Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2024

Solutions to Assignment #9
Convergent or Divergent?

1. For each of the following series, determine whether it converges or diverges. If it converges, find or approximate the sum as best you can.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$ *[1]*

SOLUTION. This is a series of positive terms, so if it converges at all, it coverges absolutely. Observe that for n large enough, we have

$$\begin{aligned}\frac{3^n}{n!} &= \frac{3}{n} \cdot \frac{3}{n-1} \cdot \frac{3}{n-2} \cdot \dots \cdot \frac{3}{4} \cdot \frac{3}{3} \cdot \frac{3}{2} \cdot \frac{3}{1} \\ &= \frac{9}{n(n-1)} \cdot \left[\frac{3}{n-1} \cdot \frac{3}{n-2} \cdot \dots \cdot \frac{3}{4}\right] \cdot \frac{9}{2} \\ &\le \frac{81}{2n(n-1)} = \frac{81}{2n^2-2n}\end{aligned}$$

since $\displaystyle\frac{3}{n-1} \cdot \frac{3}{n-2} \cdot \dots \cdot \frac{3}{4} < 1$ because each factor in this product is less than 1. As $\displaystyle\sum_{n=1}^{\infty} \frac{81}{2n^2-2n}$ converges by the Generalized p-Test, since ir has $p = 2 - 0 > 1$, it follows by the Basic Comparison Test that $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converges. Per the observation above, since it is a a series of positive terms, this means it converges absolutely.

It remains to find the sum of the series, which task we hand off to SageMath:

```
[1]: var('n')
     sum( 3^n/factorial(n), n, 0, oo)

[1]: e^3
```

Thus the series sums to $e^3 \approx 20.0855$. $\square$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n!}$ *[1]*

SOLUTION. Since $\displaystyle\sum_{n=0}^{\infty} \left|\frac{(-1)^n 3^n}{n!}\right| = \sum_{n=0}^{\infty} \frac{3^n}{n!}$, which is convergent by the solution to **a** above, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n!}$ converges absolutely.

It remains to find the sum of the series, which task we hand off to SageMath:

```
[2]: sum( (-1)^n*3^n/factorial(n), n, 0 , oo)

[2]: e^(-3)
```

Thus the series sums to $e^{-3} \approx 0.0498$. $\square$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{n!}$ *[0.5]*

SOLUTION. Observe that for n large enough, we have

$$\frac{n^n}{n!} = \frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdot \dots \cdot \frac{n}{3} \cdot \frac{n}{2} \cdot \frac{n}{1} \geq \frac{n^2}{2}$$

since $\displaystyle\frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdot \dots \cdot \frac{n}{3} \geq 1$ as soon as $n \geq 3$. It follows that

$$\lim_{n\to\infty} \frac{n^n}{n!} \geq \lim_{n\to\infty} \frac{n^2}{2} = \infty \neq 0,$$

so $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{n!}$ diverges by the Divergence Test. □

d. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^n}{n!}$ *[0.5]*

SOLUTION. Since $\displaystyle\left|\frac{(-1)^n n^n}{n!}\right| = \frac{n^n}{n!}$ for all n and $\lim_{n\to\infty} \frac{n^n}{n!} \neq 0$ by the argument in the solution to **c** above, the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^n}{n!}$ also diverges by the Divergence Test. □

e. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n^n}$ *[1]*

SOLUTION. Observe that as soon as $n \geq 4$, we have $0 < \displaystyle\frac{3^n}{n^n} = \left(\frac{3}{n}\right)^n \leq \left(\frac{3}{4}\right)^n$. Since the series $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ converges, being a geometric series with common ratio $\frac{3}{4}$ and $\left|\frac{3}{4}\right| < 1$, it follows by the Basic Comparison Test that $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n^n}$ also converges. As it is a series of positive terms, it follows that it is absolutely convergent.

It remains to find the sum of the series. Unfortunately, SageMath will only do so symbolically, and even that requires massaging what you give it. (SageMath treats 0^0 as being undefined, instead of using the convention that $0^0 = 1$.) So the best we can do is try to approximate it by computing the values of partial sums:

```
[3]: N(1 + sum( 3^n/n^n, n, 1, 10))
```

```
[3]: 7.66289462527756
```

```
[4]: N(1 + sum( 3^n/n^n, n, 1, 20))
```

```
[4]: 7.66289531150129
```

Thus the series sums to about 7.6629. □

f. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n^n}$ *[1]*

Solution. Since $\displaystyle\sum_{n=0}^{\infty} \left| \frac{(-1)^n 3^n}{n^n} \right| = \sum_{n=0}^{\infty} \frac{3^n}{n^n}$ for all n, which converges by the argument in the solution to **e** above, the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n^n}$ converges absolutely.

