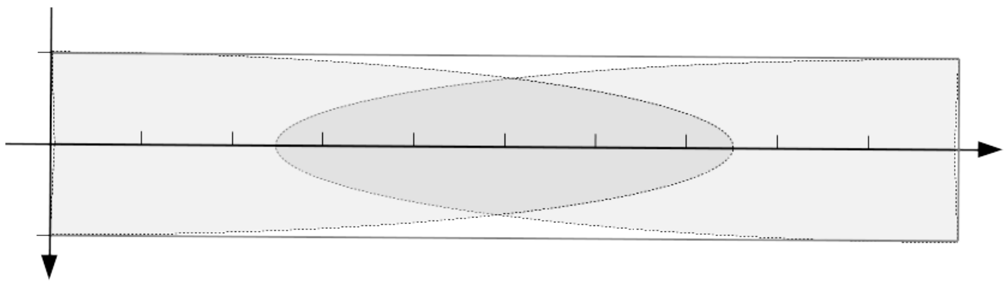
Mathematics 1120H – Calculus II: Integrals and Series
TRENT UNIVERSITY, Winter 2022

Solutions to Assignment #3
Definitely, let's integrate!

Please show all your work. As with all the assignments in this course, unless stated otherwise on the assignment, you are permitted to work together and look things up, so long as you acknowledge the sources you used and the people you worked with.

Consider the parabolas $y = a\left(1 - x^2\right)$ and $y = a\left(x^2 - 1\right) + 10$, where a is a constant such that $0 \leq a \leq 10$, as in the picture below. (Rotate the page by a right angle counterclockwise to put the axes in the picture in a more familiar orientation. :-)



1. Find the area between the x-axis and the graph of $y = a\left(1 - x^2\right)$

- **a.** by hand, *[2]*
- **b.** and again by using `SageMath`. *[2]*

Hint: To do part **b**, try the `integral` operator. (See §1.12 in *Sage for Undergraduates*.) Of course, you can find harder ways to do it ... :-)

SOLUTIONS. **a.** If $a = 0$, then the area between the graph of $y = a\left(1 - x^2\right)$ and the x-axis is 0, no matter what interval in x you pick.

If $a > 0$, then the parabola $y = a\left(1 - x^2\right)$ intersects the x-axis, otherwise known as the line $y = 0$, exactly when $1 - x^2 = 0$, *i.e.* when $x = \pm 1$. It follows that the finite region between the graph of $y = a\left(1 - x^2\right)$ and the x-axis is:

$$\int_{-1}^{1} \left[a\left(1 - x^2\right) - 0\right]\, dx = a\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = a\left(1 - \frac{1^3}{3}\right) - a\left((-1) - \frac{(-1)^3}{3}\right)$$
$$= a\left(\frac{2}{3}\right) - a\left(-\frac{2}{3}\right) = \frac{4a}{3} \qquad \square$$

b. This is just like part **a**, except that we hand off evaluating the integral to `SageMath`:

```
sage:  var("a")
a
sage:  integral(a*(1-x^2),x,-1,1)
4/3*a
```

`SageMath` agrees with the answer we obtained by hand in part **a**. Note that `a` was declared to be a variable so that `SageMath` would treat it as an unknown, especially when answering question **2** later in the same session. ■

2. Find the value of the constant a that would make the region which is both below the parabola $y = a\left(1 - x^2\right)$ and above the parabola $y = a\left(x^2 - 1\right) + 10$ have area equal to 5. *[6]*

Hint: This can be done by hand, though it will take a little while, but there may be some useful tools described in §1.8–1.9 of *Sage for Undergraduates*, at least when combined with the `integral` operator, if you would rather use `SageMath`.

Solution. We will use `SageMath`, except where it fails us. Suppose that $a > 0$.

First, the given parabolas intersect when they have the same y-values at the same x-values, *i.e.* when $a\left(1 - x^2\right) = a\left(x^2 - 1\right) + 10$. Rather than put in real work, we have `SageMath` solve the equation for x in terms of a. Conitnuing the session begun in solving question **1b** above:

```
sage:  solve( a*(1-x^2) == a*(x^2-1)+10, x)
[x == -sqrt(-5/a + 1), x == sqrt(-5/a + 1)]
```

That is, the parabolas intersect when $x = \pm\sqrt{-\frac{5}{a} + 1} = \pm\sqrt{1 - \frac{5}{a}}$.

Second, we set up the integral for the area of the region below the parabola $y = a\left(1 - x^2\right)$ and above the parabola $y = a\left(x^2 - 1\right) + 10$,

$$\int_{-\sqrt{1-\frac{5}{a}}}^{\sqrt{1-\frac{5}{a}}} \left[a\left(1 - x^2\right) - \left(a\left(x^2 - 1\right) + 10\right)\right]\, dx\,,$$

and have `SageMath` solve it in terms of a. To avoid having to do too much typing or even copy-and-pasting, we define an auxiliary variable to help deal with the limits of integration.

```
sage:  var("b")
b
sage:  b = sqrt(-5/a + 1)
sage:  integral( a*(1-x^2) - (a*(x^2-1)+10), x,-b,b)
-4/3*(((a - 5)/a)^(3/2) - 3*sqrt((a - 5)/a))*a - 20*sqrt((a - 5)/a)
```

