Darboux's Version of the Riemann Integral

(A Precise Definition of the Definite Integral)

Intuitively, the definite integral $\int_a^b f(x)\,dx$ represents the area between $y = f(x)$ and $y = 0$ for $a \le x \le b$, weighted so that area below $y = 0$ is subtracted and area above $y = 0$ is added. There are a number of ways to define the definite integral precisely. The first rigorous definition was due to Bernhard Riemann (1826-1866), who presented it in a talk in 1854, though it wasn't published until 1868. His basic idea was to approximate the area between $y = f(x)$ and $y = 0$ by rectangles. As one makes the rectangles narrower and increases their number, one can get better approximations. Taking a suitable – pretty complicated! – limit lets one use this idea to define the definite integral.

The definition developed below, due to Jean-Gaston Darboux (1842-1917), uses the same basic idea as and is equivalent to Riemann's, but is a little less complicated to deal with, especially in terms of the limits required.

PRELIMINARIES. We'll need to make a few subsidiary definitions and set up some terminology and notation. The first is actually a basic property of the real numbers.

> **Fact.** If A is a non-empty set of real numbers which has an upper bound, then A has a *least upper bound* or *supremum*, often denoted by $\sup(A)$. Similarly, if a non-empty set of real numbers has a lower bound, then A has a *greatest lower bound* or *infimum*, often denoted by $\inf(A)$.

For example, consider the set

$$A = \left\{ \frac{1}{n+1} \;\middle|\; n \ge 1 \right\} \cup \left\{ \frac{n}{n+1} \;\middle|\; n \ge 1 \right\} = \left\{ \dots, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots \right\} .$$

It is not hard to see that A has greatest lower bound $\inf(A) = 0$ and least upper bound $\sup(A) = 1$. In this case, $\inf(A)$ and $\sup(A)$ are not themselves in A. In general, they may or may not be be. For example, the interval $[-2, 3]$ includes both its greatest lower bound -2 and its least upper bound 3, the interval $[-2, 3)$ includes its greatest lower bound but not its least upper bound, the interval $(-2, 3]$ does the reverse of the last, and $(-2, 3)$ includes neither.

> **Definitions. 1.** Suppose $a < b$. A *partition* of the interval $[a, b]$ is a set of points $P = \{ t_0, t_1, t_2, \dots, t_n \}$ such that $a = t_0 < t_1 < t_2 < \cdots < t_n = b$.
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> **2.** A function $f(x)$ is *bounded* on $[a, b]$ if it is defined on $[a, b]$ and there are real numbers m and M such that $m \le f(x) \le M$ for all $a \le x \le b$.
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> **3.** Suppose $f(x)$ is bounded on $[a, b]$ and $P = \{ t_0, t_1, t_2, \dots, t_n \}$ is a partition of $[a, b]$. For each i with $1 \le i \le n$, let $m_i = \inf \{ f(x) \mid t_{i-1} \le x \le t_i \}$ and $M_i = \sup \{ f(x) \mid t_{i-1} \le x \le t_i \}$. Then the *lower sum* of $f(x)$ for P is
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> $$L(f, P) = \sum_{i=1}^{n} m_i \,(t_i - t_{i-1}) = m_1\,(t_1 - t_0) + m_2\,(t_2 - t_1) + \cdots m_n\,(t_n - t_{n-1}) \ ,$$

and the *upper sum* of $f(x)$ for P is

$$U(f,P) = \sum_{i=1}^{n} M_i\left(t_i - t_{i-1}\right) = M_1\left(t_1 - t_0\right) + M_2\left(t_2 - t_1\right) + \cdots M_n\left(t_n - t_{n-1}\right) .$$

Some comments are in order here. First, $m_i\left(t_i - t_{i-1}\right)$ and $M_i\left(t_i - t_{i-1}\right)$ are the weighted areas of rectangles such that

$$m_i\left(t_i - t_{i-1}\right) \leq \text{weighted area between } y = f(x) \text{ and } y = 0 \text{ on } \left[t_{i-1}, t_{[}\right] \leq M_i\left(t_i - t_{i-1}\right) .$$

Second, we need $f(x)$ to be bounded on $[a,b]$ when defining the upper and lower sums in order to ensure that the numbers m_i and M_i are defined. Third, we did not assume $f(x)$ was continuous. Every continuous function on a closed interval $[a,b]$ will, of course, be bounded, but so will any function that has a finite number of removable or jump discontinuities in $[a,b]$. A function with infinitely many discontinuities on $[a,b]$ might not be bounded, and a function with a vertical asymptote is guaranteed not to be.

