Mathematics 1121H – Calculus II
TRENT UNIVERSITY, Winter 2026
Solutions to the Final Examination
19:00-22:00 on Tuesday, 21 April, in the Gym.
Brought to you by Стефан Біланюк.

Instructions: Do parts **A**, **B**, and **C**, and, if you wish, part **D**. Show all your work and justify all your answers. *If in doubt about something,* **ask!**

Aids: One letter- or A4-size aid sheet with whatever you want on it, any calculator that can't communicate with other devices, one brain.

Part A. Do all three (3) of **1**–**3**. *[60 = 3 × 20 each]*

1. Evaluate any *four* (4) of the integrals **a**–**f**. *[20 = 4 × 5 each]*

a. $\displaystyle\int \frac{1}{x^2+5x+6}\,dx$ **b.** $\displaystyle\int_0^1 \left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right] dx$ **c.** $\displaystyle\int \frac{2\sin(x)\cos(x)}{1+\sin^4(x)}\,dx$

d. $\displaystyle\int_0^1 \frac{2x}{1+x^4}\,dx$ **e.** $\displaystyle\int e^x \sin(x)\,dx$ **f.** $\displaystyle\int_{-\pi/8}^{\pi/8} \arctan(2x)\,dx$

SOLUTION. **a.** We will use the method of partial fractions. Observe that $x^2+5x+6 = (x+2)(x+3)$. It follows that

$$\begin{aligned}\frac{1}{x^2+5x+6} &= \frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} \\ &= \frac{A(x+3)+B(x+2)}{(x+2)(x+3)} = \frac{(A+B)x+(3A+2B)}{x^2+5x+6}.\end{aligned}$$

Comparing the coefficients of powers of x at either end of this equation chain, we see that we must have $A+B=0$, so $B=-A$, and $3A+2B=1$. Plugging $A=-B$ into the last gives us $3A-2A=1$, so $A=1$ and $B=-A=-1$. We can now decompose the integrand to compute the integral:

$$\begin{aligned}\int \frac{1}{x^2+5x+6}\,dx &= \int \left(\frac{1}{x+2} + \frac{-1}{x+3}\right) dx = \int \frac{1}{x+2}\,dx - \int \frac{1}{x+3}\,dx \\ &= \ln(x+2) - \ln(x+3) + C = \ln\left(\frac{x+2}{x+3}\right) + C \quad \square\end{aligned}$$

b. Recall that $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. This means that

$$\int_0^1 \left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right] dx = \int_0^1 e^x\,dx = e^x\Big|_0^1 = e^1 - e^0 = e-1. \quad \square$$

c. We will use the substitution $w = \sin^2(x)$, so $dw = 2\sin(x)\cos(x)\,dx$. Then

$$\int \frac{2\sin(x)\cos(x)}{1+\sin^4(x)}\,dx = \int \frac{1}{1+w^2}\,dw = \arctan(w) + C = \arctan\left(\sin^2(x)\right) + C. \quad \square$$

d. We will use the substitution $u = x^2$, so $du = 2x\,dx$, and change the limits as we go along: $\begin{array}{ccc} x & 0 & 1 \\ u & 0 & 1 \end{array}$. [OK, so there was no change ...] Then

$$\int_0^1 \frac{2x}{1+x^4}\,dx = \int_0^1 \frac{1}{1+u^2}\,du = \arctan(u)\Big|_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4} - 0 = \frac{\pi}{4}. \quad \square$$

e. We saw this one in class ... We'll use integration by parts twice, plus a little algebra.

$$\begin{aligned} \int e^x \sin(x)\,dx &= -e^x\cos(x) - \int e^x\left(-\cos(x)\right)\,dx && \begin{array}{l}\text{Integration by parts with}\\ u = e^x \text{ and } v' = \sin(x)\text{, so}\\ u' = e^x \text{ and } v = -\cos(x).\end{array} \\ &= -e^x\cos(x) + \int e^x\cos(x)\,dx \\ &= -e^x\cos(x) + e^x\sin(x) - \int e^x\sin(x)\,dx && \begin{array}{l}\text{Integration by parts with}\\ s = e^x \text{ and } t' = \cos(x)\text{, so}\\ s' = e^x \text{ and } t = \sin(x).\end{array} \end{aligned}$$

Looking at the beginning and end of this equation chain, it follows that $2\int e^x\sin(x)\,dx = e^x\left(\sin(x) - \cos(x)\right)$, and thus $\displaystyle\int e^x\sin(x)\,dx = \frac{\sin(x)-\cos(x)}{2}e^x + C$. $\square$

f. *The hard way.* We'll use the substitution $w = 2x$, changing the limits as we go along, $\begin{array}{ccc} x & -\pi/8 & \pi/8 \\ w & -\pi/4 & \pi/4 \end{array}$, followed by integration by parts using the dummy product trick and another substitution, namely $t = 1 + w^2$, so $dt = 2w\,dw$ and $\frac{1}{2}\,dt = w\,dw$, again changing the limits as we go along, $\begin{array}{ccc} w & -\pi/4 & \pi/4 \\ t & 1+\pi^2/16 & 1+\pi^2/16 \end{array}$.

$$\begin{aligned} \int_{-\pi/8}^{\pi/8} \arctan(2x)\,dx &= \int_{-\pi/4}^{\pi/4} \arctan(w)\,dw = \int_{-\pi/4}^{\pi/4} \arctan(w)\cdot 1\,dw \\ &= w\arctan(w)\Big|_{-\pi/4}^{\pi/4} - \int_{-\pi/4}^{\pi/4} \frac{w}{1+w^2}\,dw && \begin{array}{l}\text{Integration by parts with}\\ u = \arctan(w) \text{ and } v' = 1\text{,}\\ \text{so } u' = \frac{1}{1+w^2} \text{ and } v = w.\end{array} \\ &= \frac{\pi}{4}\arctan\left(\frac{\pi}{4}\right) - \left(-\frac{\pi}{4}\right)\arctan\left(-\frac{\pi}{4}\right) - \int_{1+\pi^2/16}^{1+\pi^2/16} \frac{1}{t}\cdot\frac{1}{2}\,dt \\ &= \frac{\pi}{4}\cdot 1 + \frac{\pi}{4}\cdot(-1) - 0 \quad [\mathit{Why?}] = 0 \quad \square \end{aligned}$$

f. *The easy way.* $\arctan(2x)$ is an odd function, *i.e.* $\arctan(2(-x)) = -\arctan(2x)$ for all x. As $[-\pi/8, \pi/8]$ is symmetric about 0, it follows that $\displaystyle\int_{-\pi/8}^{\pi/8} \arctan(2x)\,dx = 0$. ■

