by 2, we have $11x=264$; wherefore $x=24$. We might have proceeded in a more concise manner, by beginning with the reduction of the three terms which contain x to the single term $\frac{11}{6}x$; and then dividing the equation $\frac{11}{6}x=44$ by 11: this would have given $\frac{1}{6}x=4$, wherefore $x=24$.

Let $\frac{2}{3}x-\frac{3}{4}x+\frac{1}{2}x=1$, then, by reduction, $\frac{5}{12}x=1$, and $x=2\frac{2}{5}$.

And generally, let $ax-bx+cx=d$; which is the same as $(a-b+c)x=d$, whence we derive $x=\frac{d}{a-b+c}$.

581. When there are terms containing x on both sides of the equation, we begin by making such terms vanish from that side from which it is most easily expunged; that is to say, in which there are fewest terms so involved.

If we have, for example, the equation $3x+2=x+10$, we must first subtract x from both sides, which gives $2x+2=10$; wherefore $2x=8$, and $x=4$.

Let $x+4=20-x$; here it is evident that $2x+4=20$; and consequently $2x=16$, and $x=8$.

Let $x+8=32-3x$, this gives us $4x+8=32$; or $4x=24$, whence $x=6$.

Let $15-x=20-2x$, here we shall have $15+x=20$, and $x=5$.

Let $1+x=5-\frac{1}{2}x$, which becomes $1+\frac{3}{2}x=5$, or $\frac{3}{2}x=4$; therefore $3x=8$; and lastly, $x=\frac{8}{3}=2\frac{2}{3}$.

If $\frac{1}{2}-\frac{1}{3}x=\frac{1}{3}-\frac{1}{4}x$, we must add $\frac{1}{3}x$, which gives $\frac{1}{2}=\frac{1}{3}+\frac{1}{12}x$; subtracting $\frac{1}{3}$, and transposing the terms, there remains $\frac{1}{12}x=\frac{1}{6}$; then multiplying by 12, we obtain $x=2$.

If $1\frac{1}{2}-\frac{2}{3}x=\frac{1}{4}+\frac{1}{2}x$, we add $\frac{2}{3}x$, which gives $1\frac{1}{2}=\frac{1}{4}+\frac{7}{6}x$; then subtracting $\frac{1}{4}$, and transposing we have $\frac{7}{6}x=1\frac{1}{4}$, whence we deduce $x=1\frac{1}{14}=\frac{15}{14}$ by multiplying by 6 and dividing by 7.

582. If we have an equation in which the unknown number x is a denominator, we must make the fraction vanish by multiplying the whole equation by that denominator.

Suppose that we have found $\frac{100}{x}-8=12$, then, adding 8, we have $\frac{100}{x}=20$; and multiplying by x, it becomes $100=20x$; lastly, dividing by 20, we find $x=5$.

Let now $\frac{5x+3}{x-1}=7$; here multiplying by $x-1$, we have $5x+3=7x-7$; and subtracting $5x$, there remains $3=2x-7$; then adding 7, we have $2x=10$; whence $x=5$.

583. Sometimes, also, radical signs are found in equations of the first degree. For example: a number x below 100 is required, such, that the square root of $100-x$ may be equal to 8, or $\sqrt{(100-x)}=8$; the square of both sides will give $100-x=64$, and adding x, we have $100=64+x$; whence again $x=100-64=36$.

Or, since $100-x=64$, we might have subtracted 100 from both sides; which would give $-x=-36$; or, multiplying by -1, $x=36$.

584. Lastly, the unknown number x is sometimes found in the exponent, of which we have already seen some examples; and, in this case, we must have recourse to logarithms.

Thus, when we have $2^x=512$, we take the logarithms of both sides; whence we obtain $x\,log.\,2=log.\,512$; and dividing by $log.\,2$, we find $x=\frac{log. 512}{log. 2}$. The tables then give,

$$x=\frac{2\cdot7092700}{0\cdot3010300}=\frac{270927}{30103}, \text{ or } x=9.$$

Let $5\times3^{2x}-100=305$, we add 100, which gives $5\times3^{2x}=405$; dividing by 5, we have $3^{2x}=81$;

and taking the logarithms, $2x\,log.\,3 = log.\,81$, then dividing by $2\,log.\,3$, we have $x = \frac{log.\,81}{2\,log.\,3}$, or $x = \frac{log.\,81}{log.\,9}$; whence $x = \frac{1\cdot9084850}{0\cdot9542425} = \frac{19084850}{9542425} = 2$.

CHAP. III.

Of the Solution of Questions relating to the preceding Chapter.

585. *Question* 1. To divide 7 into two such parts that the greater may exceed the less by 3.

Let the greater part be x, then the less will be $7-x$; so that $x=7-x+3$, or $x=10-x$; adding x, we have $2x=10$; and dividing by 2, the result is $x=5$.

The two parts therefore are 5 and 2.

Question 2. It is required to divide a into two parts, so that the greater may exceed the less by b.

Let the greater part be x, then the other will be $a-x$; so that $x=a-x+b$; adding x, we have $2x=a+b$; and dividing by 2, $x=\frac{a+b}{2}$.

Or the same may otherwise be done thus: let the greater part be x; which as it exceeds the less by b, it is evident that this is less than the other by b, and therefore must be $x-b$. Now these two parts, taken together, ought to make a; so that $2x-b=a$; adding b, we have $2x=a+b$, wherefore $x=\frac{a+b}{2}$, which is the value of the greater part; and that of the less will be $\frac{a+b}{2}-b$, or $\frac{a+b}{2}-\frac{2b}{2}$, or $\frac{a-b}{2}$.

586. *Question* 3. A father leaves 1600 pounds to be divided among his three sons in the following manner; viz. the eldest is to have 200 pounds more than the second, and the second 100 pounds more than the youngest. Required the share of each?

Let the share of the third son be x
Then the second will be - - - - $x+100$
The first son's share - - - - - - $x+300$

Now these three sums make up together 1600*l.*; we have, therefore,

$$3x+400=1600$$
$$3x=1200$$
$$\text{and } x=400$$

The share of the youngest is 400*l.*
That of the second is - - - 500*l.*
That of the eldest is - - - 700*l.*

587. *Question* 4. A father leaves to his four sons 8600*l.* and, according to the will, the share of the eldest is to be double that of the second, minus 100*l.*; the second is to receive three times as much as the third, minus 200*l.*; and the third is to receive four times as much as the fourth, minus 300*l.* What are the respective portions of these four sons?

Call the youngest son's share x
Then the third son's is - - $4x-300$
The second son's is - - - - $12x-1100$
And the eldest's - - - - - - $24x-2300$

And the sum of these four shares must make 8600*l.* we have, therefore, $41x-3700=8600$, or

$$41x=12300, \text{ and } x=300.$$

Therefore the youngest's share is	300*l.*
The third son's - - - - - - - - -	900*l.*
The second - - - - - - - - - - - -	2500*l.*
The eldest - - - - - - - - - - - - -	4900*l.*

588. *Question* 5. A man leaves 11000 crowns to be divided between his widow, two sons, and three daughters; and he intends that the mother should receive twice the share of a son, and each son to receive twice as much as a daughter. Required how much each of them is to receive?

Suppose the share of each daughter to be	x
Then each son's is consequently - - - -	$2x$
And the widow's - - - - - - - - - - - - -	$4x$

The sum of which gives $11x=11000$, and $x=1000$.

Each daughter 1000 crowns;	
So that the three receive in all	3000
Each son receives 2000;	
So that the two sons receive -	4000
And the mother receives - -	4000
Sum	11000 crowns.

589. *Question* 6. A father intends by his will, that his three sons should share his property in the following manner: the eldest is to receive 1000 crowns less than half the whole fortune; the second is to receive 800 crowns less than the third of the whole property; and the third is to have 600 crowns less than the fourth of the property. Required the sum of the whole fortune, and the portion of each son?

Let the fortune be expressed by x:

The share of the first son is $\frac{1}{2}x-1000$

That of the second - - - $\frac{1}{3}x-800$

That of the third - - - - $\frac{1}{4}x-600$.

So that the three sons receive in all $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x-2400$, and this sum must be equal to x; we have, therefore, the equation $\frac{13}{12}x-2400=x$; and subtracting x, there remains $\frac{1}{12}x-2400=0$; also adding 2400, we have $\frac{1}{12}x=2400$; and lastly multiplying by 12, we obtain $x=28800$.

The fortune therefore consists of 28800 crowns, and

The eldest of the sons receives	13400 crowns
The second - - - - - -	8800
The youngest - - - - -	6600
	28800 crowns.

590. *Question* 7. A father leaves four sons, who share his property in the following manner: the first takes the half of the fortune, minus 3000*l.*; the second takes the third, minus 1000*l.*; the third takes exactly the fourth of the property; and the fourth takes 600*l.* and the fifth part of the property. What was the whole fortune, and how much did each son receive?

Let the whole fortune be represented by x:

Then the eldest of the sons will have $\frac{1}{2}x-3000$

The second - - - - - - - - $\frac{1}{3}x-1000$

The third - - - - - - - - - $\frac{1}{4}x$

The youngest - - - - - - - - $\frac{1}{5}x+600$.

And the four will have received in all $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x+\frac{1}{5}x-3400$, which must be equal to x.

Whence results the equation $\frac{77}{60}x-3400=x$; then subtracting x, we have $\frac{17}{60}x-3400=0$; adding 3400 we obtain $\frac{17}{60}x=3400$; dividing by 17, we have $\frac{1}{60}x=200$; and multiplying by 60, gives $x=12000$.

The fortune therefore consisted of 12000*l.*

The first son received	3000
The second - - -	3000
The third - - - -	3000
The fourth - - -	3000

591. *Question* 8. To find a number such, that if we add to it its half, the sum exceeds 60 by as much as the number itself is less than 65.

Let the number be represented by x:

Then $x+\frac{1}{2}x-60=65-x$, or $\frac{3}{2}x-60=65-x$.

Now by adding x, we have $\frac{5}{2}x-60=65$; adding 60, we have $\frac{5}{2}x=125$; dividing by 5, gives $\frac{1}{2}x=25$; and multiplying by 2, we have $x=50$.

Consequently the number sought is 50.

592. *Question* 9. To divide 32 into two such parts, that if the less be divided by 6, and the greater by 5, the two quotients taken together may make 6.

Let the less of the two parts sought be x; the greater will be $32-x$; the first, divided by 6, gives $\frac{x}{6}$; the second, divided by 5, gives $\frac{32-x}{5}$; now, $\frac{x}{6}+\frac{32-x}{5}=6$: so that multiplying by 5, we have $\frac{5}{6}x+32-x=30$, or $-\frac{1}{6}x+32=30$; adding $\frac{1}{6}x$, we have $32=30+\frac{1}{6}x$; subtracting 30, there remains $2=\frac{1}{6}x$; and lastly, multiplying by 6, we have $x=12$.

So that the less part is 12, and the greater part is 20.

593. *Question* 10. To find such a number that if multiplied by 5, the product shall be as much less than 40 as the number itself is less than 12.

Let the number be x; which is less than 12 by $12-x$; then taking the number x five times, we have $5x$, which is less than 40 by $40-5x$, and this quantity must be equal to $12-x$.

We have therefore $40-5x=12-x$; adding $5x$, we have $40=12+4x$; and subtracting 12, we obtain $28=4x$; lastly, dividing by 4, we have $x=7$, the number sought.

594. *Question* 11. To divide 25 into two such parts, that the greater may be equal to 49 times the less.

Let the less part be x, then the greater will be $25-x$; and the latter divided by the former ought to give the quotient 49; we have therefore $\frac{25-x}{x}=49$; multiplying by x, we have $25-x=49x$; adding x, we have $25=50x$; and dividing by 50, gives $x=\frac{1}{2}$.

The less of the two numbers is $\frac{1}{2}$, and the greater is $24\frac{1}{2}$; dividing therefore the latter by $\frac{1}{2}$, or multiplying by 2, we obtain 49.

595. *Question* 12. To divide 48 into nine parts, so that every part may be always $\frac{1}{2}$ greater than the part which precedes it.

Let the first and least part be x, then the second will be $x+\frac{1}{2}$, the third $x+1$, &c.

Now these parts form an arithmetical progression, whose first term is x; therefore the ninth and last will be $x+4$. Adding those two terms together, we have $2x+4$; multiplying this quantity by the number of terms, or by 9, we have $18x+36$; and dividing this product by 2, we obtain the sum of all the nine parts $=9x+18$; which ought to be equal to 48. We have, therefore, $9x+18=48$; subtracting 18, there remains $9x=30$; and dividing by 9, we have $x=3\frac{1}{3}$.

The first part therefore is $3\frac{1}{3}$, and the nine parts succeed in the following order:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3\frac{1}{3} & +3\frac{5}{6} & +4\frac{1}{3} & +4\frac{5}{6} & +5\frac{1}{3} & +5\frac{5}{6} & +6\frac{1}{3} & +6\frac{5}{6} & +7\frac{1}{3}. \end{matrix}$$

Which together make 48.

596. *Question* 13. To find an arithmetical progression whose first term is 5, last term 10, and the entire sum 60.

Here we know neither the difference nor the number of terms; but we know that the first and the last term would enable us to express the sum of the progression, provided only the number of terms was given. We shall therefore suppose this number to be x, and express the sum of the progression by $\frac{15x}{2}$; we know also that this sum is 60; so that $\frac{15x}{2}=60$; $\frac{1}{2}x=4$, and $x=8$.

Now since the number of terms is 8, if we suppose the difference to be z, we have only to seek for the eighth term upon this supposition, and to make it equal to 10. The second term is $5+z$, the third is $5+2z$, and the eighth is $5+7z$; so that

$$5+7z=10$$
$$7z=5$$
$$\text{and} \quad z=\frac{5}{7}.$$

The difference of the progression therefore is $\frac{5}{7}$, and the number of terms is 8; consequently the progression is

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 + & 5\frac{5}{7} + & 6\frac{3}{7} + & 7\frac{1}{7} + & 7\frac{6}{7} + & 8\frac{4}{7} + & 9\frac{2}{7} + & 10; \end{array}$$

the sum of which is 60.

507. *Question* 14. To find such a number, that if 1 be subtracted from its double, and the remainder be doubled, from which if 2 be subtracted, and the remainder divided by 4, the number resulting from these operations shall be 1 less than the number sought.

Suppose this number to be x; the double is $2x$; subtracting 1, there remains $2x-1$; doubling this, we have $4x-2$; subtracting 2, there remains $4x-4$; dividing by 4, we have $x-1$; and this must be 1 less than x; so that

$$x-1=x-1.$$

But this is what is called an *identical equation*; and shows that x is indeterminate; or that any number whatever may be substituted for it.

598. *Question* 15. I bought some ells of cloth at the rate of 7 crowns for 5 ells, which I sold again at the rate of 11 crowns for 7 ells, and I gained 100 crowns by the transaction. How much cloth was there?

Suppose the number of ells to be x; we must first see how much the cloth cost; which is found by the following proportion:

As $5 : x :: 7 : \frac{7x}{5}$ the price of the ells.

This being the expenditure; let us now see the receipt: in order to which we must make the following proportion:

Ells. Ells. Crs.

$$\text{As } 7 : x :: 11 : \frac{11}{7}x \text{ crowns};$$

and this receipt ought to exceed the expenditure by 100 crowns; we have, therefore, this equation:

$$\frac{11}{7}x = \frac{7}{5}x + 100;$$

subtracting $\frac{7}{5}x$, there remains $\frac{6}{35}x = 100$; therefore $6x = 3500$, and $x = 583\frac{1}{3}$.

There were therefore $583\frac{1}{3}$ ells, which were bought for $816\frac{2}{3}$ crowns, and sold again for $916\frac{2}{3}$ crowns, by which means the profit was 100 crowns.

599. *Question* 16. A person buys 12 pieces of cloth for 140*l.*; of which two are white, three are black, and seven are blue: also, a piece of the black cloth costs two pounds more than a piece of the white, and a piece of blue cloth costs three pounds more than a piece of black. Required the price of each kind?

Let the price of a white piece be x pounds; then the two pieces of this kind will cost $2x$; also, a black piece costing $x+2$, the three pieces of this colour will cost $3x+6$; and lastly, a blue piece costs $x+5$, wherefore the seven blue pieces cost $7x+35$; so that the twelve pieces amount in all to $12x+41$.

Now the actual and known price of these twelve pieces is 140 pounds; we have, therefore, $12x+41 = 140$, and $12x = 99$; wherefore $x = 8\frac{1}{4}$.

A piece of white cloth costs $8\frac{1}{4}l.$

A piece of black cloth costs $10\frac{1}{4}l.$

A piece of blue cloth costs $13\frac{1}{4}l.$

600. *Question* 17. A man having bought some nutmegs, says that three nuts cost as much more than one penny as four cost him more than two pence halfpenny. Required the price of the nutmegs?

Let x be the excess of the price of three nuts above one penny, or four farthings. Now if three nuts cost $x+4$ farthings, four will cost, by condition of the question, $x+10$ farthings; but the price of three nuts gives that of four nuts in another way also, namely, by the Rule of Three. Thus,

$$3:4::x+4:\frac{4x+16}{3}.$$

So that $\frac{4x+16}{3}=x+10$; or, $4x+16=3x+30$; therefore $x+16=30$, and $x=14$.

Three nuts cost $4\frac{1}{2}d.$ and four cost $6d.$ wherefore each costs $1\frac{1}{2}d.$

601. *Question* 18. A certain person has two silver cups, and only one cover for both. The first cup weighs 12 ounces, and if the cover be put on it, it weighs twice as much as the other cup; but if the other cup be covered, it weighs three times as much as the first. Required the weight of the second cup, and that of the cover?

Suppose the weight of the cover to be x ounces; then the first cup being covered it will weigh $x+12$; and now this weight being double that of the second, this cup must weigh $\frac{1}{2}x+6$; and if it be covered, it will weigh $\frac{3}{2}x+6$; which weight ought to be the triple of 12, that is, three times the weight of the first cup. We shall therefore have the equation $\frac{3}{2}x+6=36$, or $\frac{3}{2}x=30$; wherefore $\frac{1}{2}x=10$ and $x=20$.

The cover therefore weighs 20 ounces, and the second cup weighs 16 ounces.

602. *Question* 19. A banker has two kinds of change: there must be a pieces of the first to make a crown; and b pieces of the second to make the same sum. Now a person wishes to have c pieces for a crown; how many pieces of each kind must the banker give him?

Suppose the banker gives x pieces of the first kind; it is evident that he will give $c-x$ pieces of the other kind; but the x pieces of the first are worth $\frac{x}{a}$ crown, by the proportion $a:x::1:\frac{x}{a}$; and the $c-x$ pieces of the second kind are worth $\frac{c-x}{b}$ crown, because we have $b:c-x::1:\frac{c-x}{b}$. So that,

$$\frac{x}{a}+\frac{c-x}{b}=1;$$

or $\frac{bx}{a}+c-x=b$; or $bx+ac-ax=ab$;

or, rather, $bx-ax=ab-ac$;

whence we have $x=\frac{ab-ac}{b-a}$, or $x=\frac{a(b-c)}{b-a}$;

consequently, $c-x=\frac{bc-ab}{b-a}=\frac{b(c-a)}{b-a}$.

The banker must therefore give $\frac{a(b-c)}{b-a}$ pieces of the first kind, and $\frac{b(c-a)}{b-a}$ pieces of the second kind.

Remark. These two numbers are easily found by the Rule of Three, when it is required to apply the results which we have obtained. Thus to find the first we say, $b-a:a::b-c:\frac{a(b-c)}{b-a}$; and the second number is found thus; $b-a:b::c-a:\frac{b(c-a)}{b-a}$.

It ought to be observed also, that a is less than b, and that c is also less than b, but at the same time greater than a, as the nature of the thing requires.

603. *Question* 20. A banker has two kinds of change; 10 pieces of one make a crown, and 20 pieces of the other make a crown; and a person wishes to change a crown into 17 pieces of money: how many of each sort must he have?

We have here $a=10$, $b=20$, and $c=17$, which furnishes the following proportions:

1st. $10:10::3:3$, so that the number of pieces of the first kind is 3.

2d. $10:20::7:14$, and there are 14 pieces of the second kind.

604. *Question* 21. A father leaves at his death several children, who share his property in the following manner: namely, the first receives a hundred pounds, and the tenth part of the remainder; the

second receives two hundred pounds, and the tenth part of the remainder; the third takes three hundred pounds, and the tenth part of what remains; and the fourth takes four hundred pounds, and the tenth part of what then remains; and so on. And it is found that the property has thus been divided equally among all the children. Required how much it was, how many children there were, and how much each received?

This question is rather of a singular nature, and therefore deserves particular attention. In order to resolve it more easily, we shall suppose the whole fortune to be z pounds; and since all the children receive the same sum, let the share of each be x, by which means the number of children is expressed by $\frac{z}{x}$: now this being laid down, we may proceed to the solution of the question, which will be as follows:

Sum, or property to be divided.	Order of the children.	Portion of each.	Differences.
z	1st	$x=100+\frac{z-100}{10}$	
$z-x$	2d	$x=200+\frac{z-x-200}{10}$	$100-\frac{x+100}{10}=0$
$z-2x$	3d	$x=300+\frac{z-2x-300}{10}$	$100-\frac{x+100}{10}=0$
$z-3x$	4th	$x=400+\frac{z-3x-400}{10}$	$100-\frac{x+100}{10}=0$
$z-4x$	5th	$x=500+\frac{z-4x-500}{10}$	$100-\frac{x+100}{10}=0$
$z-5x$	6th	$x=600+\frac{z-5x-600}{10}$	and so on.

We have inserted, in the last column, the differences which we obtain by subtracting each portion from that which follows; but all the portions being equal, each of the differences must be $=0$: and as it happens that all these differences are expressed exactly alike, it will be sufficient to make one of them equal to nothing, and we shall have the equation $100-\frac{x+100}{10}=0$. And multiplying by 10, we have $1000-x-100=0$, or $900-x=0$; consequently $x=900$.

We know now, therefore, that the share of each child was 900; so that taking any one of the equations of the third column, the first, for example, it becomes, by substituting the value of x $900=100+\frac{z-100}{10}$, whence we immediately obtain the value of z; for we have

$$9000=1000+z-100, \text{ or } 9000=900+z;$$

therefore $z=8100$; and consequently $\frac{z}{x}=9$.

So that the number of children was 9; the fortune left by the father was 8100 pounds; and the share of each child 900 pounds.

CHAP. IV.

Of the Resolutions of two or more Equations of the First Degree.

605. It frequently happens that we are obliged to introduce into algebraic calculations two or more un-

known quantities, represented by the letters x, y, z; and if the question is determinate, we are brought to the same number of equations as there are unknown quantities; from which it is then required to deduce those quantities. And as we consider, at present, those equations only which contain no powers of an unknown quantity higher than the first, and no products of two or more unknown quantities, it is evident that those equations have all the form

$$az+by+cx=d.$$

606. Beginning therefore with two equations, we shall endeavour to find from them the value of x and y: and in order that we may consider this case in a general manner, let the two equations be,

$$ax+by=c, \text{ and } fx+gy=h,$$

in which, a, b, c, and f, g, h, are known numbers. It is required, therefore, to obtain, from these two equations, the two unknown quantities x and y.

607. The most natural method of proceeding will readily present itself to the mind; which is, to determine, from both equations, the value of one of the unknown quantities, as for example x, and to consider the equality of those two values; for then we shall have an equation, in which the unknown quantity y will be found by itself, and may be determined by the rules which we have already given, and knowing y, we shall have only to substitute its value in one of the quantities that express x.

608. According to this rule, we obtain from the first equation, $x=\frac{c-by}{a}$, from the second, $x=\frac{h-gy}{f}$; and putting these

values equal to each other, we have this new equation:

$$\frac{c-by}{a}=\frac{h-gy}{f};$$

multiplying by a, the product is $c-by=\frac{ah-agy}{f}$; again by f, the product is $fc-fby=ah-agy$; adding agy, we have $fc-fby+agy=ah$; subtracting fc, gives $-fby+agy=ah-fc$; or $(ag-bf)y=ah-fc$; lastly dividing by $ag-bf$, we have

$$y=\frac{ah-fc}{ag-bf}.$$

In order now to substitute this value of y in one of the two values which we have found of x, as in the first, where $x=\frac{c-by}{a}$, we shall first have

$-by=-\frac{abh-bcf}{ag-bf}$; whence $c-by=c-\frac{abh-bcf}{ag-bf}$, or $c-by=\frac{acg-bcf-abh+bcf}{ag-bf}=\frac{acg+abh}{ag-bf}$; and dividing by a, $x=\frac{c-by}{a}=\frac{cg-bh}{ag-bf}$.

609. *Question* 1. To illustrate this method by examples, let it be proposed to find two numbers, whose sum may be 15, and difference 7.

Let us call the greater number x, and the less y: then we shall have

$$x+y=15, \text{ and } x-y=7.$$

The first equation gives

$$x=15-y$$

and the second, $x=7+y$;

whence results this equation, $15-y=7+y$. So that $15=7+2y$; $2y=8$, and $y=4$; by which means we find $x=11$.

So that the less number is 4, and the greater is 11.

610. *Question* 2. We may also generalise the preceding question, by requiring two numbers, whose sum may be a, and the difference b.

Let the greater of the two be expressed by x, and the less by y;

And we shall thus have

$$x+y=a, \text{ and } x-y=b;$$

and here the first equation gives $x=a-y$, and the second $x=b+y$.

