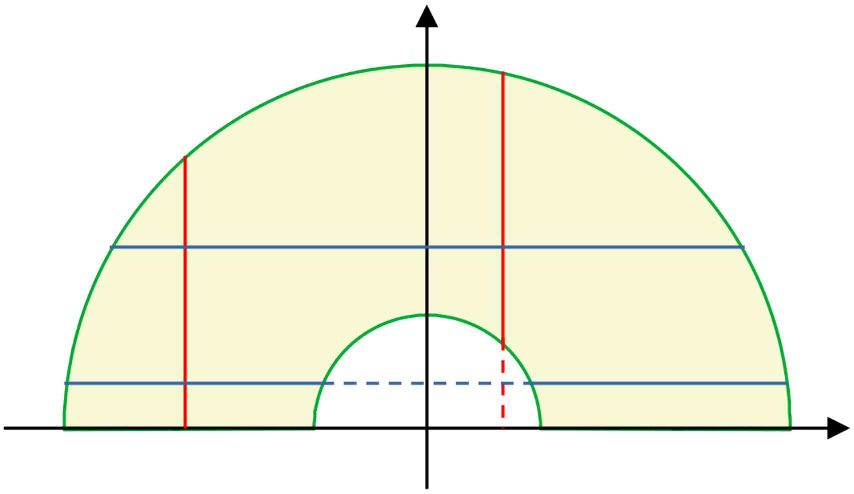
Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2024

Solutions to Assignment #6
A Centroid



Some cross-sections added.

1. Find the centroid of the region $S = \left\{ (x,y) \mid 1 \leq x^2 + y^2 \leq 9 \text{ and } y \geq 0 \right\}$. *[10]*

SOLUTION. Recall that the centroid of a two-dimensional region is its centre of mass if it has a constant density, which we may assume to be 1. The centroid has coordinates $(\bar{x}, \bar{y})$, for $\bar{x} = M_y/M$ and $\bar{y} = M_x/M$, where M is the mass of the region – *i.e.* its area because the density is constant and equal to 1 – and M_y and M_x are the moments of the region about the y- and x-axes, respectively. We compute all of these in turn:

i. As noted above, M is the area of the region. The quick way to get it is to observe that the region consists of a semicircle of radius 3 from which a semicircle of radius 1 has been removed. It follows that

$$M = \frac{1}{2}\pi 3^2 - \frac{1}{2}\pi 1^2 = \frac{9\pi}{2} - \frac{\pi}{2} = \frac{8\pi}{2} = 4\pi.$$

The slower way is to compute the area of the region using integration, in which case we have to break the integral up into three pieces because the lower boundary of the region changes definition: between -3 and -1 the lower boundary is $y = 0$, between -1 and 1 the lower boundary is $y = \sqrt{1-x^2}$, and between 1 and 3 the lower boundary is again 0. The upper boundary over the entire region, *i.e.* from -3 to 3, is $y = \sqrt{9-x^2}$. It follows that

$$M = \int_{-3}^{-1} \sqrt{9-x^2}\,dx + \int_{-1}^{1} \left(\sqrt{9-x^2} - \sqrt{1-x^2}\right)\,dx + \int_{1}^{3} \sqrt{9-x^2}\,dx.$$

Being lazy, we compute this using SageMath:

```
[1]: M = integral( sqrt( 9-x^2 ), x, -3, -1 ) + integral( sqrt( 9-x^2 ) - sqrt(
     1-x^2), x, -1, 1 ) + integral( sqrt( 9-x^2 ), x, 1, 3 )
     show(M)
```

```
4*pi
```

Were we not lazy and did this by hand, the substitutions $u = 3\sin(\theta)$ and $w = \sin(\alpha)$ would probably come in handy ...

ii. In general, the moment about the y-axis, M_y, is obtained by integrating x times the length of the vertical cross-section of the region at x. In this particular case, though, there is a shortcut: because the region is symmetric about the y-axis, we must have $M_y = 0$.

If one misses this shortcut, one can set up the integral for M_y,

$$M_y = \int_{-3}^{-1} x\sqrt{9-x^2}\,dx + \int_{-1}^{1} x\left(\sqrt{9-x^2} - \sqrt{1-x^2}\right)\,dx + \int_{1}^{3} x\sqrt{9-x^2}\,dx,$$

and evaluate it. Being lazy, we compute it using SageMath:

```
[2]: My = integral( x * sqrt( 9-x^2 ), x, -3, -1 ) + integral( x * (sqrt( 9-x^2 ) -␣
     ↪sqrt( 1-x^2)), x, -1, 1 ) + integral( x * sqrt( 9-x^2 ), x, 1, 3 )
     show(My)
```

```
0
```

If we were not lazy and did this by hand, the substitutions $u = 9 - x^2$ and $w = 1 - x^2$ would probably be useful ...

iii. In general, the moment about the x-axis, M_x, is obtained by integrating y times the length of the horizontal cross-section of the region at y. Sadly, the region is not symmetric about any horizontal line, much less the x-axis, so we will have to do this the hard way. Still, we can exploit the symmetry we do have about the y-axis to make our lives a little easier: since the region is symmetric about the y-axis, the length of each horizontal cross-section is twice the length of the part of the horizontal cross-section to the right of the y-axis. Thus the horizontal cross-section at y has length $2\left(\sqrt{9-y^2} - \sqrt{1-y^2}\right)$ for $0 \le y \le 1$ and $2\sqrt{9-y^2}$ for $1 \le y \le 3$. It follows that

$$M_x = \int_0^1 2y\left(\sqrt{9-y^2} - \sqrt{1-y^2}\right)\,dy + \int_1^3 2y\sqrt{9-y^2}\,dy.$$

Still being lazy, we have SageMath compute the integral for us:

```
[3]: var('y')
     Mx = 2 * integral( y * ( sqrt( 9-y^2 ) - sqrt( 1-y^2) ), y, 0, 1 ) + 2 *␣
     ↪integral( y * sqrt( 9-y^2 ), y, 1, 3 )
     show(Mx)
```

```
52/3
```

iv. We finish the job by calculating the coordinates he centroid has coordinates $(\bar{x}, \bar{y})$, where $\bar{x} = M_y/M$ and $\bar{y} = M_x/M$, of the centroid. Being really, truly, lazy, we have SageMath do the arithmetic:

```
[4]: Mx/M
```

```
[4]: 13/3/pi
```

```
[5]: N(Mx/M)
```

```
[5]: 1.37934284012976
```

```
[6]: My/M
```

```
[6]: 0
```

Thus the given region has its centroid at

$$(\bar{x}, \bar{y}) = \left(0, \frac{13}{3\pi}\right) \approx (0, 1.3793) \, . \quad \blacksquare$$