THE ARITHMETICA

BOOK I

PRELIMINARY

Dedication.

"Knowing, my most esteemed friend Dionysius, that you are anxious to learn how to investigate problems in numbers, I have tried, beginning from the foundations on which the science is built up, to set forth to you the nature and power subsisting in numbers.

"Perhaps the subject will appear rather difficult, inasmuch as it is not yet familiar (beginners are, as a rule, too ready to despair of success); but you, with the impulse of your enthusiasm and the benefit of my teaching, will find it easy to master; for eagerness to learn, when seconded by instruction, ensures rapid progress."

After the remark that "all numbers are made up of some multitude of units, so that it is manifest that their formation is subject to no limit," Diophantus proceeds to define what he calls the different "species" of numbers, and to describe the abbreviative signs used to denote them. These "species" are, in the first place, the various powers of the unknown quantity from the second to the sixth inclusive, the unknown quantity itself, and units.

Definitions.

A *square* ($=x^2$) is δύναμις ("power"), and its sign is a Δ with Y superposed, thus Δ^Y.

A *cube* ($=x^3$) is κύβος, and its sign K^Y.

A *square-square* ($=x^4$) is δυναμοδύναμις[1], and its sign is $\Delta^Y\Delta$.

A *square-cube* ($=x^5$) is δυναμόκυβος, and its sign ΔK^Y.

A *cube-cube* ($=x^6$) is κυβόκυβος, and its sign K^YK.

[1] The term δυναμοδύναμις was already used by Heron (*Metrica*, ed. Schöne, p. 48, 11, 19) for the fourth power of a side of a triangle.

"It is," Diophantus observes, "from the addition, subtraction or multiplication of these numbers or from the ratios which they bear to one another or to their own sides respectively that most arithmetical problems are formed"; and "each of these numbers... is recognised as an element in arithmetical inquiry."

"But the number which has none of these characteristics, but *merely has in it an indeterminate multitude of units* (πλῆθος μονάδων ἀόριστον) *is called* ἀριθμός, '*number*,' *and its sign is* ς $[=x]$."

"And there is also another sign denoting that which is invariable in determinate numbers, namely the unit, the sign being M with o superposed, thus $\overset{o}{M}$."

Next follow the definitions of the reciprocals, the names of which are derived from the names of the corresponding species themselves.

Thus

from	ἀριθμός $[x]$	we derive the term	ἀριθμοστόν $[= 1/x]$
,,	δύναμις $[x^2]$	,, ,,	δυναμοστόν $[= 1/x^2]$
,,	κύβος $[x^3]$	,, ,,	κυβοστόν $[= 1/x^3]$
,,	δυναμοδύναμις $[x^4]$	,, ,,	δυναμοδυναμοστόν $[= 1/x^4]$
,,	δυναμόκυβος $[x^5]$	,, ,,	δυναμοκυβοστόν $[= 1/x^5]$
,,	κυβόκυβος $[x^6]$	,, ,,	κυβοκυβοστόν $[= 1/x^6]$,

and each of these has the same sign as the corresponding original species, but with a distinguishing mark which Tannery writes in the form $^{\chi}$ above the line to the right.

Thus $\Delta^{Y\chi} = 1/x^2$, just as $\gamma^{\chi} = \frac{1}{3}$.

Sign of Subtraction (*minus*).

"*A minus multiplied by a minus makes a plus*[1]*; a minus multiplied by a plus makes a minus; and the sign of a minus is a truncated* Ψ *turned upside down, thus* ⋔."

Diophantus proceeds: "It is well that one who is beginning this study should have acquired practice in the addition, subtraction and multiplication of the various species. He should know how to add positive and negative terms with different coefficients to

[1] The literal rendering would be "A wanting multiplied by a wanting makes a forthcoming." The word corresponding to *minus* is λεῖψις ("wanting"): when it is used exactly as our *minus* is, it is in the dative λείψει, but there is some doubt whether Diophantus himself used this form (cf. p. 44 above). For the probable explanation of the sign, see pp. 42–44. The word for "forthcoming" is ὕπαρξις, from ὑπάρχω, to exist. Negative terms are λείποντα εἴδη, and positive ὑπάρχοντα.

other terms[1], themselves either positive or likewise partly positive and partly negative, and how to subtract from a combination of positive and negative terms other terms either positive or likewise partly positive and partly negative.

