John Wallis (1616-1703)

- worked as a cryptographer for Parliament (and, post-Restoration, the royal court) from 1643-1689
- Savilian professor of mathematics at Oxford (1649-1703)
- later also became the "Keeper of the Archives" at Oxford and royal chaplain to King Charles II

- worked on methods for teaching mutes, physics, theology, grammar, musical theory

- in math he did cryptography (but wouldn't publish his methods so that opponents wouldn't have them)

- worked in algebra and "infinitesimal analysis" Cavalieri style & Cartesian coordinates & algebra

1650s: He showed how to compute $\int_a^b p(x)dx$ where $p(x)$ is a polynomial.

He also extended this to fractional & negative power of x.

Amounts to most of the Power Rule for Integration as we know it today.

English Civil War (1642-1651) between the Parliamentarians and the Royalists resulting

1) the execution of Charles I (1649)

2) the exile of his son Charles II (1651)

3) the effective personal rule of the Lord Protector Oliver Cromwell (1653-1658) & his son Richard Cromwell (1658-1659)

4) the Restoration of 1660 of Charles II with a much more powerful Parliament

5) After he died in 1685 his brother James II became King, but was replaced by Parliament in 1688 with his daughter Mary & her husband William of Orange.

He attempted to extend his techniques to other definite integrals eg $\int_0^1 \sqrt{1-x^2}\,dx$ but with only partial success since he couldn't expand $\sqrt{1-x^2}$ as a power series. The attempts he made to compute it in other ways led to the Wallis product:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdot\,\cdots$$

$$= \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2-1}$$

He had an intuitive argument for why this worked, but a proof distantly based on these ideas is available using first-year calculus.

Consider integrals of the form $\int_0^{\pi} \sin^k(x)\,dx$.

$k=0$: $\int_0^\pi \sin^0(x)\,dx = \int_0^\pi 1\,dx = x\Big|_0^\pi = \pi,$

$k=1$: $\int_0^\pi \sin(x)\,dx = -\cos(x)\Big|_0^\pi = (-\cos(\pi)) - (-\cos(0))$

$= -(-1)-(-1) = 1+1 = 2.$

$k \geq 2$: $\int_0^\pi \sin^k(x)\,dx$

$u = \sin^{k-1}(x)$ $\quad v' = \sin(x)$

$u' = (k-1)\sin^{k-2}(x)\cos(x)$ $\quad v = -\cos(x)$

(Integration by Parts)

$$= \sin^{k-1}(x)(-\cos(x))\Big|_0^\pi - \int_0^\pi (k-1)\sin^{k-2}\cos(x)(-\cos(x))\,dx$$

$$= \cancel{\left(-\overset{0}{\overset{\shortparallel}{\sin^{k-1}(\pi)}}\cos(\pi)\right) - \left(-\overset{0}{\overset{\shortparallel}{\sin^{k-1}(0)}}\cos(0)\right)}$$
$$+ (k-1)\int_0^\pi \sin^{k-2}(x)\cos^2(x)\,dx$$

$$= (k-1)\int_0^\pi \sin^{k-2}(x)(1-\sin^2(x))\,dx$$

$$= (k-1)\int_0^\pi \sin^{k-2}(x)\,dx - (k-1)\int_0^\pi \sin^k(x)\,dx$$

$$\Rightarrow \quad k\int_0^\pi \sin^k(x)\,dx = (k-1)\int_0^\pi \sin^{k-2}(x)$$

$$\Rightarrow \quad \int_0^\pi \sin^{k-2}(x)\,dx = \frac{k}{k-1}\int_0^\pi \sin^k(x)\,dx.$$

Note that since $0 \le \sin(x) \le 1$ for $0 \le x \le \pi$,

we have $0 \le \sin^k(x) \le \sin^{k-1}(x) \le 1$ for $0 \le x \le \pi$.

This means that $0 \le \int_0^\pi \sin^k(x)\,dx \le \int_0^\pi \sin^{k-2}(x)\,dx \le \pi$.

