The Rigourization of Calculus (in the 19th Century)

People did calculus (& continue to do so) without spending much time on why it works, because it does work. It wasn't until real problems showed because of the lack of rigour that really sustained efforts to make things rigorous happened.

eg People casually summed series like

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

[converges so long as $|x|<1$]

without actually worrying about convergence most of the time. Plug in $x = 2$, to get $1 + 2 + 2^2 + 2^3 + \cdots = \frac{1}{1-2} = -1$. (Euler did stuff like this in passing and nobody had a problem with it.) Absurdities like this accumulated until really useful stuff ran into problems.

Joseph Fourier (1768-1830)

Invented "Fourier series" to solve heat transport problems

- If $f(x)$ is integrable on $[\pi,\pi]$, then the Fourier series for $f(x)$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$.

<u>Problems</u>: 0) The Fourier of $f(x)$ is periodic on $[0,2\pi]$ (up to a constant) even if $f(x)$ is not.

1) When does $f(x)$ equal its Fourier series?

2) Does the series converge at all?

This led to some major reconsiderations of what convergence meant and when it happenned...

This problem became acute when Augustin Cauchy (1789-1857), showed that (1821)

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is infinitely differentiable at 0 and every derivative $f^{(n)}(0) = 0$.

This function has the same Taylor series (which $= 0$ for all x) at 0 as $g(x) = 0$. So $f(x) \neq$ to its Taylor series (which converges for all x) even though $f(x)$ is also defined for all x (except at $x = 0$).

So even power series turned out to have problems.

Cauchy attempted to resolve all this and put in many years of effort trying to make the notion of convergence of series precise. [This required examining limits more carefully.]

by c. 1830 - Devised the Cauchy criterion for convergence of series & something pretty close to the modern definition of derivative in terms of limits.

The definition of limits was simplified and made more precise by Karl Weierstrauss (1815-1897).

Refined Cauchy's idea about limits and put them in algebraic forms, ie the ε-δ & ε-N definitions we use now.

1850's

He also went on to make the assumptions about the real numbers underlying all of this more explicit & precise.

Dropped out of university after studying math instead of law & finance. Went to teachers' college, became a full-time teacher - math, physics, botany, gymnastics - and published a lot of high-powered research at the same time. Eventually got an honorary degree (1854) and a math professorship (1856).

Integration took more effort in some ways, but did benefit considerably from the work developing a better notion of limit.

Georg Riemann (1826-1866) devised a rigourous definition of the definite integral in his 1854 "Habilitationsschrift" in which he was demonstrate his ability to think about new things as part of the qualification process to become a professor.

<u>On the hypotheses which underly geometry</u>

He gave the definition of definite integrals in terms of limits of Riemann sums (of rectangles approximating the region under a curve).

After this a lot of effort was put into devising more powerful and "nicer" definitions of integrations.