"Next, if a problem leads to an equation in which certain terms are equal to terms of the same species but with different coefficients, it will be necessary to subtract like from like on both sides, until one term is found equal to one term. If by chance there are on either side or on both sides any negative terms, it will be necessary to add the negative terms on both sides, until the terms on both sides are positive, and then again to subtract like from like until one term only is left on each side.

"This should be the object aimed at in framing the hypotheses of propositions, that is to say, to reduce the equations, if possible, until one term is left equal to one term; *but I will show you later how, in the case also where two terms are left equal to one term, such a problem is solved.*"

Diophantus concludes by explaining that, in arranging the mass of material at his disposal, he tried to distinguish, so far as possible, the different types of problems, and, especially in the elementary portion at the beginning, to make the more simple lead up to the more complex, in due order, such an arrangement being calculated to make the beginner's course easier and to fix what he learns in his memory. The treatise, he adds, has been divided into thirteen Books.

PROBLEMS

1. To divide a given number into two having a given difference.

Given number 100, given difference 40.

Lesser number required x. Therefore

$$2x + 40 = 100,$$
$$x = 30.$$

The required numbers are 70, 30.

2. To divide a given number into two having a given ratio.

Given number 60, given ratio 3 : 1.

Two numbers x, $3x$. Therefore $x = 15$.

The numbers are 45, 15.

[1] εἶδος, "species," is the word used by Diophantus throughout.

3. To divide a given number into two numbers such that one is a given ratio of the other *plus* a given difference[1].

Given number 80, ratio 3 : 1, difference 4.

Lesser number x. Therefore the larger is $3x + 4$, and $4x + 4 = 80$, so that $x = 19$.

The numbers are **61, 19.**

4. To find two numbers in a given ratio and such that their difference is also given.

Given ratio 5 : 1, given difference 20.

Numbers $5x$, x. Therefore $4x = 20$, $x = 5$, and the numbers are **25, 5.**

5. To divide a given number into two numbers such that given fractions (not the same) of each number when added together produce a given number.

Necessary condition. The latter given number must be such that it lies between the numbers arising when the given fractions respectively are taken of the first given number.

First given number 100, given fractions $\frac{1}{3}$ and $\frac{1}{5}$, given sum of fractions 30.

Second part $5x$. Therefore first part $= 3(30 - x)$.

Hence $90 + 2x = 100$, and $x = 5$.

The required parts are **75, 25.**

6. To divide a given number into two numbers such that a given fraction of the first exceeds a given fraction of the other by a given number.

Necessary condition. The latter number must be less than that which arises when that fraction of the first number is taken which exceeds the other fraction.

Given number 100, given fractions $\frac{1}{4}$ and $\frac{1}{6}$ respectively, given excess 20.

Second part $6x$. Therefore the first part is $4(x + 20)$.

Hence $10x + 80 = 100$, $x = 2$, and the parts are **88, 12.**

[1] Literally "to divide an assigned number into two in a given ratio and difference (ἐν λόγῳ καὶ ὑπεροχῇ τῇ δοθείσῃ)." The phrase means the same, though it is not so clear, as Euclid's expression (*Data*, Def. 11 and *passim*) δοθέντι μείζων ἢ ἐν λόγῳ. According to Euclid's definition a magnitude is greater than a magnitude "by a given amount (more) than in a (certain) ratio" when the remainder of the first magnitude, after subtracting the given amount, has the said ratio to the second magnitude. This means that, if x, y are the magnitudes, d the given amount, and k the ratio, $x - d = ky$ or $x = ky + d$.