Therefore $a-y=b+y$; $a=b+2y$; $2y=a-b$; lastly, $y=\frac{a-b}{2}$, and consequently

$$x=a-y=a-\frac{a-b}{2}=\frac{a+b}{2}.$$

Thus we find the greater number, or x, is $\frac{a+b}{2}$, and the less, or y, is $\frac{a-b}{2}$; or, which comes to the same, $x=\frac{1}{2}a+\frac{1}{2}b$, and $y=\frac{1}{2}a-\frac{1}{2}b$; and hence we derive the following theorem: When the sum of any two numbers is a, and their difference is b, the greater of the two numbers will be equal to half the sum *plus* half the difference; and the less of the two numbers will be equal to half the sum *minus* half the difference.

611. We may also resolve the same question in the following manner:

Since the two equations are,

$$x+y=a, \text{ and}$$
$$x-y=b;$$

if we add one to the other, we have $2x=a+b$.

$$\text{Therefore } x=\frac{a+b}{2}.$$

Lastly, subtracting the same equations from each other, we have $2y=a-b$; and therefore

$$y=\frac{a-b}{2}.$$

612. *Question* 3. A mule and an ass were carrying burdens amounting to several hundredweight. The ass complained of his, and said to the mule, I need only one hundredweight of your load, to make mine twice as heavy as yours; to which the mule answered, But if you give me a hundredweight of yours, I shall be loaded three times as much as you will be. How many hundredweight did each carry?

Suppose the mule's load to be x hundredweight, and that of the ass to be y hundredweight. If the mule gives one hundredweight to the ass, the one will have $y+1$, and there will remain for the other $x-1$; and since, in this case, the ass is loaded twice as much as the mule, we have $y+1=2x-2$.

Farther, if the ass gives a hundredweight to the mule, the latter has $x+1$, and the ass retains $y-1$; but the burden of the former being now three times that of the latter, we have $x+1=3y-3$.

Our two equations will consequently be,

$$y+1=2x-2, \text{ and } x+1=3y-3.$$

From the first $x=\frac{y+3}{2}$, and the second gives $x=3y-4$; whence we have the new equation $\frac{y+3}{2}=3y-4$, which gives $y=\frac{11}{5}$, and this also determines the value of x, which becomes $2\frac{3}{5}$.

The mule therefore carried $2\frac{3}{5}$ hundredweight, and the ass carried $2\frac{1}{5}$ hundredweight.

613. When there are three unknown numbers, and as many equations; as, for example,

$$x+y-z=8,$$
$$x+z-y=9,$$
$$y+z-x=10;$$

we begin, as before, by deducing a value of x from each, and we have, from the

1st $x=8+z-y$;
2d $x=9+y-z$;
3d $x=y+z-10$.

Comparing the first of these values with the second, and after that with the third also, we have the following equations:

$$8+z-y=9+y-z,$$
$$8+z-y=y+z-10.$$

Now, the first gives $2z-2y=1$, and the second gives $2y=18$, or $y=9$; if therefore we substitute this value of y in $2z-2y=1$, we have $2z-18=1$, or $2z=19$, so that $z=9\frac{1}{2}$; it remains, therefore, only to determine x, which is easily found $=8\frac{1}{2}$.

Here it happens, that the letter z vanishes in the last equation, and that the value of y is found immediately; but if this had not been the case, we should have had two equations between z and y, to be resolved by the preceding rule.

614. Suppose we had found the three following equations:

$$3x+5y-4z=25,$$
$$5x-2y+3z=46,$$
$$3y+5z-\ x=62.$$

If we deduce from each the value of x, we shall have from the

$$\text{1st } x=\frac{25-5y+4z}{3},$$

$$\text{2d } x=\frac{46+2y-3z}{5},$$

$$\text{3d } x=3y+5z-62.$$

Comparing these three values together, and first the third with the first,

we have $3y+5z-62=\frac{25-5y+4z}{3}$;

multiplying by 3, gives $9y+15z-186=25-5y+4z$;
so that $9y+15z=211-5y+4z$,
and $14y+11z=211$.

Comparing also the third with the second,

we have $3y+5z-62=\frac{46+2y-3z}{5}$,

or $46+2y-3z=15y+25z-310$,
which when reduced is $356=13y+28z$.

We shall now deduce, from these two new equations, the value of y:

1st $14y+11z=211$; or $14y=211-11z$,

$$\text{and } y=\frac{211-11z}{14}.$$

2d $13y+28z=356$; or $13y=356-28z$,

$$\text{and } y=\frac{356-28z}{13}.$$

These two values form the new equation

$$\frac{211-11z}{14}=\frac{356-28z}{13}, \text{ whence,}$$

$2743-143z=4984-392z$, or $249z=2241$, and $z=9$.

This value being substituted in one of the two equations of y and z, we find $y=8$; and lastly a similar substitution in one of the three values of x, will give $x=7$.

615. If there were more than three unknown quantities to determine, and as many equations to resolve, we should proceed in the same manner; but the calculations would often prove very tedious.

It is proper, therefore, to remark, that, in each particular case, means may always be discovered of greatly facilitating its resolution; which consist in introducing into the calculation, beside the principal unknown quantities, a new unknown quantity arbitrarily assumed, such as, for example, the sum of all the rest; and when a person is a little accustomed in such calculations, he easily perceives what is most proper to be done*. The following examples may serve to facilitate the application of these artifices.

616. *Question* 4. Three persons, A, B, and C, play together; and in the first game, A loses to each of the other two, as much money as each of them has; in the next game B loses to each of the other two, as much money as they then had; and lastly, in the third game A and B gain each, from C, as much money as they had before: when leaving off, they find that each has an equal sum, namely 24 guineas each. Required, with how much money each sat down to play?

Suppose that the stake of the first person was x, that of the second y, and that of the third z: also let us make the sum of all the stakes, or $x+y+z$, $=s$. Now, A losing in the first game as much money as the other two have, he loses $s-x$ (for he himself having had x, the two others must have had $s-x$); therefore there will remain to him $2x-s$; also B will have $2y$, and C will have $2z$.

So that, after the first game, each will have as follows: A$=2x-s$, B$=2y$, C$=2z$.

In the second game, B, who has now $2y$, loses as much money as the other two have, that is to say, $s-2y$; so that he has left $4y-s$. With regard to the others, they will each have double what they had; so that after the second game, the three

* M. Cramer has given, at the end of his Introduction to the Analysis of Curve Lines, a very excellent rule for determining immediately, and without the necessity of passing through the ordinary operations, the value of the unknown quantities of such questions, to any number. F. T.

persons have as follows: A $=4x-2s$, B $=4y-s$; C $=4z$.

In the third game, C, who has now $4z$, is the loser; he loses to A, $4x-2s$, and to B, $4y-s$; consequently after this game they will have:

A $=8x-4s$, B $=8y-2s$, C $=8z-s$.

Now, each having at the end of this game 24 guineas, we have three equations, the first of which immediately gives x, the second y, and the third z; farther, s is known to be 72, since the three persons have in all 72 guineas at the end of the last game; but it is not necessary to attend to this at first; since we have,

1st $8x-4s=24$, or $8x=24+4s$, or $x=3+\frac{1}{2}s$;

2d $8y-2s=24$, or $8y=24+2s$, or $y=3+\frac{1}{4}s$;

3d $8z-s=24$, or $8z=24+s$, or $z=3+\frac{1}{8}s$;

and adding these three values, we have

$$x+y+z=9+\frac{7}{8}s.$$

So that, since $x+y+z=s$, we have $s=9+\frac{7}{8}s$;

and consequently $\frac{1}{8}s=9$, and $s=72$.

If we now substitute this value of s in the expressions which we have found for x, y, and z, we shall find that before they began to play, A had 39 guineas, B 21, and C 12.

This solution shows, that by means of an expression for the sum of the three unknown quantities, we may overcome the difficulties which occur in the ordinary method.

617. Although the preceding question appears difficult at first, it may be resolved even without algebra, by proceeding inversely. For since the players, when they left off, had each 24 guineas, and, in the third game, A and B doubled their money, they must have had before that last game, as follows:

$$A=12, \quad B=12, \text{ and } C=48.$$

In the second game, A and C doubled their money; so that before that game they had;

$$A=6, \quad B=42, \text{ and } C=24.$$

Lastly, in the first game, A and C gained each as much money as they began with; so that at first the three persons had:

$$A=39, \quad B=21, \quad C=12.$$

The same result as we obtained by the former solution.

618. *Question 5.* Two persons owe conjointly 29 pistoles; they have both money, but neither of them enough to enable him, singly, to discharge this common debt; the first debtor says therefore to the second, If you give me $\frac{2}{3}$ of your money, I can immediately pay the debt: and the second answers, that he also could discharge the debt, if the other would give him $\frac{3}{4}$ of his money. Required, how many pistoles each had?

Suppose that the first has x pistoles, and that the second has y pistoles.

Then we shall first have, $x+\frac{2}{3}y=29$;

and also, $y+\frac{3}{4}x=29$.

The first equation gives $x=29-\frac{2}{3}y$,

and the second $x=\frac{116-4y}{3}$;

so that $29-\frac{2}{3}y=\frac{116-4y}{3}$.

From which equation, we obtain $y=14\frac{1}{2}$;

Therefore $x=19\frac{1}{3}$.

Hence the first person had $19\frac{1}{3}$ pistoles, and the second had $14\frac{1}{2}$ pistoles.

619. *Question* 6. Three brothers bought a vineyard for a hundred guineas. The youngest says, that he could pay for it alone, if the second gave him half the money which he had; the second says, that if the eldest would give him only the third of his money, he could pay for the vineyard singly; lastly, the eldest asks only a fourth part of the money of the youngest, to pay for the vineyard himself. How much money had each?

Suppose the first had x guineas; the second, y guineas; the third, z guineas; we shall then have the three following equations:

$$x+\frac{1}{2}y=100;$$

$$y+\frac{1}{3}z=100;$$

$$z+\frac{1}{4}x=100;$$

two of which only give the value of x, namely

1st $x=100-\frac{1}{2}y$,

3d $x=400-4z$.

So that we have the equation,

$100-\frac{1}{2}y=400-4z$, or $4z-\frac{1}{2}y=300$, which must be combined with the second, in order to determine y and z. Now, the second equation was, $y+\frac{1}{3}z=100$; we therefore deduce from it $y=100-\frac{1}{3}z$; and the equation found last being $4z-\frac{1}{2}y=300$, we have $y=8z-600$. Consequently the final equation is,

$100-\frac{1}{3}z=8z-600$; so that $8\frac{1}{3}z=700$, or $\frac{25}{3}z=700$, and $z=84$: consequently

$$y=100-28=72, \text{ and } x=64.$$

The youngest therefore had 64 guineas, the second had 72 guineas, and the eldest had 84 guineas.

620. As, in this example, each equation contains only two unknown quantities, we may obtain the solution required in an easier way.

The first equation gives $y=200-2x$, so that y is determined by x; and if we substitute this value in the second equation, we have

$$200-2x+\frac{1}{3}z=100;\ \text{therefore}\ \frac{1}{3}z=2x-100,$$
and $z=6x-300$.

So that z is also determined by x; and if we introduce this value into the third equation, we obtain $6x-300+\frac{1}{4}x=100$, in which x stands alone, and which, when reduced to
$25x-1600=0$, gives $x=64$: consequently,
$y=200-128=72$, and $z=384-300=84$.

621. We may follow the same method, when we have a greater number of equations. Suppose, for example, that we have in general;

$$u+\frac{x}{a}=n, \qquad x+\frac{y}{c}=n,$$
$$y+\frac{z}{c}=n, \qquad z+\frac{u}{d}=n;$$

or, destroying the fractions,

$$au+x=an, \qquad bx+y=bn,$$
$$cy+z=cn, \qquad dz+u=dn.$$

Here, the first equation gives immediately $x=an-au$, and, this value being substituted in the second, we have $abn-abu+y=bn$; so that $y=bn-abn+abu$; and the substitution of this value, in the third equation, gives $bcn-abcn+abcu+z=cn$;

therefore $z=cn-bcn+abcn-abcu$;

substituting this in the fourth equation, we have

$$cdn-bcdn+abcdn-abcdu+u=dn.$$

So that $dn-cdn+bcdn-abcdn=-abcdu+u$, or $(abcd-1).u=abcdn-bcdn+cdn-dn$; whence we have

$$u=\frac{abcdn-bcdn+cdn-dn}{abcd-1}=n\frac{(abcd-bcd+cd-d)}{abcd-1}.$$

And consequently,

$$x=\frac{abcdn-acdn+adn-an}{abcd-1}=n\times\frac{(abcd-acd+ad-a)}{abcd-1}.$$

$$y=\frac{abcdn-abdn+abn-bn}{abcd-1}=n\times\frac{(abcd-abd+ab-b)}{abcd-1}.$$

$$z=\frac{abcdn-abcn+bcn-cn}{abcd-1}=n\times\frac{(abcd-abc+bc-c)}{abcd-1}.$$

$$u=\frac{abcdn-bcdn+cdn-dn}{abcd-1}=n\times\frac{(abcd-bcd+cd-d)}{abcd-1}.$$

622. *Question* 7. A captain has three companies, one of Swiss, another of Swabians, and a third of Saxons. He wishes to storm with part of these troops, and he promises a reward of 901 crowns, on the following condition; namely, that each soldier of the company, which assaults, shall receive 1 crown, and that the rest of the money shall be equally distributed among the two other companies. Now it is found, that if the Swiss make the assault, each soldier of the other companies will receive half a crown; that, if the Swabians assault, each of the others will receive $\frac{1}{3}$ of a crown; and lastly, if the Saxons make the assault, each of the others will receive $\frac{1}{4}$ of a crown. Required the number of men in each company?

Let us suppose the number of Swiss to be x, that of Swabians y, and that of Saxons z. And let us also make $x+y+z=s$, because it is easy to see, that, by this, we abridge the calculation considerably. If, therefore, the Swiss make the assault, their number being x, that of the other will be $s-x$: now, the former receive 1 crown, and the latter half a crown; so that we shall have,

$$x+\frac{1}{2}s-\frac{1}{2}x=901.$$

In the same manner, if the Swabians make the assault, we have

$$y+\frac{1}{3}s-\frac{1}{3}y=901.$$

And lastly, if the Saxons mount to the assault, x, y, we have,

$$z+\frac{1}{4}s-\frac{1}{4}z=901.$$

Each of these three equations will now enable us to determine one of the unknown quantities and z;

For the first gives $x=1802-s$.
the second $2y=2703-s$,
the third $3z=3604-s$.

And if we now take the values of $6x$, $6y$, and $6z$, and write those values one above the other, we shall have,

$$6x=10812-6s,$$
$$6y=\ \ 8109-3s,$$
$$6z=\ \ 7208-2s,$$

by addition: $6s=26129-11s$, or $17s=26129$; so that $s=1537$; which is the whole number of soldiers. By this means we find,

$$x=1802-1537=265;$$
$$2y=2703-1537=1166, \text{ or } y=583;$$
$$3x=3604-1537=2067, \text{ or } z=689.$$

The company of Swiss therefore has 265 men; that of Swabians 583; and that of Saxons 689.

CHAP. V.

Of the Resolution of Pure Quadratic Equations.

623. An equation is said to be of the second degree, when it contains the square, or the second power, of the unknown quantity, without any of its higher powers; and an equation, containing likewise the third power of the unknown quantity, belongs to cubic equations, and its resolution requires particular rules. There are, therefore, only three kinds of terms in an equation of the second degree:

1. The term in which the unknown quantity is not found at all, or which is composed only of known numbers.

2. The term in which we find only the first power of the unknown quantity.

3. And that which contains the square, or the second power, of the unknown quantity.

So that x representing an unknown quantity, and the letters a, b, c, d, &c. the known quantities, the terms of the first kind will have the form a, the terms of the second kind will have the form bx, and the terms of the third kind will have the form cx^2.

625. We have already seen, how two or more terms of the same kind may be united together, and considered as a single term.

For example, we may consider the formula
$ax^2-bx^2+cx^2$ as a single term, representing it thus

$(a-b+c)x^2$; since, in fact, $(a-b+c)$ is a known quantity.

And also, when such terms are found on both sides of the sign $=$, we have seen how they may be brought to one side, and then reduced to a single term: let us take, for example, the equation,

$$2x^2-3x+4=5x^2-8x+11;$$

we first subtract $2x^2$, and there remains

$$-3x+4=3x^2-8x+11;$$

then adding $8x$, we obtain,

$$5x+4=3x^2+11;$$

lastly, subtracting 11, there remains $3x^2=5x-7$.

626. We may also bring all the terms to one side of the sign $=$, so as to leave *zero*, or 0, on the other; but it must be remembered, that when terms are transposed from one side to the other, their signs must be changed.

Thus, the above equation will assume this form, $3x^2-5x+7=0$; and, for this reason also, the following general formula represents all equations of the second degree;

$$ax^2\pm bx\pm c=0;$$

in which the sign $\pm$ is read *plus* or *minus*, and indicates, that such terms may be sometimes positive, and sometimes negative.

627. Whatever therefore be the original form of a quadratic equation, it may always be reduced to this formula of three terms. If we have, for example, the equation

$$\frac{ax+b}{cx+d}=\frac{ex+f}{gx+h}$$

we may, first, destroy the fractions; multiplying, for this purpose, by $cx+d$, which gives

$ax+b=\frac{cex^2+cfx+edx+fd}{gx+h}$, then by $gx+h$, we have

$$agx^2+bgx+ahx+bh=cex^2+cfx+edx+fd,$$

which is an equation of the second degree, and reducible to the three following terms, which we shall transpose by arranging them in the usual manner:

$$\left.\begin{matrix}ag\\-ce\end{matrix}\right\} x^2+\left\{\begin{matrix}+bg\\+ah\\-cf\\-ed\end{matrix}\right\}x+\left\{\begin{matrix}+bh\\-fd\end{matrix}\right\}=0.$$

We may exhibit this equation also in the following form, which is still more clear:

$$(ag-ce)x^2+(bg+ah-cf-ed)x+bh-fd=0.$$

628. Equations of the second degree, in which all the three of terms are found, are called *complete*, and the resolution of them is attended with greater difficulties; for which reason we shall first consider those, in which one of the terms is wanting.

Now, if the term x^2 were not found in the equation, it would not be a quadratic, but would belong to those of which we have already treated; and if the term, which contains only known numbers, were wanting, the equation would have this form, $ax^2\pm bx=0$, which being divisible by x, may be reduced to $ax\pm b=0$, which is likewise a simple equation, and belongs not to the present class.

629. But when the middle term, which contains the first power of x, is wanting, the equation as-

sumes this form, $ax^2 \pm c = 0$, or $ax^2 = \mp c$; as the sign of c may be either positive, or negative.

We shall call such an equation a *pure* equation of the second degree, since the resolution of it is attended with no difficulty: for we have only to divide by a, which gives $x^2 = \frac{c}{a}$; and taking the square root of both sides, we find $x = \sqrt{\frac{c}{a}}$; by which means the equation is resolved.

630. But there are three cases to be considered here. In the first, when $\frac{c}{a}$ is a square number (of which we can therefore really assign the root) we obtain for the value of x a rational number, which may be either integer, or fractional. For example, the equation $x^2 = 144$, gives $x = 12$. And $x^2 = \frac{9}{16}$, gives $x = \frac{3}{4}$.

The second case is, when $\frac{c}{a}$ is not a square, in which case we must therefore be contented with the sign $\sqrt{}$. If, for example, $x^2 = 12$, we have $x = \sqrt{12}$, the value of which may be determined by approximation, as we have already shown.

The third case is that, in which $\frac{c}{a}$ becomes a negative number; and then the value of x is altogether impossible and imaginary; and this result proves that the question, which leads to such an equation, is in itself impossible.

631. We shall also observe, before proceeding farther, that whenever it is required to extract the square root of a number, that root, as we have already remarked, has always two values, the one positive and the other negative. Suppose, for example, we have the equation $x^2=49$, the value of x will be not only $+7$, but also -7, which is expressed by $x=\pm 7$. So that all those questions admit of a double answer; but it will be easily perceived that in several cases, as those which relate to a certain number of men, the negative value cannot exist.

632. In such equations, also, as $ax^2=bx$, where the known quantity c is wanting, there may be two values of x, though we find only one if we divide by x. In the equation $x^2=3x$, for example, in which it is required to assign such a value of x, that x^2 may become equal to $3x$, this is done by supposing $x=3$, a value which is found by dividing the equation by x; but, beside this value, there is also another, which is equally satisfactory, namely $x=0$; for then $x^2=0$, and $3x=0$. Equations therefore of the second degree, in general, admit of two solutions, whilst simple equations admit only of one.

We shall now illustrate, by some examples, what we have said with regard to pure equations of the second degree.

633. *Question* I. Required a number, the half of which multiplied by the third, may produce 24.

Let this number be x; then by the question $\frac{1}{2}x$, multiplied by $\frac{1}{3}x$, must give 24; we shall therefore have the equation $\frac{1}{6}x^2=24$.

Multiplying by 6, we have $x^2=144$; and the extraction of the root gives $x=\pm 12$. We put $\pm$; for if $x=+12$, we have $\frac{1}{2}x=6$, and $\frac{1}{3}x=4$: now the product of these two numbers is 24; and if $x=-12$, we have $\frac{1}{2}x=-6$, and $\frac{1}{3}x=-4$, the product of which is likewise 24.

634. *Question* 2. Required a number such, that being increased by 5, and diminished by 5, the product of the sum by the difference may be 96.

Let this number be x, then $x+5$, multiplied by $x-5$, must give 96; whence results the equation,

$$x^2-25=96.$$

Adding 25, we have $x^2=121$; and extracting the root, we have $x=11$. Thus $x+5=16$, also $x-5=6$; and, lastly, $6\times 16=96$.

635. *Question* 3. Required a number such, that by adding it to 10, and subtracting it from 10, the sum, multiplied by the difference, will give 51.

Let x be this number; then $10+x$, multiplied by $10-x$, must make 51, so that $100-x^2=51$. Adding x^2, and subtracting 51, we have $x^2=49$, the square root of which gives $x=7$.

636. *Question* 4. Three persons, who had been playing, leave off; the first, with as many times 7 crowns, as the second has three crowns; and the second, with as many times 17 crowns, as the third has 5 crowns. Farther, if we multiply the money of the first by the money of the second, and the money of the second by the money of the third, and lastly, the money of the third by that of the first, the sum

of these three products will be $3830\frac{2}{3}$. How much money has each?

Suppose that the first player has x crowns; and since he has as many times 7 crowns, as the second has 3 crowns, we know that his money is to that of the second, in the ratio of $7:3$.

We shall therefore make $7:3::x:\frac{3}{7}x$, the money of the second player.

Also, as the money of the second player is to that of the third in the ratio of $17:5$, we shall say, $17:5::\frac{3}{7}x:\frac{15}{119}x$, the money of the third player.

Multiplying x, or the money of the first player, by $\frac{3}{7}x$, the money of the second, we have the product $\frac{3}{7}x^2$: then, $\frac{3}{7}x$, the money of the second, multiplied by the money of the third, or by $\frac{15}{119}x$, gives $\frac{45}{833}x^2$; and lastly, the money of the third, or $\frac{15}{119}x$, multiplied by x, or the money of the first, gives $\frac{15}{119}x^2$. Now the sum of these three products is $\frac{3}{7}x^2+\frac{45}{833}x^2+\frac{15}{119}x^2$; and reducing these fractions to the same denominator, we find their sum $\frac{507}{833}x^2$, which must be equal to the number $3830\frac{2}{3}$.

We have, therefore, $\frac{507}{833}x^2 = 3830\frac{2}{3}$.

So that $\frac{1521}{833}x^2 = 11492$, and $1521\,x^2$ being equal to 9572836, dividing by 1521, we have $x^2 = \frac{9572836}{1521}$; and taking its root, we find $x = \frac{3094}{39}$. This fraction is reducible to lower terms, if we divide by 13; so that $x = \frac{238}{3} = 79\frac{1}{3}$; and hence we conclude, that $\frac{3}{7}x = 34$, and $\frac{15}{119}x = 10$.

The first player has therefore $79\frac{1}{3}$ crowns, the second has 34 crowns, and the third 10 crowns.

Remark. This calculation may be performed in an easier manner; namely, by taking the factors of the numbers which present themselves, and attending chiefly to the squares of those factors.

It is evident, that $507 = 3 \times 169$, and that 169 is the square of 13; then, that $833 = 7 \times 119$, and $119 = 7 \times 17$: therefore $\frac{3 \times 169}{17 \times 49}x^2 = 3830\frac{2}{3}$, and if we multiply by 3, we have $\frac{9 \times 169}{17 \times 49}x^2 = 11492$. Let us resolve this number also into its factors; and we readily perceive, that the first is 4, that is to say, that $11492 = 4 \times 2873$; farther, 2873 is divisible by 17, so that $2873 = 17 \times 169$. Consequently, our equation will assume the following form $\frac{9 \times 169}{17 \times 49}x^2 = 4 \times 17 \times 169$, which, divided by 169, is reduced to $\frac{9}{17\times49}x^2=4\times17$; multiplying also by 17×49, and dividing by 9, we have $x^2=\frac{4\times289\times49}{9}$, in which all the factors are squares; whence we have, without any farther calculation, the root $x=\frac{2\times17\times7}{3}=\frac{328}{3}=79\frac{1}{3}$, as before.

637. *Question* 5. A company of merchants appoint a factor at Archangel. Each of them contributes for the trade, which they have in view, ten times as many crowns as there are partners; and the profit of the factors is fixed at twice as many crowns, *per cent.*, as there are partners. Also, if we multiply the $\frac{1}{100}$ part of his total gain by $2\frac{2}{9}$, it will give the number of partners; and that number is required.

