Mathematics 1120H – Calculus II: Integrals and Series

Trent University, Winter 2024

Assignment #4
Area versus Area

Due just before midnight on Friday, 9 February.*

Consider the region below the curve $y = \dfrac{1}{x}$ and above the x-axis for $1 \le x < \infty$, a piece of which you can colour in below.

1. Compute each of the following as best you can using SageMath.

a. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ *[0.5]* **b.** $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ *[0.5]* **c.** $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ *[0.5]* **d.** $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2}$ *[0.5]*

Hint: It is possible that you might have done something similar to some of these previously. :-)

Solution. **a.** Taken from the solutions to Assignment #3, for question **1**:

```
[3]: var('t')
     assume(t>1)
     limit( integral( 1/x, x, 1, t), t=oo )
```

```
[3]: +Infinity
```

□

b. Let's see if the sum behaves better than the corresponding integral did in Assignment #3:

```
[3]: var('n')
     sum( 1/n, n, 1, oo )
```

Sadly, it doesn't: you get a series of error messages, the last of which is "`ValueError: Sum is divergent.`" That is, it doesn't add up. We therefore try using the `assume` command:

```
[14]: var('n')
      assume(n>0)
      sum( 1/n, n, 1, oo )
```

We get pretty similar error messages again, the last of which is again "`ValueError: Sum is divergent.`" Let's try something analogous to what worked for the corresponding integral.

```
[4]: var('n')
     var('k')
     assume(k>0)
     limit( sum( 1/n, n, 1, k ), k=oo )
```

```
[4]: limit(harmonic_number(k), k, +Infinity)
```

This gives us a symbolic description of what we are trying to compute, but not an answer. We finally try a desperate last resort: computing the sum $\displaystyle\sum_{n=1}^{10^k} \frac{1}{n}$ for $k = 0, 1, \ldots 5$ to see where the limit is tending:

* You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission via Blackboard fails, please submit your work to your instructor by email or on paper.

[13]:
```
for k in range(0,6):
    show( N( sum( 1/n, n, 1, 10^k ) ) )
```

```
1.00000000000000

2.92896825396825

5.18737751763962

7.48547086055035

9.78760603604438

12.0901461298634
```

Each time we add up $\frac{1}{n}$ for n from 1 to the next power of 10, we increase the sum by about 2 or so. Since there are infinitely many powers of 10, the full sum ought to add 2 to itself infinitely often, thus summing to infinity. We can tentatively – that is, not completely confidently – conclude that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$. □

c. This a slight modification of part of the solution to question **2** on Assignment #3:

[15]:
```
integral( 1/x^2, x, 1, oo )
```

[15]: 1

□

d. The sum corresponding to the integral in **c** is also something SageMath can handle:

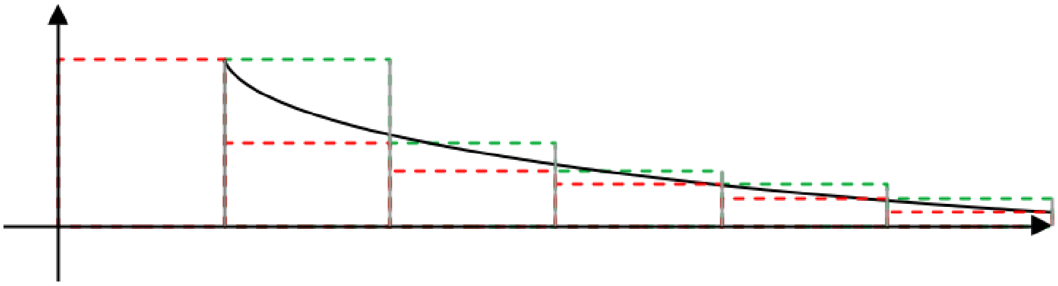
[16]:
```
var('n')
sum( 1/n^2, n, 1, oo )
```

[16]: 1/6*pi^2

■

2. Explain why the sum $\sum_{n=1}^{\infty} \frac{1}{n}$ is what it is because the integral $\int_1^{\infty} \frac{1}{x}\, dx$ is what it is. *[2.5]*

Hint: A picture may be worth 10^3 words ... Pay attention to the green (upper) dashed lines from 1 onward.



Solution. In this problem we are dealing with $y = \frac{1}{x}$ and its integer values. Consider what is happening on the interval $[n, n+1]$ for an integer $n \geq 1$. The green (upper) dashed line is the top of a rectangle of height $\frac{1}{n}$ and width 1, and hence with area $\frac{1}{n} \cdot 1 = \frac{1}{n}$. This rectangle contains the area between $\frac{1}{x}$ and the x-axis for $n \leq x \leq n+1$, so $\int_n^{n+1} \frac{1}{x}\,dx \leq \frac{1}{n}$. It follows that

$$\infty = \int_1^{\infty} \frac{1}{x}\,dx = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x}\,dx \leq \sum_{n=1}^{\infty} \frac{1}{n},$$

which is only possible if $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ too. ■

3. Explain why the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has a finite value because the integral $\int_1^{\infty} \frac{1}{x^2}\,dx$ has a finite value. *[2.5]*

Hint: Look at the picture above again. This time, pay attention to the red (lower, and between 0 and 1) dashed lines, especially from 1 onwards.