So it follows that $1 \le \dfrac{\int_0^\pi \sin^{k-2}(x)\,dx}{\int_0^\pi \sin^{k}(x)\,dx} \le \dfrac{k}{k-1} \to 1$

as $k \to \infty$, this means that (by the Squeeze Theorem) $\dfrac{\int_0^\pi \sin^{k-2}(x)\,dx}{\int_0^\pi \sin^k(x)\,dx} \to 1$.

Since $\int_0^\pi \sin^{2n}(x)\,dx = \frac{2n-1}{2n} \int_0^\pi \sin^{2n-2}(x)\,dx$

$= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \int_0^\pi \sin^{2n-4}(x)\,dx$

$= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdot \ldots \cdot \frac{1}{2} \int_0^\pi \sin^0(x)\,dx$

$$\text{and } \int_0^{\pi} \sin^{2n+1}(x)\,dx = \frac{2n}{2n+1}\int_0^{\pi} \sin^{2n-1}(x)\,dx$$

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\int_0^{\pi} \sin^{2n-3}(x)\,dx$$

$$\vdots$$

$$= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot \ldots \cdot \frac{2}{3}\int_0^{\pi} \sin^{1}(x)\,dx$$

$$\cancel{2\,\frac{2n}{2n+1}}$$

$$\text{So} \quad 1 \le \frac{\int_0^{\pi} \sin^{2n+1}(x)\,dx}{\int_0^{\pi} \sin^{2n}(x)\,dx} = \frac{\frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot \ldots \cdot \frac{2}{3}\cdot 2}{\frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot \ldots \cdot \frac{1}{2}\cdot \pi} \to 1 \text{ as } n\to\infty$$

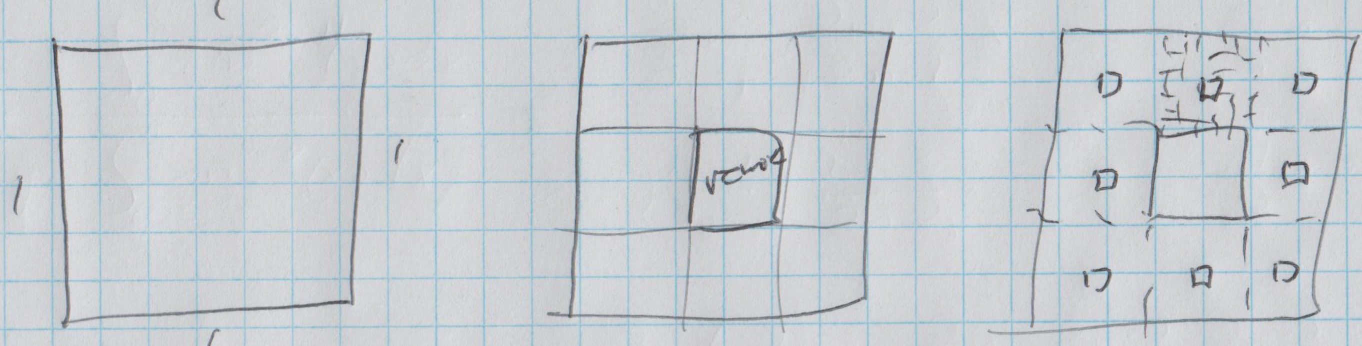
$$= \frac{2}{\pi}\left(\frac{2n}{2n+1}\cdot\frac{2n}{2n-1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-2}{2n-3}\cdot \ldots \cdot \frac{2}{3}\cdot\frac{2}{1}\right) \to 1$$

$$\therefore \frac{\pi}{2} = \frac{2n}{2n+1}\cdot\frac{2n}{2n-1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-2}{2n-3}\cdot \ldots \cdot \frac{2}{3}\cdot\frac{2}{1} \text{ as } n\to\infty$$

$$= \prod_{n=1}^{\infty} \frac{2n\cdot 2n}{(2n-1)(2n+1)} \quad \checkmark$$

The Wallis product converges more slowly to $\frac{\pi}{2}$ than the Viète product, but is much less expensive computationally (no square roots!).