Let it be x; and since, each partner has contributed $10x$, the whole capital is $10x^2$. Now, for every hundred crowns, the factor gains $2x$, so that with the capital of $10x^2$ his profit will be $\frac{1}{5}x^3$. The $\frac{1}{100}$ part of his gain is $\frac{1}{500}x^3$; multiplying by $2\frac{2}{9}$, or by $\frac{20}{9}$, we have $\frac{20}{4500}x^3$, or $\frac{1}{225}x^3$, and this must be equal to the number of partners, or x.

We have, therefore, the equation $\frac{1}{225}x^3=x$; or $x^3=225x$; which appears, at first, to be of the third degree; but as we may divide by x, it is reduced to the quadratic $x^2 = 225$, whence $x = 15$.

Hence there are fifteen partners, and each contributed 150 crowns.

CHAP. VI.

Of the Resolution of Mixt Equations of the Second Degree.

638. An equation of the second degree is said to be *mixt*, or complete, when three terms are found in it, namely, that which contains the square of the unknown quantity, as ax^2; that, in which the unknown quantity is found only in the first power, as bx; and lastly, the term which is composed of only known quantities. And since we may unite two or more terms of the same kind into one, and bring all the terms to one side of the sign $=$, the general form of a mixt equation of the second degree will be

$$ax^2 \pm bx \pm c = 0.$$

In this chapter, we shall show, how the value of x is derived from such equations: and it will be seen, that there are two methods of obtaining it.

639. An equation of the kind that we are now considering, may be reduced, by division, to such a form, that the first term may contain only the square

x^2 of the unknown quantity x. We shall leave the second term on the same side with x, and transpose the known term to the other side of the sign $=$: by which means our equation will assume the form $x^2 \pm px = \pm q$, in which p and q represent any known numbers, positive or negative; and the whole is at present reduced to determining the true value of x. We shall begin with remarking, that if x^2+px were a real square, the resolution would be attended with no difficulty, because it would only be required to take the square root of both sides.

640. But it is evident that x^2+px cannot be a square; since we have already seen, that if a root consists of two terms, for example, $x+n$, its square always contains three terms, namely, twice the product of the two parts, beside the square of each part; that is to say, the square of $x+n$ is $x^2+2nx+n^2$. Now, we have already on one side x^2+px; we may, therefore, consider x^2 as the square of the first part of the root, and in this case px must represent twice the product of x, the first part of the root by the second part: consequently, this second part must be $\frac{1}{2}p$, and in fact the square of $x+\frac{1}{2}p$, is found to be

$$x^2+px+\frac{1}{4}p^2.$$

641. Now $x^2+px+\frac{1}{4}p^2$ being a real square, which has for its root $x+\frac{1}{2}p$, if we resume our equa-

tion $x^2+px=q$, we have only to add $\frac{1}{4}p^2$ to both sides, which gives us $x^2+px+\frac{1}{4}p^2=q+\frac{1}{4}p^2$, the first side being actually a square, and the other containing only known quantities. If, therefore, we take the square root of both sides, we find $x+\frac{1}{2}p=\sqrt{(\frac{1}{4}p^2+q)}$; and subtracting $\frac{1}{2}p$, we obtain $x=-\frac{1}{2}p+\sqrt{(\frac{1}{4}p^2+q)}$; and as every square root may be taken either affirmatively or negatively, we shall have for x two values expressed thus;

$$x=-\frac{1}{2}p\pm\sqrt{(\frac{1}{4}p^2+q)}.$$

642. This formula contains the rule by which all quadratic equations may be resolved, and it will be proper to commit it to memory, that it may not be necessary to repeat, every time, the whole operation which we have gone through. We may always arrange the equation in such a manner, that the pure square x^2 may be found on one side, and the above equation have the form $x^2=-px+q$, where we see immediately that $x=-\frac{1}{2}p\pm\sqrt{(\frac{1}{4}p^2+q)}$.

643. The general rule, therefore, which we deduce from that, in order to resolve the equation $x^2=-px+q$, is founded on this consideration;

That the unknown quantity x is equal to half the coefficient or multiplier of x on the other side of the equation, *plus* or *minus* the square root of the square of this number, and the known quantity which forms the third term of the equation.

Thus, if we had the equation $x^2=6x+7$, we should immediately say, that $x=3\pm\sqrt{9+7}=3\pm4$, whence we have these two values of x, namely $x=7$, and $x=-1$. In the same manner, the equation $x^2=10x-9$, would give $x=5\pm\sqrt{25-9}=5\pm4$, that is to say, the two values of x are 9 and 1.

644. This rule will be still better understood, by distinguishing the following cases: 1st, When p is an even number; 2d, When p is an odd number; and 3d, When p is a fractional number.

1st, Let p be an even number, and the equation such, that $x^2=2px+q$; we shall, in this case, have

$$x=p\pm\sqrt{p^2+q}.$$

2d, Let p be an odd number, and the equation $x^2=px+q$; we shall here have $x=\frac{1}{2}p\pm\sqrt{\frac{1}{4}p^2+q}$; and since $\frac{1}{4}p^2+q=\frac{p^2+4q}{4}$, we may extract the square root of the denominator, and write

$$x=\frac{1}{2}p\pm\frac{\sqrt{p^2+4q}}{2}=\frac{p\pm\sqrt{p^2+4q}}{2}.$$

3d, Lastly, if p be a fraction, the equation may be resolved in the following manner. Let the equation be $ax^2=bx+c$, or $x^2=\frac{bx}{a}+\frac{c}{a}$, and we shall have, by the rule, $x=\frac{b}{2a}\pm\sqrt{\frac{b^2}{4a^2}+\frac{c}{a}}$. Now, $\frac{b^2}{4a^2}+\frac{c}{a}=$

$\frac{b^2+4ac}{4a^2}$, the denominator of which is a square; so that $x=\frac{b\pm\sqrt{b^2+4ac}}{2a}$.

645. The other method of resolving mixt quadratic equations, is to transform them into pure equations; which is done by substitution: for example, in the equation $x^2=px+q$, instead of the unknown quantity x, we may write another unknown quantity, y, such, that $x=y+\frac{1}{2}p$; by which means, when we have determined y, we may immediately find the value of x.

If we make this substitution of $y+\frac{1}{2}p$ instead of x, we have $x^2=y^2+py+\frac{1}{4}p^2$, and $px=py+\frac{1}{2}p^2$; consequently our equation will become

$$y^2+py+\frac{1}{4}p^2=py+\frac{1}{2}p^2+q;$$

which is first reduced, by subtracting py, to

$$y^2+\frac{1}{4}p^2=\frac{1}{2}p^2+q;$$

and then, by subtracting $\frac{1}{4}p^2$, to $y^2=\frac{1}{4}p^2+q$. This is a pure quadratic equation, which immediately gives

$$y=\pm\sqrt{\frac{1}{4}p^2+q}.$$

Now, since $x=y+\frac{1}{2}p$, we have

$$x=\frac{1}{2}p\pm\sqrt{\frac{1}{4}p^2+q},$$

as we found it before. It only remains therefore, to illustrate this rule by some examples.

646. *Question* 1. There are two numbers; one exceeds the other by 6, and their product is 91: what are those numbers?

If the less be x; the other will be $x+6$, and their product $x^2+6x=91$: and subtracting $6x$, there remains $x^2=91-6x$, and the rule gives

$x=-3\pm\sqrt{9+91}=-3\pm10$; so that $x=7$, or $x=-13$.

The question therefore admits of two solutions;

By one, the less number $x=7$, and the greater $x+6=13$:

By the other, the less number $x=-13$, and the greater $x+6=-7$.

647. *Question* 2. To find a number such, that if 9 be taken from its square, the remainder may be a number, as much greater than 100, as the number itself is less than 23.

Let the number sought be x; we know that x^2-9 exceeds 100 by x^2-109. And since x is less than 23 by $23-x$, we have this equation

$$x^2-109=23-x.$$

Therefore $x^2=-x+132$, and, by the rule,

$x=-\frac{1}{2}\pm\sqrt{\frac{1}{4}+132}=-\frac{1}{2}\pm\sqrt{\frac{529}{4}}=-\frac{1}{2}\pm\frac{23}{2}$. So that $x=11$, or $x=-12$.

Hence when only a positive number is required, that number will be 11, the square of which *minus* 9 is 112, and consequently greater than 100 by 12, in the same manner as 11 is less than 23 by 12.

648. *Question* 3. To find a number such, that if we multiply its half by its third, and to the product add half the number required, the result will be 30.

Suppose the number to be x, its half, multiplied by its third, will give $\frac{1}{6}x^2$; so that $\frac{1}{6}x^2+\frac{1}{2}x=30$; and multiplying by 6, we have $x^2+3x=180$, or $x^2=-3x+180$, which gives $x=-\frac{3}{2}\pm\sqrt{\frac{9}{4}+180}$ $=-\frac{3}{2}\pm\frac{27}{2}$.

Consequently either $x=12$, or $x=-15$.

649. *Question* 4. To find two numbers, the one being double the other, and such, that if we add their sum to their product, we may obtain 90.

Let one of the numbers be x, then the other will be $2x$; their product also will be $2x^2$, and if we add to this $3x$, or their sum, the new sum ought to make 90. So that $2x^2+3x=90$; or $2x^2=90-3x$; whence $x^2=-\frac{3}{2}x+45$, and thus we obtain

$$x=-\frac{3}{4}\pm\sqrt{\frac{9}{16}+45}=-\frac{3}{4}\pm\frac{27}{4}.$$

Consequently $x=6$, or $x=-7\frac{1}{2}$.

650. *Question* 5. A horse-dealer bought a horse for a certain number of crowns, and sold it again for 119 crowns, by which means his profit was as much per cent. as the horse cost him; what was his first purchase?

Suppose the horse cost x crowns; then as the dealer gains x per cent., we have this proportion;

$$\text{As } 100:x::x:\frac{x^2}{100};$$

since therefore he has gained $\frac{x^2}{100}$, and the horse originally cost him x crowns, he must have sold it for

$x+\frac{x^2}{100}$; therefore $x+\frac{x^2}{100}=119$; and subtracting x, we have $\frac{x^2}{100}=-x+119$; then multiplying by 100, we obtain $x^2=-100x+11900$. Whence, by the rule, we find $x=-50\pm\sqrt{2500+11900}=$ $-50\pm\sqrt{14400}=-50\pm120=70$.

The horse therefore cost 70 crowns, and since the horse-dealer gained 70 per cent. when he sold it again, the profit must have been 49 crowns. So that the horse must have been sold again for $70+49$, that is to say, for 119 crowns.

651. *Question* 6. A person buys a certain number of pieces of cloth: he pays for the first 2 crowns, for the second 4 crowns, for the third 6 crowns, and in the same manner always 2 crowns more for each following piece; also, all the pieces together cost him 110: how many pieces had he?

Let the number sought be x; then, by the question, the purchaser paid for the different pieces of cloth in the following manner:

for the 1, 2, 3, 4, 5 .... x pieces
he pays 2, 4, 6, 8, 10 .... $2x$ crowns.

It is therefore required to find the sum of the arithmetical progression $2+4+6+8+\ldots\ldots 2x$, which consists of x terms, that we may deduce from it the price of all the pieces of cloth taken together. The rule which we have already given for this operation requires us to add the last term to the first; and the sum is $2x+2$; which must be multiplied by the number of terms x, and the product will be $2x^2+2x$; lastly, if we divide by the difference 2 the quotient will be x^2+x, which is the sum of the progression; so that we have $x^2+x=110$; therefore $x^2=-x+110$, and $x=-\frac{1}{2}+\sqrt{\frac{1}{4}+110}=-\frac{1}{2}+\frac{21}{2}=10$.

And hence the number of pieces of cloth is 10.

652. *Question* 7. A person bought several pieces of cloth for 180 crowns; and if he had received for the same sum 3 pieces more, he would have paid 3 crowns less for each piece; how many pieces did he buy?

Let us represent the number sought by x; then each piece will have cost him $\frac{180}{x}$ crowns. Now if the purchaser had had $x+3$ pieces for 180 crowns, each piece would have cost $\frac{180}{x+3}$ crowns; and, since this price is less than the real price by three crowns, we have this equation,

$$\frac{180}{x+3}=\frac{180}{x}-3.$$

And multiplying by x, we obtain $\frac{180x}{x+3}=180-3x$; dividing by 3, we have $\frac{60x}{x+3}=60-x$; and again, multiplying by $x+3$, gives $60x=180+57x-x^2$; therefore adding x^2, we shall have $x^2+60x=180+57x$; subtracting $60x$, we shall have $x^2=-3x+180$.

The rule consequently gives,

$$x=-\frac{3}{2}+\sqrt{\frac{9}{4}+180}, \text{ or } x=-\frac{3}{2}+\frac{27}{2}=12.$$

He therefore bought for 180 crowns 12 pieces of cloth at 15 crowns the piece; and if he had got 3 pieces more, namely, 15 pieces for 180 crowns, each piece would have cost only 12 crowns, that is to say, 3 crowns less.

653. *Question* 8. Two merchants enter into partnership with a stock of 100 pounds; one leaves his money in the partnership for three months, the other leaves his for two months, and each takes out 99 pounds of capital and profit; what proportion of the stock did they separately furnish?

Suppose the first partner contributed x pounds, the other will have contributed $100-x$. Now, the former receiving 99*l.*, his profit is $99-x$, which he has gained in three months with the principal x; and since the second receives also 99*l.* his profit is $x-1$, which he has gained in two months with the principal $100-x$; it is evident also, that the profit of this second partner would have been $\frac{3x-3}{2}$, if he had remained three months in the partnership: and as the profits gained in the same time are in proportion to the principals, we have the following proportion,

$$x : 99-x :: 100-x : \frac{3x-3}{2}.$$

And the equality of the product of the extremes to that of the means, gives the equation,

$$\frac{3x^2-3x}{2} = 9900-199x+x^2;$$

then multiplying this by 2, we have $3x^2-3x=19800-398x+2x^2$; then subtracting $2x^2$, we obtain $x^2-3x=19800-398x$; and adding $3x$, gives $x^2=19800-395x$; therefore, by the rule,

$$x=-\frac{395}{2}+\sqrt{\frac{156025}{4}+\frac{79200}{4}}=-\frac{395}{2}+\frac{485}{2}=\frac{90}{2}=45.$$

Therefore the first partner contributed 45*l.* and the other 55*l.* The first having gained 54*l.* in three months, would have gained in one month 18*l.*; and the second having gained 44*l.* in two months, would have gained 22*l.* in one month: now these profits agree; for if, with 45*l.*, 18*l.* are gained in one month, 22*l.* will be gained in the same time with 55*l.*

654. *Question* 9. Two girls carry 100 eggs to market; one had more than the other, and yet the sum which they both received for them was the same. The first says to the second, If I had had your eggs, I should have received 15 pence. The other answers, If I had had yours, I should have received $6\frac{2}{3}$ pence; how many eggs did each carry to market?

Suppose the first had x eggs; then the second must have had $100-x$.

Since therefore the former would have sold $100-x$ eggs for 15 pence, we have the following proportion:

$$(100-x):15::x:\frac{15x}{100-x}.$$

Also, since the second would have sold x eggs for $6\frac{2}{3}$ pence, we readily find how much she got for $100-x$ eggs: thus,

$$\text{As } x:(100-x)::\frac{20}{3}:\frac{2000-20x}{3x}.$$

Now both the girls received the same money; we have consequently the equation, $\frac{15x}{100-x}=\frac{2000-20x}{3x}$, which becomes this,

$$25x^2=200000-4000x;$$

and lastly this,

$$x^2=-160x+8000;$$

whence we obtain

$$x=-80+\sqrt{6400+8000}=-80+120=40.$$

Hence the first girl had 40 eggs, the second had 60, and each received 10 pence.

655. *Question* 10. Two merchants sell each a certain quantity of silk; the second sells 3 ells more than the first, and they received together 35 crowns. Now the first says to the second, I should have got 24 crowns for your silk; the other answers, And I should have got for yours 12 crowns and a half. How many ells had each?

Suppose the first had x ells; then the second must have had $x+3$ ells; also, since the first would have sold $x+3$ ells for 24 crowns, he must have received $\frac{24x}{x+3}$ crowns for his x ells. And with regard to the second, since he would have sold x ells for $12\frac{1}{2}$ crowns, he must have sold his $x+3$ ells for $\frac{25x+75}{2x}$; so that the whole sum they received was

$$\frac{24x}{x+3}+\frac{25x+75}{2x}=35;$$

which equation becomes $x^2=20x-75$, whence we have $x=10\pm\sqrt{100-75}=10\pm5$.

So that the question admits of two solutions: according to the first, the first merchant had 15 ells, and the second had 18; and since the former would have sold 18 ells for 24 crowns, he must have sold his 15 ells for 20 crowns; the second, who would have sold 15 ells for 12 crowns and a half, must have sold his 18 ells for 15 crowns; so that they would have actually received 35 crowns for their commodity.

But according to the second solution, the first merchant had 5 ells, and the other 8 ells; and since the first would have sold 8 ells for 24 crowns, he must have received 15 crowns for his 5 ells; also since the second would have sold 5 ells for 12 crowns and a half, his 8 ells must have produced him 20 crowns; the sum being, as before, 35 crowns.

CHAP. VII.

Of the Extraction of the Roots of Polygon Numbers.

656. We have shown, in a preceding chapter, how polygonal numbers are to be found; and what we then called *a side*, is also called *a root*. If, therefore, we represent the root by x, we shall find the following expressions for all polygon numbers:

the III gon, or triangle, is $\frac{x^2+x}{2}$,

the IV gon, or square, - x^2,

the V gon - - - - - $\frac{3x^2-x}{2}$,

the VI gon - - - - - $2x^2-x$,

the VII gon - - - - - $\frac{5x^2-3x}{2}$,

the VIII gon - - - - - $3x^2-2x$,

the IX gon - - - - - $\frac{7x^2-5x}{2}$,

the X gon - - - - - - $4x^2-3x$,

the n gon - - - - - - $\frac{(n-2)x^2-(n-4)x}{2}$.

657. We have already satisfactorily shown, that it is easy, by means of these formulæ, to find, for any given root, any polygon number required: but when it is required reciprocally to find the side, or the root of a polygon, the number of whose sides is known, the operation is more difficult, and always requires the solution of a quadratic equation; on which account the subject deserves, in this place, to be separately considered; and in this we shall proceed regularly, beginning with the triangular numbers, and passing from them to those of a greater number of angles.

658. Let therefore 91 be the given triangular number, the side or root of which is required.

If we make this root $=x$, we must have

$\frac{x^2+x}{2}=91$; or $x^2+x=182$, and $x^2=-x+182$;

consequently,

$$x=-\frac{1}{2}+\sqrt{\frac{1}{4}+182}=-\frac{1}{2}+\sqrt{\frac{729}{4}}=-\frac{1}{2}+\frac{27}{2}=13;$$

from which we conclude, that the triangular root required is 13; that is, the triangle of 13 is 91.

659. But, in general, let a be the given triangular number, and let its root be required.

Here if we make it $=x$, we have $\frac{x^2+x}{2}=a$, or $x^2+x=2a$; therefore, $x^2=-x+2a$, and by the rule $x=-\frac{1}{2}+\sqrt{\frac{1}{4}+2a}$, or $x=\frac{-1+\sqrt{8a+1}}{2}$.

This result gives the following rule: To find a triangular root, we must multiply the given triangular number by 8, add 1 to the product, extract the root of the sum, subtract 1 from that root, and lastly, divide the remainder by 2.

660. So that all triangular numbers have this property; that if we multiply them by 8, and add unity to the product, the sum is always a square; of which the following small table furnishes some examples:

Triangles	1,	3,	6,	10,	15,	21,	28,	36,	45,	55,	&c.
8 times $+1=$	9,	25,	49,	81,	121,	169,	225,	289,	361,	441,	&c.

If the given number a does not answer this condition, we conclude, that it is not a real triangular number, or that no rational root of it can be assigned.

661. According to this rule, let the triangular root of 210 be required; we shall have $a=210$, and $8a+1=1681$, the square root of which is 41; whence we see, that the number 210 is really triangular, and that its root is $\frac{41-1}{2}=20$. But if 4 were given as the triangular number, and its root were required, we should find it $=\frac{\sqrt{33}}{2}-\frac{1}{2}$, and conse-

quently irrational; however, the triangle of this root $\frac{\sqrt{33}}{2}-\frac{1}{2}$, may be found in the following manner:

Since $x=\frac{\sqrt{33}-1}{2}$, we have $x^2=\frac{17-\sqrt{33}}{2}$, and adding x to it, the sum is $x^2+x=\frac{16}{2}=8$, and consequently the triangle, or the triangular number $\frac{x^2+x}{2}=4$.

662. The quadrangular numbers being the same thing as the squares, they occasion no difficulty. For, supposing the given quadrangular number to be a, and its required root x, we shall have $x^2=a$, and consequently, $x=\sqrt{a}$; so that the square root and the quadrangular root are the same thing.

663. Let us now proceed to pentagonal numbers. Let 22 be a number of this kind, and x its root; then we shall have $\frac{3x^2-x}{2}=22$, or $3x^2-x=44$, or $x^2=\frac{1}{3}x+\frac{44}{3}$; from which we obtain,

$x=\frac{1}{6}+\sqrt{\frac{1}{36}+\frac{44}{3}}$, or $x=\frac{1+\sqrt{529}}{6}=\frac{1}{6}+\frac{23}{6}=4$; and consequently 4 is the pentagonal root of the number 22.

664. Now let the following question be proposed: the pentagon a being given, to find its root.

Let this root be x, and we have the equation $\frac{3x^2-x}{2}=a$, or $3x^2-x=2a$, or $x^2=\frac{1}{3}x+\frac{2a}{3}$; by means of which we find $x=\frac{1}{6}+\sqrt{\frac{1}{36}+\frac{2a}{3}}$, that is,

$x = \frac{1+\sqrt{24a+1}}{6}$. Therefore, when a is a real pentagon, $24a+1$ must be a square.

Let 330, for example, be the given pentagon, the root will be $x = \frac{1+\sqrt{7921}}{6} = \frac{1+89}{6} = 15$.

665. Again, let a be a given hexagonal number, the root of which is required.

If we suppose it $= x$, we shall have $2x^2 - x = a$, or $x^2 = \frac{1}{2}x + \frac{1}{2}a$; and this gives

$$x = \frac{1}{4} + \sqrt{\frac{1}{16} + \frac{1}{2}a} = \frac{1+\sqrt{8a+1}}{4}.$$

So that, in order that a may be really a hexagon, $8a+1$ must become a square; whence we see, that all hexagonal numbers are contained in triangular; but it is not the same with the roots.

For example, let the hexagonal number be 1225, its root will be $x = \frac{1+\sqrt{9801}}{4} = \frac{1+99}{4} = 25$.

666. Suppose a an heptagonal number, of which the root is required.

Let this root be x, then we shall have $\frac{5x^2-3x}{2} = a$, or $x^2 = \frac{3}{5}x + \frac{2}{5}a$, which gives

$$x = \frac{3}{10} + \sqrt{\frac{9}{100} + \frac{2}{5}a} = \frac{3+\sqrt{40a+9}}{10};$$

therefore the heptagonal numbers have this property, that if they be multiplied by 40, and 9 be added to the product, the sum is always a square.

Let the heptagon, for example, be 2059; its root will be found $=x=\frac{3+\sqrt{82369}}{10}=\frac{3+287}{10}=29$.

667. Now let a be an octagonal number, of which the root x is required.

We shall here have $3x^2-2x=a$, or $x^2=\frac{2}{2}x+\frac{1}{3}a$,

whence results $x=\frac{1}{3}+\sqrt{\frac{1}{9}+\frac{1}{3}a}=\frac{1+\sqrt{3a+1}}{3}$.

Consequently, all octagonal numbers are such, that if they be multiplied by 3, and unity be added to the product, the sum is constantly a square.

For example, let 3816 be an octagon; its root will be $x=\frac{1+\sqrt{11449}}{3}=\frac{1+107}{3}=36$.

698. Lastly, let a be a given n-gonal number, the root of which it is required to assign; we shall then have this equation:

$\frac{(n-2)x^2-(n-4)x}{2}=a$, or $(n-2)x^2-(n-4)x=2a$,

consequently $x^2=\frac{(n-4)x}{n-2}+\frac{2a}{n-2}$; whence,

$$x=\frac{n-4}{2(n-2)}+\sqrt{\frac{(n-4)^2}{4(n-2)^2}+\frac{2a}{n-2}}, \text{ or}$$

$$x=\frac{n-4}{2(n-2)}+\sqrt{\frac{(n-4)^2}{4(n-2)^2}+\frac{8(n-2)a}{4(n-2)^2}}, \text{ or}$$

$$x=\frac{n-4+\sqrt{8(n-2)a+(n-4)^2}}{2(n-2)}.$$

This formula contains a general rule for finding all the possible polygonal roots of given numbers.

For example, let there be given the XXIV-gonal number, 3009: since a is here $=3009$ and $n=24$, we have $n-2=22$ and $n-4=20$; wherefore the root, or x, $=\frac{20+\sqrt{529584+400}}{44}=\frac{20+728}{44}=17$.

CHAP. VIII.

Of the Extraction of the Square Roots of Binomials.

669. By a *binomial** we mean a quantity composed of two parts, which are either both affected by the sign of the square root, or of which one, at least, contains that sign.

For this reason $3+\sqrt{5}$ is a binomial, and likewise $\sqrt{8}+\sqrt{3}$; and it is indifferent whether the two terms be joined by the sign $+$ or by the sign $-$. So that $3-\sqrt{5}$ and $3+\sqrt{5}$ are both binomials.

670. The reason that these binomials deserve particular attention, is, that in the resolution of quadratic equations we are always brought to quantities of this form when the resolution cannot be performed. For example, the equation $x^2=6x-4$ gives $x=3+\sqrt{5}$.

