# The Basic Substitution Rule

## The Chain Rule Reversed

The Chain Rule of differential calculus tells us how to compute the derivative of a composition of two functions $(f \circ g)(x) = f(g(x))$:

$$(f \circ g)'(x) = \frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

For example, since $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}x^2 = 2x$, the Chain Rule tells us that $\frac{d}{dx}\sin\left(x^2\right) = \cos\left(x^2\right) \cdot 2x = 2x\cos\left(x^2\right)$. As with every derivative formula, running it in reverse gives us an anti-derivative formula:

$$\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$$

The rule is rarely used in this form, however. The usual way to write and use this rule is to simplify an integral of the form $\int h(g(x)) \cdot g'(x)\, dx$ by writing $u = g(x)$, so $du = g'(x)\, dx$, and then replace the expressions $g(x)$ and $g'(x)\, dx$ by $u$ and $du$, respectively. For indefinite integrals this gives us the following.

*(Basic) Substitution Rule:* $\quad \int h(g(x)) \cdot g'(x)\, dx = \int h(u)\, du$

One normally puts the solution to an indefinite integral in terms of the original variable, so one has to substitute back at that stage: if $H(u)$ was the antiderivative of $h(u)$ above, one would finish finding the antiderivative above as follows: $\cdots = H(u) + C = H(g(x)) + C$

When dealing with definite integrals, such as $\int_a^b h(g(x)) \cdot g'(x)\, dx$, one normally replaces the limits in $x$ with their counterparts in terms of the new variable $u$, $\begin{array}{ccc} x & a & b \\ u & g(a) & g(b) \end{array}$, so

$$\int_a^b h(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} h(u)\, du = H(g(b)) - H(g(a)) \,,$$

where $H(u)$ is any antiderivative of $h(u)$. Alternatively, one can keep track of the variable the old limits belong to,

$$\int_a^b h(g(x)) \cdot g'(x)\, dx = \int_{x=a}^{x=b} h(u)\, du \,,$$

and then substitute back in terms of the original variable before using the limits of integration. In this case the calculation would work like:

$$\int_a^b h(g(x)) \cdot g'(x)\, dx = \int_{x=a}^{x=b} h(u)\, du = H(u)\big|_{x=a}^{x=b} = H(g(x))\big|_a^b = H(g(b)) - H(g(a))$$

The methods give the same answer (they'd better! :-) and which is preferable depends the particular integral and personal taste. To start with, at least, I would recommend trying both a few times and then mainly using the one you find yourself more comfortable with.

In practice, the real problem with using the Substitution Rule is identifying a suitable $g(x)$ in whatever integrand you have to deal with. Sometimes there are several apparent choices and you have to figure out which one you ought to try, sometimes whatever choices are there are hard to spot, and sometimes there are no suitable choices to make a substitution at all, so some technique other than substitution will have to be used.

**Examples, with a trick thrown in**

We will look for a composition $h(g(x))$ in the integrand, with the derivative of the inner function, $g'(x)$ as a factor of the integrand.

**1.** Let's find the antiderivative of $\displaystyle\int 2x\cos\left(x^2\right)\,dx$. There is an obvious composition, namely $\cos\left(x^2\right)$, as one factor of the integrand, with the deivative of the inner function, $\dfrac{d}{dx}x^2 = 2x$, as the other factor. To simplify the integral we therefore take $u = x^2$, so $du = 2x\,dx$, and then deal with the simplified version:

$$\int 2x\cos\left(x^2\right)\,dx = \int \cos(u)\,du = \sin(u) + C = \sin\left(x^2\right) + C$$

Note that we substituted back in terms of the original variable at the end.

**2.** This time, let's work out $\displaystyle\int x\sqrt{1+x^2}\,dx$. It would be nice to simplify the expression $\sqrt{1+x^2}$. (Anyone who tries to do this by claiming $\sqrt{1+x^2} = \sqrt{1} + \sqrt{x^2} = 1 + x$, will be *wrong*. The only time you'd get away with it is when $x = 0$; the square root function does not play nice with addition.) We'll try to simplify it by using the substitution $u = 1 + x^2$. Then $\sqrt{1+x^2} = \sqrt{u} = u^{1/2}$ which is pretty easy to handle. However, $du = \frac{d}{dx}\left(1+x^2\right)\,dx = 2x\,dx$ and we don't have $2x$ available to us in the integrand, just an $x$. We can work around this by dividing by 2 on both sides of $du = 2x\,dx$ to get $\frac{1}{2}\,du = x\,dx$ and substitute accordingly:

$$\begin{aligned}\int x\sqrt{1+x^2}\,dx &= \int \sqrt{u}\,\frac{1}{2}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{u^{3/2}}{3/2} + C\\ &= \frac{1}{3}u^{3/2} + C = \frac{1}{3}\left(1+x^2\right)^{3/2} + C\end{aligned}$$

Again, we substituted back in terms of the original variable because we were dealing with an indefinite integral and hence finding an antiderivative.

