Lecture 22

Apr. 1st, 2022

Remainder Term for a Taylor Series:

$f(x)$ at a $= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is the Taylor Polynomial at a of $f(x)$ of degree n.

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The n^{th} remainder term in this situation is $R_n(x) = f(x) - T_n(x)$.

Fact: In this situation, $R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}$ for some t between a and x. (strictly between)

ex/ $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

$* \; a=0$.

$= \lim_{n\to\infty} T_n(x)$ for all x

ie) $\lim_{n\to\infty} [\underbrace{e^x - T_n(x)}_{R_n(x)}] = 0$ so it's enough to say...

$\lim_{n\to\infty} R_n(x) = 0$

and $\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{\frac{d^{n+1}}{dx^{n+1}} e^x \big|_{x=t}}{(n+1)!}(x-0)^{n+1} = \lim_{n\to\infty} \frac{e^t}{(n+1)!} x^{n+1}$

so if $x>0$, $0<t<x$ and if $x<0$, $0>t>x$.

$0 < \left|\frac{e^t}{(n+1)!} x^{n+1}\right| < \left|\frac{e^{|x|}}{(n+1)!} x^{n+1}\right|$ since e^x is an increasing function.

so $\lim_{n\to\infty} \frac{e^t}{(n+1)!} x^{n+1} = 0 \Rightarrow \lim_{n\to\infty} R_n(x) = 0$

We can use this technology to show that e is irrational.

Suppose $e = \frac{a}{b}$ for some positive integers a and b.
ie) assume e is rational

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} = T_n(1) + R_n(1)$$

$$= \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}\right) + \frac{e^t}{(n+1)!} \cdot (1)^{n+1}$$

But $e^t < e^1 < 3$ (as e^x is increasing)

$$\Rightarrow \quad 0 < R_n(1) < \frac{3}{(n+1)!}$$

Pick $n > 3b$ then $e = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}\right) + \frac{e^t}{(n+1)!} = \frac{a}{b}$

$$0 < \frac{a}{b} - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}\right) < R_n(1) < \frac{3}{(n+1)!}$$

multiply everything by $n!$

$$\Rightarrow 0 < \underbrace{\frac{n!a}{b}}_{\text{integer}} \underbrace{- n! - \frac{n!}{1!} - \frac{n!}{2!} - \dots - \frac{n!}{n!}}_{\text{integer}} < n! R_n(1) < \frac{3n!}{(n+1)!} = \frac{3}{n+1}$$

$n \geq 3b \geq 3 \Rightarrow n+1 \geq 4$ and $\frac{3}{n+1} \leq \frac{3}{4} < 1$

So this means that $\frac{n!a}{b} - n! - \frac{n!}{1!} - \frac{n!}{2!} - \dots - \frac{n!}{n!}$ is an integer between 0 and 1, but there are no integers between 0 and 1, thus the assumption that $e = \frac{a}{b}$ is wrong