It is evident, therefore, that such quantities must

* In algebra we generally give the name *binomial* to any quantity composed of two terms, but Euler has thought proper to confine this appellation to those expressions which the French analysts call *quantities partly commensurable and partly incommensurable.* F. T.

often occur in algebraic calculations; for which reason, we have already carefully shown how they are to be treated in the ordinary operations of addition, subtraction, multiplication, and division; but we have not been able till now to show how their square roots are to be extracted; that is, so far as that extraction is possible; for when it is not, we must be satisfied with affixing to the quantity another radical sign. Thus the square root of $3+\sqrt{2}$ is written $\sqrt{3+\sqrt{2}}$.

671. And it must here be observed, in the first place, that the squares of such binomials are also binomials of the same kind; in which also one of the terms is always rational.

For, if we take the square of $a+\sqrt{b}$, we shall obtain $(a^2+b)+2a\sqrt{b}$. If therefore it were required reciprocally to take the root of the quantity $(a^2+b)+2a\sqrt{b}$, we should find it to be $a+\sqrt{b}$, and it is undoubtedly much easier to form an idea of it in this manner, than if we had only put the sign $\sqrt{}$ before that quantity. In the same manner, if we take the square of $\sqrt{a}+\sqrt{b}$, we find it $(a+b)+2\sqrt{ab}$; therefore, reciprocally, the square root of $(a+b)+2\sqrt{ab}$ will be $\sqrt{a}+\sqrt{b}$, which is likewise more easily understood than if we had been satisfied with putting the sign $\sqrt{}$ before the quantity.

672. It is here, therefore, chiefly required to assign a character, which may, in all cases, point out whether such a square root exists or not; for which purpose we shall begin with an easy quantity, requiring whether we can assign, in the sense that we have explained, the square root of the binomial $5+2\sqrt{6}$.

Suppose, therefore, that this root is $\sqrt{x}+\sqrt{y}$; the square of it is $(x+y)+2\sqrt{xy}$, which must be equal to the quantity $5+2\sqrt{6}$. Consequently, the rational part $x+y$ must be equal to 5, and the irrational part $2\sqrt{xy}$ must be equal to $2\sqrt{6}$; which last equality gives $\sqrt{xy}=\sqrt{6}$. Now, since $x+y=5$, we have $y=5-x$, and this value substituted in the equation $xy=6$, produces $5x-x^2=6$, or $x^2=5x-6$; therefore $x=\frac{5}{2}+\sqrt{\frac{25}{4}-\frac{24}{4}}=\frac{5}{2}+\frac{1}{2}=3$. So that $x=3$ and $y=2$, whence we conclude that the square root of $5+2\sqrt{6}$ is $\sqrt{3}+\sqrt{2}$.

673. As we have here found the two equations, $x+y=5$, and $xy=6$, we shall give a particular method for obtaining the values of x and y.

Since $x+y=5$, by squaring $x^2+2xy+y^2=25$; and as we know that $x^2-2xy+y^2$ is the square of $x-y$, let us subtract from $x^2+2xy+y^2=25$, the equation $xy=6$, taken four times, or $4xy=24$, in order to have $x^2-2xy+y^2=1$; whence by extraction we have $x-y=1$; and as $x+y=5$, we shall easily find $x=3$, and $y=2$: consequently, the square root of $5+2\sqrt{6}$ is $\sqrt{3}+\sqrt{2}$.

674. Let us now consider the general binomial $a+\sqrt{b}$, and supposing its square root to be $\sqrt{x}+\sqrt{y}$, we shall have the equation $(x+y)+2\sqrt{xy}=a+\sqrt{b}$; so that $x+y=a$, and $2\sqrt{xy}=\sqrt{b}$, or $4xy=b$; subtracting this square from the square of the equation $x+y=a$, or from $x^2+2xy+y^2=a^2$, there remains $x^2-2xy+y^2=a^2-b$, the square root of which is $x-y=\sqrt{a^2-b}$. Now $x+y=a$; we have therefore $x=\frac{a+\sqrt{a^2-b}}{2}$, and $y=\frac{a-\sqrt{a^2-b}}{2}$;

consequently, the square root required of $a+\sqrt{b}$ is

$$\sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

675. We admit that this expression is more complicated than if we had simply put the radical sign $\sqrt{}$ before the given binomial $a+\sqrt{b}$, and written it $\sqrt{a+\sqrt{b}}$: but the above expression may be greatly simplified when the numbers a and b are such, that a^2-b is a square; since then the sign $\sqrt{}$ which is under the radical disappears. We see also, at the same time, that the square root of the binomial $a+\sqrt{b}$ cannot be conveniently extracted, except when $a^2-b=c^2$; for in this case the square root required is $\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}$: but if a^2-b is not a perfect square, we cannot express the square root of $a+\sqrt{b}$ more simply, than by putting the radical sign $\sqrt{}$ before it.

676. The condition, therefore, which is requisite, in order that we may express the square root of a binomial $a+\sqrt{b}$ in a more convenient form, is, that a^2-b be a square; and if we represent that square by c^2, we shall have for the square root in question $\sqrt{\frac{a+c}{2}}+\sqrt{\frac{a-c}{2}}$. We must farther remark, that the square root of $a-\sqrt{b}$ will be $\sqrt{\frac{a+c}{2}}-\sqrt{\frac{a-c}{2}}$; for, by squaring this quantity, we get $a-2\sqrt{\frac{a^2-c^2}{4}}$; now, since $c^2=a^2-b$, and consequently $a^2-c^2=b$, the same square is found $=a-2\sqrt{\frac{b}{4}}=a-\frac{2\sqrt{b}}{2}=a-\sqrt{b}$.

677. When it is required, therefore, to extract the square root of a binomial, as $a \pm \sqrt{b}$, the rule is, to subtract from the square a^2 of the rational part the square b of the irrational part, to take the square root of the remainder, and calling that root c, to write for the root required $\sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}}$.

678. If the square root of $2+\sqrt{3}$ was required, we should have $a=2$ and $b=3$; wherefore $a^2-b = c^2 = 1$; so that the root sought $= \sqrt{\frac{3}{2}} + \sqrt{\frac{1}{2}}$.

Let it be required to find the square root of the binomial $11+6\sqrt{2}$. Here we shall have $a=11$, and $\sqrt{b}=6\sqrt{2}$; consequently, $b=36 \times 2=72$, and $a^2-b=49$, which gives $c=7$; and hence we conclude, that the square root of $11+6\sqrt{2}$ is $\sqrt{9}+\sqrt{2}$, or $3+\sqrt{2}$.

Required the square root of $11+2\sqrt{30}$. Here $a=11$ and $\sqrt{b}=2\sqrt{30}$; consequently, $b=4 \times 30 = 120$, $a^2-b=1$, and $c=1$; therefore the root required is $\sqrt{6}+\sqrt{5}$.

679. This rule also applies, even when the binomial contains imaginary, or impossible quantities.

Let there be proposed, for example, the binomial $1+4\sqrt{-3}$. First, we shall have $a=1$ and $\sqrt{b}=4\sqrt{-3}$, that is to say, $b=-48$, and $a^2-b = 49$; therefore $c=7$, and consequently the square root required is $\sqrt{4}+\sqrt{-3}=2+\sqrt{-3}$.

Again, let there be given $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$. First, we have $a=-\frac{1}{2}$; $\sqrt{b}=\frac{1}{2}\sqrt{-3}$, and $b=\frac{1}{4}\times -3=-\frac{3}{4}$;

whence $a^2-b=\frac{1}{4}+\frac{3}{4}=1$, and $c=1$; and the result required is $\sqrt{\frac{1}{4}}+\sqrt{-\frac{3}{4}}=\frac{1}{2}+\frac{\sqrt{-3}}{2}$, or $\frac{1}{2}+\frac{1}{2}\sqrt{-3}$.

Another remarkable example is that in which it is required to find the square root of $2\sqrt{-1}$. As there is here no rational part, we shall have $a=0$; now $\sqrt{b}=2\sqrt{-1}$, and $b=-4$; wherefore $a^2-b=4$ and $c=2$; consequently the square root required is $\sqrt{1}+\sqrt{-1}=1+\sqrt{-1}$, and the square of this quantity is found to be $1+2\sqrt{-1}-1=2\sqrt{-1}$.

680. Suppose now we have such an equation as $x^2=a\pm\sqrt{b}$, and that $a^2-b=c^2$; we conclude from this, that the value of $x=\sqrt{\frac{a+c}{2}}\pm\sqrt{\frac{a-c}{2}}$, which may be useful in many cases.

For example, if $x^2=17+12\sqrt{2}$, we shall have $x=3+\sqrt{8}=3+2\sqrt{2}$.

681. This case occurs most frequently in the resolution of equations of the fourth degree, such as $x^4=2ax^2+d$. For, if we suppose $x^2=y$, we have $x^4=y^2$, which reduces the given equation to $y^2=2ay+d$, and from this we find $y=a\pm\sqrt{a^2+d}$, therefore, $x^2=a\pm\sqrt{a^2+d}$, and consequently we have another evolution to perform. Now since $\sqrt{b}=\sqrt{a^2+d}$, we have $b=a^2+d$, and $a^2-b=-d$; if, therefore, $-d$ is a square, as c^2, that is to say, $d=-c^2$, we may assign the root required.

Suppose, in reality, that $d=-c^2$; or that the proposed equation of the fourth degree is $x^4=2ax^2-c^2$, which gives $x=\sqrt{\frac{a+c}{2}}\pm\sqrt{\frac{a-c}{2}}$.

682. We shall illustrate what we have just said by some examples.

1, Required two numbers whose product may be 105, and whose squares may together make 274.

Let us represent those two numbers by x and y; we shall then have the two equations,

$$xy=105$$
$$x^2+y^2=274.$$

The first gives $y=\frac{105}{x}$, and this value of y being substituted in the second equation, we have

$$x^2+\frac{105^2}{x^2}=274.$$

Wherefore $x^4+105^2=274x^2$, or $x^4=274x^2-105^2$.

If we now compare this equation with that in the preceding article, we have $2a=274$, and $-c^2=-105^2$; consequently, $c=105$, and $a=137$. We therefore find

$$x=\sqrt{\frac{137+105}{2}}\pm\sqrt{\frac{137-105}{2}}=11\pm4.$$

Whence $x=15$, or $x=7$. In the first case $y=7$, in the second case $y=15$; whence the two numbers sought are 15 and 7.

683. It is proper, however, to observe, that this calculation may be performed much more easily in another way. For, since $x^2+2xy+y^2$ and $x^2-2xy+y^2$ are squares, and we know the values of x^2+y^2 and of xy, we have only to take the double of this last quantity, and then to add and subtract it from the first, as follows: $x^2+y^2=274$; to which if we add $2xy=210$, we have $x^2+2xy+y^2=484$, which gives $x+y=22$.

But subtracting $2xy$, there remains $x^2-2xy+y^2=64$, whence we find $x-y=8$.

So that $2x=30$, and $2y=14$; consequently, $x=15$ and $y=7$.

The following general question is resolved by the same method.

2, Required two numbers, whose product may be m, and the sum of the squares n.

If those numbers are represented by x and y, we have the two following equations:

$$xy=m$$
$$x^2+y^2=n.$$

Now $2xy=2m$ being added to $x^2+y^2=n$, we have $x^2+2xy+y^2=n+2m$, and consequently,

$$x+y=\sqrt{n+2m}.$$

But subtracting $2xy$, there remains $x^2-2xy+y^2=n-2m$, whence we get $x-y=\sqrt{n-2m}$; we have, therefore, $x=\frac{1}{2}\sqrt{n+2m}+\frac{1}{2}\sqrt{n-2m}$ and

$$y=\frac{1}{2}\sqrt{n+2m}-\frac{1}{2}\sqrt{n-2m}.$$

684. 3, Required two numbers, such, that their product may be 35, and the difference of their squares 24.

Let the greater of the two numbers be x, and the less y: then we shall have the two equations

$$xy=35,$$
$$x^2+y^2=24;$$

and as we have not the same advantages here, we shall proceed in the usual manner. Here the first equation gives $y=\frac{35}{x}$, and, substituting this value of y in the second, we have $x^2 - \frac{1225}{x^2} = 24$. Multiplying by x^2, we have $x^4 - 1225 = 24x^2$; or $x^4 = 24x^2 + 1225$. Now the second member of this equation being affected by the sign $+$, we cannot make use of the formula given above, because having $c^2 = -1225$, c would become imaginary.

Let us therefore make $x^2 = z$; we shall then have $z^2 = 24z + 1225$, whence we obtain

$$z = 12 \pm \sqrt{144 + 1225}, \text{ or } z = 12 \pm 37;$$

consequently $x^2 = 12 \pm 37$, that is to say, either $= 49$ or $= -25$.

Or if we adopt the first value, we have $x = 7$ and $y = 5$.

The second value gives $x = \sqrt{-25}$ and

$$y = \frac{35}{\sqrt{-25}} = \sqrt{\frac{1225}{-25}} = \sqrt{-49}.$$

685. We shall conclude this chapter with the following question.

4. Required two numbers, such, that their sum, their product, and the difference of their squares, may be all equal.

Let x be the greater of the two numbers, and y the less; then the three following expressions must be equal to one another: namely, the sum $x+y$; the product xy; and the difference of the squares $x^2 - y^2$. Now if we compare the first with the second, we have $x + y = xy$, which will give a value of x; for we shall have $y = xy - x = x(y-1)$, and $x = \frac{y}{y-1}$.

Consequently, $x + y = \frac{y}{y-1} + y = \frac{y^2}{y-1}$, and $xy=\frac{y^2}{y-1}$,

that is to say, the sum is equal to the product; and to this also the difference of the squares ought to be equal. Now, we have $x^2-y^2=\frac{y^2}{y^2-2y+1}-y^2=\frac{-y^4+2y^3}{y^2-2y+1}$; so that making this equal to the quantity found $\frac{y^2}{y-1}$, we have $\frac{y^2}{y-1}=\frac{-y^4+2y^3}{y^2-2y+1}$; dividing by y^2, we have $\frac{1}{y-1}=\frac{-y^2+2y}{y^2-2y+1}$; and multiplying by $(y-1)^2$, we have $y-1=-y^2+2y$; consequently, $y^2=y+1$; which gives $y=\frac{1}{2}\pm\sqrt{\frac{1}{4}+1}=\frac{1}{2}\pm\sqrt{\frac{5}{4}}$; or $y=\frac{1\pm\sqrt{5}}{2}$, and we shall therefore have, by substitution, $x=\frac{\sqrt{5}+1}{\sqrt{5}-1}$ by using the sign +.

In order to remove the surd quantity from the denominator, multiply both terms by $\sqrt{5}+1$, and we obtain $x=\frac{6+2\sqrt{5}}{4}=\frac{3+\sqrt{5}}{2}$.

Therefore the greater of the numbers sought, or x, $=\frac{3+\sqrt{5}}{2}$; and the less, y, $=\frac{1+\sqrt{5}}{2}$.

Hence their sum $x+y=2+\sqrt{5}$; their product $xy=2+\sqrt{5}$; and since $x^2=\frac{7+3\sqrt{5}}{2}$, and $y^2=\frac{3+\sqrt{5}}{2}$, we have also the difference of the squares $x^2-y^2=2+\sqrt{5}$, being all the same quantity.

686. As this solution is very long, it is proper to remark that it may be abridged. In order to which let us begin with making the sum $x+y$ equal to the difference of the squares x^2-y^2; we shall then have $x+y=x^2-y^2$; and dividing by $x+y$, because $x^2-y^2=(x+y)\times(x-y)$, we find $1=x-y$ and $x=y+1$. Consequently, $x+y=2y+1$, and $x^2-y^2=2y+1$; farther, as the product xy, or y^2+y, must be equal to the same quantity, we have $y^2+y=2y+1$, or $y^2=y+1$, which gives, as above, $y=\frac{1+\surd 5}{2}$.

687. The preceding question leads also to the solution of the following.

5, To find two numbers, such, that their sum, their product, and the sum of their squares, may be all equal.

Let the numbers sought be represented by x and y; then there must be an equality between $x+y$, xy, and x^2+y^2.

Comparing the first and second quantities, we have $x+y=xy$, whence $x=\frac{y}{y-1}$; consequently, xy, or $x+y=\frac{y^2}{y-1}$. Now, the same quantity is equal to x^2+y^2, so that we have

$$\frac{y^2}{y^2-2y+1}+y^2=\frac{y^2}{y-1}.$$

Multiplying by y^2-2y+1, the product is

$$y^4-2y^3+2y^2=y^3-y^2, \text{ or } y^4=3y^3-3y^2;$$

and dividing by y^2, we have $y^2=3y-3$: which gives $y=\frac{3}{2}\pm\sqrt{\frac{9}{4}-3}=\frac{3+\sqrt{-3}}{2}$; consequently, $y-1=\frac{1+\sqrt{-3}}{2}$, whence results $x=\frac{3+\sqrt{-3}}{1+\sqrt{-3}}$; and multiplying both terms by $1-\sqrt{-3}$, the result is $x=\frac{6-2\sqrt{-3}}{4}$, or $\frac{3-\sqrt{-3}}{2}$.

Therefore the numbers sought are $x=\frac{3+\sqrt{-3}}{2}$, and $y=\frac{3+\sqrt{-3}}{2}$, the sum of which is $x+y=3$, their product $xy=3$; and lastly, since $x^2=\frac{3-3\sqrt{-3}}{2}$, and $y^2=\frac{3+3\sqrt{-3}}{2}$, the sum of the squares $x^2+y^2=3$, all the same quantity as required.

688. We may greatly abridge this calculation by a particular artifice, that is applicable likewise to other cases; and which consists in expressing the numbers sought by the sum and the difference of two letters, instead of representing them by distinct letters.

In our last question, let us suppose one of the numbers sought to be $p+q$, and the other $p-q$, then their sum will be $2p$, their product will be

$$p^2-q^2,$$

and the sum of their squares will be

$$2p^2+2q^2,$$

which three quantities must be equal to each other; therefore making the first equal to the second, we have $2p=p^2-q^2$, which gives $q^2=p^2-2p$.

Substituting this value of q^2 in the third quantity $(2p^2+2q)$, and comparing the result $4p^2-4p$

with the first, we have $2p=4p^2-4p$, whence $p=\frac{3}{2}$.

Consequently $q^2=-\frac{3}{4}$, and $q=\frac{\sqrt{-3}}{2}$; so that the numbers sought are $p+q=\frac{3+\sqrt{-3}}{2}$, and $p-q=\frac{3-\sqrt{-3}}{2}$, as before.

CHAP. IX.

Of the Nature of Equations of the Second Degree.

689. What we have already said sufficiently shows, that equations of the second degree admit of two solutions; and this property ought to be examined in every point of view, because the nature of equations of a higher degree will be very much illustrated by such an examination. We shall therefore retrace, with more attention, the reasons which render an equation of the second degree capable of a double solution; since they undoubtedly will exhibit an essential property of those equations.

690. We have already seen, it is true, that this double solution arises from the circumstance that the square root of any number may be taken either positively, or negatively; however, as this

principle will not easily apply to equations of higher degrees, it may be proper to illustrate it by a distinct analysis. Taking, therefore, for an example, the quadratic equation, $x^2=12x-35$, we shall give a new reason for this equation being resolvable in two ways, by admitting for x the values 5 and 7, both of which satisfy the terms of the equation.

691. For this purpose it is most convenient to begin with transposing the terms of the equation, so that one of the sides may become 0; the above equation consequently takes the form

$$x^2-12x+35=0;$$

and it is now required to find a number such, that, if we substitute it for x, the quantity $x^2-12x+35$ may be really equal to nothing; after which, we shall have to show how this may be done in two different ways.

692. Now, the whole of this consists in showing clearly, that a quantity of the form $x^2-12x+35$ may be considered as the product of two factors; thus, in fact, the quantity of which we speak is composed of the two factors $(x-5)\times(x-7)$. For, since this quantity must become 0, we must also have the product $(x-5)\times(x-7)=0$; but a product, of whatever number of factors it is composed, becomes equal to 0, only when one of those factors is reduced to 0; this is a fundamental principle to which we must pay particular attention, especially when equations of higher degrees are treated of.

693. It is therefore easily understood, that the product $(x-5)\times(x-7)$ may become 0 in two ways: first, when the first factor $x-5=0$; and

also, when the second factor $x-7=0$. In the first case $x=5$, in the second $x=7$. The reason is therefore very evident, why such an equation $x^2-12x+35=0$, admits of two solutions; that is to say, why we can assign two values of x, both of which equally satisfy the terms of the equation; which depends upon this fundamental principle, that the quantity $x^2-12x+35$ may be represented by the product of two factors.

694. The same circumstances are found in all equations of the second degree: for, after having brought all the terms to one side, we always find an equation of the following form $x^2-ax+b=0$, and this formula may be always considered as the product of two factors, which we shall represent by $(x-p)\times(x-q)$, without concerning ourselves what numbers the letters p and q represent, or whether they be negative or positive. Now, as this product must be $=0$, from the nature of our equation, it is evident that this may happen in two cases; in the first place, when $x=p$; and in the second place, when $x=q$; and these are the two values of x which satisfy the terms of the equation.

695. Let us here consider the nature of these two factors, in order that the multiplication of the one by the other may exactly produce x^2-ax+b. Now by actually multiplying them, we obtain $x^2-(p+q)x+pq$; which quantity must be the same as x^2-ax+b, therefore we have evidently $p+q=a$, and $pq=b$. Hence is deduced this very remarkable property; that in every equation of the form $x^2-ax+b=0$, the two values of x are such, that their sum is equal to a, and their product

equal to b; it therefore necessarily follows that, if we know one of the values, the other also is easily found.

696. We have at present considered the case in which the two values of x are positive, and which requires the second term of the equation to have the sign $-$, and the third term to have the sign $+$: let us also consider the cases in which either one or both values of x become negative. The first takes place, when the two factors of the equation give a product of this form $(x-p)\times(x+q)$; for then the two values of x are $x=p$, and $x=-q$; and the equation itself becomes

$$x^2+(q-p)x-pq=0;$$

the second term having the sign $+$, when q is greater than p, and the sign $-$, when q is less than p; lastly, the third term is always negative.

The second case, in which both values of x are negative, occurs, when the two factors are

$$(x+p)\times(x+q);$$

for we shall then have $x=-p$ and $x=-q$; the equation itself therefore becomes

$$x^2+(p+q)\,x+pq=0,$$

in which both the second and third terms are affected by the sign $+$.

697. The signs of the second and the third terms consequently show us the nature of the roots of any equation of the second degree. For let the equation be $x^2 \ldots\, ax \ldots\, b=0$, now, if the second and third terms have the sign $+$, the two values of x are both negative; and if the second term have the sign $-$, and the third term $+$, both values are positive; lastly, if the third term also

has the sign $-$, one of the values in question is positive. But in all cases whatever, the second term contains the sum of the two values, and the third term contains their product.

698. After what has been said, it will be very easy to form equations of the second degree containing any two given values. Thus, for example, let there be required an equation such, that one of the values of x may be 7, and the other -3. We first form the simple equations $x=7$ and $x=-3$; thence these, $x-7=0$ and $x+3=0$, which give us, in this manner, the factors of the equation required, which consequently becomes $x^2-4x-21=0$. Applying here, also, the above rule, we find the two given values of x; for if $x^2=4x+21$, we have $x=2\pm\sqrt{25}=2\pm5$, that is to say, $x=7$, or $x=-3$.

699. The values of x may also happen to be equal. Suppose, for example, an equation be required, in which both values may be 5: here the two factors will be $(x-5)\times(x-5)$, and the equation sought will be $x^2-10x+25=0$; in this equation, x appears to have only one value; but it is because x is twice found $=5$, as the common method of resolution shows; for we have $x^2=10x-25$; wherefore $x=5\pm\sqrt{0}=5\pm0$, that is to say, x is in two ways $=5$.

700. A very remarkable case sometimes occurs, in which both values of x become imaginary, or impossible; and it is then wholly impossible to assign any value for x, that would satisfy the terms of the equation. Let it be proposed, for example, to divide

the number 10 into two parts, such that their product may be 30. If we call one of those parts x, the other will be $10-x$, and their product will be $10x-x^2=30$; wherefore $x^2=10x-30$, and $x=5\pm\sqrt{-5}$, which, being an imaginary number, shows that the question is impossible.

701. It is very important, therefore, to discover some sign, by means of which we may immediately know whether an equation of the second degree is possible or not.

Let us resume the general equation $x^2-ax+b=0$. We shall have $x^2=ax-b$, and $x=\frac{1}{2}a\pm\sqrt{\frac{1}{4}a^2-b}$.

This shows, that if b be greater than $\frac{1}{4}a^2$, or $4b$ greater than a^2, the two values of x are always imaginary, since it would be required to extract the square root of a negative quantity; on the contrary, if b be less than $\frac{1}{4}a^2$, or even less than 0, that is to say, if it be a negative number, both values will be possible or real. But, whether they be real or imaginary, it is no less true, that they are still expressible, and always have this property, that their sum is equal to a, and their product equal to b. Thus in the equation $x^2-6x+10=0$, the sum of the two values of x must be 6, and the product of these two values must also be 10 by the question; hence we find, $x=3+\sqrt{-1}$, and $x=3-\sqrt{-1}$, quantities whose sum is 6, and the product 10.

702. The expression which we have just found,

may likewise be represented in a manner more general, and so as to be applied to equations of this form, $fx^2 \pm gx + h = 0$; for this equation gives

$x^2 = \mp \frac{gx}{f} - \frac{h}{f}$, and $x = \mp \frac{g}{2f} \pm \sqrt{\frac{g^2}{4f^2} - \frac{h}{f}}$, or..