2. Determine whether the series converges in any *four* (4) of **a**–**f**. *[20 = 4 × 5 each]*

a. $\displaystyle\sum_{n=0}^{\infty} 2^n e^{-n}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^3}$ **c.** $\displaystyle\sum_{n=1}^{\infty} \frac{3^n+1}{2^n-1}$

d. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{2n+1}$ **e.** $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}$ **f.** $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{n+\sqrt{n}+1}$

SOLUTION. **a.** *A hard way?* We apply the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}e^{-(n+1)}}{2^n e^{-n}}\right| = \lim_{n\to\infty}\left|2^1 e^{-1}\right| = \lim_{n\to\infty}\frac{2}{e} = \frac{2}{e} < 1$$

By the Ratio Test, it follows that the series converges. □

a. *Another hard way?* We apply the Root Test.

$$\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left|2^n e^{-n}\right|^{1/n} = \lim_{n\to\infty}\left|2e^{-1}\right| = \lim_{n\to\infty}\frac{2}{e} = \frac{2}{e} < 1$$

By the Root Test, it follows that the series converges. □

a. *The easy way?* $\displaystyle\sum_{n=0}^{\infty} 2^n e^{-n} = \sum_{n=0}^{\infty}\frac{2^n}{e^n} = \sum_{n=0}^{\infty}\left(\frac{2}{e}\right)^n$ is a geometric series with common ratio $r = \dfrac{2}{e}$. Since $|r| = \dfrac{2}{e} < 1$, the series converges. □

b. We will use the Basic Comparison Test. As $\ln(n) < n$ for all $n \geq 1$, we have

$$0 \leq \frac{\ln(n)}{n^3} = \frac{\ln(n)}{n}\cdot\frac{1}{n^2} < \frac{1}{n^2}$$

for all $n \geq 1$. Since we know from class and assignment that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, it follows by the Basic Comparison Test that $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(n)}{n^3}$ converges as well. □

c. *One way.* We will use the Basic Comparison Test. As $2^n - 1 < 2^n$ and $3^{n+1} > 3^n$ for all $n \geq 1$, we have

$$0 < \frac{3^n}{2^n} < \frac{3^n+1}{2^n-1}$$

for all $n \geq 1$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$ is a geometric series with common ratio $r = \dfrac{3}{2} > 1$, it diverges. It follows by the Basic Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 1}{2^n - 1}$ diverges as well. $\square$

c. *Another way.* We will use the Limit Comparison Test. Note that $\dfrac{3^n + 1}{2^n - 1} > 0$ and $\dfrac{3^n}{2^n} > 0$ for all $n \geq 1$. As $\displaystyle\lim_{n\to\infty} \frac{\frac{3^n+1}{2^n-1}}{\frac{3^n}{2^n}} = \lim_{n\to\infty} \frac{3^n + 1}{2^n - 1} \cdot \frac{2^n}{3^n} = \lim_{n\to\infty} \frac{1 + \frac{1}{3^n}}{1 - \frac{1}{2^n}} = \frac{1+0}{1-0} = 1$, and $0 < 1 < \infty$, it follows by the Limit Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 1}{2^n - 1}$ converges or not exactly as $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n}$ does. Since $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$ is a geometric series with common ratio $r = \dfrac{3}{2} > 1$, it diverges, and hence so does $\displaystyle\sum_{n=1}^{\infty} \frac{3^n + 1}{2^n - 1}$. $\square$

d. We will use the Divergence Test. Observe that

$$\lim_{n\to\infty} \left| \frac{(-1)^n n}{2n+1} \right| = \lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{n}{2n+1} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{2 + \frac{1}{n}} = \frac{1}{2-0} = \frac{1}{2} \neq 0.$$

It follows by the Divergence Test that the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{2n+1}$ does not converge. $\square$

e. *The hard way.* We will use the Ratio Test. Since

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} \right| = \lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n} = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1,$$

it follows by the Ratio Test that the series $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}$ converges. $\square$

e. *The easy way.* $\displaystyle\sum_{n=0}^{\infty} \frac{e^n}{n!}$ is just the Taylor series for e^x evaluated at $x = e$. Since we know from class and assignment that the Taylor series for e^x converges for all x, the given series converges. $\square$

f. We will us the Alternating Series Test. We need to check three things about the series $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{n + \sqrt{n} + 1}$ to apply this test:

i. $\cos(n\pi) = (-1)^n$ while $n + \sqrt{n} + 1 > 0$ for all $n \geq 0$, so the terms in the series do alternate sign.

ii. Since $|\cos(n\pi)| = 1$ and $(n+1)+\sqrt{n+1}+1 > n+\sqrt{n}+1$ for all $n \geq 0$, we have $\left|\dfrac{\cos((n+1)\pi)}{(n+1)+\sqrt{n+1}+1}\right| \leq \left|\dfrac{\cos(n\pi)}{n+\sqrt{n}+1}\right|$ for all $n \geq 0$, so the terms of the series are non-decreasing in absolute value.

iii. $\displaystyle\lim_{n\to\infty}\left|\frac{\cos(n\pi)}{n+\sqrt{n}+1}\right| = \lim_{n\to\infty}\frac{1}{n+\sqrt{n}+1}\begin{matrix}\to 1\\ \to\infty\end{matrix} = 0$, so the limit of (the absolute values of) the terms of the series is 0.

