3. To find two numbers such that their product is to their sum or their difference in a given ratio [cf. I. 34].

4. To find two numbers such that the sum of their squares is to their difference in a given ratio [cf. I. 32].

5. To find two numbers such that the difference of their squares is to their sum in a given ratio [cf. I. 33].

6[1]. To find two numbers having a given difference and such that the difference of their squares exceeds their difference by a given number.

Necessary condition. The square of their difference must be less than the sum of the said difference and the given excess of the difference of the squares over the difference of the numbers.

Difference of numbers 2, the other given number 20.

Lesser number x. Therefore $x+2$ is the greater, and

$$4x+4=22.$$

Therefore $x=4\frac{1}{2}$, and

the numbers are $4\frac{1}{2}$, $6\frac{1}{2}$.

7[1]. To find two numbers such that the difference of their squares is greater by a given number than a given ratio of their difference[2]. [*Difference assumed.*]

Necessary condition. The given ratio being $3:1$, the square of the difference of the numbers must be less than the sum of three times that difference and the given number.

Given number 10, difference of required numbers 2.

Lesser number x. Therefore the greater is $x+2$, and

$$4x+4=3\cdot 2+10.$$

Therefore $x=3$, and

the numbers are 3, 5.

8. To divide a given square number into two squares[3].

[1] The problems II. 6, 7 also are considered by Tannery to be interpolated from some ancient commentary.

[2] Here we have the identical phrase used in Euclid's *Data* (cf. note on p. 132 above): the difference of the squares is τῆς ὑπεροχῆς αὐτῶν δοθέντι ἀριθμῷ μείζων ἢ ἐν λόγῳ, literally "greater than their difference by a given number (more) than in a (given) ratio," by which is meant "greater by a given number than a given proportion or fraction of their difference."

[3] It is to this proposition that Fermat appended his famous note in which he enunciates what is known as the "great theorem" of Fermat. The text of the note is as follows:

"On the other hand it is impossible to separate a cube into two cubes, or a

Given square number 16.

x^2 one of the required squares. Therefore $16 - x^2$ must be equal to a square.

Take a square of the form[1] $(mx-4)^2$, m being any integer and 4 the number which is the square root of 16, *e.g.* take $(2x-4)^2$, and equate it to $16-x^2$.

Therefore $4x^2 - 16x + 16 = 16 - x^2$,

or $5x^2 = 16x$, and $x = \frac{16}{5}$.

The required squares are therefore $\frac{256}{25}$, $\frac{144}{25}$.

9. To divide a given number which is the sum of two squares into two other squares[2].

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biquadrate into two biquadrates, or generally *any power except a square into two powers with the same exponent*. I have discovered a truly marvellous proof of this, which however the margin is not large enough to contain."

Did Fermat really possess a proof of the general proposition that $x^m + y^m = z^m$ cannot be solved in rational numbers where m is any number > 2? As Wertheim says, one is tempted to doubt this, seeing that, in spite of the labours of Euler, Lejeune-Dirichlet, Kummer and others, a general proof has not even yet been discovered. Euler proved the theorem for $m=3$ and $m=4$, Dirichlet for $m=5$, and Kummer, by means of the higher theory of numbers, produced a proof which only excludes certain particular values of m, which values are rare, at all events among the smaller values of m; thus there is no value of m below 100 for which Kummer's proof does not serve. (I take these facts from Weber and Wellstein's *Encyclopädie der Elementar-Mathematik*, I_2, p. 284, where a proof of the formula for $m=4$ is given.)

It appears that the Göttingen Academy of Sciences has recently awarded a prize to Dr A. Wieferich, of Münster, for a proof that the equation $x^p + y^p = z^p$ cannot be solved in terms of positive integers not multiples of p, if $2^p - 2$ is not divisible by p^2. "This surprisingly simple result represents the first advance, since the time of Kummer, in the proof of the last Fermat theorem" (*Bulletin of the American Mathematical Society*, February 1910).

Fermat says ("Relation des nouvelles découvertes en la science des nombres," August 1659, *Oeuvres*, II. p. 433) that he proved that *no cube is divisible into two cubes* by a variety of his method of *infinite diminution* (*descente infinie* or *indéfinie*) different from that which he employed for other negative or positive theorems; as to the other cases, see Supplement, sections I., II.

[1] Diophantus' words are: "I form the square from any number of ἀριθμοί *minus* as many units as there are in the side of 16." It is implied throughout that m must be so chosen that the result may be *rational* in Diophantus' sense, *i.e.* rational and positive.

[2] Diophantus' solution is substantially the same as Euler's (*Algebra*, tr. Hewlett, Part II. Art. 219), though the latter is expressed more generally.

Required to find x, y such that

$$x^2 + y^2 = f^2 + g^2.$$

If $x \gtrless f$, then $y \lesseqgtr g$.

Put therefore $$x = f + pz, \quad y = g - qz:$$