A couple of technically useful facts about upper and lower sums of $f(x)$ on a partition P of $[a,b]$ are given in the following results.

Lemma 1. Suppose $f(x)$ is bounded on $[a,b]$ and P and Q are partitions of $[a,b]$ such that every point of P is also a point of Q. (So the extra points of Q subdivide the pieces that P divides $[a,b]$ into.) Then $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$.

Proposition 2. Suppose $f(x)$ is bounded on $[a,b]$ and P and R are any two partitions of $[a,b]$. Then $L(f,P) \leq U(f,R)$.

Corollary 3. If $f(x)$ is bounded on $[a,b]$, then

$$\begin{aligned} &\sup\{\, L(f,P) \mid P \text{ a partition of } [a,b] \,\} \\ &\leq \inf\{\, U(f,P) \mid P \text{ a partition of } [a,b] \,\} . \end{aligned}$$

The Definite Integral. We can define the fool thing at last:

Definition 4. A function $f(x)$ bounded on $[a,b]$ is said to be *integrable* on $[a,b]$ if $\sup\{\, L(f,P) \mid P \text{ a partition of } [a,b] \,\} = \inf\{\, U(f,P) \mid P \text{ a partition of } [a,b] \,\}$.

This number is the *definite integral* of $f(x)$ on $[a,b]$, denoted by $\displaystyle\int_a^b f(x)\,dx$.

That is,

$$\begin{aligned} \int_a^b f(x)\,dx &= \sup\{\, L(f,P) \mid P \text{ a partition of } [a,b] \,\} \\ &= \inf\{\, U(f,P) \mid P \text{ a partition of } [a,b] \,\} \end{aligned}$$

if the sup and inf are equal, and $\displaystyle\int_a^b f(x)\,dx$ is undefined if they are not equal. One potential problem with this definition is that the least upper and greatest lower bounds involved are not that easy to work with directly. The following result lets us work with something a little more concrete, at the cost of some epsilonics.

Theorem 4. Suppose $f(x)$ is bounded on $[a, b]$. Then $f(x)$ is integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Some Basic Properties. Armed with the definitions and facts above, one can proceed to prove the usual basic properties of the definite integral relatively easily and move on to the Fundamental Theorem of Calculus, which gives a more practical tool for computing most common definite integrals by exploting the connection with antiderivatives.

Theorem 5. Suppose $f(x)$ is continuous on $[a, b]$. Then $f(x)$ is integrable on $[a, b]$.

Theorem 6. Suppose $f(x)$ is integrable on $[a, b]$ and $[b, c]$, where $a < b < c$. Then $f(x)$ is integrable on $[a, c]$ and

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

Theorem 7. Suppose $f(x)$ is integrable on $[a, b]$ and $c \in \mathbb{R}$ is a constant. Then $g(x) = cf(x)$ is integrable on $[a, b]$ and

$$\int_a^b g(x)\,dx = \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Theorem 8. Suppose $f(x)$ is integrable on $[a, b]$ and $m \leq f(x) \leq M$ for all $x \in [a, b]$. Then

$$m(b - a) \leq \int_a^b f(x)\,dx \leq M(b - a).$$

Theorem 9. If $f(x)$ is integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$ for $x \in [a, b]$, then $F(x)$ is continuous on $[a, b]$.

Just for fun, here is one form of the Fundamental Theorem of Calculus:

Theorem 10. Suppose $f(x)$ is integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$ for $x \in [a, b]$. If $f(x)$ is continuous at some $c \in (a, b)$, then $F(x)$ is differentiable at c and $F'(c) = f(c)$.

If you want more. For a pretty detailed development of much of this material, please consult any of the four editions of *Calculus* by Michael Spivak, one of the best-written mathematics textbooks anywhere. Most of the above was ~~stolen~~ ~~borrowed~~ had creative inspiration taken from this book.

Reference

1. *Calculus* (Third Edition), by Michael Spivak, Publish or Perish, 1994. ISBN 0-914098-89-6