Mathematics 1121H – Calculus II
TRENT UNIVERSITY, Winter 2026
Solutions to Assignment #11
Taylor Series
Due on Thursday, 2 April.

1. Suppose a and r are real numbers. Find the Taylor series at 0 of $(a+x)^r$ and determine its radius of convergence. *[5]*

NOTE 1. You may find the following bit of notation handy. If r is a real number and $k \geq 1$ is an integer, then the *binomial of* r *and* k is

$$\binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}.$$

Thus $\binom{r}{1} = r$, $\binom{r}{2} = \frac{r(r-1)}{2}$, $\binom{r}{3} = \frac{r(r-1)(r-2)}{6}$, and so on. Observe that $\binom{r}{k+1} = \frac{r-(k+1)+1}{k+1} \cdot \binom{r}{k} = \frac{r-k}{k+1} \cdot \binom{r}{k}$ for $k \geq 1$. To make various formulas work nicely, we let $\binom{r}{0} = 1$. Note that when r is a positive integer, this coincides with the usual definition of binomial coefficients.

NOTE 2. Since the Taylor series at 0 of $(a+x)^r$ turns out to be equal to the function inside the radius of convergence, this gives a result due to Isaac Newton (1642-1727) – yes, *that* Newton – that is nowadays called *Newton's Binomial Theorem*.

SOLUTION. Let $f(x) = (a+x)^r$. Then

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(a+x)^r$	a^r
1	$r(a+x)^{r-1}$	ra^{r-1}
2	$r(r-1)(a+x)^{r-2}$	$r(r-1)a^{r-2}$
3	$r(r-1)(r-2)(a+x)^{r-3}$	$r(r-1)(r-2)a^{r-3}$
$\vdots$	$\vdots$	$\vdots$
n	$r(r-1)(r-2)\cdots(r-n+1)(a+x)^{r-n}$	$r(r-1)(r-2)\cdots(r-n+1)a^{r-n}$
$\vdots$	$\vdots$	$\vdots$

It follows that the Taylor series at 0 of $f(x) = (a+x)^r$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{r(r-1)(r-2)\cdots(r-n+1)a^{r-n}}{n!} x^n = \sum_{n=0}^{\infty} \binom{r}{n} a^{r-n} x^n$$

We worked out the radius of convergence in class for $a = 1$ and $r = \frac{1}{2}$ using the Ratio Test. We use it here, too.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\binom{r}{n+1} a^{r-(n+1)} x^{n+1}}{\binom{r}{n} a^{r-n} x^n} \right| &= \lim_{n\to\infty} \left| \frac{r-n}{n+1} a^{-1} x \right| = \lim_{n\to\infty} \left| \frac{r-n}{n+1} \right| \cdot \left| \frac{x}{a} \right| = \left| \frac{x}{a} \right| \lim_{n\to\infty} \left| \frac{r-n}{n+1} \right| \\
&= \left| \frac{x}{a} \right| \lim_{n\to\infty} \left| \frac{r-n}{n+1} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \right| = \left| \frac{x}{a} \right| \lim_{n\to\infty} \left| \frac{\frac{r}{n} - \frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right| = \left| \frac{x}{a} \right| \lim_{n\to\infty} \left| \frac{\frac{r}{n} - 1}{1 + \frac{1}{n}} \right| \\
&= \left| \frac{x}{a} \right| \cdot \left| \frac{0-1}{1+0} \right| = \left| \frac{x}{a} \right| \cdot 1 = \left| \frac{x}{a} \right|
\end{aligned}$$

By the Ratio Test, it follows that the Taylor series converges absolutely when $\left|\frac{x}{a}\right| < 1$, *i.e.* when $|x| < |a|$, and fails to converge when $\left|\frac{x}{a}\right| > 1$, *i.e.* when $|x| > |a|$. Thus the radius of convergence of the Taylor series of $f(x) = (a+x)^r$ is $r = |a|$.

The above works when $a \neq 0$. When $a = 0$, though, the situation is pretty simple because every derivative at 0 is 0. We leave it to the reader to work out the radius of convergence in this case, and also whether the series, when it converges, is equal to the function. □

For question **2**, you may assume that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$, and $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, and that all three series converge for all real numbers x. Also, denote the square root of -1 by i, so $i^2 = -1$.

2. Prove *Euler's Formula*: $e^{ix} = \cos(x) + i\sin(x)$ for all real numbers x. *[3]*

NOTE. Plugging in $x = \pi$ into Euler's Formula gives the equation $e^{i\pi} = -1$, which is also sometimes called Euler's Formula.

SOLUTION. The basic idea is to split up the series for e^{ix} into two series, one with all the even powers and one with all the odd powers. Note that $i^{2k} = (-1)^k$ for all $k \geq 0$. then, for all x:

$$\begin{aligned}
e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \left[\sum_{k=0}^{\infty} \frac{(ix)^{2k}}{(2k)!}\right] + \left[\sum_{k=0}^{\infty} \frac{(ix)^{2k+1}}{(2k+1)!}\right] = \left[\sum_{k=0}^{\infty} \frac{i^{2k}x^{2k}}{(2k)!}\right] + \left[\sum_{k=0}^{\infty} \frac{i^{2k+1}x^{2k+1}}{(2k+1)!}\right] \\
&= \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}\right] + \left[\sum_{k=0}^{\infty} \frac{(-1)^k i x^{2k+1}}{(2k+1)!}\right] = \left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}\right] + i\left[\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}\right] \\
&= \cos(x) + i\sin(x) \quad \square
\end{aligned}$$

3. Use Euler's Formula to prove *de Moivre's Formula:* $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$ for all real numbers x and integers n. *[2]*

SOLUTION. Here goes: $(\cos(x) + i\sin(x))^n = \left(e^{ix}\right)^n = e^{inx} = \cos(nx) + i\sin(nx)$ for all x and n by Euler's Formula, applied twice, and the properties of exponents. ■