It remains to find the sum of the series. Again, SageMath will only do so symbolically, and that only after a bit of extra work, so we try to approximate it by computing the values of partial sums:

```
[5]: N(1 + sum( (-1)^n*3^n/n^n, n, 1, 10))
[5]: -0.498038183328813
[6]: N(1 + sum( (-1)^n*3^n/n^n, n, 1, 20))
[6]: -0.498038749477707
```

Thus the series sums to about -0.4980. □

g. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$ *[0.5]*

Solution. Observe that as soon as $n \geq 5$, we have

$$0 < \frac{n!}{3^n} = \frac{n}{3} \cdot \frac{n-1}{3} \cdot \left[\frac{n-2}{3} \cdot \dots \cdot \frac{3}{3} \right] \cdot \frac{2}{3} \cdot \frac{1}{3} \geq \frac{2n(n-1)}{81},$$

since $\dfrac{n-2}{3} \cdot \dots \cdot \dfrac{3}{3} \geq 1$. It follows that $\displaystyle\lim_{n\to\infty} \frac{n!}{3^n} \geq \lim_{n\to\infty} \frac{2n(n-1)}{81} = \infty \neq 0$, so $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{3^n}$ diverges by the Divergence Test. □

h. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{(-1)^n 3^n}$ *[0.5]*

Solution. Since $\left| \dfrac{n!}{(-1)^n 3^n} \right| = \dfrac{n!}{3^n}$ for all n and $\lim_{n\to\infty} \frac{n!}{3^n} \neq 0$ by the argument in the solution to **g** above, the series $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{(-1)^n 3^n}$ also diverges by the Divergence Test. □

i. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n}$ *[1]*

Solution. Observe that for n large enough, we have

$$0 < \frac{n!}{n^n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdot \dots \cdot \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n} \leq \frac{2}{n^2}$$

since $\dfrac{n}{n} \cdot \dfrac{n-1}{n} \cdot \dfrac{n-2}{n} n-2 \cdot \dots \cdot \dfrac{3}{n} \leq 1$ as soon as $n \geq 3$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Test because $p = 2 > 1$, it follows that $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n}$ converges by the Basic Comparison Test. Since it is a series of positive terms, this means that it converges absolutely.

```
[8]: N(1 + sum( factorial(n)/n^n, n, 1, 10))
```

```
[8]: 2.87962701599508
```

```
[9]: N(1 + sum( factorial(n)/n^n, n, 1, 20))
```

```
[9]: 2.87985384815593
```

It remains to find the sum of the series. Unfortunately, SageMath will not do this one either, so the best we can do is try to approximate it by computing the values of partial sums:
Thus the series sums to about 2.8799. □

j. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{n^n}$ *[1]*

SOLUTION. ince $\left|\dfrac{(-1)^n n!}{n^n}\right| = \dfrac{n!}{n^n}$ for all n and the series $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^n}$ converges by the argument in the solution to **i** above, the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n!}{n^n}$ converges absolutely.

It remains to find the sum of the series. Once again, SageMath will not sum the entire series for us, so the best we can do is try to approximate it by computing the values of partial sums:

```
[10]: N(1 + sum( (-1)^n*factorial(n)/n^n, n, 1, 10))
```

```
[10]: 0.344269460090471
```

```
[11]: N(1 + sum( (-1)^n*factorial(n)/n^n, n, 1, 20))
```

```
[11]: 0.344168405484918
```

Thus the series sums to about 0.3442. □

k. $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{3^n}$ *[0.5]*

SOLUTION. Observe that as soon as $n \geq 4$, we have $0 < \dfrac{n^n}{3^n} = \left(\dfrac{n}{3}\right)^n \geq \left(\dfrac{4}{3}\right)^n$. It follows that

$$\lim_{n\to\infty} \frac{n^n}{3^n} \geq \lim_{n\to\infty} \left(\frac{4}{3}\right)^n = \infty \neq 0,$$

so $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{3^n}$ diverges by the Divergence Test. □

l. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^n}{3^n}$ *[0.5]*

SOLUTION. Since $\left|\dfrac{(-1)^n n^n}{3^n}\right| = \dfrac{n^n}{3^n}$ for all n and $\lim_{n\to\infty} \frac{n^n}{3^n} \neq 0$ by the argument in the solution to **g** above, the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n^n}{3^n}$ also diverges by the Divergence Test. ■

2. Does the series $\displaystyle\sum_{n=0}^{\infty}\left[\frac{1}{3n+1}-\frac{1}{3n+2}+\frac{1}{3n+3}\right]=1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}+\cdots$ converge or diverge? If it converges, does it do so conditionally or absolutely? *[1]*

SOLUTION. Observe that

$$\begin{aligned}\frac{1}{3n+1}-\frac{1}{3n+2}+\frac{1}{3n+3} &= \frac{(3n+2)(3n+3)}{(3n+1)(3n+2)(3n+3)}-\frac{(3n+1)(3n+3)}{(3n+1)(3n+2)(3n+3)} \\ &\quad+\frac{(3n+1)(3n+2)}{(3n+1)(3n+2)(3n+3)} \\ &= \frac{(3n+2)(3n+3)-(3n+1)(3n+3)+(3n+1)(3n+2)}{(9n^2+9n+2)(3n+3)} \\ &= \frac{\left(9n^2+15n+6\right)-\left(9n^2+10n+3\right)+\left(9n^2+9n+2\right)}{27n^3+54n^2+33n+6} \\ &= \frac{9n^2+14n+5}{27n^3+54n^2+33n+6}.\end{aligned}$$

Thus $\displaystyle\sum_{n=0}^{\infty}\left[\frac{1}{3n+1}-\frac{1}{3n+2}+\frac{1}{3n+3}\right]=\sum_{n=0}^{\infty}\frac{9n^2+14n+5}{27n^3+54n^2+33n+6}$. However, the latter form of the series can be seen to diverge by the Generalized p-Test because it has $p=3-2=1\not>1$.

One could also deduce that the series diverges by asking SageMath to evaluate it. While typing

```
[12]: sum( 1/(3*n+1) - 1/(3*n+2) + 1/(3*n+3), n, 0, oo)
```

will generate a bunch of error messages, the last of these will end with:

```
ValueError: Sum is divergent.
```

∎