Mathematics 1120H – Calculus II: Integrals and Series
Trent University, Winter 2022

Solutions to Assignment #10
A Power Series For $\mathbf{e^x}$
Due on Friday, 1 April.

Please show all your work. As with all the assignments in this course, unless stated otherwise on the assignment, you are permitted to work together and look things up, so long as you acknowledge the sources you used and the people you worked with.

Note. *A large part of the solutions below were presented in the first lecture on power series, on Friday, 26 March. Never let it be said that attendance, virtual or otherwise, has no rewards. :-)*

In what follows, let $f(x) = e^x$ and let $g(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

1. Determine for which values of x the series defining $g(x)$ converges conditionally, converges absolutely, or diverges. *[4]*

Solution. We will apply the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$$
$$= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} \begin{array}{l} \to |x| \\ \to \infty \end{array} = 0$$

Since the limit is 0 no matter what the value of x is, and $0 < 1$, it follows by the Ratio Test that the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely for all x. ■

In question **2**, you may assume that one can differentiate power series term-by-term for those values of x for which the power series converges.

2. Show that both $y = f(x)$ and $y = g(x)$ satisfy the differential equation $\dfrac{dy}{dx} = y$. *[2]*

Solution. For $y = f(x) = e^x$: $\dfrac{dy}{dx} = \dfrac{d}{dx}e^x = e^x = y$. For $y = g(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$:

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!}\right] = \frac{d}{dx}\left[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^k}{k!} + \cdots\right]$$
$$= \frac{d}{dx}[1] + \frac{d}{dx}[x] + \frac{d}{dx}\left[\frac{x^2}{2}\right] + \frac{d}{dx}\left[\frac{x^3}{6}\right] + \frac{d}{dx}\left[\frac{x^4}{24}\right] + \cdots + \frac{d}{dx}\left[\frac{x^k}{k!}\right] + \cdots$$
$$= 0 + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^{k-1}}{(k-1)!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = y$$

Thus $y = f(x) = e^x$ and $y = g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ both satisfy $\frac{dy}{dx} = y$. ■

3. Use your answer to **2** to help conclude that $f(x) = g(x)$. *[2]*

SOLUTION. We know from **2** that both $y = f(x) = e^x$ and $y = g(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ satisfy the differential equation $\frac{dy}{dx} = y$. In addition to this, the two functions are equal at $x = 0$:

$$f(0) = e^0 = 1 = 1 + 0 + \frac{0^2}{2} + \frac{0^3}{6} + \cdots + \frac{0^k}{k!} + \cdots = \sum_{n=0} \frac{0^n}{n!} = g(0)$$

Since the two functions satisfy the same differential equations with the same initial condition, they must be equal.† This is basically because they have the same slopes starting at the same point. ■

For question **4**, you may assume that, for all $x \in \mathbb{R}$, $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ and $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$. In addition to this you may assume that i is the square root of -1, *i.e.* $i^2 = -1$.

4. Use what you showed in answering question **3**, plus the information above, to prove *Euler's Formula*: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. *[2]*

NOTE. Plugging in $\theta = \pi$ into Euler's Formula gives the equation $e^{i\pi} = -1$, which is also sometimes called Euler's Formula.

SOLUTION. Off we go to do some algebra, some algebra, some algebra ... Note that because $i^2 = -1$, we have $i^{2k} = (-1)^k$ and $i^{2k+1} = (-1)^k i$.

$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = 1 + i\theta + \frac{i^2\theta^2}{2} + \frac{i^3\theta^3}{6} + \cdots + \frac{i^{2k}\theta^{2k}}{(2k)!} + \frac{i^{2k+1}\theta^{2k+1}}{(2k+1)!} + \cdots \\
&= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{6} + \cdots + (-1)^k \frac{\theta^{2k}}{(2k)!} + (-1)^k i \frac{\theta^{2k+1}}{(2k+1)!} + \cdots \\
&= \left[1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \cdots + (-1)^k \frac{\theta^{2k}}{(2k)!} + \cdots\right] \\
&\quad + i\left[\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots + (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} + \cdots\right] \\
&= \left[\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!}\right] + i\left[\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!}\right] = \cos(\theta) + i\sin(\theta) \quad ■
\end{aligned}$$

† This is, more or less, the uniqueness part of the Existence and Uniqueness Theorem for solutions to a (system of) differential equation(s).