$x = \frac{\mp g \pm \sqrt{g^2 - 4fh}}{2f}$; whence we conclude, that the two values are imaginary, and consequently the equation impossible, when $4fh$ is greater than g^2; that is to say, when, in the equation $fx^2 - gx + h = 0$, four times the product of the first and the last term exceeds the square of the second term: for the product of the first and the last term, taken four times, is $4fhx^2$, and the square of the middle term is g^2x^2; now, if $4fhx^2$ is greater than g^2x^2, $4fh$ is also greater than g^2, and, in that case, the equation is evidently impossible; but in all other cases, the equation is possible, and two real values of x may be assigned: it is true, they are often irrational; but we have already seen, that, in such cases, we may always find them by approximation: whereas no approximations can take place with regard to imaginary expressions, such as $\sqrt{-5}$; for 100 is as far from being the value of that root, as 1, or any other number.

703. We have farther to observe, that any quantity of the second degree, $x^2 \pm ax \pm b$, must always be resolvable into two factors, such as $(x \pm p) \times (x \pm q)$. For, if we took three factors, such as these, we should come to a quantity of the third degree; and taking only one such factor, we should not exceed the first degree.

It is therefore certain, that every equation of the second degree necessarily contains two values of x, and that it can neither have more nor less.

704. We have already seen, that when the two factors are found, the two values of x are also known, since each factor gives one of those values, by making it equal to 0. The converse also is true, *viz.* that when we have found one value of x, we know also one of the factors of the equation; for if $x=p$ represents one of the values of x, in any equation of the second degree, $x-p$ is one of the factors of that equation; that is to say, all the terms having been brought to one side, the equation is divisible by $x-p$; and farther, the quotient expresses the other factor.

705. In order to illustrate what we have now said, let there be given the equation $x^2+4x-21=0$, in which we know that $x=3$ is one of the values of x, because $\overline{3\times3}+\overline{4\times3}-21=0$; this shows, that $x-3$ is one of the factors of the equation, or that $x^2+4x-21$ is divisible by $x-3$, which the actual division proves. Thus,

$$
\begin{array}{r|l}
x-3) & x^2+4x-21\ (x+7 \\
 & \underline{x^2-3x} \\
 & 7x-21 \\
 & \underline{7x-21} \\
 & 0.
\end{array}
$$

So that the other factor is $x+7$, and our equation is represented by the product $(x-3)\times(x+7)=0$; whence the two values of x immediately follow, the first factor giving $x=3$, and the other $x=-7$.

CHAP. X.

Of Pure Equations of the Third Degree.

706. An equation of the third degree is said to be *pure*, when the cube of the unknown quantity is equal to a known quantity, and neither the square of the unknown quantity, nor the unknown quantity itself, is found in the equation; so that

$$x^3 = 125, \text{ or, more generally, } x^3 = a,\ x^3 = \frac{a}{b}, \text{ \&c.}$$

are equations of this kind.

707. And it is evident how we are to deduce the value of x from such an equation, since we have only to extract the cube root on both sides. Thus the equation $x^3 = 125$ gives $x = 5$, the equation $x^3 = a$ gives $x = \sqrt[3]{a}$, and the equation $x^3 = \frac{a}{b}$ gives

$x = \sqrt[3]{\frac{a}{b}}$, or $x = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$: therefore, to be able to resolve such equations, it is sufficient that we know how to extract the cube root of a given number.

708. But in this manner, we obtain only one value for x: as, however, every equation of the second degree has two values, there is reason to suppose that an equation of the third degree has also more than one value; and it will be deserving our attention to investigate this; and, if we find that, in such equations x must have several values, it will be very useful to determine those values.

709. Let us consider, for example, the equation $x^3=8$, with a view of deducing from it all the numbers whose cubes are 8. As $x=2$ is undoubtedly such a number, what has been said in the last chapter shows that the quantity $x^3-8=0$, must be divisible by $x-2$: let us therefore perform this division.

$$
\begin{array}{r|l}
x-2)\ x^3-8\ (x^2+2x+4 & \\
x^3-2x^2 & \\
\hline
2x^2-8 & \\
2x^2-4x & \\
\hline
4x-8 & \\
4x-8 & \\
\hline
0. &
\end{array}
$$

Hence it follows, that our equation, $x^3-8=0$, may be represented by these factors;

$$(x-2)\times(x^2+2x+4)=0.$$

710. Now the question is, to know what number we are to substitute instead of x, in order that $x^3=8$, or that $x^3-8=0$; and it is evident that this condition is answered, by supposing the product which we have just now found equal to 0: but this happens, not only when the first factor $x-2=0$, which gives us $x=2$, but also when the second factor $x^2+2x+4=0$. Let us, therefore, make $x^2+2x+4=0$; then we shall have $x^2=-2x-4$, and thence $x=-1\pm\sqrt{-3}$.

711. So that beside the case in which $x=2$, which corresponds to the equation $x^3=8$, we have two other values of x, the cubes of which are also 8; and these are,

$x=-1+\sqrt{-3}$, and $x=-1-\sqrt{-3}$, as will be evident, by actually cubing these expressions;

$-1+\sqrt{-3}$	$-1-\sqrt{-3}$
$-1+\sqrt{-3}$	$-1-\sqrt{-3}$
$1-\sqrt{-3}$	$1+\sqrt{-3}$
$-\sqrt{-3}-3$	$+\sqrt{-3}-3$
$-2-2\sqrt{-3}$ square.	$-2+2\sqrt{-3}$
$-1+\sqrt{-3}$	$-1-\sqrt{-3}$
$2+2\sqrt{-3}$	$2-2\sqrt{-3}$
$-2\sqrt{-3}+6$	$+2\sqrt{-3}+6$
8 cube.	8.

It is true, that these values are imaginary or impossible; but yet they deserve attention.

712. What we have said applies in general to every cubic equation, such as $x^3=a$; namely, that beside the value $x=\sqrt[3]{a}$, we shall always find two other values. To abridge the calculation, let us suppose $\sqrt[3]{a}=c$, so that $a=c^3$, our equation will then assume this form, $x^3-c^3=0$, which will be divisible by $x-c$, as the actual division shows:

$$\begin{array}{l} x-c)\ x^3-c^3\ (x^2+cx+c^2 \\ \quad\quad\ \ x^3-cx^2 \\ \hline \quad\quad\quad\quad cx^2-c^3 \\ \quad\quad\quad\quad cx^2-c^2x \\ \hline \quad\quad\quad\quad\quad\quad c^2x-c^3 \\ \quad\quad\quad\quad\quad\quad c^2x-c^3 \\ \hline \quad\quad\quad\quad\quad\quad\quad 0. \end{array}$$

Consequently, the equation in question may be represented by the product $(x-c)\times(x^2+cx+c^2)=0$, which is in fact $=0$, not only when $x-c=0$, or $x=c$, but also when $x^2+cx+c^2=0$. Now this expression contains two other values of x; for it gives $x^2=-cx-c^2$, and $x=-\frac{c}{2}\pm\sqrt{\frac{c^2}{4}-c^2}$, or .... $x=\frac{-c\pm\sqrt{-3c^2}}{2}$, that is to say, $x=\frac{-c\pm c\sqrt{-3}}{2}$ $=\frac{-1\pm\sqrt{-3}}{2}\times c$.

713. Now as c was substituted for $\sqrt[3]{a}$, we conclude, that every equation of the third degree, of the form $x^3=a$, furnishes three values of x expressed in the following manner:

$$1\text{st } x=\sqrt[3]{a},$$

$$2\text{d } x=\frac{-1+\sqrt{-3}}{2}\times\sqrt[3]{a},$$

$$3\text{d } x=\frac{-1-\sqrt{-3}}{2}\times\sqrt[3]{a}.$$

This shows, that every cube root has three different values; but that one only is real, or possible, the two others being impossible; which is the more remarkable, since every square root has two values, and since we shall afterwards see that a biquadratic root has four different values, that a fifth root has five values, and so on.

In ordinary calculations, indeed, we employ only the first of those values, because the other two are imaginary; as we shall show by some examples.

714. *Question* 1. To find a number, whose square multiplied by its fourth part, may produce 432.

Let x be that number; the product of x^2 multiplied by $\frac{1}{4}x$ must be equal to the number 432, that is to say, $\frac{1}{4}x^3=432$, and $x^3=1728$: whence by extracting the cube root we have $x=12$.

The number sought therefore, is 12; for its square 144, multiplied by its fourth part, or by 3, gives 432.

715. *Question* 2. Required a number such, that if we divide its fourth power by its half, and add $14\frac{1}{4}$ to the product, the sum may be 100.

Calling that number x; its fourth power will be x^4; dividing by the half, or $\frac{1}{2}x$, we have $2x^3$; and adding to that $14\frac{1}{4}$, the sum must be 100: we have therefore $2x^3+14\frac{1}{4}=100$; subtracting $14\frac{1}{4}$, there remains $2x^3=\frac{343}{4}$; dividing by 2, gives $x^3=\frac{343}{8}$, and extracting the cube root, we find $x=\frac{7}{2}$.

716. *Question* 3. Some officers being quartered in a country, each commands three times as many horsemen, and twenty times as many foot-soldiers, as there are officers; also a horseman's monthly pay amounts to as many florins as there are officers, and each foot-soldier receives half that pay; the whole monthly expense is 13000 florins. Required the number of officers.

If x be the number required, each officer will have under him $3x$ horsemen and $20x$ foot-soldiers. So that the whole number of horsemen is $3x^2$, and that of foot-soldiers is $20x^2$.

Now, each horseman receiving x florins per month, and each foot-soldier receiving $\frac{1}{2}x$ florins, therefore the pay of the horsemen, each month, amounts to $3x^3$, and that of the foot-soldiers to $10x^3$; consequently, they all together receive $13x^3$ florins, and this sum must be equal to 13000 florins; we have therefore $13x^3 = 13000$, or $x^3 = 1000$, and $x = 10$, the number of officers required.

717. *Question* 4. Several merchants enter into a partnership, and each contributes a hundred times as many sequins as there are partners; now they send a factor to Venice, to manage their capital; who gains, for every hundred sequins, twice as many sequins as there are partners, and he returns with 2662 sequins profit. Required the number of partners.

If this number is supposed $= x$, each of the partners will have furnished $100x$ sequins, and the whole capital must have been $100x^2$; now, the profit being $2x$ for 100, the capital must have produced $2x^3$; so that $2x^3 = 2662$, or $x^3 = 1331$; this gives $x = 11$, which is the number of partners.

718. *Question* 5. A country girl exchanges geese for hens, at the rate of two geese for three hens; which hens lay each $\frac{1}{2}$ as many eggs as there are geese; and the girl sells at market nine eggs for as many sous as each hen had laid eggs, receiving in all 72 sous; how many geese did she exchange?

Let the number of geese $= x$, then the number of hens which the girl will have received in exchange will be

$\frac{3}{2}x$, and each hen laying $\frac{1}{2}x$ eggs, the number of eggs will be $=\frac{3}{4}x^2$. Now, as nine eggs sell for $\frac{1}{2}x$ sous, the money which $\frac{3}{4}x^2$ eggs produce is $\frac{1}{24}x^3$, and $\frac{1}{24}x^3=72$. Consequently $x^3=24\times72=8\times3\times8\times9=8\times8\times27$, whence $x=12$; that is to say, the girl exchanged twelve geese for eighteen hens.

CHAP. XI.

Of the Resolution of Complete Equations of the Third Degree.

719. An equation of the third degree is called *complete*, when, beside the cube of the unknown quantity, it contains that unknown quantity itself, and its square; so that the general formula for these equations, bringing all the terms to one side, is

$$ax^3\pm bx^2\pm cx\pm d=0.$$

And the purpose of this chapter is to show how we are to derive from such equations the values of x, which are also called the roots of the equation. We suppose, in the first place, that every such an equation has three roots; since it has been seen, in the last chapter, that this is true even with regard to pure equations of the same degree.

720. We shall therefore first consider the equation $x^3-6x^2+11x-6=0$; and, since an equation of the second degree may be considered as the product of two factors, we may also represent an equation of the third degree by the product of three factors, which are in the present instance,

$$(x-1)\times(x-2)\times(x-3)=0;$$

since, by actually multiplying them, we obtain the given equation; for $(x-1)\times(x-2)$ gives x^2-3x+2, and multiplying this by $x-3$, we obtain $x^3-6x^2+11x-6$, which are the given quantities, and which must be $=0$. Now this happens, when the product $(x-1)\times(x-2)\times(x-3)=0$; and, as it is sufficient for this purpose, that one of the factors become $=0$, three different cases may give this result, namely, when $x-1=0$, or $x=1$; secondly, when $x-2=0$, or $x=2$; and thirdly, when $x-3=0$, or $x=3$.

We see immediately also, that if we substituted, in lieu of x, any number whatever beside one of the above three, none of the three factors would become equal to 0; and, consequently, the product would no longer be 0; which proves that our equation can have no other root than these three.

721. If it were possible, in every other case, to assign the three factors of such an equation in the same manner, we should immediately have its three roots. Let us, therefore, consider, in a more general manner, these three factors, $x-p$, $x-q$, $x-r$; now if we seek their product, the first, multiplied by the second, gives $x^2-(p+q)x+pq$, and this product, multiplied by $x-r$, makes

$$x^3-(p+q+r)x^2+(pq+pr+qr)x-pqr.$$

Here, if this formula must become $=0$, it may happen in three cases: the first is that, in which $x-p=0$, or $x=p$; the second is, when $x-q=0$, or $x=q$; the third is, when $x-r=0$, or $x=r$.

722. Let us now represent the quantity found, by the equation $x^3-ax^2+bx-c=0$; it is evident, in order that its three roots may be $x=p$, $x=q$, $x=r$, that we must have,

1st, $a=p+q+r$,
2d, $b=pq+pr+qr$, and
3d, $c=pqr$.

And we perceive, from this, that the second term contains the sum of the three roots; that the third term contains the sum of the products of the roots taken two by two; and lastly, that the fourth term consists of the product of all the three roots multiplied together.

From this last property we may deduce an important truth, which is, that an equation of the third degree can have no other rational roots than the divisors of the last term; for, since that term is the product of the three roots, it must be divisible by each of them. So that when we wish to find a root by trial, we immediately see what numbers we are to use*.

For example, let us consider the equation

* We shall find in the sequel, that this is a general property of equations of any dimension; and as this trial requires us to know all the divisors of the last term of the equation, we may for this purpose have recourse to the tables pointed out at page 27. F. T.

$x^3 = x + 6$, or $x^3 - x - 6 = 0$. Now as this equation can have no other rational roots but numbers which are factors of the last term 6, we have only the numbers 1, 2, 3, 6, to try with, and the result of these trials will be as follows:

If $x = 1$, we have $1 - 1 - 6 = -6$.
If $x = 2$, we have $8 - 2 - 6 = 0$.
If $x = 3$, we have $27 - 3 - 6 = 18$.
If $x = 6$, we have $216 - 6 - 6 = 204$.

Hence we see, that $x = 2$ is one of the roots of the given equation; and, knowing this, it is easy to find the other two; for $x = 2$ being one of the roots, $x - 2$ is a factor of the equation, and we have only to seek the other factor by means of division, as follows:

$$
\begin{array}{r l}
x-2)\; x^3 - x - 6 & (x^2 + 2x + 3 \\
x^3 - 2x^2 & \\
\hline
2x^2 - x - 6 & \\
2x^2 - 4x & \\
\hline
3x - 6 & \\
3x - 6 & \\
\hline
0. &
\end{array}
$$

Since, therefore, the formula is represented by the product $(x-2) \times (x^2 + 2x + 3)$, it will become $= 0$, not only when $x - 2 = 0$, but also when $x^2 + 2x + 3 = 0$: and, this last factor gives $x^2 = -2x - 3$, consequently $x = -1 \pm \sqrt{-2}$; and these are the other two roots of our equation, which are evidently impossible, or imaginary.

723. The method which we have explained, is applicable only when the first term x^3 is multiplied

by 1, and the other terms of the equation have integer coefficients; therefore, when this is not the case, we must begin by a preparation, which consists in transforming the equation into another form having the condition required, after which we make the trial that has been already mentioned.

Let there be given, for example, the equation

$$x^3-3x^2+\frac{11}{4}x-\frac{3}{4}=0;$$

and as it contains fourth parts, let us make $x=\frac{y}{2}$, which will give

$$\frac{y^3}{8}-\frac{3y^2}{4}+\frac{11y}{8}-\frac{3}{4}=0,$$

and, multiplying by 8, we shall obtain the equation

$$y^3-6y^2+11y-6=0,$$

the roots of which are, as we have already seen, $y=1$, $y=2$, $y=3$; whence it follows, that in the given equation, we have $x=\frac{1}{2}$, $x=1$, $x=\frac{3}{2}$.

724. Let there be an equation, where the coefficient of the first term is a whole number but not 1, and whose last term is 1; for example,

$$6x^3-11x^2+6x-1=0;$$

here, if we divide by 6, we shall have

$x^3-\frac{11}{6}x^2+x-\frac{1}{6}=0$; which equation we may clear of fractions, by the method we have just explained.

First, by making $x=\frac{y}{6}$, we shall have

$$\frac{y^3}{216}-\frac{11y^2}{216}+\frac{y}{6}-\frac{1}{6}=0;$$

and multiplying by 216, the equation will become $y^3-11y^2+36y-36=0$. But as it would be too long to make trial of all the divisors of the number 36, and as the last term of the original equation is 1, it is better to suppose, in this equation, $x=\frac{1}{z}$; for we shall then have $\frac{6}{z^3}-\frac{11}{z^2}+\frac{6}{z}-1=0$, which, multiplied by z^3, gives $6-11z+6z^2-z^3=0$, and transposing all the terms, $z^3-6z^2+11z-6=0$; where the roots are $z=1$, $z=2$, $z=3$; whence it follows that in our equation $x=1$, $x=\frac{1}{2}$, $x=\frac{1}{3}$.

725. It has been observed in the preceding articles, that in order to have all the roots in positive numbers, the signs *plus* and *minus* must succeed each other alternately; by means of which the equation takes this form, $x^3-ax^2+bx-c=0$, the signs changing as many times as there are positive roots. If all the three roots had been negative, and we had multiplied together the three factors $x+p$, $x+q$, $x+r$, all the terms would have had the sign *plus*, and the form of the equation would have been $x^3+ax^2+bx+c=0$, in which the same signs follow each other *three* times, that is, the number of negative roots.

We may conclude, therefore, that as often as the signs change, the equation has positive roots, and that as often as the same signs follow each other, the equation has negative roots; and this remark is very important, because it teaches us whether the divisors of the last term are to be taken affirmatively or negatively, when we wish to make the trial which has been mentioned.

726. In order to illustrate what has been said by an example, let us consider the equation $x^3+x^2-34x+56=0$, in which the signs are changed twice, and in which the same sign occurs only once. Here we conclude that the equation has two positive roots and one negative root; and as these roots must be divisors of the last term 56, they must be included in the numbers $\pm 1, 2, 4, 7, 8, 14, 28, 56$.

Let us, therefore, make $x=2$, and we shall have $8+4-68+56=0$; whence we conclude that $x=2$ is a positive root, and that therefore $x-2$ is a divisor of the equation, by means of which we easily find the two other roots; for actually dividing by $x-2$, we have

$$\begin{array}{l}
x-2)\ x^3+x^2-34x+56\ (x^2+3x-28 \\
x^3-2x^2 \\
\hline
3x^2-34x \\
3x^2-6x \\
\hline
-28x+56 \\
-28x+56 \\
\hline
0.
\end{array}$$

And making the quotient $x^2+3x-28=0$, we find the two other roots, which will be

$$x=-\frac{3}{2}\pm\sqrt{\frac{9}{4}+28}=-\frac{3}{2}\pm\frac{11}{2},$$ that is $x=4$; or

$x=-7$; and taking into account the root found before, namely, $x=2$, we clearly perceive that the equation has two positive and one negative root. We shall give some examples to render this still more evident.

727. *Question* 1. There are two numbers, whose difference is 12, and whose product multiplied by their sum makes 14560. What are those numbers?

Let x be the less of the two numbers, then the greater will be $x+12$, and their product will be x^2+12x, which multiplied by the sum $2x+12$, gives

$$2x^3+36x^2+144x=14560;$$

and dividing by 2, we have

$$x^3+18x^2+72x=7280.$$

Now, the last term 7280 is too great for us to make trial of all its divisors, but as it is divisible by 8, we shall make $x=2y$, because the new equation, $8y^3+72y^2+144y=7280$, after the substitution, being divided by 8, will become $y^3+9y^2+18y=910$, to solve which we need only try the divisors 1, 2, 5, 7, 10, 13, &c. of the number 910: where it is evident, that the three first, 1, 2, 5, are too small; beginning therefore with supposing $y=7$, we immediately find that number to be one of the roots; for the substitution gives $343+441+126=910$. It follows, therefore, that $x=14$; and the two other roots will be found by dividing $y^3+9y^2+18y-910$ by $y-7$, thus:

$$y-7)\ y^3+9y^2+18y-910\ (y^2+16y+130$$
$$y^3-7y^2$$
$$\overline{\quad 16y^2+18y}$$
$$16y^2-112y$$
$$\overline{\quad 130y-910}$$
$$130y-910$$
$$\overline{\quad 0.}$$

Supposing now this quotient $y^2+16y+130=0$, we shall have $y^2=-16y-130$, and thence $y=-8\pm\sqrt{-66}$; a proof that the other two roots are impossible.

The two numbers sought are therefore 14 and 26; the product of which, 364, multiplied by their sum, 40, gives 14560.

728. *Question* 2. To find two numbers whose difference is 18, and such, that their sum multiplied by the difference of their cubes, may produce 275184.

Let x be the less of the two numbers, then $x+18$ will be the greater; the cube of the first will be x^3, and the cube of the second

$$x^3+54x^2+972x+5832;$$

the difference of the cubes

$$54x^2+972x+5832=54(x^2+18x+108),$$

which multiplied by the sum $2x+18$, or $2(x+9)$, gives the product

$$108(x^3+27x^2+270x+972)=275184.$$

And dividing by 108, we have

$$x^3+27x^2+270x+972=2548, \text{ or}$$
$$x^3+27x^2+270x=1576.$$

Now the divisors of 1576 are 1, 2, 4, 8, &c. the two first of which are too small; but if we try $x=4$, that number is found to satisfy the terms of the equation.

It remains, therefore, to divide by $x-4$, in order to find the two other roots; which division gives the quotient $x^2+31x+394$; making therefore

$x^2=-31x-394$, we shall find

$$x=-\frac{31}{2}\pm\sqrt{\frac{961}{4}-\frac{1376}{4}},$$

that is, two imaginary roots.

Hence the numbers sought are 4 and 22.

729. *Question 3.* Required two numbers whose difference is 720, and such, that if the less be multiplied by the square root of the greater, the product may be 20736.

If the less be represented by x, the greater will evidently be $x+720$; and, by the question,

$$x\sqrt{x+720}=20736=8 . 8 . 4 . 81.$$

Squaring both sides, we have

$$x^2(x+720)=x^3+720x^2=8^2 . 8^2 . 4^2 . 81^2.$$

Let us now make $x=8y$; this supposition gives

$$8^3y^3+720 . 8^2y^2=8^2 . 8^2 . 4^2 . 81^2;$$

and dividing by 8^3, we have $y^3+90y^2=8 . 4^2 . 81^2$. Farther, let us suppose $y=2z$, and we shall have $8z^3+4 . 90z^2=8 . 4^2 . 81^2$; or, dividing by 8,

$$z^3+45z^2=4^2 . 81^2.$$

Again, make $z=9u$, in order to have $9^3u^3+45 . 9^2u^2=4^2 . 9^4$, because dividing now by 9^3, the equation becomes $u^3+5u^2=4^2 . 9$, or $u^2(u+5)=16 . 9=144$; where it is obvious, that $u=4$; for in this case $u^2=16$ and $u+5=9$: since, therefore, $u=4$, we have $z=36$, $y=72$, and $x=576$, which is the less of the two numbers sought; so that the greater is 1296, and the square root of this last, or 36, multiplied by the other number 576, gives 20736.

730. *Remark.* This question admits of a simpler solution; for since the square root of the greater number, multiplied by the less, must give a product equal to a given number, the greater of the two numbers must be a square. If therefore, from this consideration, we suppose it to be x^2, the other number will be x^2-720, which being multiplied by

the square root of the greater, or by x, we have $x^3-720x=20736=64.27.12$.

If we make $x=4y$, we shall have

$$64y^3-720.4y=64.27.12, \text{ or}$$
$$y^3-45y=27.12.$$

Supposing, farther, $y=3z$, we find $27z^3-135z=27.12$, or dividing by 27, $z^3-5z=12$, or $z^3-5z-12=0$. The divisors of 12 are 1, 2, 3, 4, 6, 12; the two first are too small; but the supposition of $z=3$ gives exactly $27-15-12=0$. Consequently, $z=3$, $y=9$, and $x=36$; whence we conclude, that the greater of the two numbers sought, or x^2, $=1296$, and that the less, or x^2-720, $=576$, as above.

731. *Question* 4. There are two numbers, whose difference is 12; the product of this difference by the sum of their cubes is 102144; what are the numbers?

Calling the less of the two numbers x, the greater will be $x+12$: also, the cube of the first is x^3, and of the second $x^3+36x^2+432x+1728$; the product also of the sum of these cubes by the difference 12, is

$$12(2x^3+36x^2+432x+1728)=102144;$$

and dividing successively by 12 and by 2, we have

$$x^3+18x^2+216x+864=4256, \text{ or}$$
$$x^3+18x^2+216x=3392=8.8.53.$$

If now we substitute $x=2y$, and divide by 8, we shall have $y^3+9y^2+54y=8.53=424$.

Now the divisors of 424 are 1, 2, 4, 8, 53, &c. but 1 and 2 are evidently too small; but if we make $y=4$, we find $64+144+216=424$. So that $y=4$ and $x=8$; whence we conclude that the two numbers sought are 8 and 20.

