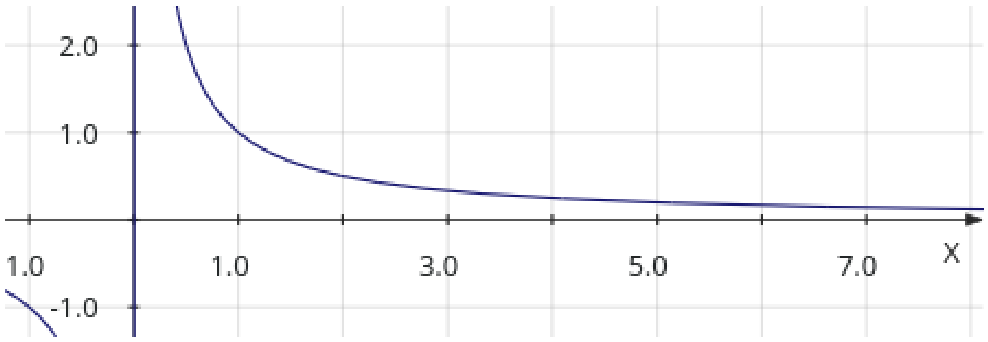
Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2024

Solutions to Assignment #3
Area versus Volume
Due just before midnight on Friday, 2 February.*

Consider the region below the curve $y = \frac{1}{x}$ and above the x-axis for $1 \le x < \infty$, a piece of which you can colour in below.



1. Compute the area of the given region, both by hand and using SageMath. *[4]*

Note. You'll probably set up an integral of the form $\int_c^\infty f(x)\,dx$ where c is a constant. This kind of "improper integral" should be computed using a limit: $\int_c^\infty f(x)\,dx = \lim_{t\to\infty} \int_c^t f(x)\,dx$. That is, work out the definite integral first, then take the limit. For more about such integrals, which will encounter in class later on, you can check out §9.8 in the textbook, or the lectures on this topic from past iterations of 1120H on the archive page at: `http://euclid.trentu.ca/math/sb/calculus/`

Solution. The area of the region below the curve $y = \frac{1}{x}$ and above the x-axis for $1 \le x < \infty$ should be – and is! – given by the integral $\int_1^\infty \frac{1}{x}\,dx$, which we compute as suggested in the note.

$$\begin{aligned}\textit{By hand: }\ \text{Area} &= \int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(x)\big|_1^t \\ &= \lim_{t\to\infty} \left[\ln(t) - \ln(1)\right] = \lim_{t\to\infty} \left[\ln(t) - 0\right] \\ &= \lim_{t\to\infty} \ln(t) = \infty\end{aligned}$$

With SageMath:

```
[3]: var('t')
     assume(t>1)
     limit( integral( 1/x, x, 1, t), t=oo )
```

```
[3]: +Infinity
```

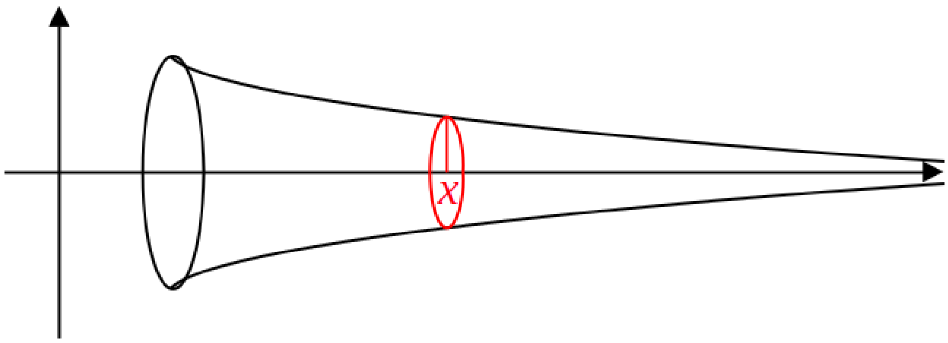
* You should submit your solutions via Blackboard's Assignments module, preferably as a single pdf. If submission via Blackboard fails, please submit your work to your instructor by email or on paper.

Note the use of the **assume** command. Without giving a SageMath a little help as to which direction you're integrating in, you will get error messages, just as in Assignment #2. Just trying `integral( 1/x, x, 1, oo )` doesn't work either, though the error messages correctly identify the integral as being "divergent".

It follows that the region has infinite area. This ought not be too much of a surprise because the region is infinite in the positive x direction. $\square$

2. Compute the volume of the solid obtained by revolving the given region about the x-axis, both by hand and using SageMath. *[4]*

SOLUTION. Here is a sketch of the solid:



The cross-section at x of this solid of revolution is a disk with radius $r = \frac{1}{x}$ and hence area $A(x) = \pi r^2 = \pi \left(\frac{1}{x}\right)^2 = \frac{\pi}{x^2}$. We compute the volume accordingly, once again using the method suggested in the note.

$$\textit{By hand:}\ \text{Volume} = \int_1^\infty A(x)\,dx = \int_1^\infty \frac{\pi}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\pi}{x^2}\,dx = \lim_{t\to\infty} \left.\frac{-\pi}{x}\right|_1^t$$
$$= \lim_{t\to\infty} \left[\frac{-\pi}{t} - \frac{\pi}{1}\right] = \lim_{t\to\infty} \left[\frac{-\pi}{t} + \pi\right] = -0 + \pi = \pi$$

With SageMath: We evaluate the integral in the same way as the one in question **1**, with the previous use of `assume(t>1)` still active:

```
[4]: limit( integral( pi/x^2, x, 1, t ), t=oo )
```

```
[4]: pi
```

In this case, though, having SageMath evaluate the integral directly works properly, since it is not divergent:

```
[5]: integral( pi/x^2, x, 1, oo )
```

```
[5]: pi
```

Thus the volume of this "infinite trumpet" is simply π. $\square$

3. There is something a little paradoxical about the (correct :-) answers to **1** and **2**. What is the paradox? Explain what's going on as best you can. *[2]*

SOLUTION. Hmm - we revolved a region with infinite area to get a solid with a finite volume of π, which seem a at least a little counterintuitive ...

The trick here is that the cross-sectional area of the solid, $A(x) = \frac{\pi}{x^2}$, shrinks faster as x increases than the corresponding cross-sectional length of the region, $y = y - 0 = \frac{1}{x}$, does. This makes it possible for the integral of the cross-sectional area that computes the volume to "add up" to much less than the integral of the cross-sectional length that computes the area of the region. We'll be seeing more examples of this kind of behavious later in the course, especially when we get to studying series. ■