It follows that the series converges by the Alternating Series Test. ■

3. Do any *four* (4) of **a**–**g**. *[20 = 4 × 5 each]*

a. Find the arc-length of the curve $y = \dfrac{2}{3}x^{3/2}$, where $0 \leq x \leq 1$.

b. Does the series $\displaystyle\sum_{n=0}^{\infty}\frac{(-e)^n}{n!}$ converge absolutely, conditionally, or not at all?

c. Find the volume of the solid obtained by revolving the region between $y = 1-x^2$ and $y = 0$, where $-1 \leq x \leq 1$, about the x-axis.

d. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2+5n+6}$.

e. Find the area of the finite region between $y = \cos(x)$ and $y = \sin(x)$, where $0 \leq x \leq 2\pi$.

f. What function of x has $\displaystyle\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$ as its Taylor series at 0?

g. Find the Taylor series at 0 of $f(x) = \dfrac{d}{dx}(xe^x)$.

SOLUTION. **a.** We plug $y = \dfrac{2}{3}x^{3/2}$, $0 \leq x \leq 1$, into the arc-length formula and integrate away. Note that $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{2}{3}x^{3/2}\right) = \dfrac{2}{3}\cdot\dfrac{3}{2}x^{1/2} = x^{1/2}$, and recall that an infinitesimal bit of arc-length is given by $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. It follows that

$$\text{arc-length} = \int_0^1 ds = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+\left(x^{1/2}\right)^2}\,dx = \int_0^1 \sqrt{1+x}\,dx$$

Substitute $u = 1+x$, so $du = dx$, and change the limits: $\begin{matrix} x & 0 & 1 \\ u & 1 & 2 \end{matrix}$

$$= \int_1^2 \sqrt{u}\,du = \int_1^2 u^{1/2}\,du = \left.\frac{u^{3/2}}{3/2}\right|_1^2 = \left.\frac{2}{3}u^{3/2}\right|_1^2 = \frac{2}{3}2^{3/2} - \frac{2}{3}1^{3/2}$$

$$= \frac{2}{3}\left(2\sqrt{2}-1\right). \quad \square$$

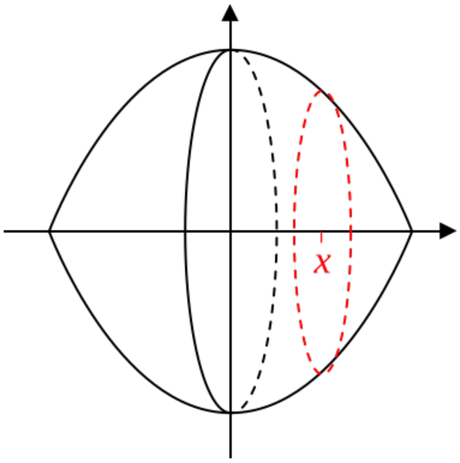
b. *One way.* We apply the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{\frac{(-e)^{n+1}}{(n+1)!}}{\frac{(-e)^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{(-e)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-e)^n}\right| = \lim_{n\to\infty}\left|\frac{-e}{n+1}\right| = \lim_{n\to\infty}\frac{e}{n+1}\begin{matrix}\to e\\ \to\infty\end{matrix} = 0<1$$

It follows by the Ratio Test that the series $\sum_{n=0}^{\infty}\frac{(-e)^n}{n!}$ converges absolutely. $\square$

b. *Another way.* By **2e**, $\sum_{n=0}^{\infty}\frac{e^n}{n!} = \sum_{n=0}^{\infty}\left|\frac{(-e)^n}{n!}\right|$ converges, so $\sum_{n=0}^{\infty}\frac{(-e)^n}{n!}$ converges absolutely.

c. We will use the disk/washer method to compute the volume of the solid. Here is a sketch of the solid, with a disk cross-section drawn in:



Since we revolved the given region about the x-axis, the disk cross-sections are perpendicular to the x-axis, so we use x as the main variable. The disk at x has radius $r = y = 1-x^2$, so the volume of the solid is:

$$\begin{aligned}\text{Volume} &= \int_{-1}^{1}\pi r^2\,dx = \int_{-1}^{1}\pi\left(1-x^2\right)^2\,dx = \int_{-1}^{1}\left(1-2x^2+x^4\right)\,dx\\ &= \left(x-\frac{2}{3}x^3+\frac{1}{5}x^5\right)\Big|_{-1}^{1} = \left(1-\frac{2}{3}1^3+\frac{1}{5}1^5\right)-\left(-1-\frac{2}{3}(-1)^3+\frac{1}{5}(-1)^3\right)\\ &= \frac{8}{15}-\left(-\frac{8}{15}\right) = \frac{16}{15}\approx 1.0667 \quad \square\end{aligned}$$

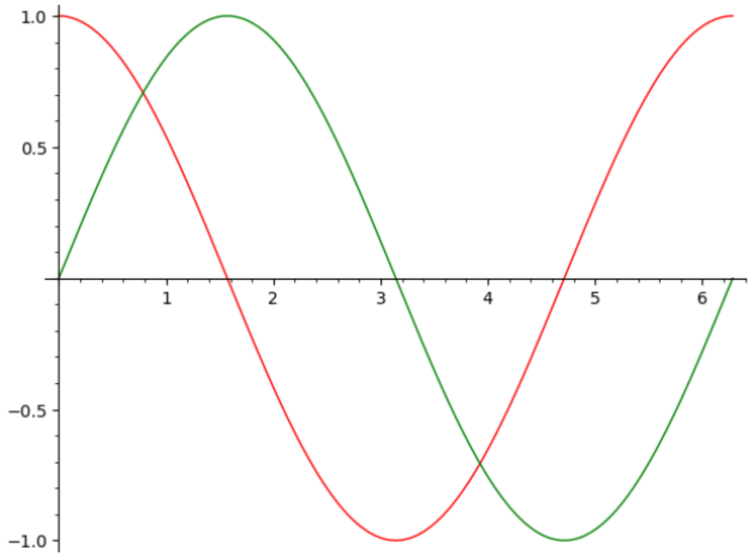
d. Just like in the solution to **1a** above, we will use partial fractions to decompose the terms; this will allow us to recast the given series as a telescoping series. Observe that $n^2+5n+6 = (n+2)(n+3)$. It follows that

$$\begin{aligned}\frac{1}{n^2+5n+6} &= \frac{1}{(n+2)(n+3)} = \frac{A}{n+2}+\frac{B}{n+3}\\ &= \frac{A(n+3)+B(n+2)}{(n+2)(n+3)} = \frac{(A+B)n+(3A+2B)}{n^2+5n+6}.\end{aligned}$$