732. *Question* 5. Several persons form a partnership, and establish a certain capital, to which each contributes ten times as many pounds as there are persons in company, and they gain 6 plus the number of partners per cent., also the whole profit is 392 pounds; required how many partners there are?

Let x be the number required; then each partner will have furnished $10x$ pounds, and conjointly $10x^2$ pounds; and since they gain $x+6$ per cent. they will have gained with the whole capital, $\frac{x^3+6x^2}{10}$, which is to be made equal to 392.

We have therefore $x^3+6x^2=3920$, consequently making $x=2y$ and dividing by 8, we have

$$y^3+3y^2=490.$$

Now the divisors of 490 are 1, 2, 5, 7, 10, &c. the three first of which are too small; but if we suppose $y=7$, we have $343+147=490$; so that $y=7$, and $x=14$.

There were therefore fourteen partners, and each of them put 140 pounds into the common stock.

733. *Question* 6. A company of merchants have a common stock of 8240 pounds; and each contributes to it forty times as many pounds as there are partners; with which they gain as much per cent. as there are partners; now on dividing the profit, it is found, after each has received ten times as many pounds as there are persons in company, that there still remains 224*l*. Required the number of merchants.

If x be made to represent the number, each will have contributed $40x$ to the stock; consequently all together will have contributed $40x^2$, which makes the stock $=40x^2+8240$; now with this sum they gain x per cent.; so that the whole gain is

$$\frac{40x^3}{100}+\frac{8240x}{100}=\frac{4}{10}x^3+\frac{824}{10}x=\frac{2}{5}x^3+\frac{412}{5}x.$$

From which sum each receives $10x$, and consequently they all together receive $10x^2$, leaving a remainder of 224; the profit must therefore have been $10x^2+224$, and we have the equation

$$\frac{2x^3}{5}+\frac{412x}{5}=10x^2+224.$$

Multiplying by 5 and dividing by 2, we have $x^3+206x=25x^2+560$, or $x^3-25x^2+206x-560=0$: the first, however, will be more convenient for trial. Here the divisors of the last term are 1, 2, 4, 5, 7, 8, 10, 14, 16, &c. and they must be taken positive, because in the second form of the equation the signs vary three times, which shows that the three roots are positive.

Here, if we first try $x=1$ and $x=2$, it is evident that the first side will become less than the second. We shall therefore make trial of other divisors.

When $x=4$, we have $64+824=400+560$, which does not satisfy the terms of the equation.

If $x=5$, we have $125+1030=625+560$, which likewise does not succeed.

But if $x=7$, we have $343+1442=1225+560$, which answers to the equation; so that $x=7$ is a root of it. Let us now seek for the other two, dividing the second form of our equation by $x-7$. Thus,

$$\begin{array}{l} x-7)\ x^3-25x^2+206x-560\ (x^2-18x+80 \\ \quad x^3-7x^2 \\ \hline \quad -18x^2+206x \\ \quad -18x^2+126x \\ \hline \qquad 80x-560 \\ \qquad 80x-560 \\ \hline \qquad\quad 0. \end{array}$$

Now making this quotient equal to nothing, we have $x^2-18x+80=0$, or $x^2=18x-80$, which gives $x=9\pm1$, so that the two other roots are $x=8$; or $x=10$.

This question therefore admits of three answers. According to the first, the number of merchants is 7; according to the second, it is 8; and, according to the third, it is 10; and the following tablet shows that all these will answer the conditions of the question:

Number of merchants	7	8	10
Each contributes $40x$ - -	280	320	400
In all they contribute $40x^2$	1960	2560	4000
The original stock was - -	8240	8240	8240
The whole stock is $40x^2$ $+8240$ - - - - - -	10200	10800	12240
With this capital they gain as much per cent. as there are partners - -	714	864	1224
Each takes from it - - - -	70	80	100
So that they all together take $10x^2$ - - - - - - -	490	640	1000
Therefore there remains -	224	224	224

CHAP. XII.

Of the Rule of Cardan, or of Scipio Ferreo.

734. When we have removed fractions from an equation of the third degree, according to the manner which has been explained, and none of the divisors of the last term are found to be a root of the equation, it is a certain proof, not only that the equation has no root in integer numbers, but also that a fractional root cannot exist, which may be proved as follows.

Let there be given the equation $x^3-ax^2+bx-c=0$, in which, a, b, c, express integer numbers. If we suppose, for example, $x=\frac{3}{2}$, we shall have $\frac{27}{8}-\frac{9}{4}a+\frac{3}{2}b-c=0$; now the first term only has 8 for the denominator; all the others being either integer numbers, or numbers divided only by 4 or by 2, and therefore cannot make 0 with the first term: and the same thing happens with every other fraction (*Appendix, note* 4).

735. As in those fractions the roots of the equation are neither integer numbers, nor fractions, they are irrational, and, as it often happens, imaginary. The manner, therefore, of expressing them, and of determining the radical signs which affect them, forms a very important point, and deserves to be carefully explained in this place. This

method, called *Cardan's Rule*, is ascribed to *Cardan*, though it properly belongs to *Scipio Ferreo*, being discovered by the latter about the year 1500 *.

736. In order to understand this rule, we must first attentively consider the nature of a cube, whose root is a binomial.

Let $a+b$ be that root; then the cube of it will be $a^3+3a^2b+3ab^2+b^3$, and we see that it is composed of the cubes of the two terms of the binomial, and beside that, of the two middle terms, $3a^2b+3ab^2$, which have the common factor $3ab$, multiplying the other factor, $a+b$; that is to say, the two terms contain thrice the product of the two terms of the binomial, multiplied by the sum of those terms.

737. Let us now suppose $x=a+b$; taking the cube of each side, we have $x^3=a^3+b^3+3ab(a+b)$: and, since $a+b=x$, we shall have the equation, $x^3=a^3+b^3+3abx$, or $x^3=3abx+a^3+b^3$, one of the roots of which we know to be $x=a+b$. Whenever, therefore, such an equation occurs, we may assign one of its roots.

For example, let $a=2$ and $b=3$; we shall then have the equation $x^3=18x+35$, which we know with certainty to have $x=5$ for one of its roots.

738. Farther, let us now suppose $a^3=p$ and $b^3=q$; we shall then have $a=\sqrt[3]{p}$ and $b=\sqrt[3]{q}$, consequently, $ab=\sqrt[3]{pq}$; therefore, whenever we meet

* This rule when first discovered by Seipio Ferreo was only for particular forms of cubics, but it was afterwards generalized by Tartalea and Cardan. See Montucla's History of the Mathematics; and also Dr. Hutton's Dictionary, article Algebra. ED.

with an equation, of the form $x^3 = 3x\sqrt[3]{pq} + p + q$, we know that one of the roots is $\sqrt[3]{p} + \sqrt[3]{q}$.

Now we can always determine p and q, in such a manner, that both $3\sqrt[3]{pq}$ and $p+q$ may be quantities equal to determinate numbers; so that we can always resolve an equation of the third degree, of the kind which we speak of*.

739. Let, in general, the equation $x^3 = fx + g$ be proposed. Here, it will be necessary to compare f with $3\sqrt[3]{pq}$, and g with $p+q$; that is, we must determine p and q in such a manner, that $3\sqrt[3]{pq}$ may become equal to f, and $p+q=g$; for we then know that one of the roots of our equation will be $x = \sqrt[3]{p} + \sqrt[3]{q}$.

740. We have therefore to resolve these two equations,

$$3\sqrt[3]{pq} = f,$$
$$p + q = g.$$

The first gives $\sqrt[3]{pq} = \frac{f}{3}$; or $pq = \frac{f^3}{27} = \frac{1}{27}f^3$, and $4pq = \frac{4}{27}f^3$. The second equation, being squared,

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* It must not be understood here, that these values of p and q are necessarily rational, or even possible surds. Suppose, for example, we have the general equations

$$3\sqrt[3]{pq} = a$$
$$p + q = b,$$

these, being reduced by the rules given for quadratics, give $p - q = (b^2 - \frac{4}{27}a^3)^{\frac{1}{2}}$: therefore, when $b^2 < \frac{4}{27}a^3$, the values of p and q fall under an imaginary form, and involve what is usually termed the irreducible case in cubic equations. ED.

gives $p^2+2pq+q^2=g^2$; and if we subtract from it $4pq=\frac{4}{27}f^3$, we have $p^2-2pq+q^2=g^2-\frac{4}{27}f^3$, and taking the square root of both sides, we have

$$p-q=\sqrt{g^2-\tfrac{4}{27}f^3}.$$

Now, since $p+q=g$, we have, by adding $p+q$ to one side of the equation, and its equal, g, to the other, $2p=g+\sqrt{g^2-\frac{4}{27}f^3}$, and, by subtracting $p-q$ from $p+q$, we have $2q=g-\sqrt{g^2-\frac{4}{27}f^3}$; consequently, $p=\frac{g+\sqrt{g^2-\frac{4}{27}f^3}}{2}$, and

$$q=\frac{g-\sqrt{g^2-\frac{4}{27}f^3}}{2}.$$

741. In a cubic equation, therefore, of the form $x^3=fx+g$, whatever be the numbers f and g, we have always for one of the roots

$$x=\sqrt[3]{\frac{g+\sqrt{g^2-\frac{4}{27}f^3}}{2}}+\sqrt[3]{\frac{g-\sqrt{g^2-\frac{4}{27}f^3}}{2}};$$

that is, an irrational quantity, containing not only the sign of the square root, but also the sign of the cube root; and this is the formula which is called *the Rule of Cardan.*

742. Let us apply it to some examples, in order that its use may be better understood.

Let $x^3=6x+9$. First, we shall have $f=6$ and $g=9$; so that $g^2=81$, $f^3=216$, and $\frac{4}{27}f^3=32$; then $g^2-\frac{4}{27}f^3=49$, and $\sqrt{g^2-\frac{4}{27}f^3}=7$.

Therefore one of the roots of the given equation is

$$x=\sqrt[3]{\frac{9+7}{2}}+\sqrt[3]{\frac{9-7}{2}}=\sqrt[3]{\frac{16}{2}}+\sqrt[3]{\frac{2}{2}}=\sqrt[3]{8}+\sqrt[3]{1}=$$

$2+1=3$.

743. Let there be proposed the equation $x^3=3x+2$. Here, we shall have $f=3$ and $g=2$; and consequently, $g^2=4$, $f^3=27$, and $\frac{4}{27}f^3=4$; which gives $\sqrt{g^2-\frac{4}{27}f^3}=0$; whence it follows, that one of the roots is

$$x=\sqrt[3]{\frac{2+0}{2}}+\sqrt[3]{\frac{2-0}{2}}=1+1=2.$$

744. It often happens, however, that, though such an equation has a rational root, that root cannot be found by the rule which we are now considering.

Let there be given the equation $x^3=6x+40$, in which $x=4$ is one of the roots. We have here $f=6$ and $g=40$, farther $g^2=1600$ and $\frac{4}{27}f^3=32$; so that $g^2-\frac{4}{27}f^3=1568$, and

$$\sqrt{g^2-\frac{4}{27}f^3}=\sqrt{1568}=\sqrt{4\,.\,4\,.\,49\,.\,2}=28\sqrt{2};$$

consequently one of the roots will be

$$x=\sqrt[3]{\frac{40+28\sqrt{2}}{2}}+\sqrt[3]{\frac{40-28\sqrt{2}}{2}} \text{ or}$$

$$x=\sqrt[3]{20+14\sqrt{2}}+\sqrt[3]{20-14\sqrt{2}};$$

which quantity is really $=4$, although, upon inspection, we should not suppose it. In fact, the cube of $2+\sqrt{2}$ being $20+14\sqrt{2}$, we have reciprocally the cube root of $20+14\sqrt{2}$ equal to $2+\sqrt{2}$; in the same manner, $\sqrt[3]{20-14\sqrt{2}}=2-\sqrt{2}$; wherefore our root $x=2+\sqrt{2}+2-\sqrt{2}=4$ *.

745. To this rule it might be objected, that it does not extend to all equations of the third degree, because the square of x does not occur in it, that is to say, the second term of the equation is wanted. But we may remark, that every complete equation may be transformed into another in which the second term is wanted, which will therefore enable us to apply the rule.

To prove this, let us take the complete equation $x^3-6x^2+11x-6=0$: where, if we take the third of the coefficient 6 of the second term, and make $x-2=y$, we shall have

$$x=y+2, \quad x^2=y^2+4y+4, \text{ and}$$
$$x^3=y^3+6y^2+12y+8;$$

Consequently,
$$\begin{aligned} x^3 &= y^3+6y^2+12y+8 \\ -6x^2 &= \quad -6y^2-24y-24 \\ +11x &= \qquad\quad +11y+22 \\ -6 &= \qquad\qquad\quad -6 \end{aligned}$$

or, $x^3-6x^2+11x-6=y^3-y$.

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* We have no general rules for extracting the cube root of these binomials, as we have for the square root; those that have been given by various authors, all lead to a mixt equation of the third degree similar to the one proposed. However, when the extraction of the cube root is possible, the sum of the two radicals which represent the root of the equation, always becomes rational; so that we may find it immediately by the method explained, Art. 722. F. T.

We have, therefore, the equation $y^3-y=0$, the resolution of which is evident, since we immediately perceive that it is the product of the factors

$$y(y^2-1)=y(y+1)\times(y-1)=0.$$

If we now make each of these factors $=0$, we have

1st $\begin{cases} y=0, \\ x=2, \end{cases}$ 2d $\begin{cases} y=-1, \\ x=1; \end{cases}$ 3d $\begin{cases} y=1, \\ x=3, \end{cases}$

that is to say, the three roots which we have already found.

746. Let there now be given the general equation of the third degree, $x^3+ax^2+bx+c=0$, of which it is required to destroy the second term.

For this purpose we must add to x the third of the coefficient of the second term, preserving the same sign, and then write for this sum a new letter, as for example y, so that we shall have $x+\frac{1}{3}a=y$, and $x=y-\frac{1}{3}a$; whence results the following calculation:

$$x=y-\frac{1}{3}a,\ x^2=y^2-\frac{2}{3}ay+\frac{1}{9}a^2,$$

$$\text{and } x^3=y^3-ay^2+\frac{1}{3}a^2y-\frac{1}{27}a^3;$$

Consequently,

$$\begin{aligned} x^3 &= y^3-ay^2+\frac{1}{3}a^2y-\frac{1}{27}a^3 \\ ax^2 &= +ay^2-\frac{2}{3}a^2y+\frac{1}{9}a^3 \\ bx &= \phantom{y^3+ay^2-\frac{2}{3}a^2}+by-\frac{1}{3}ab \\ c &= \phantom{y^3+ay^2-\frac{2}{3}a^2y-\frac{1}{3}}+c \end{aligned}$$

or, $y^3-(\frac{1}{3}a^2-b)y+\frac{2}{27}a^3-\frac{1}{3}ab+c=0$,

an equation in which the second term is wanted.

747. We are enabled, by means of this transformation, to find the roots of all equations of the third degree, as will be seen in the following example.

Let it be proposed to resolve the equation

$$x^3-6x^2+13x-12=0.$$

Here it is first necessary to destroy the second term; for which purpose, let us make $x-2=y$, and then we shall have $x=y+2$, $x^2=y^2+4y+4$, and $x^3=y^3+6y^2+12y+8$; therefore,

$$\begin{aligned} x^3 &= y^3+6y^2+12y+8 \\ -6x^2 &= \quad\; -6y^2-24y-24 \\ +13x &= \qquad\qquad +13y+26 \\ -12 &= \qquad\qquad\qquad -12 \end{aligned}$$

which gives $y^3+y-2=0$; or $y^3=-y+2$.

And if we compare this equation with the formula $x^3=fx+g$, we have $f=-1$, and $g=2$; wherefore, $g^2=4$, and $\frac{4}{27}f^3=-\frac{4}{27}$; also, $g^2-\frac{4}{27}f^3=4+\frac{4}{27}=\frac{112}{27}$, and $\sqrt{g^2-\frac{4}{27}f^3}=\sqrt{\frac{112}{27}}=\frac{4\sqrt{21}}{9}$; consequently,

$$y=\left(\frac{2+\frac{4\sqrt{21}}{9}}{2}\right)^{\frac{1}{3}}+\left(\frac{2-\frac{4\sqrt{21}}{9}}{2}\right)^{\frac{1}{3}}, \text{ or}$$

$$y=\sqrt[3]{1+\frac{2\sqrt{21}}{9}}+\sqrt[3]{1-\frac{2\sqrt{21}}{9}}, \text{ or}$$

$$y=\sqrt[3]{\frac{9+2\sqrt{21}}{9}}+\sqrt[3]{\frac{9-2\sqrt{21}}{9}}$$

$$y=\sqrt[3]{\frac{27+6\sqrt{21}}{27}}+\sqrt[3]{\frac{27-6\sqrt{21}}{27}}$$

$$y=\frac{1}{3}\sqrt[3]{27+6\sqrt{21}}+\frac{1}{3}\sqrt[3]{27-6\sqrt{21}};$$

and it remains to substitute this value in $x=y+2$.

748. In the solution of this example, we have been brought to a quantity doubly irrational; but we must not immediately conclude that the root is irrational: because the binomials $27\pm6\sqrt{21}$ might happen to be real cubes; and this is the case here; for the cube of $\frac{3+\sqrt{21}}{2}$ being $\frac{216+48\sqrt{21}}{8}=27+6\sqrt{21}$, it follows that the cube root of $27+6\sqrt{21}$ is $\frac{3+\sqrt{21}}{2}$ and that the cube root of $27-6\sqrt{21}$ is $\frac{3-\sqrt{21}}{2}$. And hence the value which we found for y becomes

$$y=\frac{1}{3}\left(\frac{3+\sqrt{21}}{2}\right)+\frac{1}{3}\left(\frac{3-\sqrt{21}}{2}\right)=\frac{1}{2}+\frac{1}{2}=1.$$

Now, since $y=1$, we have $x=3$ for one of the roots of the equation proposed, and the other two will be found by dividing the equation by $x-3$; thus,

$$\begin{array}{r|l} x-3) & x^3-6x^2+13x-12\ (x^2-3x+4 \\ & x^3-3x^2 \\ \hline & -3x^2+13x \\ & -3x^2+\ 9x \\ \hline & 4x-12 \\ & 4x-12 \\ \hline & 0. \end{array}$$

Also making the quotient $x^2-3x+4=0$, we have $x^2=3x-4$; and

$$x=\frac{3}{2}\pm\sqrt{\frac{9}{4}-\frac{16}{4}}=\frac{3}{2}\pm\sqrt{-\frac{7}{4}}=\frac{3\pm\sqrt{-7}}{2};$$

which are the other two roots, but they are imaginary.

749. It was, however, by chance, as we have remarked, that we were able, in the preceding example, to extract the cube root of the binomials that we obtained, which is the case only when the equation has a rational root; consequently, the rules of the preceding chapter are more easily employed for finding that root. But when there is no rational root, it is, on the other hand, impossible to express the root which we obtain in any other way, than according to the rule of Cardan; so that it is then impossible to apply reductions. For example, in the equation $x^3=6x+4$, we have $f=6$ and $g=4$; so that $x=\sqrt[3]{2+2\sqrt{-1}}+\sqrt[3]{2-2\sqrt{-1}}$, which cannot be otherwise expressed*.

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* In this example we have $\frac{4}{27}f^3$ less than g^2, which is the well-known *irreducible case;* a case which is so much the more remarkable, as all the three roots are then always real. We cannot, here, make use of Cardan's formula, except by applying the methods of approximation, such as transforming it into an infinite series. In the work spoken of at Note, p. 15, Lambert has given particular tables, by which we may easily find the numerical values of the roots of cubic equations, in the irreducible, as well as the other cases. For this purpose we may also employ the ordinary tables of sines. See the Spherical Astronomy of Mauduit, printed at Paris in 1765.

The reader is also referred to Bonnycastle's Trigonometry for a clear and explicit investigation of this method. We shall here only give the formulæ, for the solution of the different cases of cubic equations; which will be found useful in many cases.

1. $x^3+px-q=0$.

Here put $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}}=\tan. z$; and $\sqrt[3]{\tan.(45°-\frac{1}{2}z)}=\tan. u$.

CHAP. XIII.

Of the Resolution of Equations of the Fourth Degree.

750. When the highest power of the quantity x

Then $x=2\sqrt{\frac{p}{3}}\times\cot. 2u.$

2. $x^3+px+q=0.$

Put $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}}=\tan. z$, and $\sqrt[3]{\tan.(45^\circ-\frac{1}{2}z)}=\tan. u.$

Then $x=-2\sqrt{\frac{p}{3}}\times\cot. 2u.$

3. $x^3-px-q=0.$

This form has two cases according as $\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}}$ is less or greater than 1.

Case 1. Put $\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}}=\cos. z$, and $\sqrt[3]{\tan.(45^\circ-\frac{1}{2}z)}=\tan. u.$

Then $x=2\sqrt{\frac{p}{3}}\times\text{cosec}. 2u.$

Case 2. Put $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}}=\cos. z$, then x has the three following values,

$$x=2\sqrt{\frac{p}{3}}\times\cos.\frac{z}{3},\quad x=-2\sqrt{\frac{p}{3}}\times\cos.(60^\circ\pm\frac{z}{3}).$$

4. $x^3-px+q=0.$

This form has also two cases according as $\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}}$ is less or greater than 1.

Case 1. Put $\frac{2}{q}\left(\frac{p}{3}\right)^{\frac{3}{2}}=\cos. z$, and $\sqrt[3]{\tan.(45^\circ-\frac{1}{2}z)}=\tan. u.$

Then $x=-2\sqrt{\frac{p}{3}}\times\cos. 2u.$

Case 2. Put $\frac{q}{2}\left(\frac{3}{p}\right)^{\frac{3}{2}}=\cos. z$, then x has the three following values,

$$x=-2\sqrt{\frac{p}{3}}\times\cos.\frac{z}{3},\text{ and } x=2\sqrt{\frac{p}{3}}\times\cos.(60^\circ\pm\frac{z}{3}).$$

rises to the fourth degree, we have *equations of the fourth degree*, the general form of which is

$$x^4+ax^3+bx^2+cx+d=0.$$

We shall, in the first place, consider *pure* equations of the fourth degree, the expression for which is simply $x^4=f$, and the root of which is immediately found by extracting the biquadrate root of both sides, since we obtain $x=\sqrt[4]{f}$.

751. As x^4 is the square of x^2, the calculation is greatly facilitated by beginning with the extraction of the square root; for we shall then have $x^2=\sqrt{f}$; and, taking the square root again, we have $x=\sqrt[4]{f}$; so that $\sqrt[4]{f}$ is nothing but the square root of the square root of f.

For example, if we had the equation $x^4=2401$, we should immediately have $x^2=49$, and then $x=7$.

752. It is true this is only one root, and since there are always three roots in an equation of the third degree, so also there are four roots in an equation of the fourth degree; but the method which we have explained will actually give those four roots. For, in the above example, we have not only $x^2=49$, but also $x^2=-49$; now the first value gives the two roots $x=7$ and $x=-7$, and the second value gives $x=\sqrt{-49}$, and $x=-\sqrt{-49}=7\sqrt{-1}$, and $-7\sqrt{-1}$; which are the four biquadrate roots of 2401: and the same also is true with respect to other numbers.

753. Next to these pure equations, we shall consider those in which the second and fourth terms are wanted; which have the form $x^4+fx^2+g=0$, and may be resolved by the rule for equations of the

second degree; for if we make $x^2=y$, we have $y^2+fy+g=0$, or $y^2=-fy-g$, whence we deduce

$$y=-\frac{1}{2}f\pm\sqrt{\frac{1}{4}f^2-g}=\frac{-f\pm\sqrt{f^2-4g}}{2};$$

now $x^2=y$; so that $x=\pm\sqrt{\frac{-f\pm\sqrt{f^2-4g}}{2}}$; in which the double signs $\pm$ indicate all the four roots.

754. But whenever the equation contains all the terms, it may be considered as the product of four factors. In fact, if we multiply these four factors together, $(x-p)\times(x-q)\times(x-r)\times(x-s)$, we get the product $x^4-(p+q+r+s)x^3+(pq+pr+ps+qr+qs+rs)x^2-(pqr+pqs+prs+qrs)x+pqrs$, and this quantity cannot be equal to 0, except when one of these four factors is $=0$. Now that may happen in four ways;

1st when $x=p$; 2dly when $x=q$;
3dly when $x=r$; 4thly when $x=s$;

and consequently these are the four roots of the equation.

755. Now if we consider this formula, we observe, in the second term, the sum of the four roots multiplied by $-x^3$; in the third term, the sum of all the possible products of two roots, multiplied by x^2; in the fourth term, the sum of the products of the roots combined three by three, multiplied by $-x$; lastly, in the fifth term, the product of all the four roots multiplied together.

756. Now as the last term contains the product of all the roots, it is evident that such an equation of the fourth degree can have no rational root which is not likewise a divisor of the last term; this principle,

therefore, furnishes an easy method of determining all the rational roots, when there are any, since we have only to substitute successively for x all the divisors of the last term, till we find one which answers to the equation; for having found such a root, for example, $x=p$, we have only to divide the equation by $x-p$, after having brought all the terms to one side, and then suppose the quotient $=0$; by which we obtain an equation of the third degree, which may be resolved by the rules already given.

757. Now for this purpose it is absolutely necessary that all the terms should consist of integers, and that the first should have only unity for the coefficient; whenever, therefore, any terms contain fractions, we must begin with destroying those fractions, and this may always be done by substituting, instead of x, the quantity y, divided by a number which contains all the denominators of those fractions.