That is, the area of the finite region between the two parabolas is

$$-\frac{4}{3}\left(\left(\frac{a-5}{a}\right)^{3/2} - 3\sqrt{\frac{a-5}{a}}\right)a - 20\sqrt{\frac{a-5}{a}}\,.$$

Third, we want to find out the value of a that makes the area equal to 5, *i.e.* we want to solve the equation

$$-\frac{4}{3}\left(\left(\frac{a-5}{a}\right)^{3/2} - 3\sqrt{\frac{a-5}{a}}\right)a - 20\sqrt{\frac{a-5}{a}} = 5\,.$$

Unfortunately, at this point `SageMath` gives us an answer which is useless:

```
sage:  solve( -4/3*(((a - 5)/a)^(3/2) - 3*sqrt((a - 5)/a))*a -
....:  20*sqrt((a - 5)/a) == 5, a)
```

```
[a == -15/4*(4*sqrt((a - 5)/a) + 1)/((((a - 5)/a)^(3/2) - 3*sq
rt((a - 5)/a))]
```

[Note how `SageMath` handles input and output lines that are too long to fit in the terminal window.] We need a number, not an expression for a in terms of itself.

Fourth, we take a detour and work by hand to manipulate the equation we want to solve by hand until we have it in a form – a polynomial equation in a – that the `solve` command will almost certainly be able to handle.

$$
\begin{aligned}
& -\frac{4}{3}\left(\left(\frac{a-5}{a}\right)^{3/2}-3\sqrt{\frac{a-5}{a}}\right)a-20\sqrt{\frac{a-5}{a}}=5 \\
\implies\quad & -\frac{4a}{3}\left(\left(\frac{a-5}{a}\right)\cdot\sqrt{\frac{a-5}{a}}-3\sqrt{\frac{a-5}{a}}\right)-20\sqrt{\frac{a-5}{a}}=5 \\
\implies\quad & -\frac{4a}{3}\left(\frac{a-5}{a}-3\right)\sqrt{\frac{a-5}{a}}-20\sqrt{\frac{a-5}{a}}=5 \\
\implies\quad & \left(-\frac{4}{3}a+\frac{20}{3}+4a-20\right)\sqrt{\frac{a-5}{a}}=5 \quad\implies\quad \left(\frac{8}{3}a-\frac{40}{3}\right)\sqrt{\frac{a-5}{a}}=5 \\
\implies\quad & \frac{8}{3}(a-5)\sqrt{\frac{a-5}{a}}=5 \quad\implies\quad \frac{64}{9}(a-5)^2\frac{a-5}{a}=25 \quad\implies\quad 64(a-5)^3=225a
\end{aligned}
$$

This is probably close enough, so let's see what `SageMath` does here:

```
sage:  solve(64*(a-5)^3 == 225*a, a)
[a == -5/16*3^(1/3)*(sqrt(141) + 12)^(1/3)*(I*sqrt(3) + 1) - 5
/16*3^(2/3)*(-I*sqrt(3) + 1)/(sqrt(141) + 12)^(1/3) + 5, a ==
-5/16*3^(1/3)*(sqrt(141) + 12)^(1/3)*(-I*sqrt(3) + 1) - 5/16*3
^(2/3)*(I*sqrt(3) + 1)/(sqrt(141) + 12)^(1/3) + 5, a == 5/8*3^
(1/3)*(sqrt(141) + 12)^(1/3) + 5/8*3^(2/3)/(sqrt(141) + 12)^(1
/3) + 5]
```

We get three answers. The first two are complex numbers – the `I` that appears in them is `Sage`-speak for the "imaginary" number $i=\sqrt{-1}$ – so they are not possible answers for a real area. The last answer, however, is a real number, though not a nice-looking one:

$$a=\frac{5}{8}\cdot 3^{1/3}\cdot\left(\sqrt{141}+12\right)^{1/3}+\frac{\frac{5}{8}\cdot 3^{2/3}}{\left(\sqrt{141}+12\right)^{1/3}}+5$$

This is what a needs to be for the region between the given parabolas, $y=a\left(1-x^2\right)$ and $y=a\left(x^2-1\right)+10$, to have area equal to 5.

Finally, for those would prefer a decimal approximation to the modestly horrendous exact expression we got for a, we use `SageMath` one more time:

```
sage:  N(5/8*3^(1/3)*(sqrt(141) + 12)^(1/3) + 5/8*3^(2/3)/(sqr
....:  t(141) + 12)^(1/3) + 5)
8.04705134272099
```

Thus the a we need is a little bit bigger than 8. And that's likely more than enough! :-) ■

Question: What happens in question **2** if $a = 0$?

Answer: Left to the readers who are not yet mathed out. :-)

Note: If we had fully multiplied out the polynomial equation in a and moved everything to the left-hand side, we would have gotten the polynomial $64a^3 - 960a^2 + 4575a - 8000 = 0$, for which `SageMath` would happily give us the same possible solutions for a. Note that, in general, a cubic polynomial will have either three real roots (of which two or all three could be equal to one another), or one real root and two complex roots.