Comparing the coefficients of powers of n at either end of this equation chain, we see that we must have $A+B=0$, so $B=-A$, and $3A+2B=1$. Plugging $A=-B$ into the last gives us $3A-2A=1$, so $A=1$ and $B=-A=-1$. It follows that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n^2+5n+6} &= \sum_{n=0}^{\infty}\left[\frac{1}{n+2}-\frac{1}{n+3}\right] = \left[\frac{1}{2}-\frac{1}{3}\right]+\left[\frac{1}{3}-\frac{1}{4}\right]+\left[\frac{1}{4}-\frac{1}{5}\right]+\cdots \\ &= \frac{1}{2}+\left[-\frac{1}{3}+\frac{1}{3}\right]+\left[-\frac{1}{4}+\frac{1}{4}\right]+\left[-\frac{1}{5}+\frac{1}{5}\right]+\cdots \\ &= \frac{1}{2}+0+0+0+\cdots=\frac{1}{2}. \quad \square\end{aligned}$$

e. Between 0 and 2π, the graphs of $\cos(x)$ and $\sin(x)$ cross each other at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.



For $0 \le x \le \frac{\pi}{4}$, $\cos(x) \ge \sin(x)$, for $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$, $\cos(x) \le \sin(x)$, and for $\frac{5\pi}{4} \le x \le 2\pi$, $\cos(x) \ge \sin(x)$ again. It follows that the area of the region between the two curves is:

$$\begin{aligned}\text{Area} &= \int_0^{\pi/4}(\cos(x)-\sin(x))\,dx+\int_{pi/4}^{5\pi/4}(\sin(x)-\cos(x))\,dx \\ &\qquad +\int_{5\pi/4}^{2\pi}(\cos(x)-\sin(x))\,dx \\ &= (\sin(x)-(-\cos(x)))|_0^{\pi/4}+(-\cos(x)-\sin(x))|_{\pi/4}^{5\pi/4}+(\sin(x)-(-\cos(x)))|_{5\pi/4}^{2\pi} \\ &= (\sin(x)+\cos(x))|_0^{\pi/4}-(\cos(x)+\sin(x))|_{\pi/4}^{5\pi/4}+(\sin(x)+\cos(x))|_{5\pi/4}^{2\pi} \\ &= \left[\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)-(0+1)\right]-\left[\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)\right] \\ &\qquad +\left[(1-0)-\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)\right] \\ &= \frac{2}{\sqrt{2}}-1+\frac{4}{\sqrt{2}}+1+\frac{2}{\sqrt{2}}=\frac{8}{\sqrt{2}}\approx 5.6569 \quad \square\end{aligned}$$

f. *The real way.* The series $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ consists of all of the terms with odd indices in the Taylor series at 0 of e^x, $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, which is actually equal to e^x for all x. How do we get rid of the terms with even indices? We compare the series for e^x with the series for e^{-x}, $\displaystyle\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$. In this series, the terms with odd indices are negative and the terms with even indices are positive. It follows that in the series for $e^x - e^{-x}$, namely $\displaystyle\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right] - \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right]$ the terms with even indices cancel out and the terms with odd indices add up, *i.e.* the resulting series is $\displaystyle\sum_{k=0}^{\infty} \frac{2x^{2k+1}}{(2k+1)!}$, which adds up to $e^x - e^{-x}$. Thus the function whose Taylor series at 0 is $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ is the hyperbolic sine function, $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$. $\square$

f. *The complex way.* The series $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$ looks a lot like the Taylor series at 0 of $\sin(x)$, $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, which is actually equal to $\sin(x)$ for all x. The $(-1)^k$ that makes the sign of each term alternate in the latter series is the only difference. How to get rid of it? We can do so using complex numbers.

Observe that if $i^2 = -1$, then $(ix)^{2k+1} = i^{2k+1} x^{2k+1} = (-1)^k i x^{2k+1}$. It follows that

$$\begin{aligned}\sin(ix) &= \sum_{k=0}^{\infty} \frac{(-1)^k (ix)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k (-1)^k i x^{2k+1}}{(2k+1)!} \\ &= i \sum_{k=0}^{\infty} \frac{(-1)^{2k} x^{2k+1}}{(2k+1)!} = i \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}.\end{aligned}$$

Solving for the given series, we get $\displaystyle\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} = \frac{\sin(ix)}{i} = -i\sin(ix)$. (Observe that $(-i)i = -i^2 = -(-1) = 1$, so $\dfrac{1}{i} = -i$.) Thus the given series is the Taylor series at 0 of the function $-i\sin(ix)$. $\square$

Note. Putting the two ways above of solving part **f** together tells us that $\sinh(x) = -i\sin(ix)$.