For example, if we have the equation

$$x^4-\frac{1}{2}x^3+\frac{1}{3}x^2-\frac{3}{4}x+\frac{1}{18}=0,$$

since we find here fractions which have for denominators 2, 3, and multiples of these numbers, we shall suppose $x=\frac{y}{6}$, and shall thus have

$$\frac{y^4}{6^4}-\frac{\frac{1}{2}y^3}{6^3}+\frac{\frac{1}{3}y^2}{6^2}-\frac{\frac{3}{4}y}{6}+\frac{1}{18}=0,$$

an equation, which, multiplied by 6^4, becomes

$$y^4-3y^3+12y^2-162y+72=0.$$

If we now wish to know whether this equation has rational roots, we must write, instead of y, the divisors of 72 successively, in order to see in what cases the formula would really be reduced to 0.

758. But as the roots may as well be positive as negative, we must make two trials with each divisor; one, supposing that divisor positive, the other, considering it as negative; however, the following rule will frequently enable us to dispense with this*. Whenever the signs + and − succeed each other regularly, the equation has as many positive roots as there are changes in the signs; and as many times as the same sign recurs without the other intervening, so many negative roots belong to the equation. Now our example contains four changes of the signs, and no succession; so that all the roots are positive; and we have no need to take any of the divisors of the last term negatively.

759. Let there be given the equation

$$x^4+2x^3-7x^2-8x+12=0.$$

We see here two changes of signs, and also two successions; whence we conclude, with certainty, that this equation contains two positive and as many negative roots, which must all be divisors of the number 12. Now its divisors being 1, 2, 3, 4, 6, 12, let us first try $x=+1$, which actually produces 0; therefore one of the roots is $x=1$.

If we next make $x=-1$, we find $+1-2-7+8+12=21-9=12$: so that $x=-1$ is not one of

* This rule is general for equations of all dimensions, provided there are no imaginary roots; the French ascribe it to Descartes, the English to Harriot; but the general demonstration of it was first given by M. l'Abbé de Gua. See the Memoires de l'Academie des Sciences de Paris, for 1741. F.T.

the roots of the equation: make therefore $x=2$, and we again find the quantity $=0$; consequently, another of the roots is $x=2$; but $x=-2$, on the contrary, is found not to be a root. If we make $x=3$, we have $81+54-63-24+12=60$, so that the supposition does not answer; but $x=-3$, giving $81-54-63+24+12=0$, it is evidently one of the roots sought. Lastly, when we try $x=-4$, we likewise see the equation reduced to nothing; so that all the four roots are rational, and have the following values: $x=1$, $x=2$, $x=-3$, and $x=-4$; and, according to the rule given above, two of these roots are positive, and the two others are negative.

760. But as no root could be determined by this method, when the roots are all irrational, it was necessary to devise other expedients for expressing the roots whenever this case occurs; and two different methods have been discovered for finding such roots, whatever be the nature of the equation of the fourth degree.

But before we explain those general methods, it will be proper to give the solutions of some particular cases, which may frequently be applied with great advantage.

761. When the equation is such, that the coefficients of the terms succeed in the same manner, both in the direct and in the inverse order of the terms, as happens in the following equation*:

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* These equations may be called *reciprocal*, for they are not at all changed by substituting $\frac{1}{x}$ for x. From this property it follows, that if a, for instance, be one of the roots, $\frac{1}{a}$ will be one

$$x^4+mx^3+nx^2+mx+1=0;$$

or in this other equation, which is more general:

$$x^4+max^3+na^2x^2+ma^3x+a^4=0;$$

we may always consider such a formula as the product of two factors, which are of the second degree, and are easily resolved. In fact, if we represent this last equation by the product

$$(x^2+pax+a^2)\times(x^2+qax+a^2)=0,$$

in which it is required to determine p and q in such a manner, that the above equation may be obtained, we shall find, by performing the multiplication,

$$x^4+(p+q)ax^3+(pq+2)a^2x^2+(p+q)a^3x+a^4=0;$$

and, in order that this equation may be the same as the former, we must have,

1st $p+q=m$,

2dly $pq+2=n$,

and consequently $pq=n-2$.

Now, squaring the first of those equations, we have $p^2+2pq+q^2=m^2$; and if from this we subtract the second, taken four times, or $4pq=4n-8$, there remains $p^2-2pq+q^2=m^2-4n+8$; and taking the square root, we find $p-q=\sqrt{m^2-4n+8}$; also $p+q=m$; we shall therefore have, by addition, $2p=m+\sqrt{m^2-4n+8}$, or $p=\frac{m+\sqrt{m^2-4n+8}}{2}$; and, by subtraction, $2q=m-\sqrt{m^2-4n+8}$, or

likewise; for which reason such equations may be reduced to others of a dimension one half less. De Moivre has given, in his Miscellanea Analytica, page 71, general formulæ for the reduction of such equations, whatever be their dimension. F. T.

See also Wood's Algebra, the Complément des Elemens d'Algebra, by Lacroix, and Waring's Medit. Algeb. chap. 3.

$q=\frac{m-\sqrt{m^2-4n+8}}{2}$. Having therefore found p and q, we have only to suppose each factor $=0$, in order to determine the value of x. The first gives $x^2+pax+a^2=0$, or $x^2=-pax-a^2$, whence we obtain $x=-\frac{pa}{2}\pm\sqrt{\frac{p^2a^2}{4}-a^2}$,

or $x=-\frac{pa}{2}\pm\frac{1}{2}a\sqrt{p^2-4}$;

the second factor gives $x=-\frac{qa}{2}\pm\frac{1}{2}a\sqrt{q^2-4}$;

and these are the four roots of the given equation.

762. To render this more clear, let there be given the equation $x^4-4x^3-3x^2-4x+1=0$. We have here $a=1$, $m=-4$, $n=-3$; consequently, $m^2-4n+8=36$, and the square root of this quantity is $=6$; therefore $p=\frac{-4+6}{2}=1$, and $q=\frac{-4-6}{2}=-5$; whence result the four roots,

1st and 2d $x=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}=\frac{-1\pm\sqrt{-3}}{2}$; and

3d and 4th $x=\frac{5}{2}\pm\frac{1}{2}\sqrt{21}=\frac{5\pm\sqrt{21}}{2}$; that is, the four roots of the given equation are:

1st $x=\frac{-1+\sqrt{-3}}{2}$, 2d $x=\frac{-1-\sqrt{-3}}{2}$,

3d $x=\frac{5+\sqrt{21}}{2}$, 4th $x=\frac{5-\sqrt{21}}{2}$.

The two first of these roots are imaginary, or impossible; but the two last are possible; since we may express $\sqrt{21}$ to any degree of exactness, by means of decimal fractions. In fact, 21 being the same with 21·00000000, we have only to extract the square root, which gives $\sqrt{21}=4·5825$.

Since, therefore, $\sqrt{21}=4·5825$, the third root is very nearly $x=4·7912$, and the fourth, $x=0·2087$; it would also have been easy to have determined these roots with still more precision.

For we observe that the fourth root is very nearly $\frac{2}{10}$, or $\frac{1}{5}$, which value will answer the equation with sufficient exactness; in fact, if we make $x=\frac{1}{5}$, we find $\frac{1}{625}-\frac{4}{125}-\frac{3}{25}-\frac{4}{5}+1=\frac{31}{625}$; we ought however to have obtained 0, but the difference is evidently not great.

763. The second case in which such a resolution takes place, is the same as the first with regard to the coefficients, but differs from it in the signs, for we shall suppose that the second and the fourth terms have different signs; such, for example, as the equation

$$x^4+max^3+na^2x^2-ma^3x+a^4=0,$$

which may be represented by the product,

$$(x^2+pax-a^2)\times(x^2+qax-a^2)=0.$$

For the actual multiplication of these factors gives

$$x^4+(p+q)ax^3+(pq-2)a^2x^2-(p+q)a^3x+a^4,$$

a quantity equal to that which was given, if we suppose, in the first place, $p+q=m$, and in the second place, $pq-2=n$, or $pq=n+2$; because in this manner the fourth terms become equal of themselves: if now we square the first equation, as before, we shall have $p^2+2pq+q^2=m^2$; and if from this we subtract the second, taken four times, or $4pq=4n+8$, there will remain $p^2-2pq+q^2=m^2-4n-8$; the square root is $p-q=\sqrt{m^2-4n-8}$, and thence we obtain

$$p=\frac{m+\sqrt{m^2-4n-8}}{2};\text{ and } q=\frac{m-\sqrt{m^2-4n-8}}{2}.$$

Having therefore found p and q, we shall get by the first factor the two roots $x=-\frac{1}{2}pa\pm\frac{1}{2}a\sqrt{p^2+4}$, and by the second factor the two roots

$$x=-\frac{1}{2}qa\pm\frac{1}{2}a\sqrt{q^2+4},$$

that is, we have the four roots of the equation proposed.

764. Let there be given the equation

$$x^4-3\,.\,2x^3+3\,.\,8x+16=0.$$

Here we have $a=2$, $m=-3$, and $n=0$; so that $\sqrt{m^2-4n-8}=1$, and consequently,

$$p=\frac{-3+1}{2}=-1,\text{ and } q=\frac{-3-1}{2}=-2.$$

Therefore the two first roots are $x=1\pm\sqrt{5}$, and the two last are $x=2\pm\sqrt{8}$; so that the four roots sought will be,

1st $x=1+\sqrt{5}$, 2d $x=1-\sqrt{5}$,
3d $x=2+\sqrt{8}$, 4th $x=2-\sqrt{8}$.

Consequently, the four factors of our equation will be $(x-1-\sqrt{5})\times(x-1+\sqrt{5})\times(x-2-\sqrt{8})\times(x-2+\sqrt{8})$, and their actual multiplication produces the given equation; for the two first being multiplied together, give x^2-2x-4, and the other two give x^2-4x-4; now these products multiplied together, make $x^4-6x^3+24x+16$, which is the same equation that was proposed.

CHAP. XIV.

Of the Rule of Bombelli for reducing the Resolution of Equations of the Fourth Degree to that of Equations of the Third Degree.

765. We have already shown how equations of the third degree are resolved by the rule of Cardan; so that the principal object, with regard to equations of the fourth degree, is to reduce them to equations of the third degree. For it is impossible to resolve, generally, equations of the fourth degree without the aid of those of the third; since, when we have determined one of the roots, the others always depend on an equation of the third degree. And hence we may conclude, that the equations also of higher dimensions presuppose the resolution of all the equations of lower degrees.

766. It is now some centuries since Bombelli, an Italian, gave a rule for this purpose, which we shall explain in this chapter *.

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* This rule rather belongs to Louis Ferrari. It is improperly called the Rule of Bombelli, in the same manner as the rule discovered by Scipio Ferreo has been ascribed to Cardan. F. T.

Let there be given the general equation of the fourth degree, $x^4+ax^3+bx^2+cx+d=0$, in which the letters a, b, c, d, represent any possible numbers; and let us suppose that this equation is the same as $(x^2+\frac{1}{2}ax+p)^2-(qx+r)^2=0$, in which it is required to determine the letters p, q, and r, in order that we may obtain the equation proposed. By ordering the new equation, we shall have

$$\begin{aligned} x^4+ax^3+\tfrac{1}{4}a^2x^2&+apx+p^2 \\ +2px^2&-2qrx-r^2 \\ -q^2x^2&. \end{aligned}$$

Now, the two first terms are already the same here as in the given equation; the third term requires us to make $\frac{1}{4}a^2+2p-q^2=b$, which gives $q^2=\frac{1}{4}a^2+2p-b$; the fourth term shows that we must make $ap-2qr=c$, or $2qr=ap-c$; and, lastly, we have for the last term $p^2-r^2=d$, or $r^2=p^2-d$. We have therefore three equations which will give the values of p, q, and r.

767. The easiest method of deriving those values from them is the following: if we take the first equation four times, we shall have $4q^2=a^2+8p-4b$; which equation, multiplied by the last, $r^2=p^2-d$, gives

$$4q^2r^2=8p^3+(a^2-4b)p^2-8dp-d(a^2-4b).$$

Farther, if we square the second equation, we have $4q^2r^2=a^2p^2-2acp+c^2$. So that we have two values of $4q^4r^4$, which, being made equal, will furnish the equation

$$8p^3+(a^2-4b)p^2-8dp-d(a-b)=a^2p^2-2acp+c^2,$$

or, bringing all the terms to one side,

$$8p^3-4bp^2+(2ac-8d)p-a^2d+4bd-c^2=0,$$

an equation of the third degree, which will always give the value of p by the rules already explained.

768. Having therefore determined the three values of p by the given quantities a, b, c, d, which requires only one of those values to be found, we shall also have the values of the two other letters q and r; for the first equation will give $q=\sqrt{\frac{1}{4}a^2+2p-b}$, and the second gives $r=\frac{ap-c}{2q}$. Now these three values being determined for each given case, the four roots of the proposed equation may be found in the following manner:

This equation having been reduced to the form $(x^2+\frac{1}{2}ax+p)^2-(qx+r)^2=0$, we shall have

$$(x^2+\frac{1}{2}ax+p)^2=(qx+r)^2,$$

and extracting the root, $x^2+\frac{1}{2}ax+p=qx+r$, or $x^2+\frac{1}{2}ax+p=-qx-r$. The first equation gives $x^2=(q-\frac{1}{2}a)x-p+r$, from which we may find two roots; and the second equation, to which we may give the form $x^2=-(q+\frac{1}{2}a)x-p-r$, will furnish the two other roots.

769. Let us illustrate this rule by an example, and suppose that the equation

$$x^4-10x^3+35x^2 50x+24=0$$

was given. If we compare it with our general formula, we have $a=-10$, $b=35$, $c=-50$, $d=24$; and, consequently, the equation which must give the value of p is

$$8p^3-140p^2+808p-1540=0, \text{ or,}$$
$$2p^3-\ 35p^2+202p-\ 385=0.$$

The divisors of the last term are 1, 5, 7, 11, &c.; the first of which does not answer; but making $p=5$, we get $250-875+1010-385=0$, so that $p=5$; and if we farther suppose $p=7$, we get $686-1715+1414-385=0$, a proof that $p=7$ is the second root. It remains now to find the third root; let us therefore divide the equation by 2, in order to have $p^3-\frac{35}{2}p^2+101p-\frac{385}{2}=0$, and let us consider that the coefficient of the second term, or $\frac{35}{2}$, being the sum of all the three roots, and the two first making together 12, the third must necessarily be $\frac{11}{2}$.

We consequently know the three roots required. But it may be observed that one would have been sufficient, because each gives the same four roots for our equation of the fourth degree.

770. To prove this, let $p=5$; we shall then have

$$q=\sqrt{25+10-35}=0, \text{ and } r=\frac{-50+50}{0}=\frac{0}{0}.$$

Now, nothing being determined by this, let us take the third equation,

$$r^2=p^2-d=25-24=1,$$

so that $r=1$; our two equations of the second degree will then be:

1st, $x^2=5x-4$ 2d, $x^2=5x-6$.

The first gives the two roots

$$x=\frac{5}{2}\pm\sqrt{\frac{9}{4}}, \text{ or } x=\frac{5\pm3}{2},$$

that is to say, $x=4$ and $x=1$.

The second equation gives

$$x=\frac{5}{2}\pm\sqrt{\frac{1}{4}}=\frac{5\pm1}{2},$$

that is to say, $x=3$ and $x=2$.

But suppose now $p=7$, we shall have

$$q=\sqrt{25+14-35}=2, \text{ and } r=\frac{-70+50}{4}=-5,$$

whence result the two equations of the second degree,

1st, $x^2=7x-12$, 2d, $x^2=3x-2$;

the first gives

$$x=\frac{7}{2}\pm\sqrt{\frac{1}{4}}, \text{ or } x=\frac{7\pm1}{2},$$

so that $x=4$ and $x=3$; the second furnishes the root

$$x=\frac{3}{2}\pm\sqrt{\frac{1}{4}}=\frac{3\pm1}{2},$$

and consequently $x=2$ and $x=1$; therefore by this second supposition the same four roots are found as by the first.

Lastly, the same roots are found, by the third value of p, $=\frac{11}{2}$: for, in this case, we have

$$q=\sqrt{25+11-35}=1, \text{ and } r=\frac{-55+50}{2}=-\frac{5}{2};$$

so that the two equations of the second degree become, 1st, $x^2=6x$, 2d, $x^2=4x-3$.

Whence we obtain from the first, $x=3\pm\sqrt{1}$, that is to say, $x=4$ and $x=2$; and from the second, $x=2\pm\sqrt{1}$, that is to say, $x=3$ and $x=1$, which are the same roots we originally obtained.

771. Let there now be proposed the equation

$$x^4-16x-12=0,$$

in which $a=0$, $b=0$, $c=-16$, $d=-12$; and our equation of the third degree will be

$$8p^3+96p-256=0, \text{ or } p^3+12p-32=0,$$

and we may make this equation still more simple, by writing $p=2t$; for we have then

$$8t^3+24t-32=0, \text{ or } t^3+3t-4=0.$$

The divisors of the last term are 1, 2, 4; whence one of the roots is found to be $t=1$; therefore $p=2$, $q=\sqrt{4}=2$, and $r=\frac{16}{4}=4$. Consequently, the two equations of the second degree are

$$x^2=2x+2, \text{ and } x^2=-2x-6;$$

which give the roots

$$x=1\pm\sqrt{3}, \text{ and } x=-1\pm\sqrt{-5}.$$

772. We shall endeavour to render this resolution still more familiar, by a repetition of it in the following example. Suppose there were given the equation

$$x^4-6x^3+12x^2-12x+4=0,$$

which must be contained in the formula

$$(x^2-3x+p)^2-(qx+r)^2=0,$$

in the former part of which we have put $-3x$, because -3 is half the coefficient -6, of the given equation. This formula being expanded, gives

$x^4-6x^3+(2p+9-q^2)x^2-(6p+2qr)x+p^2-r^2=0$; which compared with our equation, there will result from that comparison the following equations:

1st, $2p+9-q^2=12$,
2d, $6p+2qr=12$,
3d, $p^2-r^2=4$.

The first gives $q^2=2p-3$;
the second, $2qr=12-6p$, or $qr=6-3p$;
the third, $r^2=p^2-4$.

Multiplying r^2 by q^2, and p^2-4 by $2p-3$, we have

$$q^2r^2=2p^3-3p^2-8p+12;$$

and if we square the value of qr, we have

$$q^2r^2=36-36p+9p^2;$$

so that we have the equation

$$2p^3-3p^2-8p+12=9p^2-36p+36, \text{ or}$$
$$2p^3-12p^2+28p-24=0, \text{ or}$$
$$p^3-6p^2+14p-12=0,$$

one of the roots of which is $p=2$; and it follows that $q^2=1$, $q=1$, and $qr=r=0$. Therefore our equation will be $(x^2-3x+2)^2=x^2$, and its square root will be $x^2-3x+2=\pm x$. If we take the upper sign, we have $x^2=4x-2$; and taking the lower sign, we obtain $x^2=2x-2$, whence we derive the four roots $x=2\pm\sqrt{2}$, and $x=1\pm\sqrt{-1}$.

CHAP. XV.

Of a new Method of resolving Equations of the Fourth Degree.

773. The rule of Bombelli, as we have seen, resolves equations of the fourth degree by means of an equation of the third degree; but since the invention of that rule, another method has been discovered of performing the same resolution; and as it is altogether different from the first, it deserves to be separately explained*.

774. We suppose that the root of an equation of the fourth degree has the form $x=\sqrt{p}+\sqrt{q}+\sqrt{r}$, in which the letters p, q, r, express the roots of an equation of the third degree, $z^3-fz^2+gz-h=0$; so that $p+q+r=f$; $pq+pr+qr=g$; and $pqr=h$. This being laid down, we square the assumed formula, $x=\sqrt{p}+\sqrt{q}+\sqrt{r}$, and we obtain

$$x^2=p+q+r+2\sqrt{pq}+2\sqrt{pr}+2\sqrt{qr};$$

and, since $p+q+r=f$, we have

$$x^2-f=2\sqrt{pq}+2\sqrt{pr}+2\sqrt{qr};$$

we again take the squares, and find $x^4-2fx^2+f^2=4pq+4pr+4qr+8\sqrt{p^2qr}+8\sqrt{pq^2r}+8\sqrt{pqr^2}$. Now $4pq+4pr+4qr=4g$, so that the equation becomes $x^4-2fx^2+f^2-4g=8\sqrt{pqr}\times(\sqrt{p}+\sqrt{q}+\sqrt{r})$; but

* This method was the invention of Euler himself. He has explained it in the sixteenth volume of the Ancient Commentaries of Petersburg. F. T.

$\sqrt{p}+\sqrt{q}+\sqrt{r}=x$, and $pqr=h$, or $\sqrt{pqr}=\sqrt{h}$; wherefore we arrive at this equation of the fourth degree, $x^4-2fx^2-8x\sqrt{h}+f^2-4g=0$, one of the roots of which is certainly $x=\sqrt{p}+\sqrt{q}+\sqrt{r}$, and in which p, q, and r, are the roots of an equation of the third degree, $z^3-fz^2+gz-h=0$.

775. The equation of the fourth degree, at which we have arrived, may be considered as general, although the second term x^3y is wanted; for we shall afterwards show, that every complete equation may be transformed into another from which the second term has been taken away.

Let there be proposed the equation

$$x^4-ax^2-bx-c=0,$$

in order to determine its root. This we must first compare with the formula, in order to obtain the values of f, g, and h; and we shall have,

1st, $2f=a$, and $f=\frac{a}{2}$;

2d, $8\sqrt{h}=b$, so that $h=\frac{b^2}{64}$;

3d, $f^2-4g=-c$, or $\frac{a^2}{4}-4g+c=0$,

or $\frac{1}{4}a^2+c=4g$;

consequently, $g=\frac{1}{16}a^2+\frac{1}{4}c$.

776. Since, therefore, the equation

$$x^4-ax^2-bx-c=0,$$

gives the values of the letters f, g, and h, so that

$f=\frac{1}{2}a$, $g=\frac{1}{16}a^2+\frac{1}{4}c$, and $h=\frac{1}{64}b^2$, or $\sqrt{h}=\frac{1}{8}b$,

we form from these values the equation of the third degree $z^3 - fz^2 + gz - h = 0$, in order to obtain its roots by the known rule. And if we suppose those roots, 1st, $z = p$, 2d, $z = q$, 3d, $z = r$, one of the roots of our equation of the fourth degree must be

$$x = \sqrt{p} + \sqrt{q} + \sqrt{r}.$$

777. This method appears at first to furnish only one root of the given equation; but if we consider that every sign $\sqrt{}$ may be taken, negatively as well as positively, we shall immediately perceive that this formula contains all the four roots.

Farther, if we chose to admit all the possible changes of the signs, we should have eight different values of x, and yet four only can exist. But it is to be observed, that the product of those three terms, or $\sqrt{pqr}$, must be equal to $\sqrt{h} = \frac{1}{8}b$, and that if $\frac{1}{8}b$ be positive, the product of the terms $\sqrt{p}$, $\sqrt{q}$, $\sqrt{r}$, must likewise be positive, so that all the variations which can be admitted are reduced to the four following:

1st, $x = \sqrt{p} + \sqrt{q} + \sqrt{r}$,
2d, $x = \sqrt{p} - \sqrt{q} - \sqrt{r}$,
3d, $x = -\sqrt{p} + \sqrt{q} - \sqrt{r}$,
4th, $x = -\sqrt{p} - \sqrt{q} + \sqrt{r}$.

In the same manner, when $\frac{1}{8}b$ is negative, we have only the four following values of x:

1st, $x = \sqrt{p} + \sqrt{q} - \sqrt{r}$,
2d, $x = \sqrt{p} - \sqrt{q} + \sqrt{r}$,
3d, $x = -\sqrt{p} + \sqrt{q} + \sqrt{r}$,
4th, $x = -\sqrt{p} - \sqrt{q} - \sqrt{r}$.

This circumstance enables us to determine the four roots in all cases; as may be seen in the following example.

778. Let there be proposed the equation of the fourth degree, $x^4-25x^2+60x-36=0$, in which the second term is wanted. Now if we compare this with the general formula, we have $a=25$, $b=-60$, and $c=36$; and after that,

$$f=\frac{25}{2},\ g=\frac{625}{16}+9=\frac{769}{16},\ \text{and}\ h=\frac{225}{4};$$

by which means our equation of the third degree becomes,

$$z^3-\frac{25}{2}z^2+\frac{769}{16}z-\frac{225}{4}=0.$$

To remove the fractions, let us make $z=\frac{u}{4}$; and we shall have $\frac{u^3}{64}-\frac{25u^2}{32}+\frac{769u}{64}-\frac{225}{4}=0$, and multiplying by the greatest denominator, we obtain

$$u^3-50u^2+769u-3600=0;$$

and we must determine the three roots of this equation; which are all three found to be positive; one of them being $u=9$, then dividing the equation by $u-9$, we find the new equation $u^2-41u+400=0$, or $u^2=41u-400$, which gives

$$u=\frac{41}{2}\pm\sqrt{\frac{1681}{4}-\frac{1600}{4}}=\frac{41\pm9}{2};$$

so that the three roots are $u=9$, $u=16$, and $u=25$.

