Lecture 21

Mar 29th, 2022

Recall: A power series is a series of the form $\sum_{n=0}^{\infty} a_n x^n$ (or $\sum_{n=0}^{\infty} a_n (x-c)^n$.)

Such a series has radius of convergence $0 \leq R \leq \infty$.
If $R=0$, the series converges only at $x=0$ (or $x=c$), otherwise, it converges absolutely for $|x|<R$ (or $|x-c|<R$) and diverges for any $|x|>R$ (or $|x-c|>R$)

* At $x=\pm R$, the series may converge or diverge.

Within the radius of convergence, you can differentiate and integrate term-by-term (R will not change).

If $\sum_{n=0}^{\infty} a_n (x-c)^n$ is a power series and we think of it as a function $f(x)$, then for each $n \geq 0$, $a_n = \frac{f^{(n)}(c)}{n!}$.

Note: $f^{(n)}(x) \equiv n^{th}$ derivative of f.

Taylor's formula:

Given $f(x)$, if it can be expanded as a power series around $x=c$, the series is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Warning: the series might not converge to $f(x)$, except at $x=c$ (but examples of this are rare).

ex/ Find Taylor Series of $f(x) = \cos(x)$ at $\emptyset$

n	$f^{(n)}(x)$	$f^{n}(0)$
0	$\cos(x)$	1
1	$-\sin(x)$	0
2	$-\cos(x)$	-1
3	$\sin(x)$	0
4	$\cos(x)$	1
$\vdots$	$\vdots$	$\vdots$

so $f^{(n)}(0) = \begin{cases} (-1)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

So the series is:

$$\frac{1}{0!}x^0 + \frac{-1}{2!}x^2 + \frac{1}{4!}x^4 + \frac{-1}{6!}x^6 + \frac{1}{8!}x^8 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{where } R = ? \text{ (use ratio test)}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!}}{\frac{(-1)^n x^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{(-1)^n x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)x^2}{(2n+2)(2n+1)} \right| = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} \quad \left[\frac{x^2}{\infty}\right] = 0.$$

By the ratio test, the series converges for all x as $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$ for all x.
ie. $R = \infty$.

If L is the limit coming from the ratio test, then you get R by solving for when $L < 1$.

Without doing all of this again, how do we get a power series for sin(x)?

$$\frac{d}{dx}\cos(x) = -\sin(x) \Rightarrow \sin(x) = -\frac{d}{dx}\cos(x)$$

$$\Rightarrow \sin(x) = -\frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$= (-1)\sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{(-1)^n x^{2n}}{(2n)!}\right)$$

$$= (-1)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\cdot\frac{d}{dx}x^{2n}$$

$$= (-1)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\cdot 2nx^{2n-1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}2nx^{2n-1}}{(2n)!}$$

*when $n=0$, the term is 0, so series should start at $n=1$.

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^{k+2}x^{2k+1}}{(2k+1)!}$$

Let $n = k+1$.

$$= \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

(usual series at 0 for sin(x)).

ex/ Taylor Series for $f(x) = x$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	x	0
1	1	1
2	0	0
3	0	0
$\vdots$	$\vdots$	$\vdots$

so the series is $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{0}{2!}x^2 + \frac{0}{3!}x^3 + \dots$

$= x$

General Fact: if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ is a polynomial, its Taylor Series at 0 is itself.

More generally, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $\sum_{n=0}^{\infty} a_n x^n$ is the Taylor series of $f(x)$.

ex/ $f(x) = \arctan(x)$

n	$f^n(x)$	$f^n(0)$
0	$\arctan(x)$	0
1	$\frac{1}{1+x^2}$	1
2	$\frac{-2x}{(1+x^2)^2}$	0
3	$\frac{-2(1+x^2)^{\cancel{2}}+2x(2\cancel{(1+x^2)}(2x))}{(1+x^2)^{\cancel{4}3}}$ $= \frac{-2(1+x^2)+8x^2}{(1+x^2)^3}$ $= \frac{6x^2-2}{(1+x^2)^3}$	-2
$\vdots$	$\vdots$	$\vdots$

* derivatives get more and more difficult

Recall: $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$ which is the sum of a geometric series with $a=1$ and $r=-x^2$

so $\frac{d}{dx}\arctan(x) = \frac{1}{1-(-x)^2} = \sum_{n=0}^{\infty}(-x^2)^n = 1 - x^2 + x^4 - x^6 + \dots$

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + \dots)\,dx$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

What is C? $\arctan(0) = C + 0 - \frac{0}{3} + \frac{0}{5} - \frac{0}{7} + \dots \Rightarrow 0 = C$

$\therefore \arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ at 0.

What are the radius and interval of convergence of $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$?

If you differentiate / integrate a power series term-by-term, the radius of convergence DOES NOT change. Convergence may change at the endpoints of the interval of convergence.

$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$ is a geometric series with $|r| = |-x^2| = |x^2| < 1$ so it converges exactly when $|x| < 1$. ie. $R = 1$.
The interval of convergence is $(-1, 1)$.

It follows that $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ has the same radius of convergence $R = 1$, but the interval may be different.

At $x = -1$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{2n+1} = -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \ldots$
which converges by the Alternating Series Test.
(converges conditionally as $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ diverges by the P-test).

A $x = 1$, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n (1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$
which converges by the Alternating Series Test.
(again conditionally, for same reason as above).

Note: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \arctan(1) = \frac{\pi}{4}$.

$\therefore$ the interval of convergence is $[-1, 1]$ for the series equal to $\arctan(x)$.