Finally, the Wallis product gives rise to the "Wallis carpet" or "Wallis sieve":

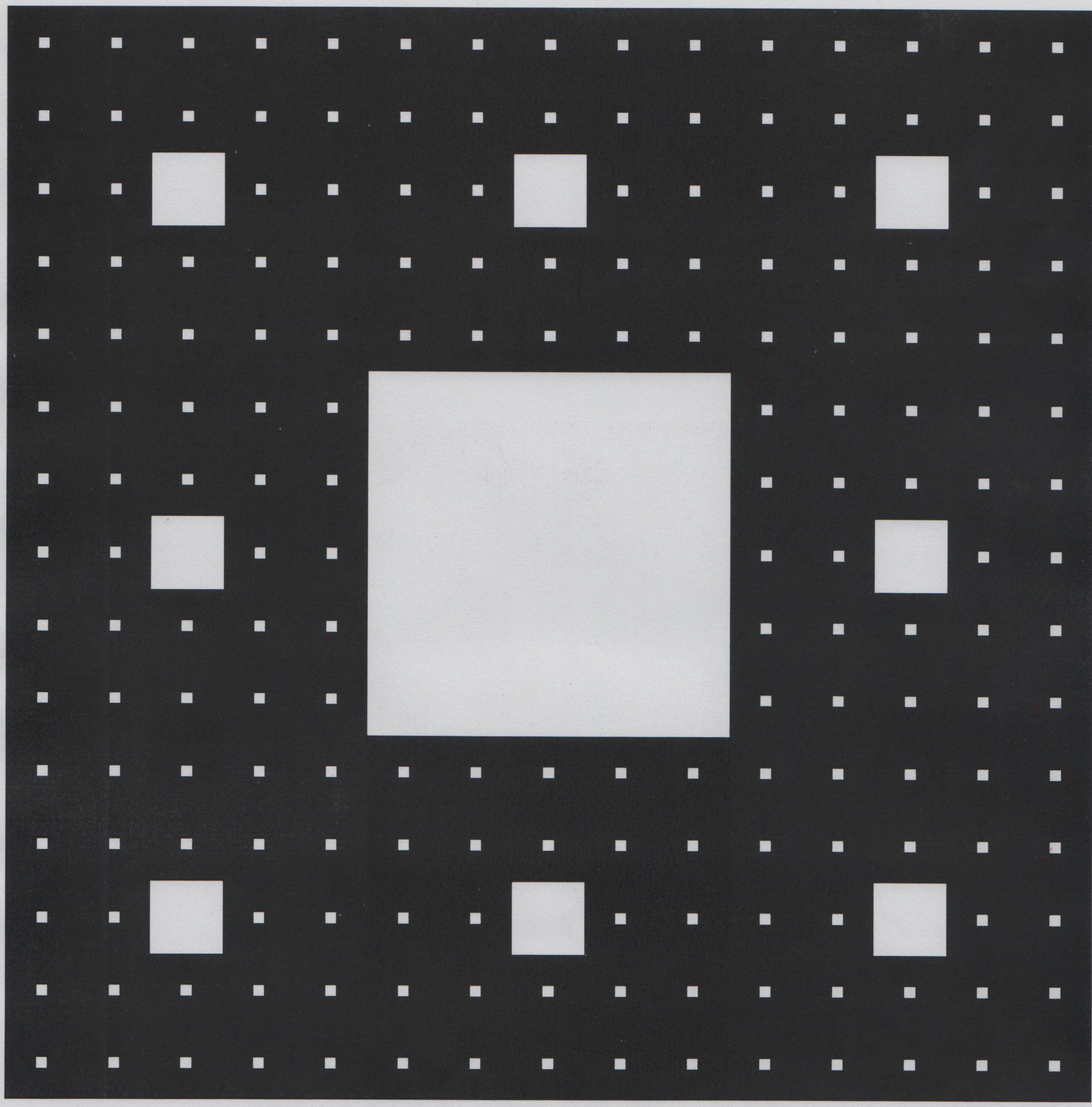


0 1 2

at step n, ~~remove~~ divide up the remaining subsquares into $(2n+1)^2$ subsquares & remove ~~each~~ the middlemost.

The limit object has area $\frac{\pi}{4}$ but has a lot of holes

that it has no interior — can't find a open disk inside it without a hole.



Step 3 of the Wallis sieve.