Solution. In this problem we are dealing with $y = \frac{1}{x^2}$ and its integer values. Consider what is happening on the interval $[n, n+1]$ for an integer $n \geq 1$. The red (lower) dashed line is the top of a rectangle of height $\frac{1}{(n+1)^2}$ and width 1, and hence with area $\frac{1}{(n+1)^2} \cdot 1 = \frac{1}{(n+1)^2}$. This rectangle is contained in the area between $\frac{1}{x^2}$ and the x-axis for $n \leq x \leq n+1$, so $\frac{1}{(n+1)^2} \leq \int_n^{n+1} \frac{1}{x^2}\,dx$. Since $\frac{1}{(n+1)^2}$ is positive for every $n \geq 1$, and given the answer to **1c**, it follows that

$$0 < \sum_{k=2}^{\infty} \frac{1}{k^2} = \sum_{n=1}^{\infty} \frac{(n+1)^2}{\leq} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^2}\,dx = \int_1^{\infty} \frac{1}{x^2} = 1,$$

so it must be the case that $1 < \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq 1 + 1 = 2$. ■

4. Explain why the limit $\lim_{k \to \infty} \left[\left(\sum_{n=1}^{k} \frac{1}{n} \right) - \ln(k+1) \right]$ exists and is between 0 and 1. *[2.5]*

Hint: Look at the picture yet again. This time, pay attention to the both the green (upper) and the red (lower) dashed lines from 1 onwards.

Solution. In this problem we are again dealing with $y = \frac{1}{x}$ and its integer values. Consider what is happening on the interval $[n, n+1]$ for an integer $n \geq 1$. The green (upper) dashed line is the top of a rectangle of height $\frac{1}{n}$ and width 1, and hence with area $\frac{1}{n} \cdot 1 = \frac{1}{n}$, while the red (lower) dashed line is the top of a rectangle of height $\frac{1}{n+1}$ and width 1, and hence with area $\frac{1}{n+1} \cdot 1 = \frac{1}{n+1}$. On the other hand, the area of the region between $\frac{1}{x}$ and the x-axis for $n \leq x \leq n+1$, given by $\int_n^{n+1} \frac{1}{x}\,dx$, is contained in larger rectangle and contains the smaller rectangle, so $\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n}$. It follows that

$$\begin{aligned}
\left(\sum_{n=1}^{k} \frac{1}{n}\right) - \ln(k+1) &= \left(\sum_{n=1}^{k} \frac{1}{n}\right) - \ln(k+1) = \left(\sum_{n=1}^{k} \frac{1}{n}\right) - \int_1^{k+1} \frac{1}{x}\,dx \\
&= \left(\sum_{n=1}^{k} \frac{1}{n}\right) - \left(\sum_{n=1}^{k} \int_n^{n+1} \frac{1}{x}\,dx\right) = \sum_{n=1}^{k} \left(\frac{1}{n} - \int_n^{n+1} \frac{1}{x}\,dx\right) \\
&\leq \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= 1 - \frac{1}{k+1}.
\end{aligned}$$

Note that $\left(\sum_{n=1}^{k} \frac{1}{n}\right) - \ln(k+1)$ must be positive because $\int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n}$ for each $n \geq 1$.

It follows from all of the above that

$$0 < \lim_{k\to\infty} \left[\left(\sum_{n=1}^{k} \frac{1}{n}\right) - \ln(k+1)\right] \leq \lim_{k\to\infty} \left[1 - \frac{1}{k+1}\right] = 1 - 0 = 1,$$

as desired. ■

5. Use SageMath to (approximately) evaluate $\lim_{k\to\infty} \left[\left(\sum_{n=1}^{k} \frac{1}{n}\right) - \ln(k+1)\right]$ as best you can. *[0.5]*

Solution. Trying to use SageMath to evaluate this limit exactly runs into the same sort of problems that getting it to handle the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ has. For example:

```
[17]: var('n')
      var('k')
      assume(k>0)
      limit( sum( 1/n, n, 1, k ) - log(k+1), k=oo )

[17]: limit(harmonic_number(k) - log(k + 1), k, +Infinity)
```

The only thing to do, sadly, is to emulate what we did in that case and see what happens when we plug in large values:

```
[19]: for k in range(0,6):
          show( N( sum( 1/n, n, 1, 10^k ) - log(10^k + 1) ) )

      0.306852819440055
```

```
0.531072981169883

0.572257000798361

0.576716081235125

0.577165669067865

0.577210664943198
```

It seems that $\lim_{k \to \infty} \left[\left(\sum_{n=1}^{k} \frac{1}{n} \right) - \ln(k+1) \right]$ is likely to be a number a little over 0.577, which fits with the conclusion of the previous problem. ■

NOTE: It is unknown whether the value of the limit in the last two questions, *usually denoted by* γ *and often called the Euler-Mascheroni constant,* is rational or irrational. If you can prove it one way or the other before the end of the term, your instructor will be *very* generous with your mark. *This constant turns up in various odd places in mathematics, including applied mathematics.*