Consequently, the roots are

$$\text{1st}\ z=\frac{9}{4},\quad \text{2d,}\ z=4,\quad \text{3d,}\ z=\frac{25}{4}.$$

And these are therefore the values of the letters

p, q, and r, that is to say, $p=\frac{9}{4}$, $q=4$, $r=\frac{25}{4}$. Now, if we consider that $\sqrt{pqr}=\sqrt{h}=-\frac{15}{2}$, and that therefore this value $=\frac{1}{8}b$ is negative, we must, agreeably to what has been said with regard to the signs of the roots $\sqrt{p}$, $\sqrt{q}$, and $\sqrt{r}$, take all those three roots negatively, or take only one of them negatively; and consequently, as $\sqrt{p}=\frac{3}{2}$, $\sqrt{q}=2$, and $\sqrt{r}=\frac{5}{2}$, the four roots of the given equation are found to be:

1st, $x=\frac{3}{2}+2-\frac{5}{2}=1$,

2d, $x=\frac{3}{2}-2+\frac{5}{2}=2$,

3d, $x=-\frac{3}{2}+2+\frac{5}{2}=3$,

4th, $x=-\frac{3}{2}-2-\frac{5}{2}=-6$.

From these roots result the four factors,

$$(x-1)\times(x-2)\times(x-3)\times(x+6)=0.$$

The first two, multiplied together, give x^2-3x+2; the product of the last two is $x^2+3x-18$; again multiplying these two products together, we obtain exactly the equation proposed.

779. It remains now to show how an equation of the fourth degree, in which the second term is found, may be transformed into another in which that term

is wanted: for which we shall give the following rule *.

Let there be proposed the general equation $y^4+ay^3+by^2+cy+d=0$. If we add to y the fourth part of the coefficient of the second term, or $\frac{1}{4}a$, and write, instead of the sum, a new letter x, so that $y+\frac{1}{4}a=x$, and consequently $y=x-\frac{1}{4}a$: we shall have

$$y^2=x^2-\frac{1}{2}ax+\frac{1}{16}a^2,\quad y^3=x^3-\frac{3}{4}ax^2+\frac{3}{16}a^2x-\frac{1}{64}a^3,$$

and lastly as follows:

$$\begin{aligned}
y^4 &= x^4-ax^3+\frac{3}{8}a^2x^2-\frac{1}{16}a^3x+\frac{1}{256}a^4\\
+ay^3 &= \quad +ax^3-\frac{3}{4}a^2x^2+\frac{3}{16}a^3x-\frac{1}{64}a^4\\
+by^2 &= \quad\quad +bx^2-\frac{1}{2}abx+\frac{1}{16}a^2b\\
+cy &= \quad\quad\quad +cx-\frac{1}{4}ac\\
+d &= \quad\quad\quad\quad +d
\end{aligned}$$

And hence by addition,

$$\left.\begin{array}{l}
x^4+0-\frac{3}{8}a^2x^2+\frac{1}{8}a^3x-\frac{3}{256}a^4\\
\quad +bx^2-\frac{1}{2}abx+\frac{1}{16}a^2b\\
\quad\quad +cx-\frac{1}{4}ac\\
\quad\quad\quad +d
\end{array}\right\}=0.$$

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* An investigation of this rule may be seen in Maclaurin's Algebra, part II. chap. 3.

We have now an equation from which the second term is taken away, and to which nothing prevents us from applying the rule before given for determining its four roots. After the values of x are found, those of y will easily be determined, since $y = x - \frac{1}{4}a$.

780. This is the greatest length to which we have yet arrived in the resolution of algebraic equations; all the pains that have been bestowed in order to resolve equations of the fifth degree, and of higher dimensions, in the same manner, or, at least, to reduce them to inferior degrees, have been unsuccessful: so that we cannot give any general rules for finding the roots of equations that exceed the fourth degree.

The only success that has attended these attempts has been the resolution of some particular cases; the chief of which is that, where a rational root takes place; for it is easily found by the method of divisors, as we know that such a root must be always a factor of the last term; the operation, in other respects, is the same as that we have taught for equations of the third and fourth degree.

781. It will be necessary, however, to apply the rule of Bombelli to an equation which has no rational roots.

Let there be given the equation $y^4 - 8y^3 + 14y^2 + 4y - 8 = 0$. Here we must begin with destroying the second term, by adding the fourth of its coefficient to y, supposing $y - 2 = x$, and substituting in the equation, instead of y, its new value $x + 2$, instead of y^2, its value $x^2 + 4x + 4$; and doing the same with regard to y^3 and y^4, we shall have,

$$\begin{array}{rl} y^4 = & x^4+8x^3+24x^2+32x+16 \\ -8y^3 = & \quad -8x^3-48x^2-96x-64 \\ +14y^2 = & \qquad +14x^2+56x+56 \\ +4y = & \qquad\qquad +4x+8 \\ -8 = & \qquad\qquad\quad -8 \\ \hline & x^4+0-10x^2-4x+8=0. \end{array}$$

This equation being compared with our general formula, gives $a=10$, $b=4$, $c=-8$; whence we conclude that $f=5$, $g=\frac{17}{4}$, $h=\frac{1}{4}$, and $\sqrt{h}=\frac{1}{2}$; that the product $\sqrt{pqr}$ will be positive; and that it is from the equation of the third degree,

$$z^3-5z^2+\frac{17}{4}z-\frac{1}{4}=0,$$

that we are to seek for the three roots p, q, r.

782. Let us first remove the fractions from this equation, by making $z=\frac{u}{2}$, and we shall thus have, after multiplying by 8, the equation

$$u^3-10u^2+17u-2=0,$$

in which all the roots are positive. Now, the divisors of the last term are 1 and 2; if we try $u=1$, we find $1-10+17-2=6$; so that the equation is not reduced to nothing: but trying $u=2$, we find $8-40+34-2=0$, which answers to the equation, and shows that $u=2$ is one of the roots; and the two others will be found by dividing by $u-2$, as usual; then the quotient $u^2-8u+1=0$ will give $u^2=8u-1$, and $u=4\pm\sqrt{15}$. And since $z=\frac{1}{2}u$, the three roots of the equation of the third degree are, 1st, $z = p = 1$,

2d, $z = q = \frac{4+\sqrt{15}}{2}$,

3d, $z = r = \frac{4-\sqrt{15}}{2}$.

783. Having therefore determined p, q, r, we have also their square roots; namely, $\sqrt{p} = 1$, $\sqrt{q} = \frac{\sqrt{8+2\sqrt{15}}}{2}$, and $\sqrt{r} = \frac{\sqrt{8-2\sqrt{15}}}{2}$.

But we have already seen, Art. 675 and 676, that the square root of $a \pm \sqrt{b}$, when $\sqrt{a^2 - b} = c$, is expressed by $\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}}$; so that, as in our case $a = 8$ and $\sqrt{b} = 2\sqrt{15}$, and consequently $b = 60$ and $c = 2$, we have $\sqrt{8+2\sqrt{15}} = \sqrt{5} + \sqrt{3}$, and $\sqrt{8-2\sqrt{15}}$ .... $= \sqrt{5} - \sqrt{3}$.

Hence we have $\sqrt{p} = 1$, $\sqrt{q} = \frac{\sqrt{5}+\sqrt{3}}{2}$, and $\sqrt{r} = \frac{\sqrt{5}-\sqrt{3}}{2}$; wherefore, since we also know that the product of those quantities is positive, the four values of x will be these:

1st, $x = \sqrt{p} + \sqrt{q} + \sqrt{r} = 1 + \frac{\sqrt{5}+\sqrt{3}+\sqrt{5}-\sqrt{3}}{2} = 1 + \sqrt{5}$,

2d, $x = \sqrt{p} - \sqrt{q} - \sqrt{r} = 1 + \frac{-\sqrt{5}-\sqrt{3}-\sqrt{5}+\sqrt{3}}{2} = 1 - \sqrt{5}$,

3d, $x = -\sqrt{p} + \sqrt{q} - \sqrt{r} = -1 + \frac{\sqrt{5}+\sqrt{3}-\sqrt{5}+\sqrt{3}}{2}$ ......
$= -1 + \sqrt{3}$,

4th, $x = -1\sqrt{p} - \sqrt{q} + \sqrt{r} = -1 + \frac{-\sqrt{5}-\sqrt{3}+\sqrt{5}-\sqrt{3}}{2}$ ...
$= -1 - \sqrt{3}$.

Lastly, as we have $y=x+2$, and the four roots of the given equation are:

1st, $y=3+\sqrt{5}$, 2d, $y=3-\sqrt{5}$,
3d, $y=1+\sqrt{3}$, 4th, $y=1-\sqrt{3}$.

CHAP. XVI.

Of the Resolution of Equations by Approximation.

784. When the roots of an equation are not rational, whether they may be expressed by radical quantities, or even if we have not that resource, as is the case with equations which exceed the fourth degree, we must be satisfied with determining their values by approximation; that is to say, by methods which are continually bringing us nearer to the true value, till at last the error being very small, it may be neglected. Different methods of this kind have been proposed, the chief of which we shall explain.

785. The first method which we shall mention, supposes that we have already determined, with tolerable exactness, the value of one root *; that we

* This is the method given by Newton at the beginning of his Method of Fluxions. When investigated, it is found subject to different imperfections; for which reason we may with advantage substitute the method given by M. de la Grange, in the Memoirs of Berlin for 1767 and 1768. F. T.

This method has since been published by La Grange, under the title of Sur la Resolution des Equations Numeriques; in which work the subject is treated in the usual masterly style of this author. Ed.

know, for example, that such a value exceeds 4, and that it is less than 5. In this case, if we suppose this value $=4+p$, we are certain that p expresses a fraction. Now as p is a fraction, and consequently less than unity, the square of p, its cube, and, in general, all the higher powers of p, will be much less with respect to unity; and, for this reason, since we require only an approximation, they may be neglected in the calculation. When we have, therefore, nearly determined the fraction p, we shall know more exactly the root $4+p$; and from that we proceed to determine a new value still more exact, and continue the same process till we come as near the truth as we desire.

786. We shall illustrate this method first by an easy example, requiring by approximation the root of the equation $x^2=20$.

Here we perceive, that x is greater than 4 and less than 5; making, therefore, $x=4+p$, we shall have $x^2=16+8p+p^2=20$; but as p^2 is very small, we shall neglect it, in order that we may have only the equation $16+8p=20$, or $8p=4$. This gives $p=\frac{1}{2}$, and $x=4\frac{1}{2}$, which already approaches nearer the true root. If, therefore, we now suppose $x=4\frac{1}{2}+p'$; we are sure that p' expresses a fraction much smaller than before, and that we may neglect p'^2 with greater propriety. We have, therefore, $x^2=20\frac{1}{4}+9p'=20$, or $9p'=-\frac{1}{4}$; and consequently, $p'=-\frac{1}{36}$; therefore $x=4\frac{1}{2}-\frac{1}{36}=4\frac{17}{36}$.

And if we wished to approximate still nearer to the true value, we must make $x=4\frac{17}{36}+p''$, and should thus have $x^2=20\frac{1}{1296}+8\frac{34}{36}p''=20$; so that $8\frac{34}{36}p''=-\frac{1}{1296}$, $322p''=-\frac{36}{1296}=-\frac{1}{36}$, and

$$p=-\frac{1}{36\times 322}=-\frac{1}{11592};$$

therefore $x=4\frac{17}{36}-\frac{1}{11592}=4\frac{4473}{11592}$, a value which is so near the truth, that we may consider the error as of no importance.

787. Now in order to generalize what we have here laid down, let us suppose the given equation to be $x^2=a$, and that we previously know x to be greater than n, but less than $n+1$. If we now make $x=n+p$, p must be a fraction, and p^2 may be neglected as a very small quantity, so that we shall have $x^2=n^2+2np=a$; or $2np=a-n^2$, and $p=\frac{a-n^2}{2n}$; consequently $x=n+\frac{a-n^2}{2n}=\frac{n^2+a}{2n}$.

Now if n approximated towards the true value, this new value $\frac{n^2+a}{2n}$ will approximate much nearer; and, by substituting it for n, we shall find the result much nearer the truth; that is, we shall obtain a new value, which may again be substituted, in order to approach still nearer; and the same operation may be continued as long as we please.

For example, let $a=2$; that is to say, let the square root of 2 be required; and as we already know a value sufficiently near, which is expressed by

n, we shall have a still nearer value of the root expressed by $\frac{n^2+2}{2n}$. Let therefore,

1st $n=1$, and we shall have $x=\frac{3}{2}$,

2d $n=\frac{3}{2}$, and we shall have $x=\frac{17}{12}$,

3d $n=\frac{17}{12}$, and we shall have $x=\frac{577}{408}$.

This last value approaches so near $\sqrt{2}$, that its square $\frac{332929}{166464}$ differs from the number 2 only by the small quantity $\frac{1}{166464}$, by which it exceeds it.

788. We may proceed in the same manner, when it is required to find by approximation cube roots, biquadrate roots, &c.

Let there be given the equation of the third degree, $x^3=a$; or let it be proposed to find the value of $\sqrt[3]{a}$.

Knowing that it is nearly n, we shall suppose $x=n\pm p$; neglecting p^2 and p^3, we shall have $x^3=n^3\pm 3n^2p=a$; so that $\pm 3n^2p=a-n^3$, and $\pm p=\frac{a-n^3}{3n^2}$; whence $x=\frac{2n^3+a}{3n^2}$. If, therefore, n is nearly $=\sqrt[3]{a}$, the quantity which we have now found will be much nearer it. But for still greater exactness, we may again substitute this new value for n, and so on.

For example, let $x^3=2$; and let it be required to determine $\sqrt[3]{2}$. Here if n is nearly the value of the number sought, the formula $\frac{2n^3+2}{3n^2}$ will express that number still more nearly; let us therefore make

1st $n=1$, and we shall have $x=\frac{4}{3}$,

2d $n=\frac{4}{3}$, and we shall have $x=\frac{91}{72}$,

3d $n=\frac{91}{72}$, and we shall have $x=\frac{162130896}{128634294}$.

789. This method of approximation may be employed, with the same success, in finding the roots of all equations.

To show this, suppose we have the general equation of the third degree, $x^3+ax^2+bx+c=0$, in which n is very nearly the value of one of the roots. Let us make $x=n\pm p$; and, since p will be a fraction, neglecting the powers of this letter which are higher than the first degree, we shall have $x^2=n^2\pm 2np$, and $x^3=n^3\pm 3n^2p$, whence we have the equation $n^3\pm 3n^2p+an^2\pm 2anp+bn\pm bp+c=0$, or $n^3+an^2+bn+c=\mp(3n^2p+2anp+bp)=\mp(3n^2+2an+b)p$; so that $\mp p=\frac{n^3+an^2b+n-c}{3n^2+2an+b}$, and $x=n-(\frac{n^3+an^2+bn+c}{3n^2+2an+b})=\frac{2n^3+an^2-c}{3n^2+2an+b}$. This value, which is more exact than the first, being substituted for n, will furnish a new value still more accurate.

790. In order to apply this operation to an example, let $x^3+2x^2+3x-50=0$, in which $a=2$, $b=3$, and $c=-50$. If n is supposed to be nearly the value of one of the roots, $x=\frac{2n^3+2n^2+50}{3n^2+4n+3}$, will be a value still nearer the truth.

Now, the assumed value of $x=3$ not being far from the true one, we shall suppose $n=3$, which gives us $x=\frac{62}{21}$; and if we were to substitute this new value instead of n, we should find another still more exact.

791. We shall give only the following example, for equations of higher dimensions than the third.

Let $x^5=6x+10$, or $x^5-6x-10=0$, where we readily perceive that 1 is too small, and that 2 is too great. Now, if $x=n$ is a value not far from the true one, and we make $x=n\pm p$, we shall have $x^5=n^5\pm 5n^4p$; and, consequently,

$$n^5\pm 5n^4p=6n\pm 6p+10;\text{ or}$$
$$\pm p(5n^4-6)=6n+10-n^5.$$

Wherefore $\pm p=\frac{6n+10-n^5}{5n^4-6}$, and $x=\frac{4n^5+10}{5n^4-6}$.

If we suppose $n=1$, we shall have $x=\frac{14}{-1}=-14$; this value is altogether inapplicable, a circumstance which arises from the approximated value of n having been taken by much too small. We shall, therefore, make $n=2$, and shall thus obtain $x=\frac{138}{74}=\frac{69}{37}$, a value which is much nearer the truth. And if we were now to substitute for n, the fraction $\frac{69}{37}$, we should obtain a still more exact value of the root x.

792. Such is the most usual method of finding the roots of an equation by approximation, and it applies successfully to all cases.

We shall however explain another method *, which

* The theory of approximation here given, is founded on the theory of what are called *recurring* series, invented by M. de Moivre. This method was given by Daniel Bernoulli, in vol. iii.

deserves attention on account of the facility of the calculation. The foundation of this method consists in determining for each equation a series of numbers, as a, b, c, &c. such, that each term of the series, divided by the preceding one, may express the value of the root with so much the more exactness, according as this series of numbers is carried to a greater length.

Suppose we have already got the terms p, q, r, s, t, &c. $\frac{q}{p}$ must express the root x with tolerable exactness; that is to say, we have nearly $\frac{q}{p}=x$. We shall have also $\frac{r}{q}=x$*, and the multiplication of the two values will give $\frac{r}{p}=x^2$. Farther, as $\frac{s}{r}=x$, we shall also have $\frac{s}{p}=x^3$; then, since $\frac{t}{s}=x$, we shall have $\frac{t}{p}=x^4$, and so on.

793. For the better explanation of this method, we shall begin with an equation of the second degree,

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of the Ancient Commentaries of Petersburg. But Euler has here presented it in rather a different point of view. Those who wish to investigate these matters, may consult chapters 13 and 17 of vol. i. of our author's Introd. in Anal. Infin.; an excellent work, in which several subjects treated of in this first part, beside others equally connected with pure mathematics, are profoundly analysed and clearly explained. F. T.

* It must only be understood here that $\frac{r}{q}$ is nearly equal to x.

$x^2 = x + 1$, and shall suppose that in the above series we have found the terms p, q, r, s, t, &c. Now, as $\frac{q}{p} = x$, and $\frac{r}{p} = x^2$, we shall have the equation $\frac{r}{p} = \frac{q}{p} + 1$, or $q + p = r$. And as we find, in the same manner, that $s = r + q$, and $t = s + r$; we conclude that each term of our series is the sum of the two preceding terms; so that having the two first terms, we can easily continue the series to any length. With regard to the two first terms, they may be taken at pleasure; if we therefore suppose them to be 0, 1, our series will be 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, &c. and such, that if we divide any term by that which immediately precedes it, we shall have a value of x so much nearer the true one, according as we have chosen a term more distant. The error, indeed, is very great at first, but it diminishes as we advance. The series of those values of x, in the order in which they are always approximating towards the true one, is as follows:

$$x = \frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \frac{144}{89}, \&c.$$

If, for example, we make $x = \frac{21}{13}$, we have $\frac{441}{169} = \frac{21}{13} + 1 = \frac{442}{169}$, in which the error is only $\frac{1}{169}$. Any of the succeeding terms will give it still smaller.

794. Let us also consider the equation $x^2 = 2x + 1$; and since, in all cases, $x = \frac{q}{p}$, and $x^2 = \frac{r}{p}$, we shall have $\frac{r}{p} = \frac{2q}{p} + 1$, or $r = 2q + p$; whence we infer that

the double of each term, added to the preceding term, will give the succeeding one. If, therefore, we begin again with 0, 1, we shall have the series,

0, 1, 2, 5, 12, 29, 70, 169, 408, &c.

Whence it follows, that the value of x will be expressed still more accurately by the following fractions:

$$x=\frac{1}{0},\ \frac{2}{1},\ \frac{5}{2},\ \frac{12}{5},\ \frac{29}{12},\ \frac{70}{29},\ \frac{169}{70},\ \frac{408}{169},\ \&c.$$

which, consequently, will always approximate nearer and nearer the true value of $x=1+\sqrt{2}$; so that if we take unity from these fractions, the value of $\sqrt{2}$ will be expressed more and more exactly by the succeeding fractions:

$$\frac{1}{0},\ \frac{1}{1},\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70},\ \frac{239}{169},\ \&c.$$

For example, $\frac{99}{70}$ has for its square $\frac{9801}{4900}$, which differs only by $\frac{1}{4900}$ from the number 2.

795. This method is no less applicable to equations which have a greater number of dimensions. If we have, for example, the equation of the third degree $x^3=x^2+2x+1$, we must make $x=\frac{q}{p}$, $x^2=\frac{r}{p}$, and $x^3=\frac{s}{p}$; we shall then have $s=r+2q+p$; which shows how, by means of the three terms p, q, and r, we are to determine the succeeding one s; and, as the beginning is always arbitrary, we may form the following series:

0, 0, 1, 1, 3, 6, 13, 28, 60, 129, &c.

from which result the following fractions for the approximate values of x:

$$x=\frac{0}{0},\ \frac{1}{0},\ \frac{1}{1},\ \frac{3}{1},\ \frac{6}{3},\ \frac{13}{6},\ \frac{28}{13},\ \frac{60}{28},\ \frac{129}{60},\ \&c.$$

The first of these values would be very far from the truth; but if we substitute in the equation instead for x $\frac{60}{28}$, or $\frac{15}{7}$, we obtain

$$\frac{3375}{343}=\frac{225}{49}+\frac{30}{7}+1=\frac{3388}{343},$$

in which the error is only $\frac{13}{343}$.

796. It must be observed, however, that all equations are not of such a nature as to admit the application of this method; and particularly when the second term is wanting, it cannot be made use of. For example, let $x^2=2$; if we wished to make $x=\frac{q}{p}$ and $x^2=\frac{r}{p}$, we should have $\frac{r}{p}=2$, or $r=2p$, that is to say, $r=0q+2p$, whence would result the series

1, 1, 2, 2, 4, 4, 8, 8, 16, 16, 32, 32, &c.

from which we can draw no conclusion, because each term, divided by the preceding, gives always $x=1$, or $x=2$. But we may obviate this inconvenience, by making $x=y-1$; for by these means we have $y^2-2y+1=2$; and if we now make $y=\frac{q}{p}$, and $y^2=\frac{r}{p}$, we shall obtain the same approximation that has been already given.

797. It would be the same with the equation

$x^3=2$; it would not furnish such a series of numbers as would express the value of $\sqrt[3]{2}$. But we have only to suppose $x=y-1$, in order to have the equation $y^3-3y^2+3y-1=2$, or $y^3=3y^2-3y+3$; and then making $y=\frac{q}{p}$, $y^2=\frac{r}{p}$ and $y^3=\frac{s}{p}$, we have $s=3r-3q+3p$, by means of which we see how three given terms determine the succeeding one.

Assuming then any three terms for the first, for example 0, 0, 1, we have the following series:

0, 0, 1, 3, 6, 12, 27, 63, 144, 324, &c.

The two last terms of this series give $y=\frac{324}{144}$ and $x=\frac{5}{4}$; and this fraction approaches sufficiently near the cube root of 2; for the cube of $\frac{5}{4}$ is $\frac{125}{64}$, and $2=\frac{128}{64}$.

798. We must farther observe, with regard to this method, that when the equation has a rational root, and the beginning of the period is chosen such, that this root may result from it, each term of the series, divided by the preceding term, will give the root with equal accuracy.

To show this, let there be given the equation $x^2=x+2$, one of the roots of which is $x=2$; as we have here, for the series, the formula $r=q+2p$, if we take 1, 2, for the first two terms, we have the series 1, 2, 4, 8, 16, 32, 64, &c. a geometrical progression whose exponent $=2$. The same property is proved by the equation of the third degree $x^3=x^2+3x+9$, which has $x=3$ for one of the roots. If we suppose the first terms to be 1, 3, 9, we shall find, by the formula, $s=r+3y+9p$, and the series 1, 3, 9, 27, 81, 243, &c. which is likewise a geometrical progression.

799. But when the beginning of the series exceeds the root, we shall not approximate towards that root at all; for when the equation has more than one root, the series gives by approximation only the greatest: and we do not find one of the less roots, unless the first terms have been properly chosen for that purpose. This will be illustrated by the following example.

Let there be given the equation $x^2=4x-3$, whose two roots are $x=1$ and $x=3$. The formula for the series is $r=4q-3p$, and if we take 1, 1 for the two first terms of the series, which consequently expresses the least root, we have for the whole series, 1, 1, 1, 1, 1, 1, 1, 1, &c. but assuming for the first terms the numbers 1, 3, which contain the greatest root, we have the series, 1, 3, 9, 27, 81, 243, 729, &c. in which all the terms express precisely the root 3. Lastly, if we assume any other beginning, provided it be such that the least term is not comprised in it, the series will continually approximate towards the greatest root 3; which may be seen by the following series:

Beginning,

0, 1, 4, 13, 40, 121, 364, &c.

1, 2, 5, 14, 41, 122, 365, &c.

2, 3, 6, 15, 42, 123, 366, 1095, &c.

2, 1, −2, −11, −38, −118, −362, 1−091, −3278, &c. in which the quotients of the division of the last terms by the preceding always approximate towards the greater root 3, and never towards the less.

800. We may even apply this method to equations which go on to infinity. The following will furnish an example:

$$x^{\infty}=x^{\infty-1}+x^{\infty-2}+x^{\infty-3}+x^{\infty-4}+, \&c.$$

The series for this equation must be such, that each term may be equal to the sum of all the preceding; that is, we must have

1, 1, 2, 4, 8, 16, 32, 64, 128, &c.

whence we see that the greater root of the given equation is exactly $x=2$; and this may be shown in the following manner. If we divide the equation by x^{∞}, we shall have

$$1=\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\frac{1}{x^4}+, \&c.$$

a geometrical progression, whose sum is found $=\frac{1}{x-1}$; so that $1=\frac{1}{x-1}$; multiplying therefore by $x-1$, we have $x-1=1$, and $x=2$.

801. Beside these two methods of determining the roots of an equation by approximation, some others have been invented, but they are all either too tedious, or not sufficiently general. The method which deserves the preference over all others, is that which we explained first; for it applies successfully to all kinds of equations: whereas the other often requires the equation to be prepared in a certain manner, without which it cannot be employed; of this we have seen a proof in different examples.

END OF VOL. I.

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