g. *One way.* Since $\displaystyle e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\displaystyle xe^x = x\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$. The Taylor series at 0 of $f(x) = \dfrac{d}{dx}\left(xe^x\right)$ should thus be $\displaystyle\frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^{n+1}}{n!}\right) = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n!}$. $\square$

g. *Another way.* Here we go:

$$\begin{aligned}
\frac{d}{dx}\left(xe^x\right) &= \left[\frac{d}{dx}x\right]e^x + x\left[\frac{d}{dx}e^x\right] = 1e^x + xe^x = \left[\sum_{n=0}^{\infty}\frac{x^n}{n!}\right] + x\left[\sum_{k=0}^{\infty}\frac{x^k}{k!}\right] \\
&= \left[\sum_{n=0}^{\infty}\frac{x^n}{n!}\right] + \left[\sum_{k=0}^{\infty}\frac{x^{k+1}}{k!}\right] \qquad \text{Now let } n = k+1 \text{ in the latter sum.} \\
&= 1 + \left[\sum_{n=0}^{\infty}\frac{x^n}{n!}\right] + \left[\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}\right] = 1 + \left[\sum_{n=1}^{\infty}\frac{x^n}{n!}\right] + \left[\sum_{n=1}^{\infty}\frac{nx^n}{n!}\right] \\
&= 1 + \sum_{n=1}^{\infty}\frac{(n+1)x^n}{n!} = \sum_{n=0}^{\infty}\frac{(n+1)x^n}{n!}
\end{aligned}$$

Note that $\dfrac{(0+1)x^0}{0!} = \dfrac{1 \cdot 1}{1} = 1.$ ■

Part B. Do any *two* (2) of **4**–**7**. *[26 = 2 × 13 each]*

4. a. Find the Taylor series at -1 of $g(x) = \sin(1+x)$, *[7]*
b. and verify that it converges to $g(x)$ for all x. *[6]*

SOLUTION. **a.** *Using Taylor's formula.* We compute the usual table of derivatives and look for patterns.

n	$g^{(n)}(x)$	$g^{(n)}(-1)$
0	$\sin(1+x)$	$\sin(0) = 0$
1	$\cos(1+x)$	$\cos(0) = 1$
2	$-\sin(1+x)$	$-\sin(0) = 0$
3	$-\cos(1+x)$	$-\cos(0) = -1$
4	$\sin(1+x)$	$\sin(0) = 0$
5	$\cos(1+x)$	$\cos(0) = 1$
6	$-\sin(1+x)$	$-\sin(0) = 0$
7	$-\cos(1+x)$	$-\cos(0) = -1$
8	$\sin(1+x)$	$\sin(0) = 0$
$\vdots$	$\vdots$	$\vdots$

It is not hard to see the pattern: when $n = 2k$ is even, $g^{(2k)}(-1) = 0$, and when $n = 2k+1$ is odd, $g^{(2k+1)}(-1) = (-1)^k$. It follows that the Taylor series at -1 of $g(x) = \sin(1+x)$ is

$$\sum_{n=0}^{\infty}\frac{g^{(n)}(-1)}{n!}\left(x-(-1)\right)^n = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(x+1)^{2k+1}. \quad \square$$

b. Recall that the difference between the nth Taylor polynomial at -1 of $g(x)$, $T_{n,-1}(x) = \displaystyle\sum_{m=0}^{n}\frac{g^{(m)}(-1)}{m!}\left(x-(-1)\right)^m$, and $g(x)$ itself is the remainder term $R_{n,-1}(x)$. Lagrange's expression for the remainder term is

$$g(x) - T_{n,-1}(x) = R_{n,-1}(x) = \frac{g^{(n}(t)}{n!}(x-(-1))^n = \frac{g^{(n}(t)}{n!}(1+x)^n\,,$$

where t is between -1 and x. From part **a** we know that $g^{(n}(t)$ is $\pm\sin(1+t)$ or $\pm\cos(1+t)$; either way $\left|g^{(n}(t)\right| \le 1$. It follows that

$$\begin{aligned}\left|g(x) - \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(x+1)^{2k+1}\right| &= \left|\sum_{m=0}^{\infty}\frac{g^{(m)}(-1)}{m!}(x-(-1))^m\right| = \left|g(x) - \lim_{n\to\infty}T_{n,-1}(x)\right| \\ &= \left|\lim_{n\to\infty}\left(g(x) - T_{n,-1}(x)\right)\right| = \left|\lim_{n\to\infty}R_{n,-1}(x)\right| \\ &\le \lim_{n\to\infty}\frac{\left|g^{(n}(t)\right|}{n!}\left|1+x\right|^n \le \lim_{n\to\infty}\frac{1}{n!}\left|1+x\right|^n = 0,\end{aligned}$$

since $n!$ grows far faster than any nth power. Thus the Taylor series at -1 of $g(x) = \sin(1+x)$ converges to $g(x)$ no matter what the value of x is. □

a & b. *Without using Taylor's formula.* We know that the Taylor series at 0 of $g(x) = \sin(x)$ is $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$, and that this series converges to $\sin(x)$ for all x. Plugging in $1+x$ for x tells us that $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(1+x)^{2k+1}$ converges to $\sin(1+x)$ for all x. Since

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(1+x)^{2k+1} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}(x-(-1))^{2k+1}$$

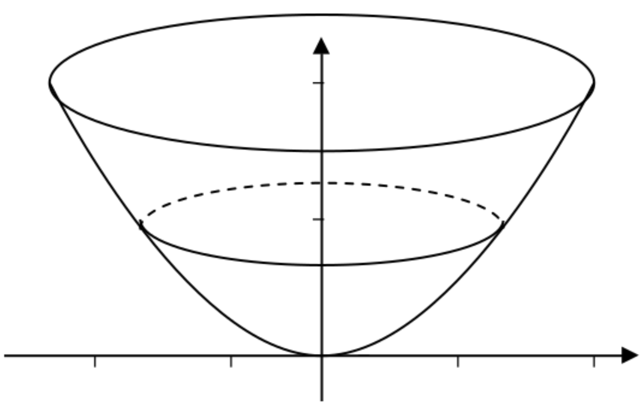
is a power series at -1 equal to $\sin(1+x)$, it must in fact also be the Taylor series at -1 of $g(x) = \sin(1+x)$. ■

5. Consider the surface obtained by revolving the curve $y = \dfrac{x^2}{2}$, $0 \le x \le 2$, about the y-axis.

a. Sketch this surface. *[3]*

b. Find the area of this surface. *[10]*

Solution. **a.** Here is a sketch of the surface:



□

b. Recall that an infinitesimal bit of arc-length is given by $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$. Note that in this case $\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{x^2}{2}\right) = x$. The bit of arc-length at x is revolved about the y-axis, that is, through a circle of radius $r = x - 0 = x$. It follows that the area of the surface obtained by revolving the curve about the y-axis is given by:

$$\text{Area} = \int_0^2 2\pi r\, ds = 2\pi \int_0^2 x\sqrt{1+x^2}\, dx$$

Substitute $w = 1 + x^2$, so $dw = 2x\, dx$ and $x\, dx = \frac{1}{2}\, dw$, changing the limits as we go: $\begin{smallmatrix} x & 0 & 2 \\ w & 1 & 5 \end{smallmatrix}$

$$= 2\pi \int_1^5 \sqrt{w} \cdot \frac{1}{2}\, dw = \pi \int_1^5 w^{1/2}\, dw = \left.\frac{\pi w^{3/2}}{3/2}\right|_1^5 = \left.\frac{2\pi w^{3/2}}{3}\right|_1^5 = \frac{2\pi 5^{3/2}}{3} - \frac{2\pi 1^{3/2}}{3}$$

$$= \frac{2\pi}{3}\left(5\sqrt{5} - 1\right) \approx 21.3217 \quad \blacksquare$$

6. Consider the power series $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \cdots$.

a. Determine the radius and interval of convergence of this power series. *[8]*
b. What function has this power series as its Taylor series? Why? *[5]*

Solution. **a.** As usual, we will try to use the Ratio Test to determine the radius of convergence of this power series.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{((n+1)+1)\,x^{n+1}}{(n+1)x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+2)\,x^{n+1}}{(n+1)x^n}\right| = \lim_{n\to\infty}\frac{n+2}{n+1}|x|$$

$$= |x| \lim_{n\to\infty}\frac{n+2}{n+1}\cdot\frac{1/n}{1/n} = |x| \lim_{n\to\infty}\frac{1+\frac{2}{n}}{1+\frac{1}{n}} = |x| \cdot \frac{1+0}{1+0} = |x| \cdot 1 = |x|$$

It follows by the Ratio Test that the given power series converges absolutely when $|x| < 1$ and does not converge when $|x| > 1$.

To determine the interval of convergence, it remains to check whether the series converges when $|x| = 1$, *i.e.* when $x = -1$ and when $x = 1$. Suppose $|x| = 1$. Then

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}|(n+1)x^n| = \lim_{n\to\infty}(n+1)|x|^n = \lim_{n\to\infty}(n+1)1^n = \lim_{n\to\infty}(n+1) = \infty \neq 0,$$

so, by the Divergence Test, the series does not converge when $|x| = 1$. Thus the interval of convergence of the given power series is $(-1, 1)$. $\square$

b. Observe that each term $(n+1)x^n$ of the series is the derivative of x^{n+1}. Thus, for x within the radius of convergence, we have

$$\sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=0}^{\infty}\frac{d}{dx}x^{n+1} = \frac{d}{dx}\sum_{n=0}^{\infty}x^{n+1} = \frac{d}{dx}\left(\frac{x}{1-x}\right) \quad \text{[Sum of a geometric series!]}$$

$$= \frac{\left[\frac{d}{dx}x\right](1-x) - x\left[\frac{d}{dx}(1-x)\right]}{(1-x)^2} = \frac{1(1-x) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}.$$

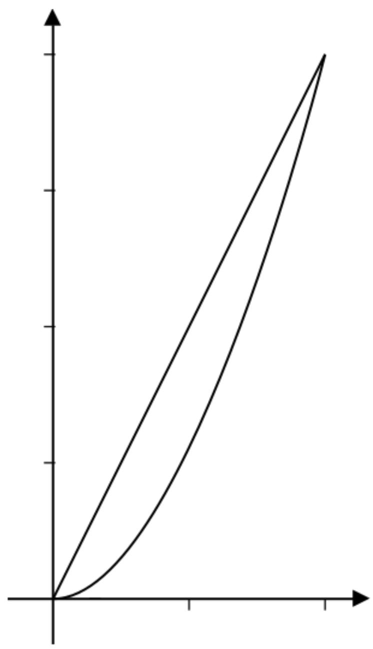
Since $f(x) = \dfrac{1}{(1-x)^2}$ is actually equal to the series $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$ when it converges, this series is the Taylor series (at 0) of $f(x)$. ■

7. Consider the region below $y = 2x$ and above $y = x^2$, for $0 \le x \le 2$.

a. Sketch the region and find its area. *[3]*

b. Sketch the solid obtained by revolving this region about the line $x = -1$ and find its volume. *[10]*

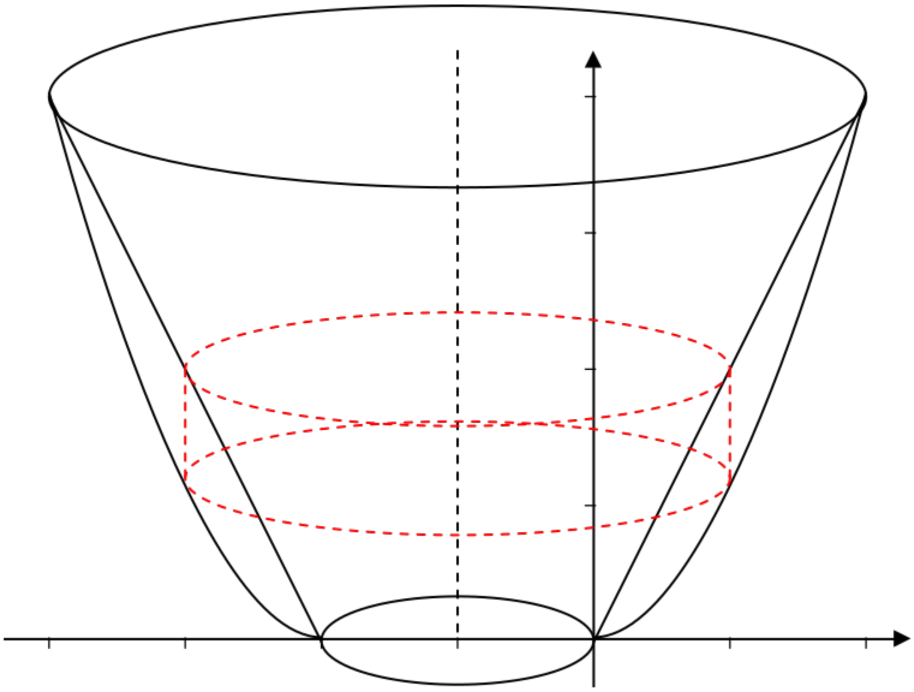
SOLUTION. **a.** Here is a sketch of the region:



Note that $2x = x^2$ exactly when $x = 0$ or $x = 2$, and that between 0 and 2 we have $x^2 < 2x$. It follows that the area of the given region is:

$$\begin{aligned}\text{Area} &= \int_0^2 (\text{upper} - \text{lower})\, dx = \int_0^2 \left(2x - x^2\right)\, dx = \left.\left(x^2 - \frac{x^3}{3}\right)\right|_0^2 \\ &= \left(2^2 - \frac{2^3}{3}\right) - \left(0^2 - \frac{0^3}{3}\right) = \left(4 - \frac{8}{3}\right) - 0 = \frac{12}{3} - \frac{8}{3} = \frac{4}{3} \quad \square\end{aligned}$$

b. *Using cylindrical shells.* Here is a sketch of the solid with a cylindrical shell drawn in:

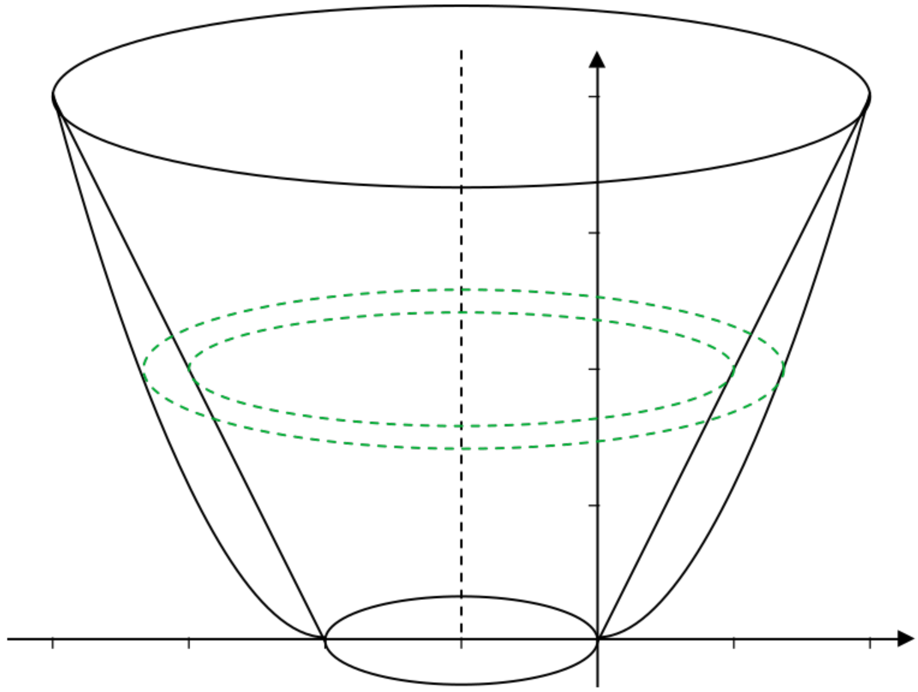


Since we revolved around a vertical line, the shells are perpendicular to the x-axis and parallel to the y-axis, so we will use x as our basic variable. Note that the cylindrical shell at x has radius $r = x - (-1) = x + 1$ and height $h = 2x - x^2$. Note also that the possible x values for the region are between 0 and 2.

It follows that the volume of the solid is:

$$\begin{aligned}
\text{Volume} &= \int_0^2 2\pi r h \, dx = 2\pi \int_0^2 (x+1)\left(2x - x^2\right)\, dx = 2\pi \int_0^2 \left(2x + x^2 - x^3\right)\, dx \\
&= 2\pi \left(x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^2 = 2\pi\left(2^2 + \frac{2^3}{3} - \frac{2^4}{4}\right) - 2\pi\left(0^2 + \frac{0^3}{3} - \frac{0^4}{4}\right) \\
&= 2\pi\left(4 + \frac{8}{3} - \frac{16}{4}\right) - 2\pi \cdot 0 = 2\pi\left(4 + \frac{8}{3} - 4\right) - 0 = 2\pi \cdot \frac{8}{3} = \frac{16\pi}{3} \quad \square
\end{aligned}$$

b. *Using washers.* Here is a sketch of the solid with a washer drawn in:



Since we revolved around a vertical line, the washers are perpendicular to the y-axis and parallel to the x-axis, so we will use y as our basic variable. Note that the washer at y has outer radius $R = x - (-1) = x + 1 = \sqrt{y} + 1$, since it comes from the curve $y = x^2$, and inner radius $r = x - (-1) = \frac{y}{2} - (-1) = \frac{y}{2} + 1$, since it come from the line $y = 2x$. Note also that the possible y values for the region are between 0 and 4.