This **trick of adjusting the expression $du = g'(x)\,dx$ to fit what is actually available in the integrand** is a very common one. Multiplying or dividing by a non-zero constant is quite safe in this context, but it is not safe to add or subtract anything on both sides (ironically, except for zero). Also, any non-constant expression on the $du$ side must be in terms of $u$, but not $x$, and any non-constant expression on the $dx$ side must be in terms of $x$, but not $u$. No expressions mixing $u$s and $x$s on either side!

**3.** Let's try a definite integral next, say $\displaystyle\int_{3\pi/2}^{2\pi} \frac{\sin(x)}{1+\cos^2 x}\,dx$. We'll use the substitution $u = \cos(x)$, so $du = -\sin(x)\,dx$ and thus $(-1)\,du = \sin(x)\,dx$. We'll also change the limits as we go along; since $\cos(3\pi/2) = 0$ and $\cos(2\pi) = 1$, this change can be summarized as $\begin{array}{ccc} x & 3\pi/2 & 2\pi \\ u & 0 & 1 \end{array}$. On to the computation:

$$\int_{3\pi/2}^{2\pi} \frac{\sin(x)}{1+\cos^2 x}\,dx = \int_0^1 \frac{-1}{1+u^2}\,du = -\arctan(u)\big|_0^1$$
$$= (-\arctan(1)) - (-\arctan(0)) = \left(-\frac{\pi}{4}\right) - (-0) = -\frac{\pi}{4}$$

Notice that we used the adjustment trick again, since we didn't have the negative sign in the original intehrand that we needed for the substitution we used. This example does point out the fact that we need to be able to evaluate the antiderivative we eventually find when computing a definite integral. If you didn't know – or have someone tell you – what $\arctan(1)$ and $\arctan(0)$ are, you'd probably have to work to look things up or haul out a calculator or computer to get anywhere with this integral.

**4.** One more definite integral, this time $\displaystyle\int_0^{\ln(2)} \frac{1-e^x}{1+e^x}\,dx$. We'll try to simplify this using the more-or-less obvious substitution $u = e^x$. Then $du = e^x\,dx$ and we run into a problem: there is no $e^x$ we can immediately isolate in the integrand. We can't use the numerator $1+e^x$ because it isn't equal to $e^x$ (well, unless $1 = 0$ :-). However, we can use the adjust the $du = g'(x)\,dx$ trick: since $du = e^x\,dx$, we have $dx = \dfrac{1}{e^x}\,du = \dfrac{1}{u}\,du$. Note that we have put everything on the $du$ side in terms of $u$!

If this seems confusing, there is an alternate, and perhaps somewhat better, way of handling this part of the process. Instead of working out $dx$ in terms of $u$ as we did, we could have proceeded as follows: we want to substitute $u = e^x$, so let's solve for $x$ first, $x = \ln(u)$, and then take the derivative, $\dfrac{dx}{du} = \dfrac{d}{du}\ln(u) = \dfrac{1}{u}$, so $dx = \frac{1}{u}\,du$. A number of substitutions seem to work more easily with this kind of approach.

We'll also change the limits as we go along: $\begin{array}{ccc} x & 0 & \ln(2) \\ u & 1 & 2 \end{array}$ Now we're off to the races:

$$\int_0^{\ln(2)} \frac{1-e^x}{1+e^x}\,dx = \int_1^2 \frac{1-u}{1+u}\cdot\frac{1}{u}\,du = \int_1^2 \frac{1-u}{u+u^2}\,du$$

... right up until we try to integrate the allegedly simplified integral on the right. This requires a bit algebraic trickery you're not responsible for yet (look up "partial fractions") so I will simply tell you that $\dfrac{1-u}{u+u^2} = \dfrac{-2}{1+u} + \dfrac{1}{u}$. Plugging this in for the integrand in the last one on the right gives:

$$\int_1^2 \frac{1-u}{u+u^2}\,du = \int_1^2 \left(\frac{-2}{1+u} + \frac{1}{u}\right)du = \int_1^2 \frac{-2}{1+u}\,du + \int_1^2 \frac{1}{u}\,du$$

We'll use another substitution, $w = 1 - u$, so $dw = (-1)\,du$ and thus $(-1)\,dw = du$, in the first of the integral on the right.

$$\begin{aligned}
\int_1^2 \frac{-2}{1+u}\,du + \int_1^2 \frac{1}{u}\,du &= \int_{u=1}^{u=2} \frac{-2}{w}(-1)\,dw + \ln(u)\Big|_1^2 = 2\ln(w)\Big|_{u=1}^{u=2} + (\ln(2) - \ln(1)) \\
&= 2\ln(1+u)\Big|_1^2 + (\ln(2) - 0) = (2\ln(3) - 2\ln(2)) + \ln(2) \\
&= 2\ln(3) - \ln(2) = \ln\left(3^2\right) - \ln(2) = \ln(9) - \ln(2) = \ln\left(\frac{9}{2}\right)
\end{aligned}$$

Whew!

All in all, the Substitution Rule is a tool for simplifying the integrals we are dealing with. It isn't always a useful tool – good luck trying to do anything useful with it in $\displaystyle\int x\cos(x)\,dx$, for example – and it does require you to be able to handle the simplified integral. That may require additional use of the Substitution Rule itself, or variants of it (like trigonometric substitutions, which we'll do in a while), or entirely different techniques, such as integration by parts, which will be next on our list.