Mathematics 1121H – Calculus II
Trent University, Winter 2026

Solutions to Assignment #9
Series Business IV
Due on Friday, 20 March.

Recall from Assignment #6 that the harmonic series, $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$, diverges; and recall from Assignment #4 that the alternating harmonic series, $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$, converges.

1. Determine whether the harmonic series given a repeated $++--$ pattern of signs,

$$1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}-\frac{1}{12}+\cdots,$$

converges or not. *[3]*

Solution. There are several ways to show that this series converges. Here is one:
Observe that

$$\begin{aligned}
&1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{1}{10}-\frac{1}{11}-\frac{1}{12}+\cdots\\
&=1+\left[\frac{1}{2}-\frac{1}{3}\right]-\left[\frac{1}{4}-\frac{1}{5}\right]+\left[\frac{1}{6}-\frac{1}{7}\right]-\left[\frac{1}{8}-\frac{1}{9}\right]+\left[\frac{1}{10}-\frac{1}{11}\right]-\cdots\\
&=1+\frac{3-2}{2\cdot 3}-\frac{5-4}{4\cdot 5}+\frac{7-6}{6\cdot 7}-\frac{9-8}{8\cdot 9}+\frac{11-10}{10\cdot 11}-\cdots\\
&=1+\frac{1}{2\cdot 3}-\frac{1}{4\cdot 5}+\frac{1}{6\cdot 7}-\frac{1}{8\cdot 9}+\frac{1}{10\cdot 11}-\cdots=1+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)(2n+1)}.
\end{aligned}$$

So the given series converges or not exactly as $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)(2n+1)}$ does. This series, however, converges by the Alternating Series Test:

- The individual terms alternate in sign because $(-1)^{n+1}$ alternates in sign as n increases while $(2n)(2n+1)$ is always positive.
- The absolute values of the individual terms are decreasing since

$$\begin{aligned}
\left|\frac{(-1)^{(n+1)+1}}{((2(n+1))\,(2(n+1)+1)}\right| &= \frac{1}{((2(n+1))\,(2(n+1)+1)}\\
&< \frac{1}{(2n)(2n+1)} = \left|\frac{(-1)^{n+1}}{(2n)(2n+1)}\right|.
\end{aligned}$$

Make the denominator bigger, the fraction gets smaller ...

- The limit of (the absolute values of) the individual terms is 0:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(2n)(2n+1)}\right| = \lim_{n\to\infty}\frac{1}{(2n)(2n+1)}\begin{matrix}\to 1\\ \to\infty\end{matrix} = 0$$

It follows that the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)(2n+1)}$, and hence also the given series, converges by the Alternating Series Test. □

2. Find a_n for each $n \geq 0$ such that $\displaystyle\sum_{n=0}^{\infty} a_n x^n = \left(\frac{1}{1-x}\right)^2$ when the series converges. *[3]*

HINT. What series is $\dfrac{1}{1-x}$ the sum of?

SOLUTION. $\dfrac{1}{1-x}$ is the sum of the geometric series $\displaystyle\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots$, which converges exactly when $|x| < 1$. It follows that

$$\begin{aligned}
\left(\frac{1}{1-x}\right)^2 &= \left(\sum_{k=0}^{\infty} x^k\right)^2 = \left(1 + x + x^2 + x^3 + \cdots\right)^2 \\
&= \left(1 + x + x^2 + x^3 + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right) \\
&= 1\left(1 + x + x^2 + x^3 + \cdots\right) + x\left(1 + x + x^2 + x^3 + \cdots\right) \\
&\quad + x^2\left(1 + x + x^2 + x^3 + \cdots\right) + x^3\left(1 + x + x^2 + x^3 + \cdots\right) \\
&\quad + \cdots \\
&= \left(1 + x + x^2 + x^3 + \cdots\right) + \left(x + x^2 + x^3 + x^4 + \cdots\right) \\
&\quad + \left(x^2 + x^3 + x^4 + x^5 \cdots\right) + \left(x^3 + x^4 + x^5 + x^6 \cdots\right) \\
&\quad + \cdots \\
&= 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty}(n+1)x^n .
\end{aligned}$$

That is, $a_n = n + 1$ for $n \geq 0$. □

3. Let $\mathbb{M} = \{\, n \in \mathbb{N} \mid n \text{ is not divisible by 2, 3, or 5} \,\}$. Show that $\displaystyle\sum_{m\in\mathbb{M}}\frac{1}{m}$ diverges. *[1]*

HINT. $30k + 1 \in \mathbb{M}$ for every integer $k \geq 0$. Note that the given series is the harmonic series with a lot of the terms simply omitted.

Solution. Since $30k+1 \in \mathbb{M}$ and $\dfrac{1}{30k+1} \geq \dfrac{1}{30k+30}$ for every integer $k \geq 0$,

$$\begin{aligned}\sum_{m \in \mathbb{M}} \frac{1}{m} &\geq \sum_{k=0}^{\infty} \frac{1}{30k+1} \geq \sum_{k=0}^{\infty} \frac{1}{30k+30} = \frac{1}{30} \sum_{k=0}^{\infty} \frac{1}{k+1} \\ &= \frac{1}{30}\left[\frac{1}{0+1} + \frac{1}{1+1} + \frac{1}{2+1} + \frac{1}{3+1} + \cdots\right] \\ &= \frac{1}{30}\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right] = \frac{1}{30} \sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{30} \cdot \infty = \infty.\end{aligned}$$

That is, $\sum_{m \in \mathbb{M}} \dfrac{1}{m}$ diverges because the harmonic series does. □

4. What is your favourite series? Explain why in a short poem. *[1]*

Solution. There is a reference to work by Pythagoras – not in geometry! – in this poem.

My fave is the series harmonic,
It encodes relationships sonic.
Terms tend to zero,
The sum is a hero,
Having drunk the infinity tonic. ■