Mathematics 1121H – Calculus II

Trent University, Winter 2026

Solutions to Assignment #10
A Choice and an Approximation

Due on Friday, 27 March.

Instructions. Do question **1** and *one* (1) of questions **2** or **3**.

The *Taylor polynomial of degree $n \geq 0$ at a* of a function $f(x)$ is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Recall that the average value of a function $f(x)$ over an interval $[b,c]$ is $\frac{1}{c-b}\int_b^c f(x)\,dx$.

1. What is the least n such that the Taylor polynomial of degree n at 0 of $\sin(x)$ makes $T_n(x) - \sin(x)$ have an average value over $[0,\pi]$ that is between -0.1 and 0.1? *[6]*

Hint. SageMath has a handy `taylor` operator that computes the Taylor polynomial of degree n at a point a of any suitably differentiable function.

Solution. Here's a code snippet that should be readily adaptable to similar problems:

```
[1]: var('x,n,i')
f = function('f')(x)
f(x) = sin(x)
a = 0  # point at which the Taylor polynomial is computed
n = 0  # initial degree of the Taylor polynomial
b = 0  # left-hand endpoint of the interval
c = pi # right-hand endpoint of the interval
i = integral( taylor(f(x),x,a,n)-f(x), x, b, c )/(c-b)
       # integral of T_n(x) - f(x) over [b,c]
print(n, N(i))
t = 0.01   # desired tolerance
while( i < -t or i == -t or i == t or i > t ):
    n = n + 1
    i = integral( taylor(f(x),x,a,n) - f(x), x, b, c )/(c-b)
    print(n, N(i))
```

```
0 -0.636619772367581
1 0.934176554427315
2 0.934176554427315
3 -0.357751640585177
4 -0.357751640585177
5 0.0672756993943804
6 0.0672756993943804
7 -0.00763236677071868
```

Thus the least n that meets the given requirements is $n = 7$. □

NOTE. For the curious, the Taylor polynomial of degree 7 at 0 of $\sin(x)$ is:

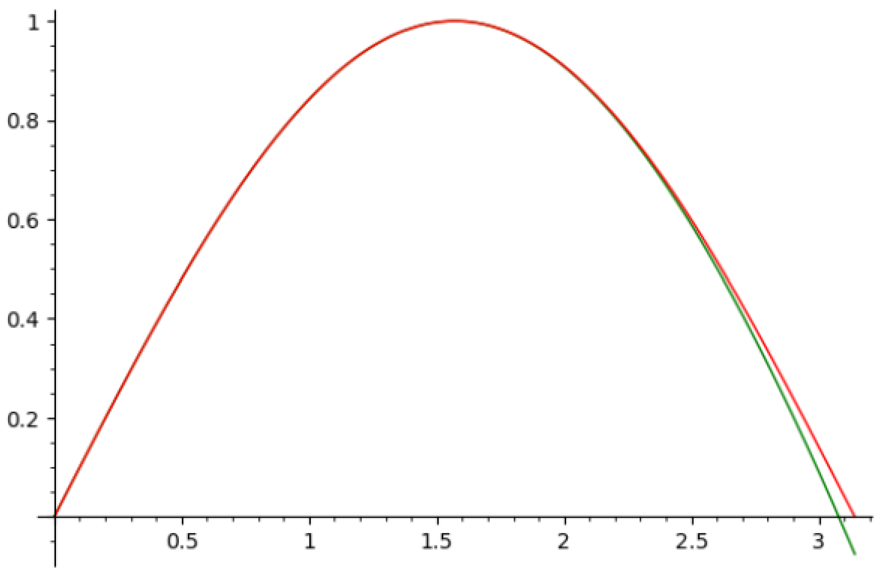
```
[2]: show( taylor( sin(x), x, 0, 7 ) )
```

```
-1/5040*x^7 + 1/120*x^5 - 1/6*x^3 + x
```

This polynomial is pretty close to $\sin(x)$ over the interval $[0, \pi]$:

```
[4]: plot( taylor(sin(x),x,0,7), 0, pi, color='green' ) + plot( sin(x), 0, pi,
     ↪color='red' )
```

[4]:



Recall that $\sum_{n=0}^{\infty} a_n$ is *absolutely convergent* if $\sum_{n=0}^{\infty} |a_n|$ is convergent. Absolutely convergent series must be convergent. Series that are convergent but not absolutely convergent are said to be *conditionally convergent*.

2. Explain why a conditionally convergent series can be made to add to any value between $-\infty$ and ∞ inclusive by rearranging the terms of the series, without discarding any of the terms or adding new ones. *[4]*

HINT. A conditionally convergent series must still pass the Divergence Test. Also, it must have infinitely many positive terms and infinitely many negative terms. (Why?) The positive terms must add up to ∞ and the negative terms must add up to $-\infty$. (Why?)