It follows that the volume of the solid is:

$$\begin{aligned}
\text{Volume} &= \int_0^4 \pi \left(R^2 - r^2\right)\, dy = \pi \int_0^4 \left((\sqrt{y}+1)^2 - \left(\frac{y}{2}+1\right)^2 \right)\, dx \\
&= \pi \int_0^4 \left((y + 2\sqrt{y} + 1) - \left(\frac{y^2}{4} + y + 1\right)\right)\, dx = \pi \int_0^4 \left(2y^{1/2} - \frac{y^2}{4}\right)\, dx \\
&= \pi \left(\frac{2y^{3/2}}{3/2} - \frac{y^3}{12} \right)\Bigg|_0^4 = \pi \left(\frac{4 \cdot 4^{3/2}}{3} - \frac{4^3}{12} \right) - \pi \left(\frac{4 \cdot 0^{3/2}}{3} - \frac{0^3}{12} \right) \\
&= \pi \left(\frac{4 \cdot 8}{3} - \frac{64}{20} \right) - \pi \cdot 0 = \pi \left(\frac{32}{3} - \frac{16}{3} \right) - 0 = \frac{16\pi}{3} \quad \blacksquare
\end{aligned}$$

Part C. Do *one* (1) of **8** or **9**. *[14]*

8. Let $\Gamma(x) = \displaystyle\int_0^\infty t^{x-1}e^{-t}\,dt$ for $x > 0$. *[This is the Gamma function.]*

a. Show that $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$. *[8]*

b. Show that $\Gamma(1) = 1$. *[3]*

c. Explain why $\Gamma(n+1) = n!$ for all integers $n \geq 0$. *[3]*

SOLUTION. **a.** Suppose $x > 0$. We will use integration by parts, with $u = t^{(x+1)-1} = t^x$ and $v' = e^{-t}$, so $u' = xt^{x-1}$ and $v = -e^{-t}$. Note that if $x > 0$, then $x+1 > 0$, too.

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^{(x+1)-1}e^{-t}\,dt = \int_0^\infty t^x e^{-t}\,dt = t^x\left(-e^{-t}\right)\Big|_0^\infty - \int_0^\infty xt^{x-1}\left(-e^{-t}\right)\,dt \\
&= \left(\lim_{t\to\infty}(-1)t^x e^{-t}\right) - \left(\lim_{t\to0^+}(-1)t^x e^{-t}\right) - (-1)x\int_0^\infty t^{x-1}e^{-t}\,dt \\
&= 0 - 0 + x\Gamma(x) = x\Gamma(x)
\end{aligned}$$

Note that $\displaystyle\lim_{t\to\infty}(-1)t^x e^{-t} = \lim_{t\to\infty}\frac{-t^x}{e^t} = 0$ since e^t heads to ∞ as $t\to\infty$ much faster than any power of t, and $\displaystyle\lim_{t\to0^+}(-1)t^x e^{-t} = -0e^0 = 0$. $\square$

b. We plug $x = 1$ into the definition of $\Gamma(x)$ and see what happens:

$$\begin{aligned}
\Gamma(1) &= \int_0^\infty t^{1-1}e^{-t}\,dt = \int_0^\infty t^0 e^{-t}\,dt = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty \\
&= \left(-e^{-\infty}\right) - \left(e^{-0}\right) = (-0) - (-1) = 1 \quad \square
\end{aligned}$$

c. By part **b**, $\Gamma(0+1) = \Gamma(1) = 1 = 0!$. (Note that it is not the case that $0 > 0$, so part **a** does not apply.)

For $n > 0$, observe that by repeated application of part **a**,

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) = n\Gamma\left((n-1)+1\right) = n(n-1)\Gamma(n-1) = \cdots \\
&= n(n-1)\cdots 2\cdot\Gamma(0+1) = n(n-1)\cdots 2\cdot 1 = n!. \quad \blacksquare
\end{aligned}$$

9. Prove the p-Test: For $p \in \mathbb{R}$, $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only if $p > 1$. *[14]*

SOLUTION. It is enough to show that $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. We will use the Integral Test to do this. Observe that if p is a real number, then $f(x) = x^{-p} = \dfrac{1}{x^p}$ is a continuous function for (at least) $x > 0$, and hence integrable on $[0,\infty)$. Since we also have that $f(n) = \dfrac{1}{n^p}$ for integers $n \geq 1$, the Integral Test tells that the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only if the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges.

On the one hand, suppose that $p > 1$. Then

$$\int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx = \left.\frac{x^{-p+1}}{-p+1}\right|_1^\infty = \lim_{a\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^a = \lim_{a\to\infty}\left(\frac{a^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right)$$
$$= \frac{0}{-p+1} - \frac{1}{-p+1} = \frac{1}{p-1} \qquad (a^{-p+1} \to 0 \text{ as } a \to \infty \text{ since } -p+1 < 0.)$$

Since the integral converges to a finite value when $p > 1$, it follows by the Integral Test that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

On the other hand, suppose that $p < 1$. The calculation above can be repeated exactly until near the end: $a^{-p+1} \to \infty$ as $a \to \infty$ since $-p+1 > 0$ in this case. This means that the integral diverges, and hence so does the series.

On our third hand [!?], suppose that $p = 1$. Then the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, *i.e.* the harmonic series, which we know from assignment and class diverges. (We could do this part using the Integral Test as well; note that $\int \frac{1}{x}\,dx = \ln(x)$ and $\ln(x) \to \infty$ when $x \to \infty$.)

Thus $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges exactly when $p > 1$. ■

[Total = 100]

Part D. Bonus problems! If you feel like it and have the time, do one or both of these.

$\sqrt{\mathbf{100}}$. A certain math student asserted that $1 + 2 + 4 + 8 + \cdots = -1$. What exactly is the error they are likely to have made? *[1]*

Solution. $1 + 2 + 4 + 8 + \cdots$ is a geometric series with first term $a = 1$ and common ratio $r = 2$. Since $|r| = 2 > 1$, this series does not converge, but our intrepid student probably plugged $a = 1$ and $r = 2$ into the summation formula $\frac{a}{1-r}$, which works only when $|r| < 1$, anyway. This would give $\frac{a}{1-r} = \frac{1}{1-2} = \frac{1}{-1} = -1$. ■

$\sqrt{\mathbf{121}}$. Write a short poem touching on calculus or mathematics in general. *[1]*

Solution. You're on your own here! :-) ■

Enjoy the summer!

P.S.: You may keep this question sheet. (It could be a souvenir, become a paper airplane, be used to help start a campfire, whatever! :-) The solutions to this exam will be posted to Blackboard and the course archive page in a day or three.