SOLUTION. We will answer the questions posed in the hint first.

If a series has only finitely many positive (respectively, negative) terms, then all terms past some point are negative (respectively, positive) or zero, in which case the series converges absolutely if it converges at all. Thus a conditionally convergent series must have infinitely many positive terms and infinitely many negative terms.

If the negative terms of a series and the positive terms of a series each add up to a finite number, then each converge absolutely and hence so does the original series. (Think about it!) If the negative terms add up to a finite number and the positive terms add up to ∞, the series as a whole must add up to ∞, and hence does not converge. Similarly, if the negative terms add up to $-\infty$ and the positive terms add to a finite number, the series as a whole must add up to $-\infty$, and hence does not converge. It follows that in a conditionally convergent series – *i.e.* a series that converges, but not absolutely – the positive terms must add up to ∞ and the negative terms must add up to $-\infty$.

Suppose that we have a conditionally convergent series $\sum_{n=0}^{\infty} a_n$ and want a sum of S. As the series converges, it must survive the Divergence Test, so $\lim_{n\to\infty} |a_n| = 0$. Let b_0, b_1, b_2, ... be a list of the positive terms of the series, and let c_0, c_1, c_2, ... be a list of the negative terms of the series, each set listed in the order that they occur in the original series in each case.

We now make a new series using the follow process.

Step 0. Set the initial partial sum to 0.

Step 1. Add just as many of the b_is that have not been used yet, taking them in the order in which they occur, to the partial sum as you *need* to make the partial sum be greater than S.

Step 2. Add just as many of the c_js that have not been used yet, taking them in the order in which they occur, to the partial sum as you *need* to make the partial sum be less than S.

Step 3. Go back to step 1.

The process continues for infinitely many steps because the positive terms add up to ∞, so step 1 always works, and the negative terms add up to $-\infty$, so step 2 always works. Thus the process eventually uses all of the b_is and all of the c_is (Why?), so the new series has all the same terms, though likely in a different order, as the original series.

Note that the partial sum after each step overshoots or undershoots, respectively, S after each step 1 or step 2, respectively, by no more than the absolute value of the last term added. This difference approaches 0 since, as observed above, $\lim_{n\to\infty} |a_n| = 0$. It follows that the new series converges to S. $\square$

Recall from class that the Root Test is the following assertion:

Root Test. Suppose $\sum_{n=0}^{\infty} a_n$ is a series and $\lim_{n\to\infty} |a_n|^{1/n} = L$. Then:

1. If $L < 1$, then the series converges absolutely.
2. If $L > 1$, then the series does not converge.
3. If $L = 1$, the test tells us nothing, *i.e.* the series may converge or not.

3. Prove cases 1 and 2 of the Root Test. *[4]*

Solution. Suppose $\lim_{n\to\infty} |a_n|^{1/n} = L$. By definition, this means that for any $\varepsilon > 0$, there is an N such that for all $n \geq N$, $L - \varepsilon < |a_n|^{1/n} < L + \varepsilon$.

Case 1. ($L < 1$) Let $\varepsilon = \dfrac{1-L}{2}$, so $L + \varepsilon < 1$; note that since $L < 1$ we have $0 < \varepsilon$. Let N be such that for all $n \geq N$, $L - \varepsilon < |a_n|^{1/n} < L + \varepsilon$. It follows that for all $n \geq N$, $|a_n| < (L + \varepsilon)^n$.

Since the geometric series $\displaystyle\sum_{n=N}^{\infty} (L+\varepsilon)^n$ has a common ratio $r = L + \varepsilon$ that has an absolute value < 1, it converges. By the Basic Comparison Test, it follows that $\displaystyle\sum_{n=N}^{\infty} |a_n|$, and hence also $\displaystyle\sum_{n=0}^{\infty} |a_n|$, converges. Thus $\displaystyle\sum_{n=0}^{\infty} a_n$ converges absolutely.

Case 2. ($L > 1$) Let $\varepsilon = \dfrac{L-1}{2}$, so $L - \varepsilon > 1$; note that since $L > 1$ we have $0 < \varepsilon$. Let N be such that for all $n \geq N$, $L - \varepsilon < |a_n|^{1/n} < L + \varepsilon$. It follows that for all $n \geq N$, $|a_n| > (L - \varepsilon)^n$.

Since $L - \varepsilon > 1$, $\displaystyle\lim_{n\to\infty} |a_n| \geq \lim_{n\to\infty} (L-\varepsilon)^n = \infty \neq 0$, so $\displaystyle\sum_{n=0}^{\infty} |a_n|$, and hence also $\displaystyle\sum_{n=0}^{\infty} a_n$, fails the Divergence Test and thus does not converge. (Note that $\displaystyle\lim_{n\to\infty} |a_n| = 0$ if and only if $\displaystyle\lim_{n\to\infty} a_n = 0$.) ■